Scalar conservation laws and Isentropic Euler system with stochastic forcing

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Nice 2015

Plan of the talk

1. Isentropic Euler system with stochastic forcing and shallow water equations
2. Invariant measure: system and equation
3. Martingale solutions to the stochastic Isentropic Euler system
   - 3.a. Parabolic Approximation: uniqueness, a priori bounds
   - 3.b. Time splitting approximation
   - 3.c. From the parabolic to the hyperbolic system.

Isentropic Euler system with stochastic forcing

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis, let $\mathbb{T}$ be the one-dimensional torus, let $T > 0$ and set $Q_T := \mathbb{T} \times (0, T)$. We study the system

$$d\rho + (\rho u)_x dt = 0, \qquad \text{in } Q_T, \tag{1a}$$

$$d(\rho u) + (\rho u^2 + p(\rho))_x dt = \Phi(\rho, u) dW(t), \qquad \text{in } Q_T, \tag{1b}$$

$$\rho = \rho_0, \quad \rho u = \rho_0 u_0, \qquad \text{in } \mathbb{T} \times \{0\}, \tag{1c}$$

where p follows the γ-law

$$p(\rho) = \kappa \rho^\gamma, \quad \kappa = \frac{\theta^2}{\gamma}, \quad \theta = \frac{\gamma - 1}{2}, \tag{2}$$

for $\gamma > 1$, W is a cylindrical Wiener process and $\Phi(\rho = 0, u) = 0$.

References

- Deterministic equations, [Di Perna 83, Lions, Perthame, Tadmor 94 and Lions, Perthame, Souganidis 96] in particular.
- Scalar stochastic first-order equations, [Hofmanová 13] in particular.
- Stochastic compressible Navier Stokes: [Feireisl, Maslowski, Novotny 13, Breit Hofmanová 14, Breit Feireisl Hofmanová 15, Smith 15].
- Stochastic first-order systems of conservation laws: [Kim 2011, *On the stochastic quasi-linear symmetric hyperbolic system*], [Audusse, Boyaval, Goutal, Jodeau, Ung 2015, *Numerical simulation of the dynamics of sedimentary river beds with a stochastic Exner equation*]

Stochastic forcing term

Our hypotheses on the stochastic forcing term $\Phi(\rho,u)W(t)$ are the following ones. We assume that $W=\sum_{k\geq 1}\beta_k e_k$ where the β_k are independent brownian processes and $(e_k)_{k\geq 1}$ is a complete orthonormal system in a Hilbert space $\mathfrak{U}$. For each $\rho\geq 0, u\in\mathbb{R}$, $\Phi(\rho,u)\colon\mathfrak{U}\to L^2(\mathbb{T})$ is defined by

$$\Phi(\rho,u)e_k=\sigma_k(\cdot,\rho,u)=\rho\sigma_k^*(\cdot,\rho,u), \tag{3}$$

where $\sigma_k^*(\cdot,\rho,u)$ is a 1-periodic continuous function on $\mathbb{R}$. More precisely, we assume $\sigma_k^*\in C(\mathbb{T}_x\times\mathbb{R}_+\times\mathbb{R})$ and the bound

$$\mathbf{G}(x,\rho,u):=\left(\sum_{k\geq 1}|\sigma_k(x,\rho,u)|^2\right)^{1/2}\leq A_0\rho\left[1+u^2+\rho^{2\theta}\right]^{1/2}. \tag{4}$$

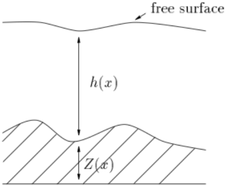
$x\in\mathbb{T}$, $\rho\geq 0$, $u\in\mathbb{R}$, where A_0 is some non-negative constant

Example: shallow water equations with stochastic topography

(Deterministic) evolution of shallow water flow described in terms of the height $h(x)$ of the water above x and the speed $u(x)$ of a column of water:

$$\begin{cases} h_t + q_x = 0, \\ q_t + \left(\dfrac{q^2}{h} + g\dfrac{h^2}{2}\right)_x + ghZ_x = 0. \end{cases}$$

Here $q := hu$ is the charge of the colum of water, g is the acceleration of the gravity and $x \mapsto Z(x)$ a parametrization of the graph of the bottom.



Example: shallow water equations with stochastic topography

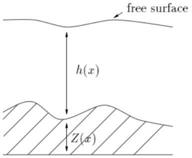
Stochastic evolution:

$$\begin{cases} h_t + q_x = 0, \\ dq + \left(\dfrac{q^2}{h} + g\dfrac{h^2}{2}\right)_x dt + ghdZ_x = 0, \end{cases}$$

with

$$Z(x,t) = \sum_k \sigma_k \left(\cos(2\pi kx)\beta_k^\flat(t) + \sin(2\pi kx)\beta_k^\sharp(t)\right).$$

$\rightarrow$ fluid dynamics forced *via* the motion of the ground



Energy evolution

Stochastic evolution:

$$\begin{cases} h_t + q_x = 0, \\ dq + \left(\dfrac{q^2}{h} + g\dfrac{h^2}{2}\right)_x dt + ghdZ_x = 0. \end{cases}$$

Energy:

$$e = \frac{q^2}{2h} + g\frac{h^2}{2}.$$

For smooth (C^1) solutions:

$$\frac{d}{dt}\mathbb{E}\int_{\mathbb{T}} e(x,t)dx = \frac{1}{2}\|\sigma\|^2_{l^2(\mathbb{N})}\mathbb{E}\int_{\mathbb{T}} h(x,t)dx = \mathrm{Cst}.$$

$\rightarrow$ Input of energy by noise VS Dissipation in shocks

Large-time behaviour and invariant measure

Example: shallow water equations with stochastic topography

Choose the initial condition (h_0, q_0) to

$$\left\{\begin{array}{l} h_t + q_x = 0, \\ dq + \left(\dfrac{q^2}{h} + g\dfrac{h^2}{2}\right)_x dt + ghdZ_x = 0, \end{array}\right.$$

at random according to a law μ_0 (this a probability measure on a space of functions $\mathbb{T} \to \mathbb{R}_+ \times \mathbb{R}$).

- Question 1: how to choose μ_0 such that, for all $t > 0$,

$$\mu_t := \mathrm{Law}(h_t, q_t) = \mu_0$$

  (*i.e.* μ_0 invariant measure) ?

- Question 2: what is μ_t for large t?

Deterministic case

Total height and total charge are the characteristic parameters of the invariant measure (a Dirac mass here).

Theorem [Chen Frid 99] Let $\rho, u \in L^\infty(\mathbb{R}_+ \times \mathbb{T})$ such that $\mathbf{U} = \begin{pmatrix} \rho \\ q \end{pmatrix}$ is a weak entropy solution to the isentropic Euler system. Then, for every $1 \leq p < +\infty$,

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T |\mathbf{U}(t) - \bar{\mathbf{U}}|^p dt = 0,$$

where $\bar{\mathbf{U}}$ is the constant state $\begin{pmatrix} \bar{\rho} \\ \bar{q} \end{pmatrix}$ characterized by

$$\bar{\rho} = \int_{\mathbb{T}} \rho_0(x) dx, \quad \bar{q} = \int_{\mathbb{T}} q_0(x) dx.$$

Stochastic shallow water equation

Integrate over $\mathbb{T} \times (0,t)$:

$$\begin{cases} \int_{\mathbb{T}} h(x,t)dx = \int_{\mathbb{T}} h(x,0)dx, \\ \int_{\mathbb{T}} q(x,t)dx = \int_{\mathbb{T}} q(x,0)dx - \int_0^t \int_{\mathbb{T}} ghdZ_x(x,t)dx, \end{cases}$$

which gives

$$\begin{cases} \int_{\mathbb{T}} h(x,t)dx = \int_{\mathbb{T}} h(x,0)dx, \\ \mathbb{E} \int_{\mathbb{T}} q(x,t)dx = \mathbb{E} \int_{\mathbb{T}} q(x,0)dx. \end{cases}$$

Numerical experiment

Observation of the scalar function

$$T \mapsto \frac{1}{T}\int_0^T \langle \mu_t, \varphi\rangle dt,$$

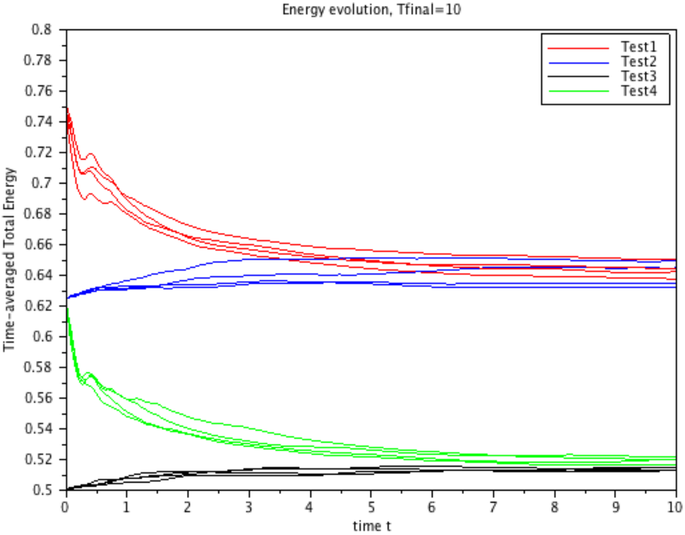
for

$$\varphi(h,q) = \text{energy} = \frac{q^2}{2h} + g\frac{h^2}{2}$$

and four sets of data, different realizations.

Numerical experiment

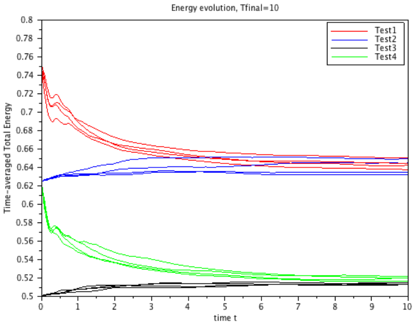
Test



Data

Test 1 and 2: $\int_0^1 h_0(x)dx = 1$, $\int_0^1 q_0(x)dx = \frac{1}{2}$.

Test 3 and 4: $\int_0^1 h_0(x)dx = 1$, $\int_0^1 q_0(x)dx = 0$.



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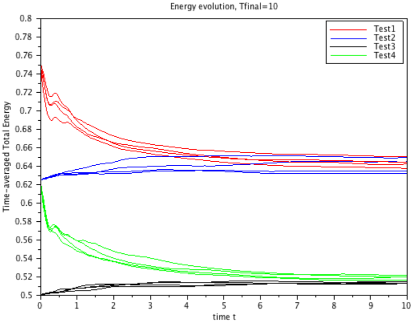

$\rightarrow$ Proof???

Large time behavior in scalar conservation laws with stochasting forcing

First-order scalar conservation law with stochastic forcing

Scalar equation in dimension N:

$$\partial_t u + \operatorname{div}_x(A(u)) = f(x,t), \quad f = \text{time white-noise}$$

First-order scalar conservation law with stochastic forcing

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Example: the periodic stochastic inviscid Burgers Equation

$$\partial_t u + \partial_x(u^2/2) = \mathrm{Re} \sum_k \sigma_k e^{2i\pi kx} \dot{B}_k(t),$$

where the $B_k(t)$'s are independent Brownian motions on $\mathbb{C} \simeq \mathbb{R}^2$.

First-order scalar conservation law with stochastic forcing

Resolution of the Cauchy Problem

- Dimension $N = 1$, Burgers Equation, additive noise, E, Khanin, Mazel, Sinai 2000,
- Dimension $N = 1$, additive noise, Kim 2003,
- Dimension $N = 1$, multiplicative noise, Feng-Nualart 2008,
- Dimension $N \geq 1$, additive noise, Vallet-Wittbold 2009,
- Dimension $N \geq 1$, multiplicative noise, Debussche-Vovelle 2010,
- Dimension $N \geq 1$, multiplicative noise, Chen-Ding-Karlsen 2012,
- Dimension $N = 1$, Burgers Equation, additive noise, Fractionnal Brownian Motion, Sausserau-Stoica 2012,
- Dimension $N \geq 1$, multiplicative noise, Bauzet-Vallet-Wittbold, 2012, 2014.

First-order scalar conservation law with stochastic forcing

Invariant Measure

- E, Khanin, Mazel, Sinai, Annals of maths. 2000,
- Sausserau-Stoica 2012,
- Boritchev 2013,
- Debussche-Vovelle, PTRF 2014, ($N \geq 1$, "general" fluxes).
- Dirr-Souganidis (Hamilton-Jacobi with stochastic forcing) 2005.
- Non-compact setting ($x \in \mathbb{R}$) : Bakhtin, Cator, Khanin 2014, Bakhtin 2014 (Poisson Noise)

First-order scalar conservation law with rough flux

Equation

$$du + \mathrm{div}_x(A(x,u) \circ dz) = 0, \quad z = (z_1, \ldots, z_d) \text{ rough path.}$$

- Lions, Perthame, Souganidis 2013, 2014,
- Gess Souganidis 2014,
- Hofmanová 2015 $du + \mathrm{div}_x(A(x,u) \circ dz) = g(x,u)dW(t)$.

Note also: Lions, Perthame, Souganidis 2013, averaging lemma for

$$\partial_t f(t,x,\xi) + \dot{B}(t) \circ \xi \nabla_x f(t,x,\xi) = g(t,x,\xi).$$

Large-time behavior

Stochastic scalar first-order conservation law

$$du(t) + \mathrm{div}(A(u(t)))dt = \Phi(x)dW(t),$$

under the condition (non-stationarity of $a := A'$)

$$\sup_{\alpha\in\mathbb{R},\beta\in\mathbb{S}^{N-1}} |\{\xi \in \mathbb{R}; |\alpha + \beta\cdot a(\xi)| < \varepsilon\}| \to 0$$

and the structure condition (this ensures the conservation of mass)

$$\Phi(x)dW(t) = \mathrm{div}_x(\cdot).$$

Existence of invariant measure for sub-cubic fluxes

For $m \in \mathbb{R}$, let $L^1_m := \left\{ u \in L^1(\mathbb{T}^N); \int_{\mathbb{T}^N} u(x)dx = m \right\}.$

Theorem (Debussche-Vovelle 14)

Under the condition (non-stationarity of $a := A'$)

$$\sup_{\alpha \in \mathbb{R}, \beta \in \mathbb{S}^{N-1}} |\{\xi \in \mathbb{R}; |\alpha + \beta \cdot a(\xi)| < \varepsilon\}| \leq \varepsilon^b,$$

and (sub-cubic flux)

$$|a'(\xi)| \leq C(1 + |\xi|),$$

there exists an invariant measure μ_m on L^1_m. It is supported in $L^p(\mathbb{T}^N)$ for $p < 2 + \frac{b}{2}$ if $N = 1$ and $p < \frac{N}{N-1}$ if $N > 1$.

Uniqueness for sub-quadratic fluxes

Theorem (Debussche-Vovelle, 14)

Assume (sub-quadratic flux)

$$|a'(\xi)| \leq C,$$

then the invariant measure is unique and ergodic on L^1_m.

Entropy solutions to the isentropic Euler system

Entropy

A couple $(\eta, H)\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ of continuous functions is an entropy - entropy flux pair if η is convex and

$$\nabla\eta(\mathbf{U})^* D\mathbf{F}(\mathbf{U}) = \nabla H(\mathbf{U})^*, \quad \forall \mathbf{U} \in \mathbb{R}_+^* \times \mathbb{R}.$$

Formally, by the Itō Formula, solutions to (1) satisfy

$$d\mathbb{E}\eta(\mathbf{U}) + \mathbb{E}H(\mathbf{U})_x dt = \frac{1}{2}\mathbb{E}\partial^2_{qq}\eta(\mathbf{U})\mathbf{G}^2(\mathbf{U})dt, \tag{5}$$

for all entropy - entropy flux pair (η, H)

Entropy for isentropic Euler

cf. Lions, Perthame, Tadmor 1994: let $g \in C^2(\mathbb{R})$ be a convex function, let

$$\eta(\mathbf{U}) = \int_{\mathbb{R}} g(\xi)\chi(\rho, \xi - u)d\xi,$$
$$H(\mathbf{U}) = \int_{\mathbb{R}} g(\xi)[\theta\xi + (1-\theta)u]\chi(\rho, \xi - u)d\xi,$$

where

$$\chi(\mathbf{U}) = c_\lambda(\rho^{2\theta} - u^2)_+^\lambda, \quad \lambda = \frac{3-\gamma}{2(\gamma-1)}, \quad c_\lambda = \left(\int_{-1}^1 (1-z^2)_+^\lambda \, dz\right)^{-1}.$$

Then (η, H) is an entropy - entropy flux pair.

Pathwise entropy solution

Definition Let $\rho_0, u_0 \in L^2(\mathbb{T})$ satisfy $\rho_0 \geq 0$ a.e. and

$$\int_{\mathbb{T}} \rho_0(1 + u_0^2 + \rho_0^{2\theta})dx < +\infty.$$

A predictable process $\mathbf{U} \in L^2(\Omega; C([0,T]; W^{-2,2}(\mathbb{T})))$ is said to be a (pathwise-) weak entropy solution to (1) if

$$\mathbb{E} \operatorname*{ess-sup}_{0\leq t\leq T} \int_{\mathbb{T}} \rho(x,t)(1 + u(x,t)^2 + \rho(x,t)^{2\theta})dx < +\infty$$

and if, for every g-entropy-entropy flux (η, H) pair associated to a convex subquadratic function g, $\mathbb{P}$-almost surely, for all non-negative $\varphi \in C^1(\overline{Q_T})$ such that $\varphi \equiv 0$ on $\mathbb{T} \times \{t = T\}$,

$$\iint_{Q_T} [\eta(\mathbf{U})\partial_t\varphi + H(\mathbf{U})\partial_x\varphi]\, dxdt + \int_{\mathbb{T}} \eta(\mathbf{U}_0)\varphi(0)dx$$
$$+ \int_0^T \int_{\mathbb{T}} \eta'(\mathbf{U})\mathbf{\Psi}(\mathbf{U})\varphi\, dxdW(t) + \frac{1}{2}\iint_{Q_T} \mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)^2\partial_{qq}^2\eta(\mathbf{U}_\varepsilon)\varphi dxdt \geq 0.$$

Martingale solution

Definition A sextuplet

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{U}})$$

is a martingale weak entropy solution to (1) if, after the substitution

$$\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W\big) \leftarrow \big(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}\big), \tag{6}$$

$\tilde{\mathbf{U}}$ is a pathwise weak entropy solution to (1).

Martingale solution

Theorem Let $p \in \mathbb{N}$ satisfy $p \geq 4 + \frac{1}{2\theta}$. Let

$$\rho_0 \colon \mathbb{T} \to \mathbb{R}_+, \quad q_0 \colon \mathbb{T} \to \mathbb{R}$$

satisfy

$$\int_{\mathbb{T}} \rho_0 (1 + u_0^{4p} + \rho_0^{4\theta p}) dx < +\infty.$$

Then there exists a martingale solution to (1).

Parabolic approximation

Viscous isentropic Euler system

For $\varepsilon > 0$, the (non-physical...) regularization of (1) is

$$d\mathbf{U}_\varepsilon + \partial_x \mathbf{F}(\mathbf{U}_\varepsilon)dt = \varepsilon\partial_x^2\mathbf{U}_\varepsilon dt + \mathbf{\Psi}^\varepsilon(\mathbf{U}_\varepsilon)dW(t), \tag{7a}$$

$$\mathbf{U}_{\varepsilon|t=0} = \mathbf{U}_{\varepsilon 0}. \tag{7b}$$

where $\mathbf{U}$, $\mathbf{F}(\mathbf{U})$, $\mathbf{\Psi}^\varepsilon(\mathbf{U})$ are defined by

$$\mathbf{U} = \begin{pmatrix}\rho\\ q\end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + p(\rho)\end{pmatrix}, \quad \mathbf{\Psi}^\varepsilon(\mathbf{U}) = \begin{pmatrix} 0 \\ \Phi^\varepsilon(\mathbf{U})\end{pmatrix}.$$

Singularities

Note that the flux $\mathbf{F}(\mathbf{U})$ is singular at $\rho = 0$ and (strictly) superlinear with respect to ρ and q.

For $R > 1$, let D_R denote the set of $\mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}$ such that

$$R^{-1} \leq \rho \leq R, \quad |q| \leq R.$$

For a random function $\mathbf{U}(x,t)$ with values in $\mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}$, we say that $\mathbf{U} \in D_R$ w.h.p. ("with high probability") if, for all $\alpha > 0$, there exists $R > 0$ such that the event "$\mathbf{U}(x,t) \in D_R$ for all (x,t)" has probability greater then $1 - \alpha$.

Pathwise solution

Definition Let $\mathbf{U}_0 \in L^\infty(\mathbb{T})$ satisfy $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$, where $c_0 > 0$. Let $T > 0$. A process $(\mathbf{U}(t))_{t\in[0,T]}$ with values in $(L^2(\mathbb{T}))^2$ is said to be a $\mathrm{bounded\ solution}$ to (7) if it is a predictable process such that

1. almost surely, $\mathbf{U} \in C([0,T]; L^2(\mathbb{T}))$,
2. $\mathbf{U} \in D_R$ w.h.p.
3. almost surely, for all $t \in [0,T]$, for all test function $\varphi \in C^2(\mathbb{T};\mathbb{R}^2)$, the following equation is satisfied:

$$\langle \mathbf{U}(t), \varphi\rangle = \langle \mathbf{U}_0, \varphi\rangle + \int_0^t \langle \mathbf{F}(\mathbf{U}), \partial_x\varphi\rangle + \varepsilon\langle \mathbf{U}, \partial_x^2\varphi\rangle\, ds + \int_0^t \langle \mathbf{\Psi}^\varepsilon(\mathbf{U})\, dW(s), \varphi\rangle. \quad (8)$$

ε-Noise

For each given $\mathbf{U}$, $[\Phi^\varepsilon(\mathbf{U})e_k](x) = \sigma_k^\varepsilon(x, \mathbf{U})$ where σ_k^ε is a continuous function of its arguments. We assume

$$\mathbf{G}^\varepsilon(x, \mathbf{U}) := \left(\sum_{k\geq 1} |\sigma_k^\varepsilon(x, \mathbf{U})|^2 \right)^{1/2} \leq A_0 \rho \left[1 + u^2 + \rho^{2\theta}\right]^{1/2},$$

for all $x \in \mathbb{T}, \mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}$. We will also assume that $\mathbf{G}^\varepsilon$ is supported in an invariant region: there exists $\varkappa_\varepsilon > 0$ such that

$$\mathrm{supp}(\mathbf{G}^\varepsilon) \subset \mathbb{T}_x \times \Lambda_{\varkappa_\varepsilon},$$

where

$$\Lambda_\varkappa = \{\mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}; -\varkappa \leq z \leq w \leq \varkappa\},$$

where z, w are the Riemann invariants, $w = u + \rho^\theta$, $z = u - \rho^\theta$.

ε-Noise 2

Eventually, we will assume that the following Lipschitz condition is satisfied:

$$\sum_{k\geq 1} |\sigma_k^\varepsilon(x,\mathbf{U}_1) - \sigma_k^\varepsilon(x,\mathbf{U}_2)|^2 \leq C(\varepsilon, R)|\mathbf{U}_1 - \mathbf{U}_2|^2,$$

for all $x \in \mathbb{T}, \mathbf{U}_1, \mathbf{U}_1 \in D_R$.

Existence- Uniqueness

Theorem Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_0$ a.e. in $\mathbb{T}$, for a positive constant c_0. Then the problem (7) admits a unique bounded solution $\mathbf{U}_\varepsilon$.

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Uniqueness: mild formulation and property $U_\varepsilon \in D_R$ w.h.p. More precisely, we show that two bounded solutions $\mathbf{U}^1_\varepsilon$, $\mathbf{U}^2_\varepsilon$ satisfy

$$\mathbb{E} \sup_{t\in[0,T]} \|\mathbf{U}^1_\varepsilon(t\wedge\tau_R^{1,2}) - \mathbf{U}^2_\varepsilon(t\wedge\tau_R^{1,2})\|^2_{L^2(\mathbb{T})} \leq C(\varepsilon,R,T)\|\mathbf{U}^1_{\varepsilon 0} - \mathbf{U}^2_{\varepsilon 0}\|^2_{L^2(\mathbb{T})},$$

where $\tau_R^{1,2}$ is the stopping time

$$\tau_R^{1,2} = \inf\left\{t\in[0,T]; \mathbf{U}^1(t) \text{ or } \mathbf{U}^2(t) \notin D_R\right\}.$$

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where $\tau_R^{1,2}$ is the stopping time

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Existence...

A priori bounds for bounded solutions to the Parabolic approximation

Improved regularity

Proposition Let $\mathbf{U}_{\varepsilon 0} \in W^{1,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_0$ a.e. in $\mathbb{T}$, for a positive constant c_0. Then, for all $\alpha \in [0, 1/4)$, $s \in [0, 1)$, $\mathbf{U}_\varepsilon(\cdot \wedge \tau_R)$ has a modification whose trajectories are almost surely in $C^\alpha([0, T]; L^2(\mathbb{T}))$ and such that

$$\mathbb{E}\|\mathbf{U}_\varepsilon(\cdot \wedge \tau_R)\|^2_{C^\alpha([0,T];L^2(\mathbb{T}))} \leq C(R, \varepsilon, T, \alpha, \mathbf{U}_{\varepsilon 0}),$$
$$\sup_{t\in[0,T]} \mathbb{E}\|\mathbf{U}_\varepsilon(t \wedge \tau_R)\|^2_{W^{s,2}(\mathbb{T})} \leq C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0}),$$

where τ_R is the exit time from D_R,

$$\tau_R = \inf \{t \in [0, T]; \mathbf{U}(t) \notin D_R\},$$

and $C(R, \varepsilon, T, \alpha, \mathbf{U}_{\varepsilon 0})$ and $C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0})$ are constants depending on R, T, ε, $\|\mathbf{U}_{\varepsilon 0}\|_{W^{1,2}(\mathbb{T})}$ and α/s respectively.

Entropy bounds (independent on ε)

Set

$$\Gamma_\eta(\mathbf{U}) = \int_{\mathbb{T}} \eta(\mathbf{U}(x))dx.$$

Proposition Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_0$ a.e. in $\mathbb{T}$, for a positive constant c_0. For $m \in \mathbb{N}$, let η_m denote the entropy associated to $\xi \mapsto \xi^{2m}$. Then for all $m \in \mathbb{N}$,

$$\mathbb{E} \sup_{t\in[0,T]} \int_{\mathbb{T}} \left(|u_\varepsilon|^{2m} + |\rho_\varepsilon|^{2\theta m}\right) \rho_\varepsilon dx = \mathcal{O}(1),$$

where $\mathcal{O}(1)$ depends on T, γ, on the constant A_0, on m and on $\mathbb{E}\Gamma_\eta(\mathbf{U}_{\varepsilon 0})$ for $\eta \in \{\eta_0, \eta_{2m}\}$.

Weighted gradient bounds (independent on ε)

Proposition Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_0$ a.e. in $\mathbb{T}$, for a positive constant c_0. Then for all $m \in \mathbb{N}$,

$$\varepsilon \mathbb{E} \iint_{Q_T} \left(|u_\varepsilon|^{2m} + \rho_\varepsilon^{2m\theta} \right) \rho_\varepsilon^{\gamma-2} |\partial_x \rho_\varepsilon|^2 dx dt = \mathcal{O}(1),$$

and

$$\varepsilon \mathbb{E} \iint_{Q_T} \left(|u_\varepsilon|^{2m} + \rho_\varepsilon^{2m\theta} \right) \rho_\varepsilon |\partial_x u_\varepsilon|^2 dx dt = \mathcal{O}(1),$$

where $\mathcal{O}(1)$ depends on T, γ, on the constant A_0, on m and on $\mathbb{E}\Gamma_\eta(\mathbf{U}_{\varepsilon 0})$ for $\eta \in \{\eta_0, \eta_{2m+2}\}$.

L^∞-bounds (*dependent* on $\varkappa_\varepsilon$)

Proposition: the region $\Lambda_{\varkappa_\varepsilon}$ is an invariant region for (7): if $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$, then almost surely, for all $t \in [0,T]$, $\mathbf{U}_\varepsilon(t) \in \Lambda_{\varkappa_\varepsilon}$. In particular, almost surely,

$$\|u^\varepsilon\|_{L^\infty(Q_T)} \leq 2\varkappa_\varepsilon, \quad \|\rho^\varepsilon\|^\theta_{L^\infty(Q_T)} \leq 2\varkappa_\varepsilon.$$

Positivity of ρ_ε

Theorem Let $m > 6$. Let $u \in L^m(Q_T) \cap L^2(0,T;H^1(\mathbb{T}))$ and $\rho_0 \in L^2(\mathbb{T})$. Let $\rho \in C([0,T];L^2(\mathbb{T}))$ be the generalized solution of the problem

$$\partial_t \rho + \partial_x(\rho u) - \partial_x^2 \rho = 0 \text{ in } Q_T, \tag{9}$$

with initial condition

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbb{T}. \tag{10}$$

Assume $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ where c_0 is a positive constant. Then there exists a constant $c > 0$ depending on c_0, T, m and

$$\iint_{Q_T} \rho |\partial_x u|^2 dxdt \quad \text{and} \quad \|u\|_{L^m(Q_T)}$$

only, such that

$$\rho \geq c$$

a.e. in Q_T.

Existence of a bounded solution to the Parabolic approximation

Approximation by splitting-in-time

Let $\tau > 0$. Set $t_k = k\tau$, $k \in \mathbb{N}$. We solve alternatively the deterministic, parabolic part of (7) on time intervals $[t_{2k}, t_{2k+1})$ and the stochastic part of (7) on time intervals $[t_{2k+1}, t_{2k+2})$, *i.e.*

- for $t_{2k} \le t < t_{2k+1}$,

$$\partial_t \mathbf{U}^\tau + 2\partial_x \mathbf{F}(\mathbf{U}^\tau) = 2\varepsilon\partial_x^2 \mathbf{U}^\tau \qquad \text{in } Q_{t_{2k},t_{2k+1}}, \tag{11a}$$

$$\mathbf{U}^\tau(t_{2k}) = \mathbf{U}^\tau(t_{2k}-) \qquad \text{in } \mathbb{T}, \tag{11b}$$

- for $t_{2k+1} \le t < t_{2k+2}$,

$$d\mathbf{U}^\tau = \sqrt{2}\mathbf{\Psi}^{\varepsilon,\tau}(\mathbf{U}^\tau)dW(t) \qquad \text{in } Q_{t_{2k+1},t_{2k+2}}, \tag{12a}$$

$$\mathbf{U}^\tau(t_{2k+1}) = \mathbf{U}^\tau(t_{2k+1}-) \qquad \text{in } \mathbb{T}. \tag{12b}$$

Bounds

Proposition Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_0$ a.e. in $\mathbb{T}$, for a positive constant c_0. Then $\mathbf{U}^\tau$ satisfies the a priori bounds derived on $\mathbf{U}_\varepsilon$, uniformly with respect to $\tau \in (0,1)$:

- Improved regularity
- Entropy bounds
- Gradient bounds
- L^∞-bounds
- Bound from below on the density

Entropy balance law

Let

$$\mathbf{1}_{\text{det}} = \sum_{k\geq 0} \mathbf{1}_{[t_{2k},t_{2k+1})}, \quad \mathbf{1}_{\text{sto}} = 1 - \mathbf{1}_{\text{det}}.$$

Let (η, H) be an entropy - entropy flux pair associated to a C^2, convex subquadratic function g. Then, almost surely, for all $t \in [0,T]$, for all test-function $\varphi \in C^2(\mathbb{T})$,

$$\begin{aligned}
&\langle \eta(\mathbf{U}^\tau(t)), \varphi\rangle \\
=&\langle \eta(\mathbf{U}_0), \varphi\rangle + 2\int_0^t \mathbf{1}_{\text{det}}(s)\left[\langle H(\mathbf{U}^\tau), \partial_x\varphi\rangle + \varepsilon\langle \eta(\mathbf{U}^\tau), \partial_x^2\varphi\rangle\right] ds \\
&- 2\varepsilon\int_0^t \mathbf{1}_{\text{det}}(s)\langle \eta''(\mathbf{U}^\tau)\cdot(\mathbf{U}_x^\tau, \mathbf{U}_x^\tau), \varphi\rangle ds \\
&+ \sqrt{2}\int_0^t \mathbf{1}_{\text{sto}}(s)\langle \eta'(\mathbf{U}^\tau)\mathbf{\Psi}^{\varepsilon,\tau}(\mathbf{U}^\tau)\, dW(s), \varphi\rangle \\
&+ \int_0^t \mathbf{1}_{\text{sto}}(s)\langle \mathbf{G}^{\varepsilon,\tau}(\mathbf{U}^\tau)^2\partial_{qq}^2\eta(\mathbf{U}^\tau), \varphi\rangle ds.
\end{aligned}$$

Weak solution

If $g(\xi) = \alpha + \beta\xi$, then $\eta(\mathbf{U}) = \alpha\rho + \beta q$. Consequence:

almost surely, for all $t \in [0, T]$, for all test-function $\varphi \in C^2(\mathbb{T}; \mathbb{R}^2)$,

$$\begin{aligned}
&\big\langle \mathbf{U}^\tau(t), \varphi \big\rangle \\
=&\big\langle \mathbf{U}_0, \varphi \big\rangle + 2\int_0^t \mathbf{1}_{\mathrm{det}}(s) \left[\big\langle \mathbf{F}(\mathbf{U}^\tau), \partial_x \varphi \big\rangle + \varepsilon \big\langle \mathbf{U}^\tau, \partial_x^2 \varphi \big\rangle \right] ds \\
&+ \sqrt{2} \int_0^t \mathbf{1}_{\mathrm{sto}}(s) \big\langle \mathbf{\Psi}^{\varepsilon,\tau}(\mathbf{U}^\tau), \varphi \big\rangle \, dW(s)
\end{aligned}$$

Path spaces

Recall that $W = \sum_{k\geq 1} \beta_k e_k$ where the β_k are independent brownian processes and $(e_k)_{k\geq 1}$ is a complete orthonormal system in a Hilbert space $\mathfrak{U}$.

Let $\mathfrak{U}_0 \supset \mathfrak{U}$ be defined by

$$\mathfrak{U}_0 = \left\{ v = \sum_{k\geq 1} \alpha_k e_k;\ \sum_{k\geq 1} \frac{\alpha_k^2}{k^2} < \infty \right\},$$

with the norm $\|v\|_{\mathfrak{U}_0}^2 = \sum_{k\geq 1} \frac{\alpha_k^2}{k^2}$, $v = \sum_{k\geq 1} \alpha_k e_k$. The embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is then an Hilbert-Schmidt operator and, almost surely, $W \in \mathcal{X}_W := C([0,T];\mathfrak{U}_0)$.

The path space of $\mathbf{U}^\tau$ is $\mathcal{X}_{\mathbf{U}} = C([0,T];L^2(\mathbb{T}))$. Set $\mathcal{X} = \mathcal{X}_{\mathbf{U}} \times \mathcal{X}_W$.

Tightness - 1

Let

$$X_k^\tau(t) = \sqrt{2}\int_0^t \mathbf{1}_{\mathrm{sto}}(s)d\beta_k(s)$$

and set

$$W^\tau(t) = \sum_{k\geq 1} X_k^\tau(t)e_k.$$

Lemma The $\mathcal{X}_W$-valued process W^τ converges in law to W when $\tau \to 0$.

Tightness - 2

Let us denote by $\mu^\tau_{\mathbf{U}}$ the law of $\mathbf{U}^\tau$ on $\mathcal{X}_{\mathbf{U}}$. The joint law of $\mathbf{U}^\tau$ and W^τ on $\mathcal{X}$ is denoted by μ^τ.

Proposition Let $\mathbf{U}_0 \in (W^{2,2}(\mathbb{T}))^2$ be such that $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ for a given constant $c_0 > 0$. Assume $\mathbf{U}_0 \in \Lambda_{\varkappa_\varepsilon}$. Then the set $\{\mu^{\tau_n}\}$ is tight and therefore relatively weakly compact in the set of probability measures on $\mathcal{X}$.

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Corollary (Skorohod) There exists a probability space $(\tilde{\Omega}^\varepsilon, \tilde{\mathcal{F}}^\varepsilon, \tilde{\mathbb{P}}^\varepsilon)$, a sequence of $\mathcal{X}$-valued random variables $(\tilde{\mathbf{U}}^{\tau_n}, \tilde{W}^{\tau_n})_{n\in\mathbb{N}}$ and a $\mathcal{X}$-valued random variable $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$ such that, up to a subsequence,

1. the laws of $(\tilde{\mathbf{U}}^{\tau_n}, \tilde{W}^{\tau_n})$ and $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$ under $\tilde{\mathbb{P}}^\varepsilon$ coincide with μ^{τ_n} and μ_ε respectively,
2. $(\tilde{\mathbf{U}}^{\tau_n}, \tilde{W}^{\tau_n})$ converges $\tilde{\mathbb{P}}^\varepsilon$-almost surely to $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$ in the topology of $\mathcal{X}$.

Identification of the limit $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$

Let $(\tilde{\mathcal{F}}_t^\varepsilon)$ be the $\tilde{\mathbb{P}}^\varepsilon$-augmented canonical filtration of the process $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$, *i.e.*

$$\tilde{\mathcal{F}}_t^\varepsilon = \sigma\big(\sigma\big(\varrho_t\tilde{\mathbf{U}}_\varepsilon, \varrho_t\tilde{W}_\varepsilon\big) \cup \big\{N \in \tilde{\mathcal{F}}^\varepsilon;\ \tilde{\mathbb{P}}^\varepsilon(N) = 0\big\}\big), \quad t \in [0,T],$$

where ϱ_t is the operator of restriction to the interval $[0,t]$ defined as follows: if E is a Banach space and $t \in [0,T]$, then

$$\begin{aligned} \varrho_t : C([0,T];E) &\longrightarrow C([0,t];E) \\ f &\longmapsto f|_{[0,t]}. \end{aligned}$$

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Proposition The sextuplet

$$\big(\tilde{\Omega}^\varepsilon, \tilde{\mathcal{F}}^\varepsilon, (\tilde{\mathcal{F}}_t^\varepsilon), \tilde{\mathbb{P}}^\varepsilon, \tilde{W}_\varepsilon, \tilde{\mathbf{U}}_\varepsilon\big)$$

is a martingale bounded solution to (7).

Final step in the proof of existence of a bounded solution to the Parabolic Approximation

Gyöngy-Krylov argument [Gyöngy, Krylov, 96]:

Existence of a martingale solution
& Uniqueness of pathwise solutions (extended a little bit...)
$\Rightarrow$ Existence of pathwise solutions.

Martingale solution to isentropic Euler equations

Back to the original problem

$$d\mathbf{U} + \partial_x \mathbf{F}(\mathbf{U})dt = \mathbf{\Psi}(\mathbf{U})dW(t),$$

$$\mathbf{U}_{|t=0} = \mathbf{U}_0.$$

Or: how to pass to the limit in the parabolic approximation?

Back to the original problem

$$d\mathbf{U} + \partial_x \mathbf{F}(\mathbf{U})dt = \mathbf{\Psi}(\mathbf{U})dW(t),$$

$$\mathbf{U}_{|t=0} = \mathbf{U}_0.$$

Or: how to pass to the limit in the parabolic approximation?
Three steps:

- **Step 1.** Estimates on $\mathbf{U}_\varepsilon$,
- **Step 2.** Convergence in the sense of (random) Young measures,
- **Step 3.** Reduction to a Dirac Mass (outside the vacuum) of the Young measure obtained at the limit.

Remarks

- No L^∞-bounds.
- When $\gamma > 2$, div-curl lemma requires $\varepsilon^{\frac{1}{\gamma-2}}\varkappa_\varepsilon$ bounded with ε
- Martingale argument: with densely defined martingales only, *cf.* [Hofmanová 2013].

Open questions

- Parabolic approximation with additive noise
- Proof of the existence of invariant measures (with Khaled Saleh)
- Well-balanced schemes (with Khaled Saleh)
- Convergence of the stochastic compressible Navier-Stokes system
- Uniqueness of weak entropy solutions (in law ?), uniqueness of weak entropy solutions in the deterministic case

Thank you for your attention