Invariant measures of semilinear SPDEs

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Consider the following ODE

$$\begin{cases} du(t) = f(u(t))dt, \quad t \geq 0, \\ u(0) = x \in \mathbb{R}^n \end{cases}$$

$f : \mathbb{R}^n \to \mathbb{R}^n$ (f is Lipschitz), for all $x \in \mathbb{R}^n$ there exists a unique global solution $u(t,x)$ with

$$u(t+s,x) = u(t,u(s,x)), \quad 0 \leq s \leq t$$

$$\|u(t,x) - u(t,y)\| \leq \|x-y\| e^{Lip(f)t}, \quad t \geq 0,\ x,\ y \in \mathbb{R}^n.$$

Set

$$P_t\varphi(x) = \varphi(u(t,x)), \quad \varphi \in C_b(\mathbb{R}^n)$$

$(P_t)_{t\geq 0}$ is a family of bounded linear operators with $P_t\varphi \in C_b(\mathbb{R}^n)$ for any $\varphi \in C_b(\mathbb{R}^n)$.
Moreover,

$$u(t+s,x) = u(t,u(s,x)), \quad 0 \leq s \leq t \Longrightarrow P_{t+s} = P_t P_s$$

Suppose that for some $x_0 \in \mathbb{R}^n$, $\lim_{t\to+\infty} u(t,x_0) = 0$. Then for every $\varphi \in C_b(H)$ we have

$$\lim_{t\to+\infty} P_t\varphi(x_0) = \lim_{t\to+\infty} \varphi(u(t,x_0)) = \varphi(0).$$

In particular for any $s \geq 0$

$$\lim_{t\to+\infty} P_t P_s\varphi(x_0) = \lim_{t\to+\infty} P_s\varphi(u(t,x_0)) = P_s\varphi(0).$$

But

$$\lim_{t\to+\infty} P_t P_s\varphi(x_0) = \lim_{t\to+\infty} P_{t+s}\varphi(x_0) = \varphi(0)$$

Thus for all $s \geq 0$ and $\varphi \in C_b(\mathbb{R}^n)$

$$P_s\varphi(0) = \varphi(0) \quad \Longleftrightarrow \quad \langle \delta_0, P_s\varphi \rangle = \langle \delta_0, \varphi \rangle$$

δ_0 is invariant for P_t

$$\{f(x_0) = 0\} \Rightarrow \quad \delta_{x_0} \text{ is invariant for } P_t.$$

Aim:

- Existence and properties of invariant measures corresponding to dynamical systems governed by some class of SPDEs.
- Uniqueness results for associated Kolmogorov operators.

Heat equation

$$\begin{cases} du(t) = \Delta u(t)dt, & t \geq 0. \\ u(0) = u_0 \in L^2(\mathbb{R}^n) \end{cases}$$

For all $u_0 \in L^2(\mathbb{R}^n)$, there exists a unique solution which is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and such that $u(0,x) = u_0(x)$ for all $x \in \mathbb{R}^n$

$$u(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy = e^{t\Delta} u_0$$

Additional term (inhomogeneous term)

$$\frac{du(t,x)}{dt} = \Delta u(t,x) + g(t,x).$$

Variations constants formula

$$u(t,\cdot) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta} g(s,\cdot)\, ds$$

Stochastic heat equation

If the term g is a random process wet get a stochastic heat equation.

$$\frac{du(t,x)}{dt} = \Delta(u(t,x))dt + \underbrace{\dot{W}(t,x)}_{Noise}$$

where W_t is a Wiener Process (Brownian motion) on the state space H

$$\text{Formally} \quad u(t,\cdot) = e^{t\Delta}u_0 + \underbrace{\int_0^t e^{(t-s)\Delta} W(ds,dy)}_{Stochastic\ Integral}$$

Definition: A Wiener Process on H is a family of continuous random variables $(W_t)_{t\geq 0}$ on H having the following properties

- $W_0 = 0$ and $W_t - W_s$ is a Gaussian random variable with law $\mathcal{N}(0,(t-s)Q)$, for $0 \leq s < t$, Q positif, symmetric operator with $\mathrm{Tr}(Q) = \sum_{k=1}^{+\infty} \lambda_k < +\infty$ $(\lambda_k := \langle Qe_k, e_k\rangle)$.
- If $n \in \mathbb{N}$ and $0 < t_1 < \cdots < t_n$, the random variables

$$W_{t_1},\ W_{t_2} - W_{t_1}, \cdots, W_{t_n} - W_{t_{n-1}}$$

are independent.

$$W_t = \sum_{k=1}^{+\infty} \sqrt{\lambda_k}\beta_k(t)e_k, \qquad \beta_k \text{ are iid real BM}$$

If $Q = \mathrm{Id}$, $(W_t)_{t\geq 0}$ is a cylindrical Wiener process.

Examples of Motivation:
Ornstein-Uhlenbeck equation:

$$dX(t) = AX(t)dt + CdW_t,$$

where A and C some linear operators on some Hilbert space H.
Stochastic reaction diffusion equations

$$dX(t) = \Big(\Delta X(t) + f(X(t))\Big)dt + CdW_t$$

- on $L^2(0,1)$
- $f(t) = a_{2n+1}t^{2n+1} + \ldots + a_1 t$ with $a_{2n+1} < 0$

stochastic Cahn hilliard equations

$$\begin{cases} dX(t) = \Big(-\Delta^2 X(t) - \Delta F(X(t))\Big)dt + CdW_t, \quad t \geq 0. \\ X(0) = X_0 \end{cases}$$

Stochastic Burgers equation

$$\begin{cases} dX(t) = \Big(\Delta X(t) + \partial_\xi \Big(X(t)^2\Big)\Big) dt + C dW_t, \quad t \geq 0. \\ X(0) = X_0 \end{cases}$$

Abstract formulation

$$\begin{cases} dX(t) = \Big(AX(t) + F(X(t))\Big) dt + C dW_t, \quad t \geq 0. \\ X(0) = X_0 \in H. \end{cases}$$

- A is the generator of C_0-semigroup $(e^{tA})_{t\geq 0}$ on H

  $$\Updownarrow$$

  Well posdness $\frac{du}{dt} = Au, \; u(0) \in D(A)$.
- F is a nonlinear function.

- Strong solution

$$X(t) = X(0) + \int_0^t \left(AX(s) + f(X(s))\right) ds + \underbrace{\int_0^t C(X(s))\, dW_s}_{\text{Itô Integral}}.$$

- Weak solution

$$\begin{aligned}\langle X(t), \eta\rangle =& \langle X(0), \eta\rangle + \int_0^t \left(\langle AX(s) + f(X(s)), \eta\rangle\right) ds \\ &+ \int_0^t \langle C(X(s)), \eta\rangle\, dW_s. \quad \eta \in D(A^*)\end{aligned}$$

- Mild solution

$$X(t) = e^{tA}X(0) + \int_0^t e^{(t-s)A} f(X(s)) ds + \int_0^t e^{(t-s)A} C(X(s)) dW_s.$$

Invariant measures for Markov process:

Let $X(t)_{t\geq 0}$ be a continuous Markov process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with values in E (separable Banach space).

Markov transition probability

$$P(t,x,A) = \mathbb{P}(X(t) \in A | X(0) = x), \quad A \in \mathcal{B}(E).$$

Chapman-Kolmogorov equation

$$P(t+s,x,A) = \int_E P(t,y,A)P(s,x,dy),$$

Transition semigroup

$$P_t f(x) = \mathbb{E}(f(X(t))|X(0) = x) = \int_E f(y)P(t,x,dy), \quad f \in \mathcal{B}_b(E)$$

a probability measure μ is invariant for P_t if

$$\int_E P_t f(x)\,\mu(dx) = \int_E f(x)\,\mu(dx), \quad f \in B_b(E).$$

μ is stationary state for the transition probabilities $P(t, x, \cdot)$:

$$\mathbb{P}_{X(t_0)} = \mu \Longrightarrow \mathbb{P}_{X(t)} = \mu, \text{ for all } t \geq t_0.$$

$$\begin{aligned} P(t, x, A) &= P(t_0 + (t - t_0), x, A) = \int_E P(t - t_0, y, A) P(t_0, x, dy) \\ &= \int_E P(t - t_0, y, A)\mu(dy) = \int_E P_{t-t_0} \mathbf{1}_A \mu(dy) \\ &= \int_E \mathbf{1}_A \mu(dy) = \mu(A) \end{aligned}$$

Existence of invariant measures:

Definition: (Feller property): The transition semigroup P_t is

- *Feller:* $P_t f \in C_b(E)$ for all $f \in C_b(E)$
- *strong Feller:* $P_t f \in C_b(E)$ for all $f \in B_b(E)$.

Recall the transition semigroup corresponding to $X(t,x)$

$$P_t f(x) = \mathbb{E}\big(f(X(t,x))\big)$$

Sufficient condition for the Feller property:

$$\lim_{n\to+\infty} \|x_n - x\|_H = 0, \Longrightarrow \lim_{n\to+\infty} \|X(t,x_n) - X(t,x)\|_H = 0 \quad \text{in prob.}$$

This holds in particular if we have

$$\mathbb{E}\big(\|X(t,x_n) - X(t,x)\|_H^2\big) \leq C_t \|x_n - x\|_H^2.$$

Mean occupation time measures

$$\mu_T(A) := \frac{1}{T}\int_0^T \mathbb{P}(X(t,x) \in A)\, dt, \quad A \in \mathcal{B}(E).$$

Theorem: (Krylov-Bogoliubov) Assume that for some $T_0 > 0$

- The transition semigroup $(P_t)_{t\geq 0}$ is Feller.
- The family $(\mu_T)_{t\geq T_0}$ is tight, i.e., for all $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset E$ with $\mu_T(K_\varepsilon) > 1-\varepsilon$, $T \geq T_0$. Then there exists an invariant measure μ for P_t. Moreover, every limit μ_∞ of some weakly convergent subsequence $(\mu_{T_n})_{n\geq 1}$ with $T_n \to +\infty$, is an invariant measure.

Sufficient condition for the tightness :

Lyapunov function

$$V: \ H \to \mathbb{R} \cup +\infty, \quad \{V \leq \alpha\} \text{ is compact}$$

with

$$\sup_{T \geq T_0} \int V \, d\mu_T < \infty \Leftrightarrow \sup_{T \geq T_0} \frac{1}{T} \int_0^T \mathbb{E}(V(X(t))) \, dt < \infty.$$

$$\Longrightarrow \quad \text{tightness of } \{\mu_T\}_{T \geq T_0}.$$

Ergodic measures:

Definition: Let μ an invariant measure for P_t. The set $A \in \mathcal{E}$ is invariant with respect to P_t if for all $t \geq 0$

$$P_t \mathbf{1}_A = \mathbf{1}_A \quad \mu a.s.$$

The measure μ is ergodic if $\mu(A) \in \{0, 1\}$ for any invariant set A.

Theorem: If μ is an ergodic measure for P_t. Then for any $f \in L^2(E, \mu)$,

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t P_s f \, ds = \int_E f(x) \mu(dx) \quad \text{in } L^2(H, \mu).$$

Application: If μ and ν are two ergodic measures with respect to P_t and if $\mu \neq \nu$, then μ and ν are singular.

Let $A \in \mathcal{B}(E)$ such that $\mu(A) \neq \nu(A)$. There exists $t_n \to \infty$

$$\lim_{n \to +\infty} \frac{1}{t_n} \int_0^{t_n} P_s \mathbf{1}_A(x)\, ds = \mu(A) \quad \text{for any } x \in M \text{ with } \mu(M) = 1.$$

$$\lim_{n \to +\infty} \frac{1}{t_n} \int_0^{t_n} P_s \mathbf{1}_A(x)\, ds = \nu(A) \quad \text{for any } x \in N \text{ with } \mu(N) = 1.$$

We have $M \cap N = \emptyset$ and $\mu(M) = \nu(N) = 1$.

Remark: If μ is the unique invariant measure for P_t, then μ is ergodic.

Uniqueness of μ (Sufficient conditions:)

Proposition: Suppose that P_t is Feller. Let $\mu_{t,x}$ denote the distribution of $X(t,x)$. If μ_t converges weakly to μ as $t \to +\infty$ for all $x \in H$, then μ is the unique invariant measure for P_t.

Proof: For $\varphi \in C_b(H)$

$$P_t\varphi(x) = \mathbb{E}(\varphi(X(t,x))) = \int_H \varphi(y)\, d\mu_{t,x}(y) \longrightarrow \int_H \varphi(y)\, d\mu(y).$$

Thus for any fixed $s > 0$

$$P_{t+s}\varphi(x) \longrightarrow \int_H \varphi(y)\, d\mu(y) \quad \text{as } t \to +\infty.$$

$$P_{t+s}\varphi(x) = P_t(P_s\varphi)(x) \longrightarrow \int_H P_s\varphi(y)\, d\mu(y) \quad \text{as } t \to +\infty.$$

$$\text{Hence for all } s > 0 \quad \int_H P_s\varphi(y)\, d\mu(y) = \int_H \varphi(y)\, d\mu(y)$$

Uniqueness: For any invariant measure ν of P_t, we have

$$\int_H \varphi(x)\, d\nu(x) = \int_H P_t\varphi(x)\, d\nu(x) \quad \forall\, t \geq 0$$
$$= \lim_{t\to+\infty} \int_H P_t\varphi(x)\, d\nu(x) = \int_H \int_H \varphi(y)\, d\mu(y)\, d\nu(x)$$
$$= \int_H \varphi(y)\, d\mu(y)$$

Thus for all $\varphi \in C_b(H)$

$$\int_H \varphi(x)\, d\nu(x) = \int_H \varphi(y)\, d\mu(y)$$

Hence the uniqueness of μ.

Definition: $X(t,x)$ is irreducible if for all $t>0$, $x\in E$

$$\mathbb{P}\big(X(t,x)\in O\big)>0 \quad \text{for any open set } O \text{ in } E$$

$$P_t\mathbf{1}_O>0 \quad \text{for any open set } O \text{ in } E$$

(The process $X(t,x)$ visits the whole space, for all choice of the initial state x.)

Theorem: (Doob's Theorem) Let μ an invariant measure for P_t. If P_t is irreducible and strong Feller, then μ is the unique invariant measure for P_t. Moreover, μ is equivalent to all transition probability $\mu_{X(t,x)}$, $t>0$.

Consider the linear stochastic equation on a separable Hilbert space H

$$\begin{cases} \mathrm{d}X(t) & = AX(t)\,\mathrm{d}t + C\,\mathrm{d}W(t), \\ X(0) & = x \in H. \end{cases}$$

Hypothesis $(\mathbf{H_1})$:

- $A: \ D(A) \subset H \to H$ generates a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ on H.
- C is a bounded linear operator on H.
- $(W_t)_{t\geq 0}$ is a cylindrical Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$.
- The linear operator $Q_t: \ H \to H$ defined by
$$Q_t x = \int_0^t e^{tA} CC^* e^{tA^*} x \, ds$$
is a trace class operator.

Remark: In particular $\mathrm{Tr}(Q_t) < +\infty$ if $\mathrm{Tr}(CC^*) < +\infty$.

Write for an orthonormal basis $(e_k)_{k\geq 1}$ of H

$$W_t = \sum_{k=1}^{+\infty} \beta_k(t) e_k$$

where β_k are independent real Brownian motions.
Under hypothesis $\mathbf{(H_1)}$ the linear equation has a unique mild solution

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} C \; \mathrm{d}W_s,$$

which is a Gaussian, Markov process in H. (Ornstein-Uhlenbeck process).
The stochastic convolution

$$W_{A,C} = \int_0^t e^{(t-s)A} C \; \mathrm{d}W_s \text{ is an } H\text{-valued process.}$$

$$W_{A,C}(t) = \sum_{k=1}^{+\infty} \int_0^t e^{(t-s)A} C e_k \, d\beta_k(s).$$

The series converges in $L^2(\Omega, \mathbb{P}, H)$:

$$\begin{aligned}
\mathbb{E}\|W_{A,C}(t)\|^2 &= \sum_{k=1}^{+\infty} \int_0^t \|e^{(t-s)A} C e_k\|^2 \, ds \\
&= \int_0^t \sum_{k=1}^{+\infty} \|e^{(t-s)A} C e_k\|^2 \, ds \\
&= \int_0^t \mathrm{Tr}\left(e^{tA} C C^* e^{tA^*}\right) ds) \\
&= \mathrm{Tr}(Q_t) < +\infty.
\end{aligned}$$

The corresponding transition semigroup is

$$\begin{aligned} R_t f(x) &= \mathbb{E} f(X(t,x)), \quad f \in \mathcal{B}_b(H) \\ &= \mathbb{E} f\left(e^{tA}x + \int_0^t e^{(t-s)A} C \, \mathrm{d}W_s\right). \end{aligned}$$

By using

$$W_{A,C} = \int_0^t e^{(t-s)A} C \, \mathrm{d}W_s = \mathcal{N}(0, Q_t).$$

We get the (Mehler formula)

$$\begin{aligned} R(t)f(x) &= \int_H f(y) \, \mathcal{N}(e^{tA}x, Q_t) \, dy \\ &= \int_H f(e^{tA}x + y) \, \mathcal{N}(0, Q_t) \, dy. \end{aligned}$$

Strong Feller property:

Hypothesis $(\mathbf{H_2})$:

$$e^{tA}(H) \subset Q_t^{\frac{1}{2}}(H), \quad \text{for all } t > 0. \qquad \text{(Range condition)}$$

Theorem: Under $(\mathbf{H_2})$, the semigroup R_t maps $\mathcal{B}_b(H)$ to $C_b^\infty(H)$. In particular R_t is strong Feller.

$$(\mathbf{H_2}) \Longrightarrow \Lambda_t = Q_t^{-\frac{1}{2}} e^{tA}, \;\; t > 0 \text{ is a bounded operator}$$

Cameron-Martin's theorem: If $\mu_1 = \mathcal{N}(a, Q)$, and $\mu_2 = \mathcal{N}(0, Q)$ are two Gaussian measures with $a \in Q^{\frac{1}{2}}(H)$. Then μ_1 and μ_2 are equivalent and

$$\frac{d\mu_1}{d\mu_2} = e^{-\frac{1}{2}\|Q^{-\frac{1}{2}}a\|^2 + \langle Q^{-\frac{1}{2}}a, Q^{-\frac{1}{2}}x\rangle}, \quad x \in H.$$

$$\rho(x,y) = \frac{d\mathcal{N}(e^{tA}x, Q_t)}{d\mathcal{N}(0, Q_t)}(y) = \exp\left(-\frac{1}{2}|\Lambda_t x|^2 + \langle \Lambda_t x, Q_t^{-\frac{1}{2}} y\rangle\right).$$

Thus,

$$R(t)\varphi(x) = \int_H \varphi(y)\rho(x,y)\mathcal{N}(0, Q_t)(dy).$$

$$\begin{aligned}\langle DR(t)\varphi(x), h\rangle &= \int_H \langle \Lambda_t h, Q_t^{-\frac{1}{2}}(y - e^{tA}x)\rangle \varphi(y)\mathcal{N}(e^{tA}x, Q_t)(dy) \\ &= \int_H \langle \Lambda_t h, Q_t^{-\frac{1}{2}} y\rangle \varphi(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy).\end{aligned}$$

$$\|DR_t\varphi\|_\infty \leq \|\Lambda_t\|\|\varphi\|_\infty$$

Irreducibility:

Theorem: If Ker $C = \{0\}$, then the transition semigroup R_t is irreducible.

Proof: For all $t > 0$, $R_t \mathbf{1}_O > 0$ for any open non-empty set O of H? equivalently $\mathbb{P}(X(t,x) \in O) > 0$.

But $X(t,x)$ is a Gaussian process with law $\mathcal{N}(e^{tA}x, Q_t)$, and remark that

$$\text{Ker } C = \{0\} \Longrightarrow \text{Ker } Q_t = \{0\} \quad \text{for all } t > 0$$

Thus, $\mathcal{N}(e^{tA}x, Q_t)$ is a nondegenerate Gaussian measure, hence it has a full support. This means that $\mathcal{N}(e^{tA}x, Q_t)(O) > 0$ for any open non-empty set O of H

Invariant measure:

If μ is invariant for R_t then

$$\int_H R_t f(x)\,\mu(dx) = \int_H f(x)\,\mu(dx) \qquad \text{for all } f \in C_b(H),\ t \geq 0\,.$$

Given $h \in H$, and set $f_h = e^{i\langle h,x\rangle}$, then $R_t f_h$ coincides with the characteristic functional of the measure $\mathcal{N}(e^{tA}x, Q_t)$ and we have

$$R_t f_h(x) = e^{i\langle h, e^{tA}x\rangle - \frac{1}{2}\langle Q_t h, h\rangle}, \quad x \in H.$$

Hence μ is an invariant measure for R_t if and only if its characteristic functional $\hat{\mu}$ is given by

$$\hat{\mu}(h) = \hat{\mu}(e^{tA^*}h)e^{-\frac{1}{2}\langle Q_t, h, h\rangle}, \quad h \in H$$

Theorem: Suppose that

$$\sup_{t>0} \mathrm{Tr}(Q_t) < +\infty$$

Then there exists an invariant measure for R_t.

Proof: Set

$$Q_\infty x = \int_0^{+\infty} e^{tA} CC^* e^{tA^*} x \, ds$$

Then, $\mathrm{Tr}(Q_\infty) < +\infty$ and the Gaussian measure $\mu = \mathcal{N}(0, Q_\infty)$ satisfies the condition

$$\hat{\mu}(h) = \hat{\mu}(e^{tA^*} h) e^{-\frac{1}{2}\langle Q_t, h, h\rangle}, \quad h \in H$$

We have $\hat{\mu}(h) = e^{-\frac{1}{2}\langle Q_\infty, h, h\rangle}$ and

$$\hat{\mu}(e^{tA^*} h) = e^{-\frac{1}{2}\langle Q_\infty e^{tA^*} h, e^{tA^*} h\rangle} = e^{-\frac{1}{2}\langle e^{tA} Q_\infty e^{tA^*} h, h\rangle}$$

$$= e^{-\frac{1}{2}\langle Q_\infty, h, h\rangle} e^{\frac{1}{2}\langle Q_t, h, h\rangle}$$

$\|e^{tA}\| \le Me^{-\alpha t}, \quad \alpha > 0 \Longrightarrow \sup_{t>0} \mathrm{Tr}(Q_t) < +\infty.$

Theorem: $\lim_{t\to+\infty} R_t\varphi(x) = \int_H \varphi(y)\mu(dy)$ for any $\varphi \in C_b(H)$.

Proof: Set $M_t = \int_0^t e^{sA}C\,dW_s, \quad \mathcal{L}\big(W_{A,C}(t)\big) = \mathcal{L}\big(M_t\big)$. We have

$$\lim_{t\to+\infty} M_t = M_\infty = \int_0^{+\infty} e^{sA}C\,dW_s \quad \text{in } L^2(\Omega, H).$$

$$(\mathbb{E}\|M_{s+t} - M_s\|^2 = \mathrm{Tr}(e^{sA}Q_te^{sA^*}) \le M^2e^{-2\alpha s}\,\mathrm{Tr}(Q_\infty).)$$

$$\begin{aligned} R_t\varphi(x) &= \mathbb{E}\big(\varphi(X(t,x))\big) = \mathbb{E}\big(\varphi(e^{tA}x + W_{A,C}(t))\big) \\ &= \mathbb{E}\big(\varphi(e^{tA}x + M_t)\big). \end{aligned}$$

$e^{tA}x \to 0$, and $M_t \to M_\infty = \mathcal{N}(0, Q_\infty)$ in law.

Semilinear equations

$$\begin{cases} dX(t) = \Big(AX(t) + F(X(t))\Big)dt + \sqrt{Q}dW_t, \quad t \geq 0. \\ X(0) = x \in H. \end{cases}$$

- $A^* = A, \quad \langle Ax, x\rangle \leq -\omega\|x\|^2.$
- A^{-1} is compact, $\quad \langle Qx, x\rangle > 0.$
- F is a nonlinear vector-field.

(A_1)

$$\int_0^t s^{-2\alpha}\|e^{sA}\sqrt{Q}\|_{HS}^2\,ds<\infty,\quad 0<\alpha<\frac{1}{2}.$$

$$T\in\mathcal{L}_{HS}(H)\Longleftrightarrow \|T\|_{HS}^2=\sum_{k=1}^{\infty}\|Te_k\|^2<+\infty,$$

(A_2) $F(V_\gamma)\subset V_\gamma$, locally Lipschitz continuous and bounded on bounded sets of V_γ. $0<\gamma<\alpha$.

$$V_\gamma := (D((-A)^\gamma),\|\cdot\|_\gamma).$$

(typical example : $H=L^2(\Omega), A=\Delta \quad V_\gamma = H^{2\gamma}(\Omega)$).

(A_3) There exists $a:\ \mathbb{R}^+\longrightarrow\mathbb{R}^+$

$$\langle F(y+w),y\rangle\le a(\|w\|_\gamma)(1+\|y\|_\gamma^2)\quad \text{for } y,\ w\in V_\gamma.$$

Comments on $(\mathbf{A_1})$: $\int_0^t s^{-2\alpha}\|e^{sA}\sqrt{Q}\|_{HS}^2\, ds < \infty, \quad 0 < \alpha < \frac{1}{2}$.
The stochastic integral $M_t = \int_0^t \Phi(s)\, dW_s$ is continuous (Doob martingale inequality). However, the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A}\sqrt{Q}\, dW_s$$

will not be in general pathwise continuous.
Factorization method (Da Prato, Kwapien, Zabczyk 87): If A generates an analytic semigroup and

$$\int_0^t s^{-2\alpha}\|e^{sA}\sqrt{Q}\|_{HS}^2\, ds < \infty, \quad 0 < \alpha < \frac{1}{2}.$$

then W_A has a modification that is Hölder-continuous with values in V_γ for any $\gamma < \alpha$.

$$\int_0^t s^{-2\alpha} \|e^{sA}\sqrt{Q}\|_{HS}^2\, ds < \infty \quad \Rightarrow \quad W_A(t) = \int_0^t e^{(t-s)A}\sqrt{Q}\, dW_s$$

has a continuous version in V_γ.
By using hypotheses ($\mathbf{A_2}$) and ($\mathbf{A_3}$) and a fixed point principle, we get existence and uniqueness of the mild solution for all initial conditions $x \in V_\gamma$.

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(X(s))ds + \int_0^t e^{(t-s)A}\sqrt{Q} dW_s, \quad t \geq 0.$$

Transition semigroup:

$$P_t f(x) := \mathbb{E} f(X(t,x)), \quad t \geq 0,\ f \in \mathcal{B}_b(H),$$

Under hypothesis $(\mathbf{A_1})$ we have

$$M := \sup_{t\geq 0} \mathbb{E}(\| W_A(t)\|_\alpha^2) = \int_0^\infty \|(-A)^\alpha e^{tA}\sqrt{Q}\|_{HS}^2 \, dt < \infty.$$

In particular,

$$\sup_{t\geq 0} \mathbb{E}(\| W_A(t)\|_\gamma^2) < \infty, \quad \text{and} \quad \sup_{t\geq 0} \mathbb{E}\left(e^{\varepsilon_0 \| W_A(t)\|_\gamma^2}\right) < \infty$$

for all $\gamma \in [0, \alpha]$, ε_0 small enough.

Write

$$\int_0^\infty \|(-A)^\alpha e^{tA}\sqrt{Q}\|_{HS}^2\,dt = \sum_{k=0}^{\infty}\int_k^{k+1} \|(-A)^\alpha e^{tA}\sqrt{Q}\|_{HS}^2\,dt$$

$$= \sum_{k=0}^{\infty}\int_0^{1} \|(-A)^\alpha e^{tA}e^{kA}\sqrt{Q}\|_{HS}^2\,dt$$

$$\leq \sum_{k=0}^{\infty}\|e^{kA}\|^2\int_0^{1} \|(-A)^\alpha e^{tA}\sqrt{Q}\|_{HS}^2\,dt$$

$$\leq \sum_{k=0}^{\infty}e^{-2\omega k}\int_0^{1} \|(-A)^\alpha e^{tA/2}\|^2\|e^{tA/2}\sqrt{Q}\|_{HS}^2\,dt$$

$$\text{(Using hypothesis } (\mathbf{A_1})) \leq 2^{2\alpha}\sum_{k=0}^{\infty}e^{-2\omega k}\int_0^{1} t^{-2\alpha}\|e^{tA/2}\sqrt{Q}\|_{HS}^2\,dt < +\infty.$$

Invariant measure:

Lyapunov type conditions:

(A_4) $\exists\ \Psi:\ H \to \mathbb{R}_+$ Fréchet-differentiable, $\Theta:\ V_{\frac{1}{2}} \to \mathbb{R}_+$,

$$\lim_{\|y\|_{V_{\frac{1}{2}}} \to +\infty} \Theta(y) = +\infty$$

and β, $\delta \in \mathbb{R}^+$, such that

$$\langle Ay + F(y+w), D\Psi(y)\rangle \leq -\Theta(y) + \beta e^{\varepsilon_0 \|w\|^2_{V_\gamma}} + \delta$$

for all $y \in D(A)$, $w \in V_\gamma$.

(Da Parto, Zabczyk). If there exists $\alpha_1 > 0$, α_2, $\alpha_3 \in \mathbb{R}$

$$\langle Ay + F(y+w), y\rangle \leq -\alpha_1\|y\|^2 + \alpha_2\|w\|^2 + \alpha_3$$

for $y \in D(A)$, $w \in H$.

$$\Psi(y) = \frac{1}{2}\|y\|^2 \qquad \text{and} \qquad \Theta(y) = \alpha_1\|y\|^2$$

(Goldys, Maslowski). If there exist $\alpha_1 > 0$, α_2, $\alpha_3 \in \mathbb{R}$, $s \geq 2$

$$\langle Ay + F(y+w), y\rangle \leq -\alpha_1\|y\|^2_{V_{\frac{1}{2}}} + \alpha_2\|w\|^s_{V_\gamma} + \alpha_3$$

for $y \in D(A)$, $w \in V_\gamma$.

$$\Psi(y) = \frac{1}{2}\|y\|^2 \qquad \text{and} \qquad \Theta(y) = \alpha_1\|y\|^2_{V_{\frac{1}{2}}}$$

Define

$$\left\{\mu_T := \frac{1}{T}\int_0^T \mu_{X(t,x)}\, dt,\ T \geq 1\right\}$$

Proposition: The family of measures $\{\mu_T\}_{t\geq 1}$ is tight on H.

$$\Downarrow$$

Existence of an invariant measure μ for $(P_t)_{t\geq 0}$ if it is a Feller semigroup .

Proof:

$$Y(t) := X(t) - W_A(t)\,.$$

$$\mathbb{E}\left(\Psi(Y(t)) + \int_0^t \Theta(Y(s))\,\mathrm{d}s\right) \leq C\,(t+1)\,.$$

$$\begin{aligned}\frac{d}{dt}\Psi(Y(t)) &= \langle AY(t) + F(Y(t) + W_A(t)), D\Psi(Y(t))\rangle \\ &\leq -\Theta(Y(t)) + \beta e^{\varepsilon_0 \|W_A(t)\|^2_{V_\gamma}} + \delta.\end{aligned}$$

Therefore

$$\Psi(Y(t)) + \int_0^t \Theta(Y(s))\, \mathrm{d}s \leq \Psi(x) + \beta \int_0^t e^{\varepsilon_0 \|W_A(s)\|^2_{V_\gamma}}\, \mathrm{d}s + \delta t.$$

By taking the expectation, we obtain that

$$\mathbb{E}\left(\Psi(Y(t)) + \int_0^t \Theta(Y(s))\, \mathrm{d}s\right) \leq C\,(t+1) \quad \text{for } t \geq 0 \text{ and } C > 0.$$

To prove tightness, take $\varepsilon > 0$. Since Θ and $w \mapsto \|w\|_{V_\gamma}$ both are coercive on V_γ, there exists $R_\varepsilon > 0$ such that

$$\varepsilon \left(\Theta(y) + \|w\|_{V_\gamma}^2 \right) \geq 1$$

for $w,\ y \in V_\gamma$ with $\|w + y\|_{V_\gamma} \geq R_\varepsilon$. Consequently

$$\begin{aligned} \mu_T(H \setminus \overline{B}(0, R_\varepsilon)) &= \mathbb{E} \left(\frac{1}{T} \int_0^T \mathbf{1}_{\{\|X(s)\|_{V_\gamma} \geq R_\varepsilon\}} \, \mathrm{d}s \right) \\ &\leq \varepsilon \mathbb{E} \left(\frac{1}{T} \int_0^T \Theta(Y(s)) + \|W_A(s)\|_{V_\gamma}^2 \, \mathrm{d}s \right) \\ &\leq \varepsilon \left(C \left(1 + \frac{1}{T} \right) + M \right) \leq \varepsilon \left(2C + M \right) \end{aligned}$$

Theorem: *(ES, Stannat: JEE 08')*
Assume that there exist α_1, α_2, $\alpha_3 > 0$, $s \geq 2$

$$\langle Ay+F(y+w), y\rangle \leq -\alpha_1\|y\|^2_{V_{\frac{1}{2}}} + \alpha_2\|w\|^s_{V_\gamma} + \alpha_3, y \in D(A), w \in V_\gamma.$$

$$\int_H \|x\|^2_{V_\gamma}\,\mu(dx) < \infty, \qquad \int_H e^{\varepsilon\|x\|^{\frac{4}{s}}}\,\mu(dx) < \infty\,,$$

Proof: $\rho : V_\gamma \to \mathbb{R}_+$, be a continuous function such that

$$\rho(y+w) \leq c_1\Theta(y) + c_2 e^{\varepsilon_0\|w\|^2_{V_\gamma}} + c_3 \qquad y, w \in V_\gamma.$$

Then

$$\int_H \rho\, d\mu < \infty.$$

$$P_t f(x) := \mathbb{E} f(X(t,x)), \quad t \geq 0,\ f \in C_b(H),$$

Kolmogorov equation

$$\frac{d}{dt} P_\varphi = L_F \varphi$$

$$L_F \varphi(x) = \frac{1}{2} \operatorname{Tr}(Q D^2 \varphi(x)) + \langle x, AD\varphi(x)\rangle + \langle F(x), D\varphi(x)\rangle,$$

$$\varphi \in \mathcal{F}C_b^2(D(A)) := \{\varphi \in C_b^2(H),\ \varphi(x) = f_m(\langle x, e_1\rangle, \ldots, \langle x, e_m\rangle), \quad f_m \in C_b^2(\mathbb{R}^m)\}.$$

$$\int_H P_t\varphi \, d\mu = \int_H \varphi \, d\mu \quad \text{for all } \varphi \in \mathcal{B}_b(H)$$

$$P_t \dashrightarrow \widetilde{P}_t \quad \text{on } L^1(H,\mu).$$

$\widetilde{P}_t$ is C_0-sgp on $L^1(H,\mu)$ with generator $(\widetilde{L}_F, D(\widetilde{L}_F))$

Uniqueness of $\widetilde{P}_t$?

$$L_F\varphi(x) = \frac{1}{2}\operatorname{Tr}_H\left(QD^2\varphi(x)\right) + \langle x, AD\varphi(x)\rangle + \langle F(x), D\varphi(x)\rangle$$

$$\text{invariance of } \mu \Longrightarrow \int L_F\varphi \, d\mu = 0 \qquad \forall\, \varphi \in \mathcal{F}C_b^2$$

$\Longrightarrow (L_F, \mathcal{F}C_b^2(D(A)))$ is **dissipative** on $L^1(H, \mu)$

$\exists \lambda > 0$ with $(\lambda - L_F)(\mathcal{F}C_b^2(D(A)))$ is dense (**range condition**)

$$\Downarrow$$

$(\bar{L}_F, D(\bar{L}_F))$ generates a C_0-semigroup of contrations on $L^1(H, \mu)$

$$\Downarrow$$

L^1-Uniqueness

$$du(t,x) = \left(\frac{d^2 u}{dx^2}(t,x) + f(u(t,x)) \right) \, dt + (-A)^{-\beta} \, dW_t(x),$$

on $L^2([0,L])$, $\quad 0 < \beta < \frac{1}{2}$.

$$A = \frac{d^2}{dx^2}, \quad D(A) = H_0^1(I) \cap H^2(I).$$

$$A^* = A, \quad \langle Ax, x \rangle < 0.$$

$$e_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x \right), n \geq 1,$$

$$Ae_n = \lambda e_n, \quad \lambda_n = -\left(\frac{\pi}{L} \right)^2 n^2.$$

$$f(t) = a_{2n+1}t^{2n+1} + \ldots + a_1 t, \quad a_{2n+1} < 0.$$

$$F(u)(x) = f(u(x))\,, u \in \mathcal{B}(I)\,.$$

F is local Lipschitz and bounded on bounded sets of V_γ, $\gamma > \frac{1}{4}$.

$$\|uv\|_{V_\gamma} \le c_\gamma \|u\|_{V_\gamma} \cdot \|v\|_{V_\gamma}.$$

$$\|F(u) - F(v)\|_{V_\gamma} \le C_\gamma \|u - v\|_{V_\gamma} \left(1 + \|u\|_{V_\gamma} + \|v\|_{V_\gamma}\right)^{2n}$$

$$\langle F(u+v), u^{2r+1}\rangle \leq \frac{a_{2n+1}}{2}\int u^{2(r+n+1)}\,dx+\alpha_2\|v\|_{V_\gamma}^{(2n+1)2(r+n+1)}+\alpha_3\,.$$

$$\langle Au+F(u+v),u\rangle \leq -\|u\|^2_{V_{\frac{1}{2}}}+\alpha_2\|v\|_{V_\gamma}^{2(2n+1)(n+1)}+\alpha_3,$$

$$\Downarrow$$

$$\begin{cases}\text{Existence of } \mu.\\ \int \|x\|^2_{V_\gamma}\,\mu(dx)<\infty.\\ \int e^{\epsilon\|x\|^{\frac{2}{(2n+1)(n+1)}}}\,\mu(dx)<\infty \text{ for } \varepsilon \text{ small}\,.\end{cases}$$

Cahn-Hilliard equations

$$\begin{cases} dX(t) = \Big(AX(t) + (-A)^{\frac{1}{2}} F(X(t))\Big) dt + dW_t, \quad t \geq 0 \\ X(0) = x \end{cases}$$

$(\mathbf{H_1^s})$ $F:\ V_{\frac{1}{4}} \to V_{\frac{1}{4}}$ continuous, $F(V_{\frac{1}{2}}) \subseteq V_{\frac{1}{2}}$,

$$\|F(x)\|_{\frac{1}{4}} \leq a(1 + \|x\|^r_{\frac{1}{4}}), \quad x \in V_{\frac{1}{4}}.$$

$(\mathbf{H_2^s})$

$$\mathrm{Range}(I - F) = H, \quad \langle F(u) - F(v), u - v\rangle \leq 0, \quad u,\ v \in V_{\frac{1}{4}}.$$

$$L_F\varphi(x) = L\varphi(x) + \langle F(x), (-A)^{\frac{1}{2}} D\varphi(x)\rangle, \quad \varphi \in \mathcal{F}C_b^2(D(A))$$

$$L\varphi(x) := \frac{1}{2}\operatorname{Tr} D^2\varphi(x) + \langle x, AD\varphi(x)\rangle$$

$$\updownarrow$$

$$\begin{cases} dZ(t) = AZ(t)dt + dW_t. \\ Z(0) = x \in H. \end{cases}$$

1. Find a probability measure μ such that $\int_H L_F\varphi(x)\,\mu(dx) = 0$, $\varphi \in \mathcal{F}C_b^2(D(A))$.
2. Maximal dissipativity of L_F in $L^1(H,\mu)$.

$\exists$ F_α global Lipschitz, dissipative, $\quad F_\alpha(x) \to F(x)$ pointwise.

$$\|F_\alpha(x)\| \leq a(1 + \|x\|^r_{\frac{1}{4}}), \quad a > 0,\ x \in V_{\frac{1}{4}}.$$

Consider

$$(E_\alpha) \quad \begin{cases} dX_\alpha(t) = \Big(AX_\alpha(t) + (-A)^{\frac{1}{2}} F_\alpha(X(t))\Big)dt + dW_t, & t \geq 0. \\ X_\alpha(0) = x \in H. \end{cases}$$

From

$$\int_0^t s^{-2\alpha} \|e^{sA}\sqrt{Q}\|_{HS}^2 \, ds < \infty, \quad 0 < \alpha < \frac{1}{2}$$

we get existence of the solution $X_\alpha(t,x)$ in $V_{\frac{1}{4}}$.

$$X_\alpha(t) = e^{tA}x + \int_0^t e^{(t-s)A}(-A)^{\frac{1}{2}} F_\alpha(X_\alpha(s))ds + \int_0^t e^{(t-s)A} dW_s, \quad t \geq 0.$$

$$P_t^\alpha f(x) := \mathbb{E} f(X_\alpha(t,x)), \quad t \geq 0, \ f \in \mathcal{B}_b(H).$$

Theorem: *(ES: JMAA 09')*

- $(P_t^\alpha)_{t\geq 0}$ is strong Feller semigroup ($P_t^\alpha \varphi \in C_b(H)$, $\varphi \in \mathcal{B}(H)$).
- $(P_t^\alpha)_{t\geq 0}$ is irreducible ($P_t^\alpha \varphi > 0$, $\varphi \geq 0$).
- There exists a unique invariant measure μ_α for $(P_t^\alpha)_{t\geq 0}$.

$$\int_H P_t^\alpha f(x)\,\mu_\alpha(dx) = \int_H f(x)\,\mu_\alpha(dx), \qquad f \in C_b(H),\ t \geq 0\,.$$

Moreover,

$$\int_H \|x\|^2_{\frac{1}{4}}\, \mu_\alpha(dx) \leq \kappa, \quad \int_H \|x\|^4\, \mu_\alpha(dx) \leq \theta$$

κ and θ are independent of α.

Proof: Suppose that $F_\alpha \in C^2_b(H,H)$ and consider the approximation problem

$$\begin{cases} dX_n(t) = \Big(AX_n(t) + A_n F_\alpha(X(t))\Big)dt + dW_t, & t \geq 0. \\ X_n(0) = x \end{cases}$$

where $A_n := (-A)^{\frac{1}{2}} nR(n,A) = -n(-A)^{-\frac{1}{2}} AR(n,A)$.

The solution $X_n(t,x)$ is differentiable with respect to x, for $h \in H$

$$DX_n(t,x)\cdot h = \eta_n^h(t,x)$$

$$\begin{cases} \frac{d}{dt}\eta_n^h = A\eta_n^h + (-A)^{\frac{1}{2}} DF_\alpha(X(t,x))\cdot \eta_n^h, \\ \eta_n^h(0,x) = h \in H. \end{cases}$$

$$|\eta_n^h| \le e^{t\frac{(Lip(F_\alpha))^2}{4}}|h|, \quad t \ge 0.$$

Bismut-Elworty formula

$$\langle DP_t^n\varphi(x), h\rangle = \frac{1}{t}\mathbb{E}\left(\varphi(X_n(t,x))\int_0^t \langle \eta_n^h, dW_s\rangle\right) \quad \text{for all } \varphi \in C_b^2(H).$$

For $\varphi \in C_b^2(H)$

$$
\begin{aligned}
|DP_t^n\varphi(x), h\rangle|^2 &\le t^{-2}\|\varphi\|_\infty^2 \int_0^t |\eta_n^h s|^2\, ds \\
&\le t^{-2}\|\varphi\|_\infty^2 \int_0^t e^{\frac{Lip(F_\alpha)^2}{2}s}|h|^2\, ds \\
&\le t^{-2}\|\varphi\|_\infty^2 \frac{2}{Lip(F_\alpha)^2}(e^{\frac{Lip(F_\alpha)^2}{2}t} - 1)|h|^2 \quad \text{for all } n \in \mathbb{N}.
\end{aligned}
$$

Letting $n \to \infty$ we get

$$
|DP_t\varphi(x)| \le t^{-1}\frac{\sqrt{2}}{Lip(F_\alpha)}(e^{\frac{Lip(F_\alpha)^2}{2}t} - 1)^{\frac{1}{2}}\|\varphi\|_\infty.
$$

$$
|P_t\varphi(x) - P_t\varphi(y)| \le t^{-1}\frac{\sqrt{2}}{Lip(F_\alpha)}(e^{\frac{Lip(F_\alpha)^2}{2}t} - 1)^{\frac{1}{2}}\|\varphi\|_\infty|x-y|.
$$

$$\mathrm{Var}(P_t(x,\cdot)-P_t(y,\cdot)) = \sup_{\{\varphi\in C_b(H):\ \|\varphi\|_\infty\le 1\}} |P_t\varphi(x)-P_t\varphi(y)|$$
$$= \sup_{\{\varphi\in C_b^2(H):\ \|\varphi\|_\infty\le 1\}} |P_t\varphi(x)-P_t\varphi(y)|$$

Therefore, for all $x,\ y\in H$

$$\mathrm{Var}(P_t(x,\cdot)-P_t(y,\cdot)) \le t^{-1}\frac{\sqrt{2}}{L}(e^{\frac{Lip(F_\alpha)^2}{2}t}-1)^{\frac{1}{2}}|x-y|.$$

For $\varphi\in B_b(H)$ we have

$$|P_t\varphi(x)-P_t\varphi(y)| = \left|\int_H \varphi(u)(P_t(x,du)-P_t(y,du))\right|$$

Consequently for $\varphi \in B_b(H)$

$$\begin{aligned}
|P_t\varphi(x) - P_t\varphi(y)| &= \left| \int_H \varphi(u)(P_t(x,du) - P_t(y,du)) \right| \\
&\leq \|\varphi\|_\infty \operatorname{Var}(P_t(x,\cdot) - P_t(y,\cdot)) \\
&\leq t^{-1} \frac{\sqrt{2}}{Lip(F_\alpha)} (e^{\frac{Lip(F_\alpha)^2}{2} t} - 1)^{\frac{1}{2}} \|\varphi\|_\infty |x-y|.
\end{aligned}$$

For general F_α, proceed by approximation:

$$F_\alpha^n = \int_{\mathbb{R}^n} \rho_n(\xi - P_n x) F_\alpha \left(\sum_{i=1}^n \xi_i e_i \right) \, d\xi.$$

$$|P_t\varphi(x) - P_t\varphi(y)| \leq t^{-1} \frac{\sqrt{2}}{Lip(F_\alpha)} (e^{\frac{Lip(F_\alpha)^2}{2} t} - 1)^{\frac{1}{2}} \|\varphi\|_\infty |x-y|.$$

Existence of invariant measure

Set

$$Y(t) := X(t) - W_A(t)\,.$$

$$Y'(t) = AY(t) + (-A)^{\frac{1}{2}} F_\alpha(Y(t) + W_A(t))\,,\, t > 0\,.$$

$$\mathbb{E}\left(\|(-A)^{-\frac{1}{4}}(Y(t))\|^2 + \alpha \int_0^t \|Y(s)\|_{\frac{1}{4}}^2 \,\mathrm{d}s \right) \leq C\,(t+1) \quad \text{for } t \geq 0.$$

Use energy estimate in $V_{-\frac{1}{4}}$

$$
\begin{aligned}
\frac{d}{dt}\|Y(t)\|^2_{V_{-\frac{1}{4}}} &= \langle AY(t) + (-A)^{\frac{1}{2}} F_\alpha(Y(t) + W_A(t)), (-A)^{-\frac{1}{2}}(Y(t))\rangle \\
&= -\|Y(t)\|^2_{\frac{1}{4}} + \langle F_\alpha(Y(t) + W_A(t)) - F(W_A(t)), Y(t)\rangle \\
&+ \langle F_\alpha(W_A(t)), Y(t)\rangle \\
&\leq -\|Y(t)\|^2_{\frac{1}{4}} + \langle F_\alpha(W_A(t)), Y(t)\rangle \\
&\leq -\|Y(t)\|^2_{\frac{1}{4}} + \sigma\|Y(t)\|^2 + \frac{1}{4\sigma} c(1 + \|W_A(t)\|^2).
\end{aligned}
$$

Krylov-Bogoliubov's theorem $\Rightarrow$ Existence of μ_α. with

$$
\int_H \|x\|^2_{\frac{1}{4}}\, \mu_\alpha(dx) \leq \kappa, \quad \int_H \|x\|^4\, \mu_\alpha(dx) \leq \theta
$$

κ and θ are independent of α.

Uniqueness of μ_α follows by Doob's Theorem.

$$V_{\frac{1}{4}} \hookrightarrow H \text{ compact.}$$

$$\Downarrow$$

There exists a probability measure μ on $\mathcal{B}(H)$ such that

$$\mu_\alpha \rightharpoonup \mu.$$

$$\int_H \|x\|^2_{\frac{1}{4}} \, \mu(dx) \le \kappa, \quad \int_H \|x\|^4 \, \mu(dx) \le \theta.$$

Theorem: *(ES, Stannat: JDE 09')*

- μ is infinitesimal invariant for L_F.
- L_F is maximal dissipative on $L^1(H,\mu)$.

Proof:

$$\int L_F\varphi(x)\,\mu(dx) = 0, \quad \varphi \in \mathcal{F}C_b^2(D(A)).$$

$$\int L_{F_\alpha}\varphi(x)\,\mu_\alpha(dx) = 0, \quad \lim_{n\to+\infty}\int \psi(x)\mu_{\alpha_n}(dx) = \int \psi(x)\mu(dx),$$

for ψ continuous, $\|\psi(x)\| \leq C(1+\|x\|^2)$.

- Maximal dissipativity

$$(\lambda - \overline{L_F})(D(\overline{L_F})) \quad \text{is dense in } L^1(H,\mu)?$$

Define

$$F_{\alpha,\beta}(x) := \int_H e^{\beta A} F_\alpha(e^{\beta A}x + y)\mathcal{N}_{0,Q_\beta}(dy)$$

$$Q_\beta x := \int_0^\beta e^{2sA}x\, ds.$$

$$F_{\alpha,\beta} \in C^\infty(H), \text{ dissipative}, \quad F_{\alpha,\beta}(x) \to F(x) \text{ pointwise}$$

$$\lim_{\alpha,\, \beta \to 0} \|F(x) - F_{\alpha,\beta}(x)\|_{\frac{1}{4}} = 0 \quad \text{in } L^2(H,\mu).$$

For $f \in \mathcal{F}C_b^2(D(A))$

$$(\lambda - L_{F_{\alpha,\beta}})\varphi_{\alpha,\beta} = f, \quad \lambda > 0$$

$$\varphi_{\alpha,\beta} = \int_0^{+\infty} e^{-\lambda t}\mathbb{E}(f(X_{\alpha,\beta}(t,x)))\ \mathrm{d}t.$$

Since $F_{\alpha,\beta}$ is regular and dissipative

$$\varphi_{\alpha,\beta} \in C_b^1(H), \quad K := \sup_{\alpha,\beta>0} \|D\varphi_{\alpha,\beta}(x)\|_{\frac{1}{4}} < +\infty.$$

$$\varphi_{\alpha,\beta} \in D(\overline{L_F})$$

If $\varphi_{\alpha,\beta}$ is a Cauchy sequence in $L^2(H,\mu)$

$$\lim_{\alpha,\beta\to 0} \varphi_{\alpha,\beta} = \varphi \text{ in } L^1(H,\mu).$$

$$\begin{aligned} L_F\varphi_{\alpha,\beta} &= L_{F_{\alpha,\beta}}\varphi_{\alpha,\beta} + \langle (-A)^{\frac{1}{4}}(F - F_{\alpha,\beta}), (-A)^{\frac{1}{4}} D\varphi_{\alpha,\beta}\rangle \\ &= \lambda\varphi_{\alpha,\beta} - f + \langle (-A)^{\frac{1}{4}}(F - F_{\alpha,\beta}), (-A)^{\frac{1}{4}} D\varphi_{\alpha,\beta}\rangle \end{aligned}$$

$$\xrightarrow[\alpha,\beta\to 0]{L^1(H,\mu)} \lambda\varphi - f.$$

$$\Rightarrow \quad \varphi \in D(\overline{L_F}), \quad (\lambda - \overline{L_F})\varphi = f.$$

$$\Rightarrow \quad \mathcal{F}C_b^2(D(A)) \subset (\lambda - \overline{L_F})(D(\overline{L_F})).$$

- $\varphi_{\alpha,\beta} \in D(\overline{L_F})$.

$$\int_H \varphi_{\alpha,\beta}(x)\overline{L_F}\varphi_{\alpha,\beta}(x)\mu(dx) = -\frac{1}{2}\int_H \langle D\varphi_{\alpha,\beta}(x), D\varphi_{\alpha,\beta}(x)\rangle \mu(dx).$$

- $$\overline{L_F}(\varphi_{\alpha,\beta}^2) = 2\varphi_{\alpha,\beta}\overline{L_F}\varphi_{\alpha,\beta} + \langle D\varphi_{\alpha,\beta}, D\varphi_{\alpha,\beta}\rangle.$$

$\varphi_{\alpha,\beta}$ is a Cauchy sequence in $L^2(H,\mu)$?

Set

$$\varphi_{\alpha,\beta}^{\gamma,\delta} = \varphi_{\alpha,\beta} - \varphi_{\gamma,\delta}, \quad \alpha,\ \beta,\ \gamma,\ \delta > 0.$$

$$\lambda \int_H (\varphi^{\gamma,\delta}_{\alpha,\beta}(x))^2 \mu(dx) + \frac{1}{2}\int_H \langle D\varphi^{\gamma,\delta}_{\alpha,\beta}(x), D\varphi^{\gamma,\delta}_{\alpha,\beta}(x)\rangle\, \mu(dx)$$
$$= \int_H (\lambda - \overline{L_F})\varphi^{\gamma,\delta}_{\alpha,\beta}(x)\, \varphi^{\gamma,\delta}_{\alpha,\beta}(x)\, \mu(dx)$$

$$\int_H (\lambda - \overline{L_F})\varphi^{\gamma,\delta}_{\alpha,\beta}(x)\, \varphi^{\gamma,\delta}_{\alpha,\beta}(x)\, \mu(dx)$$
$$= \int_H \langle (-A)^{\frac{1}{4}}(F_{\alpha,\beta}(x) - F(x)), (-A)^{\frac{1}{4}} D\varphi_{\alpha,\beta}(x)\rangle \varphi^{\gamma,\delta}_{\alpha,\beta}(x)\, \mu(dx)$$
$$- \int_H \langle (-A)^{\frac{1}{4}}(F_{\gamma,\delta}(x) - F(x)), (-A)^{\frac{1}{4}} D\varphi_{\gamma,\delta}(x)\rangle \varphi^{\gamma,\delta}_{\alpha,\beta}(x)\, \mu(dx)$$
$$\leq 2K\frac{\|f\|_\infty}{\lambda}\Big(\|(-A)^{\frac{1}{4}}(F_{\alpha,\beta} - F)\|_{L^2(H,\mu)} + \|(-A)^{\frac{1}{4}}(F_{\gamma,\delta} - F)\|_{L^2(H,\mu)}\Big).$$

Let $I = [0,1]$, $A = -\frac{d^4}{dx^4}$: (Neumann bilaplacian). $A \leq 0$, $A = A^*$ on $\dot{H}$ with domain

$$D(A) := \left\{ v \in \dot{H} \cap H^4(I) : \frac{dv}{dx}(x) = \frac{d^3 v}{dx}(x) = 0,\ x = 0,\ 1 \right\},$$

$$\dot{H} := \left\{ v \in L^2(I) : \int_0^1 v(x)\, dx = 0 \right\}.$$

$$D((-A)^{\frac{1}{2}}) = V_{\frac{1}{2}} = \{u \in \dot{H}^2(I),\ \tfrac{dv}{dx}(0) = \tfrac{dv}{dx}(1) = 0\}\ ,\ V_{\frac{1}{4}} = \dot{H}^1(I).$$

$$f(t) = -t^3.$$

$$du(t,x) = \left(-\frac{d^4}{dx^4} u(t,x) - \frac{d^2}{dx^2} f(u(t,x)) \right) dt + dW(t,x)$$

Set

$$F(u)(x) = -u^3(x), \quad u \in V_{\frac{1}{4}}.$$

$$F_\alpha(u) := f_\alpha(u), \quad u \in V_{\frac{1}{4}}, \qquad f_\alpha(t) = \frac{-t^3}{1+\alpha t^2}.$$

F_α is dissipative, Lipschitz,

$$\|F_\alpha(x)\| \leq a(1 + \|x\|_{\frac{1}{4}}^r), \quad x \in V_{\frac{1}{4}}.$$

$$du_\alpha(t,x) = \left(-\frac{d^4}{dx^4} u_\alpha(t,x) - \frac{d^2}{dx^2} F_\alpha(u_\alpha(t,x))\right) dt + dW_t$$

$$\int \|v^m\|^2_{\frac{1}{4}}\, \mu_\alpha(dv) < \delta, \quad \text{for all } v \in V_{\frac{1}{4}},\ m \geq 1.$$

$$\Downarrow$$

$$\begin{cases} \text{Existence of } \mu, \\ \int_H L_F\, \varphi(x)\, \mu(dx) = 0, \quad \varphi \in \mathcal{F}C_b^2(D(A)), \\ \int_H \|F\|^2_{\frac{1}{4}}\, \mu(dx) < +\infty. \end{cases}$$

Burgers equations:

$$\begin{cases} dX(t) = \Big(AX(t) + B(X(t))\Big)dt + CdW_t, & t \geq 0. \\ X(0) = X_0 \end{cases}$$

on the space $L^2(0,1)$

- $Au = D_\xi^2 u$, $u \in D(A)$, $D(A) = H^2(0,1) \cap H_0^1(0,1)$.
- $B(u) = D_\xi(u^2)$, $x \in H_0^1(0,1)$.
- $(W_t)_{t\geq 0}$ is H- valued cylindrical Wiener process.
- C is a bounded linear operator.

(Da Prato, Debussche, Temam 94), *(Da Prato, Gatarek 95)*, *(Gyongy 98)*

For every initial valued $X_0 \in L^2(0,1)$. There exists a unique mild solution

$$X(t) = e^{tA}X_0 + \int_0^t e^{(t-s)A}B(X(s))ds + \int_0^t e^{(t-s)A}CdW_s, \quad t \geq 0.$$

with a unique invariant measure μ.
Moment estimates of μ?

Lyapunov condition of polynomial type: There exist positive constants α, β, γ, δ and $s \geq 2$ such that

$$\langle Ay + B(y+w), y \rangle \leq -\alpha \|y\|_{\gamma_1}^2 + \beta \|w\|_{\gamma_2}^s \cdot \|y\|_{\gamma_1}^2 + \gamma \|w\|_{\gamma_2}^s + \delta$$

for all $y \in D(A)$, $w \in V_{\gamma_2}$.

Theorem: *(ES, Stannat: JFA 10')* Suppose A and $Q = CC^*$ are simultaneously diagonalizable and that $(-A)^{\gamma_2+\varepsilon} C$ is Hilbert-Schmidt for some $\varepsilon > 0$. Then μ satisfies the moment estimate

$$\int \left(1 + \|x\|_{\gamma}^2\right) \|x\|^{2p} \, \mu(dx), \quad p \geq 0, \gamma < \gamma_1.$$

Proof:

$$\langle Ay + B(y+w), y\rangle \leq -\alpha_1 \|y\|^2_{\frac{1}{2}} + \alpha_2 \|w\|^s_{\gamma_0} \cdot \|y\|^2_{\frac{1}{2}} + \alpha_3 \|w\|^s_{\gamma_0} + \alpha_4$$

$\delta \in (0, \frac{1}{2})$, $\gamma \in \mathbb{R}$.
for $\lambda > 0$ let

$$W_{A-\lambda}(t) := \int_0^t e^{(t-s)(A-\lambda)} C\, dW(s), \quad \lambda > 0.$$

Consider the following decomposition

$$X(t) = Y_\lambda(t) + W_{A-\lambda,C}(t)$$

$$dY_\lambda(t) = \Big(AY_\lambda(t) + \lambda W_{A-\lambda}(t)\Big)dt + B(Y_\lambda(t) + W_{A-\lambda}(t))dt$$

Pathwise Control on the Stochastic Convolution

$$\sup_{0\leq t\leq T} \|W_{A-\lambda}(t)\|_\gamma^2 \leq C_\delta^2 \sum_{k=1}^{+\infty} \frac{\lambda_k^{2\gamma}}{(\lambda+\lambda_k)^{2\delta}} M_k(\delta, T)^2.$$

Here,

$$M_k(\delta, T) := \sup_{0\leq s<t\leq T} \frac{|\beta_k(t)-\beta_k(s)|}{|t-s|^\delta}, \quad k\geq 1$$

ind. r.v., having finite moments of any order.
$(\lambda_k)_{k\geq 1}$ and $(q_k)_{k\geq 1}$ the eigenvalues of $-A$ and Q resp.

$$\|W_{A-\lambda}(t)\|_\gamma^2 = \sum_{k=1}^{+\infty} \lambda_k^{2\gamma} \left(\int_0^t e^{-(\lambda+\lambda_k)(t-s)} \sqrt{q_k}\, d\beta_k(s)\right)^2,$$

$$\frac{1}{2}\frac{d}{dt}\|Y_\lambda(t)\|^2 \leq -\frac{\alpha}{2}\|Y_\lambda(t)\|_{\gamma_1}^2 + R_\lambda(t).$$

Where

$$R_\lambda(t) = \delta + \gamma\|W_{A-\lambda}(t)\|_{\gamma_2}^s + \frac{\lambda^2}{2\alpha}\|W_{A-\lambda}(t)\|_{-\gamma_1}^2$$

$$\lambda = \left(\frac{4}{\alpha}(\beta M_T(\gamma_2, s) + 1)\right)^{\frac{1}{\varepsilon}s}.$$

Conclude moment estimate for

$$X(t) = Y_\lambda(t) + W_{A-\lambda,C}(t).$$

Multiplicative Noise:

$$\begin{cases} du(t) = \Big(Au(t) + F(u(t))\Big)dt + g(u(t))dW_t, & \text{for } t \geq 0, \\ u(0) = x \in H, \end{cases}$$

- g is Lipschitz from H into $\mathcal{L}_{HS}(H)$ with Lipschitz constant L.
- F is locally Lipschitz, dissipative and there exists a continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}$, with $\lim_{r\to+\infty} \frac{\rho(r^2)}{r^2} = -\infty$ such that

$$\langle F(u), u\rangle \leq \rho(\|u\|_0^2), \quad u \in V_\gamma.$$

$$\Rightarrow \quad \forall\, \lambda > 0 \quad \exists\, K_\lambda \geq 0 \quad \langle F(v), v\rangle \leq -\lambda\|v\|_0^2 + K_\lambda.$$

Theorem: *((ES, Scheutzow, Toelle, van Gaans)* For every Lipschitz diffusion term g. The solution

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(u(s))ds + \int_0^t e^{(t-s)A} g(u(s))dW_s, \quad t \geq 0,$$

has an invariant measure μ. *(Appl. Math. Optim. (2013))*.

Application: Reaction diffusion equations with multiplicative noise.

Thank you!