Modèles de neurones en champ-moyen avec interactions spatiales

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Travail en commun avec Wilhelm Stannat (TU Berlin)

[L., Stannat, 2014]

[L., Stannat, 2015, arXiv :1502.00532]

Dynamics of large population of interacting neurons

- Consider a population a N interacting neurons, each of them described by its action potential V_i, and possibly by others variables (eg. gating variables, etc.) $\rightarrow$ possibly very complex dynamics
- The graph of interaction is unknown, random and a priori very complex too. Common assumption : interaction on the complete graph $\leftrightarrow$ mean-field models

Two main types of models :

- Models of interacting diffusions (Hodgkin-Huxley, FitzHugh-Nagumo [Baladron, Fasoli, Faugeras, Touboul, '12, Faugeras, MacLaurin, '14, etc.]),
- Jump processes (integrate-and-fire [Delarue, Inglis, Rubenthaler, Tanré, '15, Inglis, Talay, '15.], Hawkes processes [Brémaud-Massoulié, '96, Reynaud-Bouret, Rivoirard, Grammont, Tuleau-Malot, '14., Chevallier, '15])

Mean-field diffusions within random environment

We consider the system of N stochastic differential equations

$$d\theta_{i,t} = c(\theta_{i,t}, \omega_i)\,dt + \frac{1}{N}\sum_{j=1}^{N}\Gamma(\theta_{i,t}, \omega_i, \theta_{j,t}, \omega_j)\,dt + dB_{i,t}, \quad i = 1, \cdots, N,$$

- $c(\cdot)$: local dynamics of the particle θ_i
- $\Gamma()$: interaction kernel between two particles,
- $\{B_i\}_i$: IID standard Brownian motions (thermal noise).
- $\{\omega_i\}_i$: IID random variables with law μ (local inhomogeneity of the particles, random environment), independent with the thermal noise.

Example 1 : The Kuramoto model

- $\theta_i \in \mathbb{S} := \mathbb{R}/2\pi$ (phase oscillators),
- $c(\theta, \omega) = \omega$,
- $\Gamma(\theta, \omega, \tilde{\theta}, \tilde{\omega}) = K \sin(\tilde{\theta} - \theta)$

$$\mathrm{d}\theta_{i,t} = \omega_i \, \mathrm{d}t + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_{j,t} - \theta_{i,t}) \, \mathrm{d}t + \sigma \, \mathrm{d}B_{i,t}, \quad i = 1, \cdots, N,$$

Remarks

- The real parameter of the model is K/σ^2. In the following we take $\sigma = 1$.
- The model is invariant by rotation : if $\{\theta_j(t)\}_{j=1\ldots N}$ is solution, so is $\{\theta_j(t) + \psi\}_{j=1\ldots N}$, for all $\psi \in \mathbb{S}$.
- When $\omega_i \equiv 0$, the process is reversible under the invariant measure $\pi_{N,K}$ (Hamiltonian Mean-Field model, HMF or XY model)

$$\pi_{N,K}(\mathrm{d}\theta) \propto \exp\left(\frac{K}{N} \sum_{i,j=1}^{N} \cos(\theta_i - \theta_j) \right) \mathrm{d}\theta$$

Example 2 : FitzHugh-Nagumo oscillators for neuronal dynamics

- Dynamics of one neuron $\theta = (V, W) \in \mathbf{R}^2$:

$$c(V, W) = \begin{pmatrix} \frac{1}{\varepsilon}\left(V - V^3/3 + W + I\right) \\ aW + bV, \end{pmatrix}$$

  V : membrane potential, W *recovery variable.* [✎ Berglund, Gentz - 2007, Berglund, Landon 2012]

- Random environment $\omega = (a, b)$: inhomogeneous discrimination between inhibited/excited neurons.
- Γ : synaptic coupling between neurons [✎ Baladron, Fasoli, Faugeras, Touboul - 2012]

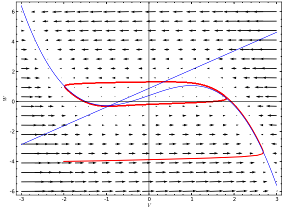
Main features

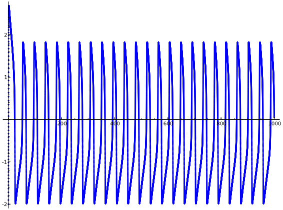
- The dynamics c is not globally-Lipschitz
- Dynamics with polynomial bounds
- But c is monotone :

$$\left\langle c(\theta, \omega) - c(\tilde{\theta}, \omega), \theta - \tilde{\theta} \right\rangle \leqslant -C \left\| \theta - \tilde{\theta} \right\|^2.$$

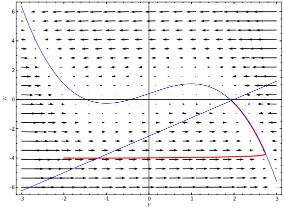
Depending on the values of the local disorder $\omega = (a, b)$, one obtains two different behaviors

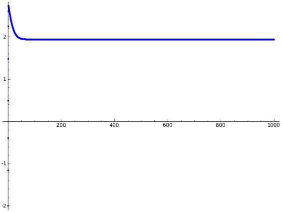
- Excitation :





- Inhibition :





Propagation of chaos and large population behavior

All the statistical information is contained in the disordered empirical measure of the system

$$\nu_{N,t} := \frac{1}{N}\sum_{j=1}^{N} \delta_{(\theta_{j,t},\omega_j)},\ t \in [0,T]$$

The system can be rewritten as

$$d\theta_{i,t} = c(\theta_{i,t},\omega_i)\,dt + \int \Gamma(\theta_{i,t},\omega_i,\theta,\omega)\nu_{N,t}(d\theta,\,d\omega)\,dt + dB_{i,t}, \quad i = 1,\cdots,N.$$

Taking formally the limit as $N \to \infty$ (i.e. suppose that $\theta_i \to \bar{\theta}$ and $\nu_N \to \nu$), one obtains (propagation of chaos) the nonlinear process[✎ Sznitman '91] :

$$d\bar{\theta}_t = c(\bar{\theta}_t,\omega)\,dt + \int \Gamma(\bar{\theta}_t,\omega,\tilde{\theta},\tilde{\omega})\nu_t(d\tilde{\theta},\,d\tilde{\omega})\,dt + dB_t.$$

with $\nu_t =$ law of $(\bar{\theta}_t,\omega)$.
Writing $\nu_t(d\theta,\,d\omega) = q_t(\theta,\omega)\,d\theta\mu(d\omega)$, it is the weak solution of the McKean-Vlasov equation

$$\partial_t q_t(\theta,\omega) = \frac{1}{2}\Delta q_t(\theta,\omega) - \mathrm{div}_\theta\left(q_t(\theta,\omega)\left(c(\theta,\omega) + \int \Gamma(\theta,\omega,\tilde{\theta},\tilde{\omega})q_t(\tilde{\theta},\tilde{\omega})\,d\tilde{\theta}\mu(d\tilde{\omega})\right)\right).$$

Known results

1. Quenched and averaged law of large numbers [✎ Gärtner, Oelschläger '87, Dai Pra, den Hollander '96, L. '11],
2. Quenched and averaged fluctuations of ν_N around its mean-field limit : [✎ Fernandez, Méléard '97], [L. '11]
3. Averaged large deviations as $N \to \infty$: [✎ Dai Pra, den Hollander '96]
4. Linear and nonlinear stability results of the Kuramoto model [✎ Bertini Giacomin Pakdaman '10], [Giacomin, Pakdaman, Pellegrin, Poquet, '12], [Giacomin, Pakdaman, Pellegrin, '12], [Giacomin, L. Poquet, '14]
5. Long-time dynamics for the Kuramoto model and effect of quenched disorder [✎ Bertini, Giacomin, Poquet, '13], [L. Poquet, '15]

Generalization to spatially-extended mean-field systems

Motivation : understand the behavior as $N \to \infty$ of similar systems where :

- the interaction no longer follows the complete graph (random graphs), or
- the interaction is still mean-field but the strength of interaction is not uniform along the population

Intuition

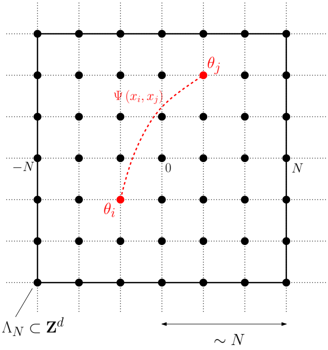
If each particle has enough connections, one should keep the same mean-field properties (law of large numbers, fluctuations, large deviations).

Mean-field diffusions with spatial interaction

Each particle θ_i has a fixed position x_i in $\mathbb{R}^d$,

$$x_i := \frac{i}{2N},\ i \in \Lambda_N,$$
$$\Lambda_N := \{-N, \dots, N\}^d.$$

The interaction between θ_i and θ_j depends on a spatial kernel $\Psi(x_i, x_j)$.



$$d\theta_{i,t} = c(\theta_{i,t}, \omega_i)\,dt + \frac{1}{|\Lambda_N|} \sum_{\substack{j \in \Lambda_N \\ j \neq i}} \Gamma\left(\theta_{i,t}, \omega_i, \theta_{j,t}, \omega_j\right) \Psi(x_i, x_j)\,dt + dB_{i,t},$$

with a (possibly irregular) spatial kernel $\Psi(\cdot, \cdot)$.

Remark

It is possible to impose periodic boundary conditions on Λ_N.

Existing models

1. Kuramoto-type oscillators with phase-lag and space : $\Gamma(\theta,\tilde{\theta}) = \sin(\tilde{\theta} - \theta + \alpha)$ with regular kernels [✎ Kuramoto, Battogtokh, '02]

$$\Psi(x,y) \propto \exp\left(-\kappa|x-y|\right),$$
$$\Psi(x,y) \propto 1 + A\cos(x-y).$$

[✎ Abrams, Strogatz - Int. Journ. Bifurc. Chaos, '06] or general kernels [✎ O. Omel'chenko - Nonlin., '13], [M. Wolfrum, O. Omel'chenko, S. Yanchuk, Y. Maistrenko, Chaos, '11], [O. Omel'chenko, M. Wolfrum, C. Laing, Chaos '14], etc.

2. P-nearest neighbor model : each particle interacts only with a fixed-proportion RN (with $R \in]0,1]$) of its nearest neighbors :

$$\Psi(x,y) := \frac{1}{R^d}\mathbf{1}_{[-R,R]^d}(|x-y|).$$

[✎ I. Omelchenko, B. Riemenschneider, P. Hövel, Y. Maistrenko, E. Schöll, Phys. Rev E, '12] (Rössler system), [✎ A. Vüllings, J. Hizanidis, I. Omelchenko, P. Hövel, '14] (FitzHughNagumo oscillators), and many others. . .

3. Kuramoto oscillators with polynomial decay (α-Hamiltonian Mean-Field model, α-HMF)

$$\Psi(x,y) := \frac{1}{|x-y|^{\alpha}}$$

[✎ Gupta, Potters, Ruffo - Phys. Rev. E '12], [Gupta, Campa, Ruffo - '12], [Gupta, Campa, Ruffo - Kuramoto model of synchronization : equilibrium and nonequilibrium aspects, '14]

Large population behavior of spatially-extended systems

We focus here on the rigorous derivation of the large population behavior for spatially extended models, with special attention given to the singular spatial kernels, namely the P-nearest neighbor model and the model with polynomial decay.

1. Law of large numbers for the empirical measure and propagation of chaos ?
2. Well-posedness of the McKean-Vlasov equation ?
3. Fluctuations around its limit ?

The empirical measure and the McKean-Vlasov equation

Look at

$$d\theta_{i,t} = c(\theta_{i,t}, \omega_i)\,dt + \frac{1}{|\Lambda_N|} \sum_{\substack{j \in \Lambda_N \\ j \neq i}} \Gamma\left(\theta_{i,t}, \omega_i, \theta_{j,t}, \omega_j\right) \Psi(x_i, x_j)\,dt + dB_{i,t}.$$

The empirical measure of the system is given by

$$\nu_{N,t} = \frac{1}{|\Lambda_N|} \sum_{i \in \Lambda_N} \delta_{(\theta_{i,t}, \omega_i, x_i)}.$$

Its natural limit is given by $\nu_t(\,d\theta,\, d\omega,\, dx) = q_t(\theta, \omega, x)\,d\theta \mu(\,d\omega)\,dx$ where q solves

$$\partial_t q_t = \frac{1}{2}\Delta_\theta q_t - \mathrm{div}_\theta \left(q_t \left\{ c(\cdot) + \int \Gamma(\cdot, \bar\theta, \bar\omega)\Psi(\cdot, \bar x) q_t(\bar\theta, \bar\omega, \bar x)\, d\bar\theta\, d\mu(\bar\omega)\, d\bar x \right\} \right)$$

Problem : is (the weak formulation of) this equation well-posed (c unbounded, Ψ possibly singular) ?

Law of large numbers in P-nearest neighbor case : $\Psi(x,y) = \frac{1}{R^d}\mathbf{1}_{|x-y|\leqslant R}$

Theorem (L., Stannat - 2014)

For all $d \geqslant 1$, $R \in]0,1]$, under some regularity assumptions on c and Γ and some moments hypothesis on the initial condition and the disorder,

- *The weak formulation of the McKean-Vlasov equation with space is well-posed,*
- *The empirical measure ν_N converges in law as $N \to \infty$, in $\mathcal{C}([0,T], \mathcal{M}_1(\mathbb{R}^m \times \mathbb{R}^n \times [-1/2, 1/2]^d))$ to the unique solution ν to the McKean-Vlasov equation.*

Moreover, there exists a constant C (depending only on T, R, c and Γ) such that

$$\sup_{0 \leqslant t \leqslant T} d(\nu_{N,t}, \nu_t) \leqslant \frac{C}{N^{1\wedge\frac{d}{2}}},$$

for some distance d of Wasserstein type.

Remarks

- The same estimates holds for any regular spatial weight Ψ,
- In small dimension ($1 \leqslant d \leqslant 2$), we obtain the Gaussian scaling $N^{\frac{d}{2}}$,
- In higher dimension ($d \geqslant 3$), the speed of convergence is governed by the spatial constraints.

Law of large numbers in the polynomial case : $\Psi(x,y) = \frac{1}{|x-y|^{\alpha}}$, $(\alpha < d)$

Theorem (L., Stannat - 2014)

For all $d \geqslant 1$, $\alpha < d$, under some regularity assumptions on c and Γ and some moments hypothesis on the initial condition and the disorder, one has a similar result concerning the well-posedness of the McKean-Vlasov equation and the convergence of the empirical measure.
Moreover, one has the following estimate : for all $\alpha \leqslant \gamma < \frac{d}{2}$,

$$\sup_{0 \leqslant t \leqslant T} d(\nu_{N,t}, \nu_t) \leqslant C \begin{cases} \frac{1}{N^{\gamma}}, & \text{if } \alpha \in \left[0, \frac{d}{2}\right), \\ \frac{\ln N}{N^{\frac{d}{2}}}, & \text{if } \alpha = \frac{d}{2}, \\ \frac{\ln N}{N^{d-\alpha}}, & \text{if } \alpha \in \left(\frac{d}{2}, d\right). \end{cases}$$

for some distance d of Wasserstein type.

Remark

For small spatial constraints ($\alpha < \frac{d}{2}$), we retrieve (approximately) the Gaussian scaling, whereas for strong spatial contraints, a nontrivial scaling appears.

Question : Are the scaling found the correct ones ?

Idea of proof

Introduce the nonlinear process associated to the particle system

$$\mathrm{d}\theta_t = c(\theta_t, \omega)\,\mathrm{d}t + \int \Gamma(\theta_t, \omega, \overline{\theta}, \overline{\omega})\Psi(x,y)\nu_t(\mathrm{d}\overline{\theta}, \mathrm{d}\overline{\omega}, \mathrm{d}y) + \mathrm{d}B_t.$$

and define its propagator

$$P_{s,t}f(\theta, \omega, x) := \mathbf{E}_{\theta_s=\theta}(f(\theta_t, \omega, x)).$$

$P_{s,t}$ satisfies a Backward Kolmogorov equation which enables to get rid of the second derivative :

$$\left\langle f\,,\nu_{N,t} - \nu_T \right\rangle = \left\langle P_{0,t}f\,,\nu_{N,0} - \nu_0 \right\rangle + \frac{1}{|\Lambda_N|}\sum_k \int_0^t \partial_\theta (P_{s,t}f)(\theta_{k,s}, \omega_k, x_k)\,\mathrm{d}B_{k,s}$$
$$+ \frac{1}{|\Lambda_N|}\sum_k \int_0^t \partial_\theta (P_{s,t}f)(\theta_{k,s}, \omega_k, x_k)\left\langle \Gamma(\theta_{k,s}, \omega_k, \cdot)\Psi(x_k, \cdot)\,, \nu_{N,s} - \nu_s \right\rangle \mathrm{d}s.$$

Key point : find a space $\mathcal{C}$ of regular (Lipschitz) functions that is stable under $P_{s,t}$ + Gronwall Lemma.

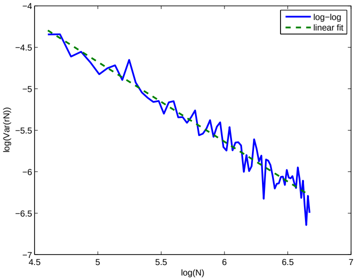
Fluctuations of the order parameter in the Kuramoto model

The correct order parameter in the Kuramoto model is

$$r_{N,t} := \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_{j,t}} \to_{N\to\infty} r_t := \int e^{i\theta} q_t(\mathrm{d}\theta, \mathrm{d}\omega).$$

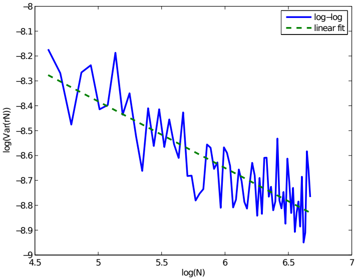
Log-log computation of $\left(\mathbb{E}(r_{N,t}-r_t)^2\right)^{1/2}$ w.r.t. N :

$\alpha = 0$



Result : decay as ≈ 0.481

$\alpha = 0.9$



Result : decay as ≈ 0.134

Fluctuations around the limit I

Suppose for simplicity $d = 1$. Looking at the fluctuations of the empirical measure ν_N around its limit ν is to look at the (distribution-valued) process

$$\eta_N = a_N(\nu_N - \nu).$$

For which renormalization a_N ?
One wants to compute the speed of convergence of the particle system

$$\mathrm{d}\theta_{i,t} = c(\theta_{i,t}, \omega_i)\,\mathrm{d}t + \frac{1}{|\Lambda_N|} \sum_{\substack{j\in\Lambda_N \\ j\neq i}} \Gamma\left(\theta_{i,t}, \omega_i, \theta_{j,t}, \omega_j\right) \Psi(x_i, x_j)\,\mathrm{d}t + \mathrm{d}B_{i,t}$$

to the nonlinear process $\bar{\theta}_t = \bar{\theta}_t(\omega, x)$

$$\mathrm{d}\bar{\theta}_t = c(\bar{\theta}_t, \omega)\,\mathrm{d}t + \int \Gamma\left(\bar{\theta}_t, \omega, \tilde{\theta}, \tilde{\omega}\right) \Psi(x, \tilde{x}) \nu_t(\,\mathrm{d}\tilde{\theta},\, \mathrm{d}\tilde{\omega},\, \mathrm{d}\tilde{x})\,\mathrm{d}t + \mathrm{d}B_t,$$

Fluctuations around the limit II

One has two scales of convergence :

- The convergence of the empirical distribution of the empirical measure of the Brownian motion and the initial condition : typical scale is $\sqrt{N}$
- The convergence with respect to the space variable :

$$\frac{1}{|\Lambda_N|} \sum_{k \in \Lambda_N} \frac{1}{\left|\frac{k}{2N}\right|^{\alpha}} \to_{N \to \infty} \int_{-1/2}^{1/2} \frac{1}{|x|^{\alpha}} \, \mathrm{d}x.$$

It is related to the speed of convergence of the Riemann sum related to $x \mapsto \frac{1}{|x|^{\alpha}}$ towards the corresponding integral : typical scale is $N^{1-\alpha}$.

Phase transition ($d = 1$)

One should take the smallest renormalization in all cases

- For $0 \leqslant \alpha < \frac{1}{2}$, the proper renormalization is $a_N := \sqrt{N}$,
- For $\frac{1}{2} < \alpha < 1$, the proper renormalization is $a_N := N^{1-\alpha}$.

Intuition

- When $\alpha < \frac{1}{2}$, the spatial constraints are too weak, the randomness should prevail and everything should happen like we were in the full mean-field case ($\alpha = 0$) : the limit of η_N should be random (Gaussian),
- The case $\alpha > \frac{1}{2}$ corresponds to a strong spatial attenuation. What we see at first order should not be the Gaussian fluctuations but the constraints due to the spatial kernel : the limit of η_N should be deterministic.

Remark

This scaling was first noticed by [Firpo, Ruffo - J Phys A, 2001] in the case of the α-Hamiltonian Mean-Field model.

Fluctuation process

Question : is it the right scaling ? Do we have an actual Central Limit Theorem ? Let us go back to $\eta_N = a_N(\nu_N - \nu)$. Strategy to study η_N :

1. Write a semi-martingale decomposition for η_N,
2. Prove tightness of the process η_N in a certain Sobolev space of regularity of type H^{-k},
3. Identify and prove uniqueness of the limit.

[✎ Fernandez, Méléard - SPA, 1997], [Meleard, Roelly - SPA, 1987], [Jourdain, Méléard - Ann. IHP 1998], [Oelschläger, 1987], [L. 2011]

Problem : here, if one looks at η_N only, we need to consider test functions $(\theta, \omega, x) \mapsto f(\theta, \omega, x)$ that are singular w.r.t. the space variable x (in order to capture singularities such as $x \mapsto \frac{1}{|x|^\alpha}$).
This does not work w.r.t the usual tightness criteria.

Two-particle fluctuation process

Idea : build an auxiliary fluctuation process that carries the singularity in space. This process takes into account the mutual fluctuations of two particles (θ_i, θ_j) instead of one.

$$\left\langle \mathcal{H}_{N,t}\,,\, g \right\rangle :=$$

$$a_N \left(\frac{1}{|\Lambda_N|^2} \sum_{i,j \in \Lambda_N} \Psi(x_i, x_j) g(\theta_{i,t}, \theta_{j,t}) - \frac{1}{|\Lambda_N|} \sum_{i \in \Lambda_N} \int \Psi(x_i, \tilde{x}) g(\theta_{i,t}, \tilde{\theta}) \, \mathrm{d}\nu_t \right)$$

Now, it is the process itself that carries the singularity in space, not the test functions.

Remark

This process gives the correct renormalization in space : if $g \equiv 1$, it boils down to

$$a_N \left(\frac{1}{|\Lambda_N|^2} \sum_{i,j \in \Lambda_N} \Psi(x_i, x_j) - \frac{1}{|\Lambda_N|} \sum_{i \in \Lambda_N} \int \Psi(x_i, \tilde{x}) \, \mathrm{d}\tilde{x} \right)$$

which is exactly of order 1, when $\alpha > \frac{1}{2}$.

Semi-martingale decomposition of $(\eta_N, \mathcal{H}_N)$

Proposition (L., Stannat - 2015)

One has the following decomposition :

$$\left\langle \eta_{N,t}\,,\,f \right\rangle = \left\langle \eta_{N,0}\,,\,f \right\rangle + \int_0^t \left\langle \eta_{N,s}\,,\,L_s^{(1)} f \right\rangle \mathrm{d}s + \int_0^t \left\langle \mathcal{H}_{N,s}\,,\,\Phi[f] \right\rangle \mathrm{d}s + \mathcal{M}_{N,t}^{(\eta)} f,$$

$$\left\langle \mathcal{H}_{N,t}\,,\,g \right\rangle = \left\langle \mathcal{H}_{N,0}\,,\,g \right\rangle + \int_0^t \left\langle \mathcal{H}_{N,s}\,,\,L_s^{(2)} g \right\rangle \mathrm{d}s + \int_0^t F_{N,s} g \,\mathrm{d}s + \mathcal{M}_{N,t}^{(\mathcal{H})} g,$$

where $L^{(1)}$ and $L^{(2)}$ are explicit linear operators, with smooth coefficients.

Sub-critical fluctuations ($\alpha < \frac{1}{2}$)

Theorem (L., Stannat - 2015)

Suppose $\alpha < \frac{1}{2}$. Under regularity assumptions c and Γ and moment conditions on the disorder and initial condition, there exist a Sobolev space $\mathbf{H}$ such that η_N converges as $N \to \infty$ in $\mathcal{C}([0,T],\mathbf{H})$ to the unique solution of the following linear stochastic partial differential equation

$$\eta_t = \eta_0 + \int_0^t L_s^{(1)*} \eta_s \, ds + \mathcal{M}_t^{(\eta)}, \quad t \in [0,T],$$

where θ_0 and $\mathcal{M}^{(\eta)}$ are independent Gaussian processes.

Remarks

- In the case $\alpha < \frac{1}{2}$, the process $\mathcal{H}_N$ vanishes as $N \to \infty$.
- This generalizes the result of [Fernandez, Méléard - SPA, 1997], [L. 2011] where the case $\alpha = 0$ was considered.
- This is an averaged result. A quenched result is also true (see [L. 2011]).

Supercritical fluctuations ($\alpha > \frac{1}{2}$)

Theorem (L. Stannat - 2015)

Suppose $\alpha > \frac{1}{2}$*. Under regularity assumptions on* c *and* Γ *and moment conditions on the disorder and initial condition, there exist Sobolev spaces* $(\mathbf{H}_1, \mathbf{H}_2)$ *such that the couple* $(\eta_N, \mathcal{H}_N)$ *converges in law in* $\mathcal{C}([0,T], \mathbf{H}_1 \oplus \mathbf{H}_2)$ *to the unique solution of the system of coupled PDEs*

$$\begin{cases} \eta_t = \int_0^t L_s^{(1),*} \eta_s \, ds + \int_0^t \Phi^* \mathcal{H}_s \, ds, \\ \mathcal{H}_t = \mathcal{H}_0 + \int_0^t L_s^{(2),*} \mathcal{H}_s \, ds, \end{cases} \quad t \in [0,T],$$

where $\mathcal{H}_0$ *is explicit.*

Remark

Unfortunately, our assumptions do not cover the FitzHugh-Nagumo case.

On the dynamical properties of the continuous Kuramoto model with space

Look at the stationary states of the Kuramoto equation with space

$$0=\frac{1}{2}\partial_\theta^2 q(\theta,x)-K\partial_\theta\left(q(\theta,x)\int\sin(\tilde{\theta}-\theta)\Psi(x,\bar{x})q(\tilde{\theta},\tilde{x})\,\mathrm{d}\tilde{\theta}\,\mathrm{d}\tilde{x}\right).$$

[✎ Gupta, Campa, Ruffo - 2014] : introduce magnetization parameters at position x

$$m_X(x):=\int\cos(\theta)q(\theta,x)\,\mathrm{d}\theta,\ m_Y(x):=\int\sin(\theta)q(\theta,x)\,\mathrm{d}\theta,$$

and their spatial transform

$$\hat{m}_X(x):=\int\Psi(x,y)m_X(y)\,\mathrm{d}y,\ \hat{m}_Y(x):=\int\Psi(x,y)m_Y(y)\,\mathrm{d}y.$$

Then, any stationary solution can be written as

$$q(\theta,x)=\frac{1}{Z(x)}\exp(e^{2K\cos(\theta)\hat{m}_X(x)+2K\sin(\theta)\hat{m}_Y(x)})$$

where the magnetization parameters satisfy the consistency relation

$$\sqrt{m_X(x)^2+m_Y(x)^2}=\frac{I_1}{I_0}\left(2K\sqrt{\hat{m}_X(x)^2+\hat{m}_Y(x)^2}\right).$$

Conclusion

- For $\alpha > \frac{1}{2}$, what is the next term in the asymptotic development of ν_N ?
- What if one chooses positions x randomly ? Positions depending on time ?
- What are the dynamical properties of the Kuramoto model with space ? Synchronisation ? Stability of synchronized profiles ? Chimera states ?
- What about random graphs ? Is there a mean-field limit for the empirical measure too ?

Merci de votre attention !

- E. Luçon and W. Stannat. *Mean field limit for disordered diffusions with singular interactions*. Ann. Appl. Probab., 24(5) :1946–1993, 2014.
- E. Luçon and W. Stannat. *Transition from Gaussian to non-Gaussian fluctuations for mean-field diffusions in spatial interaction*, arXiv : 1502.00532, 2015.