Malliavin Calculus for the generalized PAM equation

Paul Gassiat

CEREMADE, Université Paris Dauphine

Nice, November 24, 2015

Joint work with G. Cannizzaro and P. Friz (TU Berlin)

We consider the stochastic PDE

$$\begin{cases} u(0) = u_0 \\ (\partial_t - \Delta)u = f(u) \cdot \xi \quad \text{on } (0,\infty) \times \mathbb{T}^2, \end{cases} \tag{gPAM}$$

where $f \in C^\infty$, $u_0 \in L^\infty$, and $\xi = \xi(x)$ is spatial white noise on the torus.

In this talk I will present results on the Malliavin differentiability of u, and as an application prove that the value at a fixed point $u(t,x)$ admits a density wrt the Lebesgue measure.

(gPAM) is a singular SPDE

$$(\partial_t - \Delta)u = f(u) \cdot \xi \text{ on } (0,\infty) \times \mathbb{T}^2, \quad u(0) = u_0 \qquad \text{(gPAM)}$$

Basic problem when trying to solve this equation :

ξ has Hölder regularity $\alpha = -1 - \varepsilon$. By properties of $(\partial_t - \Delta)$, u (hence $f(u)$) will have regularity (at best) $\alpha + 2$.

For the product $f(u) \cdot \xi$ to make sense, one would need $\alpha + (\alpha + 2) > 0$, which is not true, hence the equation is ill-posed.

Nevertheless, this problem has been solved by Hairer and Gubinelli-Imkeller-Perkowski, giving a good notion of solution for this PDE. (And many other singular SPDEs, such as (KPZ), (Φ^4_3), ...)

Why consider gPAM ?

- The linear case ($f(u) = u$) is the classical Parabolic Anderson Model

$$(\partial_t - \Delta)u = u \cdot \xi.$$

  It has been extensively studied when either the state space is discrete ($\mathbb{Z}^2$), or the noise is smooth. The case of white noise can then appear as limit of such models.
- It is the simplest example of PDE which can (and should) be solved by the theory of regularity structures. (Simple here means fewer nonlinear terms to give sense to.) It is therefore natural to start there...

Malliavin calculus and rough paths

Previous works using Malliavin calculus with T.Lyons' rough path theory in the case of **ordinary** (stochastic/rough) differential equations. Extend the usual results for SDEs

$$dY_t = \sum_i V^i(Y_t) dX_t$$

to a large class of Gaussian driving signals (fractional Brownian motion,...)

- Cass-Friz (2010) : Existence of a density under Hörmander's Lie bracket condition,
- Cass-Hairer-Litterer-Tindel (2015) : Smoothness of densities.

One should then also be able to combine pathwise techniques for **partial** differential equations (i.e. regularity structures or paracontrolled distributions) with Malliavin calculus.

Outline

1. Preliminaries
   - Solution theory for gPAM (regularity structures)
   - Malliavin calculus

2. Main results
   - Malliavin differentiability : the extended regularity structure
   - Application : density for value at fixed point

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We want to solve the equation

$$(\partial_t - \Delta)u = f(u) \cdot \xi \text{ on } (0,\infty) \times \mathbb{T}^2, \quad u(0) = u_0 \qquad \text{(gPAM)}$$

where $\xi = \xi(x)$ is spatial white noise.
Problem : what is meant by the product $f(u) \cdot \xi$?

Theorem (Hairer, Gubinelli-Imkeller-Perkowski (2013))

Assume $f \in C^4$. Let ξ_ε be smooth approximations of ξ. Then there exists constants C_ε such that if u_ε solves

$$(\partial_t - \Delta)u_\varepsilon = f(u_\varepsilon) \cdot \xi_\varepsilon - C_\varepsilon f(u_\varepsilon) f'(u_\varepsilon), \quad u_\varepsilon(0) = u_0$$

$\mathbb{P}$-a.s. there exists a (random) time $T > 0$ such that $u^\varepsilon \to_{\varepsilon\to 0} u$ on $[0,T) \times \mathbb{T}^2$. In addition, the limit u does not depend on the choice of approximations ξ_ε.

General idea

We first rewrite

$$(\partial_t - \Delta)u = f(u) \cdot \xi \text{ on } (0,\infty) \times \mathbb{T}^2, \quad u(0) = u_0$$

in integral form as

$$u = K * (f(u) \cdot \xi) + \mathcal{G}u_0 \text{ on } (0,\infty) \times \mathbb{T}^2,$$

where K is the heat kernel, and $\mathcal{G}u_0(t,\cdot) = K_t * u_0$.
Now the idea of the theory of regularity structures (applied to (gPAM)) can be summarized as follows:

1. We make the ansatz that locally, u admits a Taylor-like expansion (of order $1+\varepsilon$) in function of usual polynomials 1, x_i, and of $K * \xi$,
2. In order to make sense of $f(u) \cdot \xi$, it then suffices (at least locally) to make sense of $(K * \xi) \cdot \xi$.

Basic ingredients of the theory

- The **regularity structure** T. Vector space generated by symbols ($\sim$ abstract monomials) :

$$\text{usual monomials } 1, X_i, \ldots$$

$$\text{additional symbols } : \Xi, \mathcal{I}\Xi, \Xi \cdot \mathcal{I}\Xi, \Xi \cdot X_i, \ldots$$

T is equipped with a grading

$$|1| = 0, |X| = 1, \ldots, |\Xi| = \alpha, |\Xi \cdot \mathcal{I}\Xi| = 2 + 2\alpha, \ldots$$

$(\alpha = -1 - \kappa)$.

Basic ingredients of the theory

- The **model** : $\Pi \in \mathcal{M}$.

$$\Pi : (x, \tau) \in \mathbb{R}_+ \times \mathbb{T}^2 \times T \mapsto \Pi_x \tau \in \mathcal{S}'$$

Gives a concrete meaning to symbols

$$\Pi_x 1 = 1, (\Pi_x X)(y) = (y - x), ...$$

$$\Pi \Xi \leftrightarrow \xi, \quad \Pi(\Xi \cdot \mathcal{I}\Xi) \leftrightarrow (K * \xi) \cdot \xi,$$

must satisfy analytic conditions, namely if τ is a symbol,

$$\Pi_x(\tau) \text{ of "order" } |\tau| \text{ at } x,$$

and some algebraic conditions, such as

$$\Pi_x \Xi = \Pi_y \Xi (\equiv \xi)$$

$$\Pi_x(\Xi \cdot \mathcal{I}\Xi) - \Pi_y(\Xi \cdot \mathcal{I}\Xi) = (K * \xi(x) - K * \xi(y))\xi, \ldots$$

$\mathcal{M}$ is a complete (nonlinear) metric space.

Basic ingredients of the theory

Recall

$$f \in C^\gamma \Leftrightarrow f(x) = f(y) + Df(y)\cdot(x-y) + \ldots + \frac{D^{\lfloor\gamma\rfloor} f(y)}{\lfloor\gamma\rfloor!}(y-x)^{\otimes\lfloor\gamma\rfloor} + O(|x-y|^\gamma).$$

- **Modelled distributions** : $U \in \mathcal{D}^\gamma = \mathcal{D}^\gamma(\Pi)$.
  Functions : $(\mathbb{R}_+ \times \mathbb{T}^2) \to T$ satisfying some Hölder-type conditions.
  For instance (writing $U(x) = \sum_{\tau\in\mathcal{F}} U_\tau(x)\tau$) :

$$\begin{aligned} U_1(x) =& U_1(x) + U_{X_i}(y)(x_i - y_i) + \ldots \\ &+ U_{\mathcal{I}\Xi}(y)(K * \xi(x) - K * \xi(y)) + \ldots + O(|x-y|^\gamma) \end{aligned}$$

$$\begin{aligned} U_\Xi(x) =& U_\Xi(y) + ((K * \xi)(x) - (K * \xi)(y)) U_{\Xi\cdot\mathcal{I}\Xi}(y) \\ &+ \ldots + O(|x-y|^{\gamma-\alpha}). \end{aligned}$$

$\mathcal{D}^\gamma$ is a Banach space.

Solving the equation

We can then solve the equation in two steps :

1. **Probabilistic step** : The models Π^ε given by

$$\Pi^\varepsilon \Xi = \xi_\varepsilon, \quad \Pi^\varepsilon(\Xi \cdot \mathcal{I}\Xi) = (K * \xi_\varepsilon)\xi_\varepsilon - C_\varepsilon$$

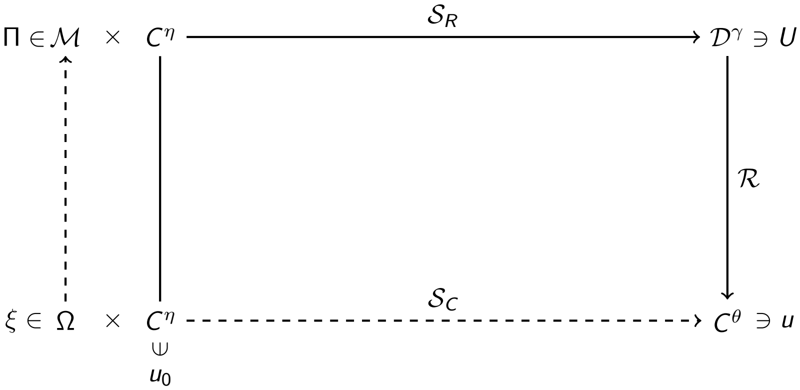
   converge to a model Π as $\varepsilon \to 0$, $\mathbb{P}$-almost surely.

2. **Analytic step** Given a model Π, we solve for $U \in \mathcal{D}^\gamma$

$$U = \mathcal{K}(F(U) \cdot \Xi) + \mathcal{G}u_0,$$

   and the map $(\Pi, u_0) \mapsto U$ is continuous.

Summary


$\Pi \in \mathcal{M} \quad \times \quad C^\eta$
$\mathcal{S}_R$
$\mathcal{D}^\gamma \ni U$
$\mathcal{R}$
$\xi \in \Omega \quad \times \quad C^\eta$
$\mathcal{S}_C$
$C^\theta \ni u$
$\Cup$
u_0

Malliavin calculus

$\mathbb{P}$ a Gaussian measure on Ω, with Cameron-Martin space $\mathcal{H}$ ($\subset \Omega$).

(In our case : we can take $\mathbb{P}$ as a measure on $\Omega = C^{\alpha}(\mathbb{T}^2)$, and $\mathcal{H} = L^2(\mathbb{T}^2)$.)

We then say that a r.v. $F : \Omega \to \mathbb{R}^N$ is in $\mathcal{C}^1_{\mathcal{H}-loc}$ on Ω_0 if for $\mathbb{P}$-a.e. $\omega \in \Omega_0$,

$$h \mapsto F(\omega + h) \text{ is Frechet-differentiable in a neighbourhood of } 0.$$

We then call $DF(\omega)$ ($\in \mathcal{H}$) the derivative at 0.

Theorem (Bouleau-Hirsch criterion)

Assume that F is in $\mathcal{C}^1_{\mathcal{H}-loc}$, and that

$$\mathbb{P} - a.e.\, \omega, \text{ the map } h \in \mathcal{H} \mapsto \langle DF(\omega), h \rangle \in \mathbb{R}^N \text{ is surjective.}$$

Then F admits a density w.r.t. the Lebesgue measure (on $\mathbb{R}^N$).

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The result

Theorem (Cannizzaro-Friz-G.)

Let u be the solution to (gPAM), with explosion time T_∞. Fix $(t,x) \in (0,\infty) \times \mathbb{T}^2$. Then $F = u(t,x)$ is $\mathcal{C}^1_{\mathcal{H}-loc}$ on $\{t < T_\infty\}$, with derivative given by

$$\langle DF, h \rangle = v^h(t,x), \ \textit{where } v^h = \lim_{\varepsilon} v^h_\varepsilon,$$

$$(\partial_t - \Delta) v^h_\varepsilon = f(u_\varepsilon) h_\varepsilon + v^h_\varepsilon \left(f'(u_\varepsilon)\xi_\varepsilon - C_\varepsilon (ff'' + (f')^2)(u_\varepsilon) \right), \quad v^h_\varepsilon(0) = 0.$$

(Recall that $u = \lim_\varepsilon u_\varepsilon$, where $(\partial_t - \Delta) u_\varepsilon = f(u_\varepsilon) \cdot \xi_\varepsilon - C_\varepsilon f(u_\varepsilon) f'(u_\varepsilon)$.)

Extended regularity structure

Idea of the argument:

Given a noise $\xi (= \omega)$ and $h \in \mathcal{H}$, we want to make sense at the same time of (gPAM), and of

$$(\partial_t - \Delta) u^h = f(u^h)(\xi + h),$$

$$(\partial_t - \Delta) v^h = f(u)h + v^h f'(u)\xi.$$

In order to do so we expand our regularity structure : T^H $(\supset T)$ now contains all symbols where instances of Ξ may be replaced by H, i.e.

$$\Xi, H, \Xi \cdot \mathcal{I}H, H \cdot \mathcal{I}\Xi, H \cdot \mathcal{I}H, \ldots$$

Extended model

Proposition

Given a model Π on T and $h \in \mathcal{H}$, there exists a unique model Π^h on T^H such that :

$$\Pi^h = \Pi \text{ on } T, \;\; \Pi^h(H) = h, \;\; \Pi^h(H\mathcal{I}\Xi) = h \cdot \Pi(\mathcal{I}\Xi), \ldots$$

Idea of proof :
Comes from the fact that multiplication is well-defined on $C^\beta \times H^\gamma$ (resp. Hölder and Sobolev spaces), provided $\beta + \gamma > 0$, with suitable Hölder-type estimates, such as :

$$\xi \in C^\alpha, K * h \in H^2$$

$$\Rightarrow \xi \cdot (K * h - (K * h)(x)) \text{ of order } \alpha + 2 - \frac{d}{2} - \varepsilon(\geqslant 2\alpha + 2) \text{ at } x.$$

And then letting U^h, V be the solutions to

$$U^h = \mathcal{K}(F(U^h)\cdot(\Xi + H)) + Ku_0,$$

$$V^h = \mathcal{K}\left(F(U)\cdot H + V^h \cdot F'(U)\cdot \Xi\right),$$

(depending continuously on $(\Pi,\ h,\ u_0)$),
one proves that

$$\mathbb{P} - \text{a.e. } \xi, \forall h, U^h(\Pi(\xi)) = U(\Pi(\xi + h)),$$

and for $u = \mathcal{R}U$, $u^h = \mathcal{R}U^h$, $v^h = \mathcal{R}V^h$,

$$\left\| u^h - u - v^h \right\|_{C^\theta} = o\left(\|h\|_{\mathcal{H}}\right).$$

Application : density for value at fixed point

We obtain the following absolute continuity result :

Theorem (Cannizzaro-Friz-G.)

Assume $f \geqslant 0$, and $f(u_0)$ is not identically 0.
Then for each $(t, x) \in (0, \infty) \times \mathbb{T}^2$, the law of $u(t, x)$ conditionally on $\{t < T_\infty\}$ is absolutely continuous with respect to the Lebesgue measure.

Proof :
It is enough to show that $\mathbb{P}$-a.e., for some $h \in \mathcal{H}$,
$\langle D(u(t, x)), h\rangle = v^h(t, x) \neq 0$.
In fact : we show that if h is such that $f(u)h \geqslant 0$ and is not identically 0, then $v^h(t, x) > 0$.
One notes that $v^h(t, x) = \int_0^t w^s(t, x)ds$, with

$$w^s(s, x) = f(u(s, x))h(x), \quad (\partial_t - \Delta)w^s = w^s(f'(u)\xi) \text{ on } (s, T) \times \mathbb{T}^2.$$

We conclude with a strong maximum principle :

Proposition

Let w be the solution to a linear heat equation

$$(\partial_t - \Delta)w = w\widetilde{\xi}, \quad w(0,\cdot) = w_0$$

where $\widetilde{\xi}$ is such that the theory of regularity structures applies. Then

$$w_0 \geqslant 0,\ w_0 \text{ not identically } 0 \ \Rightarrow w(t,\cdot) > 0 \text{ for all } t > 0.$$

Proof follows an idea due to Mueller : write in integral form

$$w = K * (w\widetilde{\xi}) + Kw_0,$$

and using the estimates from the theory, the first term is negligible for small t. Then iterate...

Conclusion

- It is possible to combine the tools of regularity structures and Malliavin calculus.

- This is only a first result, much more left to do :
    - treat more general SPDEs (i.e. go beyond "level 2"),
    - density for N-dimensional marginals,
    - smoothness of densities,
    - ...