J. Fluid Mech. (2022), *vol.* 950, A33, doi:10.1017/jfm.2022.830

Explicit Dirichlet–Neumann operator for water waves

Didier Clamond†

Université Côte d'Azur, CNRS UMR 7351, Laboratoire J. A. Dieudonné, Parc Valrose, 06108 Nice CEDEX 2, France

(Received 30 January 2022; revised 13 September 2022; accepted 27 September 2022)

An explicit expression for the Dirichlet–Neumann operator for surface water waves is presented. For non-overturning waves, but without assuming small amplitudes, the formula is first derived in two dimensions, and subsequently extrapolated to higher dimensions and with a moving bottom. Although described here for water waves, this elementary approach could be adapted to many other problems having similar mathematical formulations.

Key words: general fluid mechanics, surface gravity waves

1. Introduction

In this short paper, we consider the classical problem of gravity waves propagating at the (non-overturning) free surface of a homogeneous non-viscous fluid in irrotational motion over an impermeable (uneven but non-overturning) seabed. Mathematically, in two dimensions without obstacles (i.e. for a simply connected fluid domain extending to infinity in all horizontal directions), this leads to the system of equations (for $x \in \mathbb{R}$, $t \in \mathbb{R}$ or $t \geqslant t_0$) (Wehausen & Laitone 1960)

$$\partial_x^2 \phi + \partial_y^2 \phi = 0 \quad \text{for } -d(x) \leqslant y \leqslant \eta(x,t), \tag{1.1}$$

$$\partial_y \phi + (\partial_x d)(\partial_x \phi) = 0 \quad \text{at } y = -d(x), \tag{1.2}$$

$$\partial_y \phi - \partial_t \eta - (\partial_x \eta)(\partial_x \phi) = 0 \quad \text{at } y = \eta(x,t), \tag{1.3}$$

$$\partial_t \phi + g\eta + \tfrac{1}{2}(\partial_x \phi)^2 + \tfrac{1}{2}(\partial_y \phi)^2 = 0 \quad \text{at } y = \eta(x,t), \tag{1.4}$$

where $\phi(x, y, t)$ is a velocity potential such that $u \stackrel{\text{def}}{=} \partial_x \phi$ is the horizontal velocity and $v \stackrel{\text{def}}{=} \partial_y \phi$ is the vertical one, $g > 0$ is the acceleration due to gravity (directed downwards), with (x, y) respectively the horizontal and upward vertical Cartesian coordinates, and t is

† Email address for correspondence: didier.clamond@univ-cotedazur.fr

© The Author(s), 2022. Published by Cambridge University Press

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

the time. Here $y = \eta(x, t)$, $y = 0$ and $y = -d(x)$ are, respectively, the equations of the free surface, of the still-water level and of the bottom; and $h(x, t) \stackrel{\text{def}}{=} \eta(x, t) + d(x)$ is the total water depth. Physically, (1.1) means that the motion is irrotational and isochoric, (1.2) and (1.3) characterise the impermeability of the bottom and of the free surface, while (1.4) expresses that the pressure at the free surface equals the constant atmospheric pressure (set to zero without loss of generality). Capillarity and other surface effects can be considered but they do not affect the analysis below, so they are of no interest here. Also, extensions of (1.1)–(1.4) in higher dimensions and/or moving bottoms are straightforward; these generalisations are considered at the end of the present paper. However, further generalisations (e.g. overturning surface and/or bottom, submerged obstacles, floating bodies, lateral solid boundaries, rough bottom) are beyond the scope of the present study; they require *ad hoc* investigations.

A Dirichlet–Neumann (or Dirichlet-to-Neumann) operator (DNO) takes as input a function expressed at a point of the domain boundary and outputs its (outward) normal derivative at the same point. Here, the DNO producing the (non-unitary outgoing) normal derivative at the free surface is $G(\phi_s) \stackrel{\text{def}}{=} [\partial_y\phi - (\partial_x\eta)(\partial_x\phi)]_{y=\eta}$, where $\phi_s(x, t) \stackrel{\text{def}}{=} \phi(x, \eta, t)$ denotes the velocity potential at the free surface. Fulfilling the Laplace equation (1.1) and the bottom impermeability condition (1.2), the DNO is a homogeneous linear function of ϕ_s, i.e. $G(\phi_s) = \mathscr{G}\phi_s$, where $\mathscr{G}$ is a self-adjoint positive-definite pseudo-differential operator depending nonlinearly on η and d (Craig & Sulem 1993; Craig *et al.* 2005). The operator $\mathscr{G}$ is a fundamental mathematical object because it 'encodes' the domain geometry, the kinematics of the fluid motion and the bottom impermeability; moreover, it appears explicitly in the Hamiltonian formulation (Zakharov 1968) of the equations (1.1)–(1.4). Understandably, $\mathscr{G}$ has been the subject of many mathematical studies – see Lannes (2013) and Nicholls & Reitich (2001) for details – and it is at the heart of several rigorous investigations on water waves (e.g. Alazard, Burq & Zuily 2012; Alazard & Baldi 2015). Knowledge of the mathematical features of the DNO is certainly important, but its explicit construction is at least as important, in particular for practical applications.

For a flat horizontal free surface and bottom, the fluid domain is a strip and the DNO is easily obtained analytically, e.g. via a Fourier transform. For a wavy surface and bottom, the DNO can be constructed as a perturbation of the strip, assuming small amplitudes. This is the route followed in two dimensions by Craig & Sulem (1993) and in three dimensions by Craig & Groves (1994) for flat seabeds, then extended to varying bottoms (Craig *et al.* 2005); these authors provided recurrence relations for computing the DNO to an arbitrary order of their perturbative expansion. For small perturbations of the flat surface and seabed, other series representations of the DNO are available in the literature (Dommermuth & Yue 1987; West *et al.* 1987). Although all these series are formally equivalent, this is not necessarily the case with their truncations at the same order, as outlined by Schäffer (2008). Moreover, such expansions are badly conditioned, so prone to large numerical errors and instabilities (Wilkening & Vasan 2015). An explicit formulation of the DNO in expected to facilitate various reformulations for more efficient computations, for example, but this is not the scope of the present paper.

The main purpose of this paper is to show how explicit DNOs can be derived and, via a few examples, to show their interest for analytic manipulations. Although some indications on potential issues and remedies with numerical computations are briefly discussed, it is not the purpose here to derive the most effective way to compute numerically a DNO.

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

The paper is organised as follow. In § 2, an explicit DNO is derived in two dimensions via rather elementary algebra. This DNO being in complex form, a real reformulation is introduced in § 3 in order to facilitate analytical approximations. Some approximations for small amplitudes in finite depth and for finite amplitudes in shallow water are then derived in § 4. The DNO is extended to higher dimensions in § 5, and its generalisation for moving bottoms is provided in § 6. Finally, a summary and perspectives are briefly drawn in § 7.

2. Two-dimensional Dirichlet–Neumann operator

Let ψ be the streamfunction harmonic conjugate of the velocity potential ϕ (Milne-Thomson 2011). These two functions are related by the Cauchy–Riemann relations $\phi_x = \psi_y = u$ and $\phi_y = -\psi_x = v$. Thus, the complex potential $f \stackrel{\text{def}}{=} \phi + \mathrm{i}\psi$ is a holomorphic function of $z \stackrel{\text{def}}{=} x + \mathrm{i}y$, with $z = z_s \stackrel{\text{def}}{=} x + \mathrm{i}\eta$ at the free surface and $z = z_b \stackrel{\text{def}}{=} x - \mathrm{i}d$ at the bottom. (As general notation, subscripts 's' and 'b' denote quantities written, respectively, at the free surface and at the bottom.) The seabed being impermeable and static, it is a streamline where $\psi = \psi_b$ is constant. Without loss of generality, we then choose $\psi_b = 0$ for simplicity.

For any complex abscissa z_0, the Taylor expansion around $z_0 = 0$ is (omitting temporal dependences for brevity)

$$f(z - z_0) = \exp[-z_0 \partial_z] f(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n z_0^n}{n!} \frac{\partial^n f(z)}{\partial z^n}. \tag{2.1}$$

For instance, taking $z_0 = \mathrm{i}h = \mathrm{i}(d + \eta)$, the relation (2.1) written at the free surface becomes

$$f(z_s - \mathrm{i}h) = \exp[-\mathrm{i}h\partial_{z_s}] f(z_s), \tag{2.2}$$

with the formal operator $\exp[-\mathrm{i}h\partial_{z_s}] \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (n!)^{-1} (-\mathrm{i}h)^n \partial_{z_s}^n$ together with $\partial_{z_s} \stackrel{\text{def}}{=} (1 + \mathrm{i}\eta_x)^{-1}\partial_x$, $\partial_{z_s}^2 = (1 + \mathrm{i}\eta_x)^{-1}\partial_x(1 + \mathrm{i}\eta_x)^{-1}\partial_x$, etc. (Throughout this paper, we use the classical convention that any operator acts on everything it multiplies on its right, unless parentheses enforce otherwise.)

It should be noticed that exponents denote differential compositions, so h^n is the nth power of the function $h = d + \eta$, while $\partial_{z_s}^n$ is the nth iteration of the differential operator ∂_{z_s}. Therefore, for example, if h is not constant then $h^2 = h(x)^2 \neq h(h(x))$, $h^2\partial_{z_s}^2 \neq (h\partial_{z_s})^2 = h\partial_{z_s} h\partial_{z_s}$, $\exp[-\mathrm{i}h\partial_{z_s}] \neq \sum_{n=0}^{\infty} (n!)^{-1} (-\mathrm{i}h\partial_{z_s})^n$ and the operator inverse of $\exp[-\mathrm{i}h\partial_{z_s}]$ is not $\exp[\mathrm{i}h\partial_{z_s}]$ (but $\exp[\mathrm{i}h\partial_{z_b}]$ as shown in § 3).

Since $z_s - \mathrm{i}h = x - \mathrm{i}d = z_b$ then $f(z_s - \mathrm{i}h) = f(z_b) = \phi_b$ is real (recall that $\psi_b = \operatorname{Im} f_b = 0$ by definition), while $f(z_s) = \phi_s + \mathrm{i}\psi_s$ is complex. Therefore, the imaginary part of (2.2), i.e.

$$0 = \operatorname{Re}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\psi_s + \operatorname{Im}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\phi_s, \tag{2.3}$$

yields at once

$$\psi_s = -\left(\operatorname{Re}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\right)^{-1} \operatorname{Im}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\phi_s. \tag{2.4}$$

The equation for the free-surface impermeability being $\partial_t \eta = \mathscr{G}\phi_s = -\partial_x \psi_s = v_s - u_s \partial_x \eta$, an explicit definition of the DNO is obtained directly from (2.4) as

$$\mathscr{G} = \partial_x \left(\operatorname{Re}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\right)^{-1} \operatorname{Im}\{\exp[-\mathrm{i}h\partial_{z_s}]\}. \tag{2.5}$$

Formula (2.5) provides an explicit expression for the DNO, i.e. $\mathscr{G}$ appears only on the left-hand side. It is the main result of this paper, which can be generalised to higher

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

dimensions and for moving bottoms (see below). It is also suitable to derive various approximations, in particular high-order shallow-water approximations without assuming small amplitudes (see § 4.3 below; in fact, this goal was the original motivation for deriving (2.5)).

For applications, it is convenient to introduce an operator $\mathscr{J}$ such that $\mathscr{G} = -\partial_x \mathscr{J} \partial_x$, so

$$\mathscr{J} = -\left(\mathrm{Re}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\right)^{-1} \mathrm{Im}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\partial_x^{-1}. \tag{2.6}$$

Since $\mathscr{G}$ is a self-adjoint positive-definite operator (Lannes 2013), so is $\mathscr{J}$. Further, it is also convenient to introduce the operators $\mathscr{R}$ and $\mathscr{I}$ defined by

$$\mathscr{R} \stackrel{\text{def}}{=} \mathrm{Re}\{\exp[-\mathrm{i}h\partial_{z_s}]\}, \quad \mathscr{I} \stackrel{\text{def}}{=} -\mathrm{Im}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\partial_x^{-1}, \tag{2.7a,b}$$

so $\mathscr{J} = \mathscr{R}^{-1}\mathscr{I}$.

Although explicit, the formulae (2.5) and (2.6) are not quite in closed form since they involve series (via the definition of the exponential operator) and operator inversion. Additional relations, suitable for practical applications, are thus derived below.

3. Auxiliary relations

With different choices of z and z_0, the Taylor expansion (2.1) provides various relations of practical interest. Several variants of (2.5) can then be derived, their convenience depending on the problem at hand.

With the choice $z = z_b$ and $z_0 = -\mathrm{i}h = -\mathrm{i}(d+\eta)$, the relation (2.1) becomes

$$f(z_b + \mathrm{i}h) = f(z_s) = \exp[\mathrm{i}h\partial_{z_b}]f(z_b), \tag{3.1}$$

so a comparison with (2.2) yields at once

$$\left(\exp[-\mathrm{i}h\partial_{z_s}]\right)^{-1} = \exp[\mathrm{i}h\partial_{z_b}] \quad \Longleftrightarrow \quad \left(\exp[\mathrm{i}h\partial_{z_b}]\right)^{-1} = \exp[-\mathrm{i}h\partial_{z_s}]. \tag{3.2}$$

With this relation, the operator involving ∂_{z_s} in the DNO (2.5) can be replaced by one involving ∂_{z_b}. This is somewhat convenient in constant depth because, then, $\partial_{z_b} = \partial_x$. However, two operators then need to be inverted instead of one with (2.5), so further simplifications are desirable.

Taking $z = x$ together with $z_0 = -\mathrm{i}\eta$ and $z_0 = \mathrm{i}d$, (2.1) yields

$$f(x + \mathrm{i}\eta) = f(z_s) = \exp[\mathrm{i}\eta\partial_x]\, f(x), \quad f(x - \mathrm{i}d) = f(z_b) = \exp[-\mathrm{i}d\partial_x]\, f(x), \tag{3.3a,b}$$

and the elimination of $f(x)$ between these two relations, together with (2.2), yields

$$\exp[-\mathrm{i}h\partial_{z_s}] = \exp[-\mathrm{i}d\partial_x]\,(\exp[\mathrm{i}\eta\partial_x])^{-1} = \exp[-\mathrm{i}d\partial_x]\,(\exp[-\mathrm{i}\eta\partial_x])^{\dagger}\,(1 + \mathrm{i}\eta_x)\,, \tag{3.4}$$

where a † denotes the adjoint operator. (For any complex function γ of a single real variable x, the operator $\exp[\gamma\,\partial_x] \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}(n!)^{-1}\gamma^n\partial_x^n$ has for Hermitian adjoint $(\exp[\gamma\,\partial_x])^{\dagger} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}(n!)^{-1}\partial_x^n(-\gamma^*)^n$, a star denoting the complex conjugate. We then have $(\exp[\gamma\,\partial_x])^{-1} = (\exp[\gamma^*\partial_x])^{\dagger}(1+\gamma_x)$.) We thus have relations that allow us to avoid the computation of the ∂_z operators, moreover without inversions. The operator $\mathscr{R}$ remains to be inverted, however.

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

For a real or complex function γ depending on a single real variable x, let the operators and their Hermitian adjoints be

$$\mathscr{C}_\gamma \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, \gamma^{2n} \partial_x^{2n}, \quad \mathscr{S}_\gamma \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, \gamma^{2n+1}\, \partial_x^{2n+1}, \tag{3.5a,b}$$

$$\mathscr{C}_\gamma^\dagger = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, \partial_x^{2n} \gamma^{*2n}, \quad \mathscr{S}_\gamma^\dagger = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\, \partial_x^{2n+1}\, \gamma^{*2n+1}. \tag{3.6a,b}$$

We then have

$$\exp[-\mathrm{i}d\partial_x] = \mathscr{C}_d - \mathrm{i}\,\mathscr{S}_d, \quad (\exp[-\mathrm{i}\eta\,\partial_x]\,)^\dagger = \mathscr{C}_\eta^\dagger + \mathrm{i}\,\mathscr{S}_\eta^\dagger, \tag{3.7a,b}$$

and the relation (3.4) is split into real and imaginary parts as

$$\mathrm{Re}\{\exp[-\mathrm{i}h\partial_{z_s}]\} = \mathscr{C}_d\,\mathscr{C}_\eta^\dagger + \mathscr{S}_d\,\mathscr{S}_\eta^\dagger - \mathscr{C}_d\,\mathscr{S}_\eta^\dagger \eta_x + \mathscr{S}_d\,\mathscr{C}_\eta^\dagger \eta_x, \tag{3.8}$$

$$\mathrm{Im}\{\exp[-\mathrm{i}h\partial_{z_s}]\} = \mathscr{C}_d\,\mathscr{S}_\eta^\dagger - \mathscr{S}_d\,\mathscr{C}_\eta^\dagger + \mathscr{C}_d\,\mathscr{C}_\eta^\dagger \eta_x + \mathscr{S}_d\,\mathscr{S}_\eta^\dagger \eta_x. \tag{3.9}$$

With the operator relation $\eta_x = \partial_x\eta - \eta\partial_x$ (resulting from the Leibniz rule), we have

$$\mathscr{C}_\eta^\dagger \eta_x = \partial_x^{-1}\,\mathscr{S}_\eta^\dagger\,\partial_x - \mathscr{S}_\eta^\dagger, \quad \mathscr{S}_\eta^\dagger \eta_x = \mathscr{C}_\eta^\dagger - \partial_x^{-1}\,\mathscr{C}_\eta^\dagger\,\partial_x, \tag{3.10a,b}$$

so the relations (3.8) and (3.9) yield

$$\mathscr{R} = \mathscr{C}_d\partial_x^{-1}\,\mathscr{C}_\eta^\dagger\,\partial_x + \mathscr{S}_d\partial_x^{-1}\,\mathscr{S}_\eta^\dagger\,\partial_x, \quad \mathscr{I} = \mathscr{S}_d\partial_x^{-1}\,\mathscr{C}_\eta^\dagger - \mathscr{C}_d\partial_x^{-1}\,\mathscr{S}_\eta^\dagger. \tag{3.11a,b}$$

The latter relations are particularly convenient to derive analytic approximations and to extrapolate the DNO to higher dimensions, as shown below.

4. Approximate Dirichlet–Neumann operators

From the explicit DNO (2.5) and the relations derived in the previous section, several approximations of practical interest can be easily obtained. We consider here only two special cases.

4.1. *Infinitesimal waves in arbitrary depth*

Assuming that the free surface η remains close to zero, one can formally expand the DNO in increasing order of nonlinearities in η (Craig & Sulem 1993; Craig *et al.* 2005). Thus, writing $\mathscr{G} = \mathscr{G}_0 + \mathscr{G}_1 + \mathscr{G}_2 + \cdots$ and similarly for $\mathscr{R}$, $\mathscr{I}$ and $\mathscr{J}$, one obtains at once from (3.11) the following:

$$\mathscr{R}_0 = \mathscr{C}_d, \quad \mathscr{R}_1 = -\mathscr{S}_d\eta\partial_x, \quad \mathscr{R}_2 = -\tfrac{1}{2}\,\mathscr{C}_d\partial_x\eta^2\partial_x, \quad \text{etc.}, \tag{4.1a–c}$$

$$\mathscr{I}_0 = \mathscr{S}_d\partial_x^{-1}, \quad \mathscr{I}_1 = \mathscr{C}_d\eta, \quad \mathscr{I}_2 = -\tfrac{1}{2}\,\mathscr{S}_d\partial_x\eta^2, \quad \text{etc.} \tag{4.2a–c}$$

The relation

$$\begin{aligned}\mathscr{R}^{-1} &= [\mathscr{R}_0 + \mathscr{R}_1 + \mathscr{R}_2 + \cdots]^{-1} = [1 + \mathscr{R}_0^{-1}\,\mathscr{R}_1 + \mathscr{R}_0^{-1}\,\mathscr{R}_2 + \cdots]^{-1}\mathscr{R}_0^{-1} \\ &= [1 - \mathscr{R}_0^{-1}\,\mathscr{R}_1 - \mathscr{R}_0^{-1}\,\mathscr{R}_2 + \mathscr{R}_0^{-1}\,\mathscr{R}_1\,\mathscr{R}_0^{-1}\,\mathscr{R}_1 + \cdots]\mathscr{R}_0^{-1}\end{aligned} \tag{4.3}$$

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

then yields, after some algebra,

$$\left.\begin{aligned} \mathscr{J}_0 = \mathscr{C}_d^{-1}\,\mathscr{S}_d\,\partial_x^{-1}, \quad \mathscr{J}_1 = \eta + \mathscr{J}_0\partial_x\eta\partial_x\mathscr{J}_0, \\ \mathscr{J}_2 = \tfrac{1}{2}\,\partial_x\eta^2\partial_x\mathscr{J}_0 + \mathscr{J}_0\partial_x\eta\partial_x\,\mathscr{J}_1 - \tfrac{1}{2}\,\mathscr{J}_0\partial_x^2\eta^2, \quad \text{etc.,} \end{aligned}\right\} \tag{4.4}$$

hence

$$\left.\begin{aligned} \mathscr{G}_0 = -\partial_x\,\mathscr{C}_d^{-1}\,\mathscr{S}_d, \quad \mathscr{G}_1 = -\partial_x\eta\partial_x - \mathscr{G}_0\eta\,\mathscr{G}_0, \\ \mathscr{G}_2 = \tfrac{1}{2}\,\partial_x^2\eta^2\,\mathscr{G}_0 - \mathscr{G}_0\eta\,\mathscr{G}_1 - \tfrac{1}{2}\,\mathscr{G}_0\partial_x\eta^2\partial_x, \quad \text{etc.} \end{aligned}\right\} \tag{4.5}$$

In constant depth, the expansion of Craig & Sulem (1993) is, as expected, recovered by introducing the operator $\mathscr{D} \stackrel{\text{def}}{=} \mathrm{i}\partial_x$, i.e. replacing ∂_x by $-\mathrm{i}\mathscr{D}$. With a variable bottom, the expansion of Craig *et al.* (2005) is also recovered, except for the definition of $\mathscr{G}_0$. Indeed, Craig *et al.* (2005) define $\mathscr{G}_0$ with an expansion for small amplitudes of the bottom corrugation (i.e. $\max|d(x) - \bar{d}|$ is small, where $\bar{d}$ is the mean depth), and they provide a recursion formula for computing this series. In (4.5), $\mathscr{G}_0$ is defined explicitly for arbitrary (non-overturning) bottom and no additional expansions are required.

4.2. *Remarks*

For higher-order approximations, the recursion formula of Craig & Sulem (1993) can be used verbatim with $\mathscr{G}_0$ defined here in (4.5). This approach is convenient for the derivation of (rather low-order) analytical approximations. However, with numerical computations, this recursion is prone to cancellation errors leading to large numerical errors and instabilities (Wilkening & Vasan 2015). This problem is more pronounced in higher dimensions (W. Craig, personal communication 2005).

These difficulties come mostly from the expansion of the inverse operator $\mathscr{R}^{-1}$. For numerical computations, this expansion should be avoided to obtain $B = \mathscr{R}^{-1}A$ (for some functions A and B). It is generally more efficient to solve $\mathscr{R}B = A$ via an iterative procedure. This is a similar problem to the resolution of linear systems of equations, for which iterative methods are often more efficient (Isaacson & Keller 1994). For the DNO, the relation (3.11*a*) (see also relation (5.11)) shows that $\mathscr{R}$ behaves (roughly) like a cosh function, so $\mathscr{R}^{-1}$ behaves like a sech function. The Maclaurin series of $\cosh(z)$ having an infinite radius of convergence, while that of $\operatorname{sech}(z)$ converges only for $|z| < \pi/2$, this provides an informal/heuristic argument showing why $B = \mathscr{R}^{-1}A$ should not be computed but $\mathscr{R}B = A$ should be solved instead. With other representations (than truncated Taylor series) of $\mathscr{R}^{-1}$, the computation of $B = \mathscr{R}^{-1}A$ may be efficient, however.

For linear waves in the context of a highly variable bathymetry, the improvements of the DNO expansion proposed by Andrade & Nachbin (2018) could be exploited to reformulate the explicit DNO in a more effective form for numerical computations. However, when speed and high numerical accuracy are required, the DNO perturbation expansions are not competitive (specially for steep waves) and boundary integral formulations should be preferred (Clamond & Grue 2001; Fructus *et al.* 2005; Fructus & Grue 2007).

4.3. *Long waves in shallow water*

For long waves in shallow water, the characteristic wavelength L_c is much larger than the characteristic depth d_c, so $\sigma \stackrel{\text{def}}{=} d_c/L_c \ll 1$ is a 'shallowness' dimensionless small parameter. The horizontal derivative ∂_x is then of first order in shallowness and the DNO can be expanded is power series of σ, without assuming small amplitude for the waves

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

and/or for the bottom corrugation. Thus, we do not need to explicitly introduce scalings to assess the order of terms – it is sufficient to count the number of derivatives. For instance, $\partial_x^3\eta$, $(\partial_x^2\eta)(\partial_x\eta)$ and $(\partial_x\eta)^3$ are all of third order in shallowness, as well as $\partial_x^3 d$, $(\partial_x^2 d)(\partial_x d)$ and $(\partial_x d)^3$.

We then have the shallow-water even-terms expansions $\mathscr{R} = \mathscr{R}_0 + \mathscr{R}_2 + \mathscr{R}_4 + \cdots$ (and similarly for $\mathscr{I}$ and $\mathscr{J}$), so, from (3.11),

$$\mathscr{R}_0 = 1, \quad \mathscr{R}_2 = -\tfrac{1}{2}\, d^2\partial_x^2 - d\partial_x\eta\partial_x - \tfrac{1}{2}\,\partial_x\eta^2\partial_x, \quad \text{etc.}, \tag{4.6a,b}$$

$$\mathscr{I}_0 = h, \quad \mathscr{I}_2 = -\tfrac{1}{6}\, d^3\partial_x^2 - \tfrac{1}{2}\, d^2\partial_x^2\eta - \tfrac{1}{2}\, d\partial_x^2\eta^2 - \tfrac{1}{6}\,\partial_x^2\eta^3, \quad \text{etc.}, \tag{4.7a,b}$$

hence, after some algebra,

$$\mathscr{J}_0 = h, \quad \mathscr{J}_2 = \tfrac{1}{2}\, h^2 d_{xx} + h h_x d_x - h d_x^2 + \tfrac{1}{3}\,\partial_x h^3\partial_x, \quad \text{etc.} \tag{4.8a,b}$$

Note that $\mathscr{J}_0$ and $\mathscr{J}_2$ are obviously self-adjoint, as they should be.

It should be emphasised that these approximations were obtained directly from the explicit DNO, considering weak variations in x (i.e. long waves in shallow water) but without assuming small amplitudes of the free surface and of the seabed (i.e. there are no restrictions on the magnitude of $|\eta|$ and $|d(x) - \bar{d}|$, $\bar{d}$ being the mean depth).

5. Dirichlet–Neumann operator in higher dimensions

It is rather straightforward to extrapolate the DNO given by (2.5) to three (and more) spatial dimensions. In higher dimensions, the holomorphic functions cannot be used but series representations remain. This feature is exploited here to obtain an explicit expression for the DNO in an arbitrary number of dimensions.

With $\boldsymbol{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ referring to the 'horizontal' coordinates, the mathematical problem is then posed in the $(N+1)$-dimensional Cartesian $(\boldsymbol{x}, y)$-space, with y the 'upward vertical' coordinate. Obviously, only the two-dimensional (i.e. $N = 1$) and three-dimensional (i.e. $N = 2$) cases are of physical interest for water waves. Let $\boldsymbol{\nabla} \stackrel{\text{def}}{=} (\partial_{x_1}, \ldots, \partial_{x_N})$, $\varDelta \stackrel{\text{def}}{=} \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\nabla}$ and $\mathscr{D} \stackrel{\text{def}}{=} (-\varDelta)^{1/2}$ denote, respectively, the horizontal gradient, Laplacian and semi-Laplacian operators.

The DNO is naturally extended in higher dimensions by extrapolating the relation $\mathscr{G} = -\partial_x\mathscr{R}^{-1}\mathscr{I}\partial_x$, the operators $\mathscr{R}$ and $\mathscr{I}$ having to be redefined. In the two-dimensional case, these operators are defined via complex expressions in § 2. In order to extend these operators to higher dimensions, one must consider their real form (3.11), so their extrapolation is natural.

One-dimensional operators involving only even-order derivatives have straightforward extensions in higher dimensions by replacing the second-order horizontal derivative ∂_x^2 by

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

the horizontal Laplacian $\varDelta$. For instance,

$$\mathscr{C}_d \mapsto \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, d^{2n} \varDelta^n \stackrel{\text{def}}{=} \cosh(d\,\mathscr{D}), \tag{5.1}$$

$$\mathscr{C}_\eta^\dagger \mapsto \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\, \varDelta^n \eta^{2n} \stackrel{\text{def}}{=} \cosh(\mathscr{D}\,\eta), \tag{5.2}$$

$$\mathscr{S}_d \partial_x^{-1} \mapsto \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, d^{2n+1} \varDelta^n \stackrel{\text{def}}{=} \sinh(d\,\mathscr{D})\mathscr{D}^{-1}, \tag{5.3}$$

$$\partial_x^{-1} \mathscr{S}_\eta^\dagger \mapsto \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\, \varDelta^n \eta^{2n+1} \stackrel{\text{def}}{=} -\mathscr{D}^{-1} \sinh(\mathscr{D}\,\eta). \tag{5.4}$$

It should be emphasised that, as in the one-dimensional case, the operators do not commute, so, for example, $\cosh(d\mathscr{D}) \neq \cosh(\mathscr{D}d)$ and $\cosh(d\mathscr{D})^{-1} \neq \operatorname{sech}(d\mathscr{D})$, the equalities holding only in constant depth because then $d\mathscr{D} = \mathscr{D}d$.

A natural extension of $\mathscr{I}\,\partial_x$ is thus $\mathscr{I}\,\nabla$ with

$$\mathscr{I} \mapsto \sinh(d\,\mathscr{D})\mathscr{D}^{-1}\cosh(\mathscr{D}\,\eta) + \cosh(d\,\mathscr{D})\mathscr{D}^{-1}\sinh(\mathscr{D}\,\eta). \tag{5.5}$$

In order to find the extension of $\partial_x \mathscr{R}^{-1}$, the operator $\mathscr{R}$ given by (3.11*a*) is rewritten as

$$\mathscr{R} = \mathscr{C}_d \partial_x{}^{-1}[\mathscr{C}_\eta^\dagger + \partial_x\, \mathscr{C}_d^{-1}(\mathscr{S}_d\, \partial_x^{-1})\partial_x(\partial_x^{-1}\, \mathscr{S}_\eta^\dagger)]\partial_x. \tag{5.6}$$

Thus, we have the natural extension

$$\partial_x \mathscr{R}^{-1} \mapsto [\cosh(\mathscr{D}\eta) + \mathscr{G}_0\, \mathscr{D}^{-1}\sinh(\mathscr{D}\eta)]^{-1} \nabla \cdot \cosh(d\mathscr{D})^{-1}, \tag{5.7}$$

where $\cosh(d\mathscr{D})^{-1}$ is the inverse operator of $\cosh(d\mathscr{D})$, and

$$\mathscr{G}_0 \stackrel{\text{def}}{=} -\nabla \cdot \cosh(d\mathscr{D})^{-1} \sinh(d\mathscr{D})\mathscr{D}^{-1}\, \nabla. \tag{5.8}$$

Therefore, the DNO becomes at once

$$\mathscr{G} = -[\cosh(\mathscr{D}\eta) + \mathscr{G}_0\, \mathscr{D}^{-1}\sinh(\mathscr{D}\eta)]^{-1} \nabla \cdot \cosh(d\mathscr{D})^{-1} \mathscr{I}\, \nabla. \tag{5.9}$$

In order to avoid misinterpretations of the formula (5.9), it is worth re-emphasising here that: (i) any operator acts on everything it multiplies on its right, so (5.9) should be applied successively leftward starting from the furthest right; and (ii) exponents denote operator compositions, so an exponent -1 means an operator inversion.

Note that $\mathscr{G} \to \mathscr{G}_0$ as $\eta \to 0$. Moreover, processing as in § 4.1 for infinitesimal waves, one finds the expansion of Craig *et al.* (2005), except for $\mathscr{G}_0$ that is defined implicitly by Craig *et al.* (2005) but explicitly here.

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

5.1. *Constant depth*

In constant depth, d commuting then with both $\mathscr{D}$ and ∇, we have the simplified relation $\mathscr{G}_0 = \mathscr{D}\tanh(d\mathscr{D})$, while $\mathscr{I}$ and $\mathscr{G}$ become (see Appendix A for details)

$$\mathscr{I} = \mathscr{D}^{-1}[\sinh(d\mathscr{D})\cosh(\mathscr{D}\eta) + \cosh(d\mathscr{D})\sinh(\mathscr{D}\eta)] = \mathscr{D}^{-1}\sinh(\mathscr{D}h), \tag{5.10}$$

$$\begin{aligned}\mathscr{G} &= -[\cosh(d\mathscr{D})\cosh(\mathscr{D}\eta) + \sinh(d\mathscr{D})\sinh(\mathscr{D}\eta)]^{-1}\nabla\cdot\mathscr{I}\,\nabla \\ &= -\cosh(\mathscr{D}h)^{-1}\,\mathscr{D}^{-1}\,\nabla\cdot\sinh(\mathscr{D}h)\nabla.\end{aligned} \tag{5.11}$$

A better-conditioned formulation, avoiding the computation of $\mathscr{D}$, is

$$\mathscr{G} = -[\operatorname{sech}(d\mathscr{D})\cosh(\mathscr{D}h)]^{-1}\,\nabla\cdot[\operatorname{sech}(d\mathscr{D})\operatorname{sinhc}(\mathscr{D}h)h]\nabla, \tag{5.12}$$

with

$$\operatorname{sech}(d\mathscr{D}) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}\,d^{2n}\,\mathscr{D}^{2n}, \quad \operatorname{sinhc}(\mathscr{D}h) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\,\mathscr{D}^{2n}h^{2n}, \tag{5.13a,b}$$

where E_n are the Euler numbers (Abramowitz & Stegun 1965) (since d and $\mathscr{D}$ commute, we have $\operatorname{sech}(d\mathscr{D}) = \cosh(d\mathscr{D})^{-1}$). The relation (5.12) involving only even powers of $\mathscr{D}$, only Laplacian and gradient operators need to be evaluated, i.e. the computation of the non-local operator $\mathscr{D}$ can be avoided.

As the DNO appears in Hamiltonian formulations of water waves, its functional variations are crucial to derive the equations of motion and to investigate stability (Fazioli & Nicholls 2010). Thanks to the explicit DNO (5.11), these variations can be obtained quite effortlessly. Indeed, with the relations (cf. Appendix A)

$$\cosh(\mathscr{D}(h+\delta h)) = \cosh(\mathscr{D}h) + \mathscr{D}\sinh(\mathscr{D}h)\delta h + O(\delta h^2), \tag{5.14}$$

$$\sinh(\mathscr{D}(h+\delta h)) = \sinh(\mathscr{D}h) + \mathscr{D}\cosh(\mathscr{D}h)\delta h + O(\delta h^2), \tag{5.15}$$

the first variation of the DNO is obtained at once as

$$\begin{aligned}\mathscr{G}(h+\delta h) = \mathscr{G}(h) &- \cosh(\mathscr{D}h)^{-1}\,\nabla\cdot\cosh(\mathscr{D}h)\delta h\,\nabla \\ &+ \cosh(\mathscr{D}h)^{-1}\mathscr{D}\sinh(\mathscr{D}h)\delta h\,\mathscr{G}(h) + O(\delta h^2),\end{aligned} \tag{5.16}$$

or

$$\begin{aligned}\mathscr{G}(h+\delta h) = \mathscr{G}(h) &- \cosh(\mathscr{D}h)^{-1}\,\nabla\cdot\cosh(\mathscr{D}h)\delta h\nabla + \mathscr{G}(h)\delta h\,\mathscr{G}(h) \\ &- \cosh(\mathscr{D}h)^{-1}\nabla\cdot\cosh(\mathscr{D}h)(\nabla h)\delta h\,\mathscr{G}(h) + O(\delta h^2).\end{aligned} \tag{5.17}$$

Similarly, higher-order functional variations of $\mathscr{G}$ can be easily obtained. This is one illustration of the advantage of dealing with an explicit DNO.

5.2. *Remarks*

Since the multi-dimensional DNO was derived by extrapolating from the two-dimensional case, one can then naturally ask if (5.9) is a correct expression.

First, we note that the explicit expression of the DNO is not unique. For instance, as in two dimensions and as suggested by the Taylor expansion around $\eta = 0$, the DNO could also be written as $\mathscr{G} = -\nabla\cdot\mathscr{J}\,\nabla$ for some operator $\mathscr{J}(d, \eta, \mathscr{D}, \nabla)$ to be specified.

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

In two dimensions (i.e. for $N=1$), one can exploit the theory of holomorphic functions to directly check that the explicit DNO (5.9) is a correct one. This procedure is simply the reverse of the derivations made in §§ 2 and 3. This is not possible in higher dimensions (i.e. $N>1$) because holomorphic functions cannot be used. The validity of (5.9) was then checked by expanding it *à la* Craig & Sulem, checking that the two expansions match. (This is detailed in Appendix A for constant depth.)

6. Moving bottom

We consider finally the generalisation of a moving bottom, i.e. $d=d(x,t)$. Of course, for simplicity, we begin with the two-dimensional case, the generalisation to higher dimensions being straightforward.

When $\partial_t d \neq 0$, the bottom is no longer a streamline, so the streamfunction is not zero at the seabed, i.e. $\psi_b=\psi_b(x,t)\neq 0$. The lower boundary condition (1.2) becomes $\partial_t d = \partial_x \psi_b = -v_b - u_b \partial_x d$. With a moving bottom, the relations (2.1), (2.2) and (3.1) still hold, but (2.3) becomes

$$\psi_b = \mathrm{Re}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\psi_s + \mathrm{Im}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\phi_s. \tag{6.1}$$

The condition for the bottom impermeability yielding $\psi_b = \partial_x^{-1}\partial_t d$, the relation (6.1) gives

$$\psi_s = \mathrm{Re}\{\exp[-\mathrm{i}h\partial_{z_s}]\}^{-1}(\partial_x^{-1}\partial_t d - \mathrm{Im}\{\exp[-\mathrm{i}h\partial_{z_s}]\}\phi_s). \tag{6.2}$$

The relation (6.2) shows that the Dirichlet-to-Neumann transformation at the free surface is no longer a homogeneous linear function of ϕ_s. The impermeability of the free surface is then $\partial_t \eta = G(\phi_s)$, the generalised DNO G being

$$G(\phi_s) = \mathscr{G}\,\phi_s - \partial_x\,\mathrm{Re}\{\exp[-\mathrm{i}h\,\partial_{z_s}]\}^{-1}\partial_x^{-1}\,\partial_t d, \tag{6.3}$$

where $\mathscr{G}$ is given by (2.5). Note that $\partial_x^{-1}\partial_t d$ is not uniquely defined due to the antiderivative, uniqueness being enforced by the definitions of the mean water level and of the frame of reference.

In higher dimensions, with $\partial_x^{-1} = \partial_x\partial_x^{-2} \mapsto \nabla\Delta^{-1} = -\mathscr{D}^{-2}\nabla$, the DNO obviously becomes

$$G(\phi_s) = [\cosh(\mathscr{D}\,\eta) + \mathscr{G}_0\,\mathscr{D}^{-1}\sinh(\mathscr{D}\,\eta)]^{-1}\nabla\cdot\cosh(d\,\mathscr{D})^{-1}(\mathscr{D}^{-2}\nabla\partial_t d - \mathscr{I}\nabla\phi_s), \tag{6.4}$$

with $\mathscr{I}$ and $\mathscr{G}_0$ being defined, respectively, by (5.5) and (5.8).

7. Discussion

Using elementary algebra, we obtained explicit formulae for the DNOs involved in water wave problems. We first derived the DNO for two-dimensional waves over a static (uneven) bottom. We then extrapolated the formula to higher dimensions and generalised the formula for moving bottoms. The latter generalisation is interesting for its applications, such as tsunami generation (Iguchi 2011), but also because it shows that extensions to fluids stratified into several homogeneous layers is possible (Craig, Guyenne & Kalisch 2005; Constantin & Ivanov 2019). The DNO is also used in some water wave problems with vorticity (Constantin, Ivanov & Martin 2016; Groves & Horn 2020), and the derivation of an explicit DNO for rotational waves is conceivable.

In this short paper, the focus is on the DNO at the free surface assuming a given bottom shape and motion. Obviously, one can as easily obtain the DNO at the bottom from an

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

assumed free surface, which, in particular, should find applications in bottom detection from free-surface measurements (Fontelos *et al.* 2017).

The explicit DNOs derived here are expressed with pseudo-differential operators formally defined in terms of series. Such definition supposes sufficient regularity of the free surface and the bottom, regularity yet to be specified by rigorous mathematical analysis. When these regularity conditions are not met, other more general representations of the operators should be used instead, such as integral formulations. Once these operators are properly defined, the explicit DNO should then be usable verbatim, allowing the investigation of rough bottoms and waves with angular crests, for example.

The main purpose of this paper is to show how explicit DNOs can be derived and, via examples, to show their interest for analytic manipulations. Although some indications on potential issues and remedies with numerical computations are briefly discussed, it is not the purpose here to derive the most effective way to compute the DNO numerically. For special functions, their definitions via power series are often not suitable for accurate fast computations, at least not in every case and without extra knowledge (e.g. periodicity, symmetries, locations of singularities). The situation is similar with DNOs defined via series, with the substantial extra difficulty that they involve non-commutative algebra.

DNOs appear in many fields of research in physics (acoustics, elasticity, electromagnetism, etc.) and, more generally, in the theory of partial differential equations. The use of a DNO is not restricted to problems involving the Laplace equation; it is also commonly employed in close relatives, such as the Helmholtz equation. The elementary formal approach presented here could thus be adapted in these contexts.

Declaration of interests. The author reports no conflict of interest.

Author ORCIDs.
Didier Clamond https://orcid.org/0000-0003-0543-8995.

Appendix A. Some operator relations in constant depth

In constant depth, the algebra is significantly simplified because d commutes with both $\mathscr{D}$ and ∇. As mentioned at the end of § 5, we then have $\mathscr{G}_0 = \mathscr{D}\tanh(d\mathscr{D})$. We also have, from the definition of the operators,

$$\begin{aligned}\cosh(d\mathscr{D})\cosh(\mathscr{D}\eta) &= \left(\sum_{i=0}^{\infty}\frac{d^{2i}\,\mathscr{D}^{2i}}{(2i)!}\right)\left(\sum_{j=0}^{\infty}\frac{\mathscr{D}^{2j}\eta^{2j}}{(2j)!}\right) = \sum_{i,j=0}^{\infty}\frac{d^{2i}\,\mathscr{D}^{2i+2j}\eta^{2j}}{(2i)!(2j)!}\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\mathscr{D}^{2i}\,d^{2i-2j}\,\eta^{2j}}{(2j)!(2i-2j)!} = \sum_{i=0}^{\infty}\sum_{j\ \text{even}}^{2i}\frac{\mathscr{D}^{2i}\,d^{2i-j}\eta^{j}}{j!(2i-j)!},\end{aligned}\tag{A1}$$

$$\begin{aligned}\sinh(d\mathscr{D})\sinh(\mathscr{D}\eta) &= \sum_{i,j=0}^{\infty}\frac{d^{2i+1}\,\mathscr{D}^{2i+2j+2}\eta^{2j+1}}{(2i+1)!(2j+1)!} = \sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\frac{\mathscr{D}^{2i}d^{2i-2j-1}\eta^{2j+1}}{(2j+1)!(2i-2j-1)!}\\ &= \sum_{i=1}^{\infty}\sum_{j\ \text{odd}}^{2i-1}\frac{\mathscr{D}^{2i}d^{2i-j}\eta^{j}}{j!(2i-j)!}.\end{aligned}\tag{A2}$$

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

Since $|n!| = \infty$ for all negative integers n, the summation $\sum_{i=1}^{\infty}$ in (A2) can be replaced by $\sum_{i=0}^{\infty}$. Thus, we have

$$\cosh(d\mathscr{D})\cosh(\mathscr{D}\eta) + \sinh(d\mathscr{D})\sinh(\mathscr{D}\eta) = \sum_{i=0}^{\infty} \frac{\mathscr{D}^{2i}}{(2i)!} \sum_{j=0}^{2i} \frac{(2i)!\, d^{2i-j}\eta^j}{j!(2i-j)!} = \sum_{i=0}^{\infty} \frac{\mathscr{D}^{2i}\,(d+\eta)^{2i}}{(2i)!} = \cosh(\mathscr{D}h). \tag{A3}$$

Similarly, one can easily derive the relations

$$\sinh(d\mathscr{D})\cosh(\mathscr{D}\eta) + \cosh(d\mathscr{D})\sinh(\mathscr{D}\eta) = \sinh(\mathscr{D}h), \tag{A4}$$

$$\cosh(\eta\mathscr{D})\sinh(\mathscr{D}d) + \sinh(\eta\mathscr{D})\cosh(\mathscr{D}d) = \sinh(h\mathscr{D}), \tag{A5}$$

$$\cosh(\eta\mathscr{D})\cosh(\mathscr{D}d) + \sinh(\eta\mathscr{D})\sinh(\mathscr{D}d) = \cosh(h\mathscr{D}), \tag{A6}$$

and, obviously,

$$\cosh(\mathscr{D}h) \pm \sinh(\mathscr{D}h) = \exp(\pm\mathscr{D}h). \tag{A7}$$

Note that, as $\mathscr{D}$ commutes with d, but not with η and h, these relations are not valid for uneven bottoms and, in constant depth, $\sinh(\mathscr{D}h) \neq \sinh(h\mathscr{D})$ for example. However, for varying bottoms, similar relations can be easily obtained if η is constant.

Taylor expansions around $\eta = 0$ yield

$$\sinh(\mathscr{D}h) = \sinh(d\mathscr{D})[1 + \tfrac{1}{2}\mathscr{D}^2\eta^2 + \cdots] + \cosh(d\mathscr{D})[\mathscr{D}\eta + \tfrac{1}{6}\mathscr{D}^3\eta^3 + \cdots], \tag{A8}$$

$$\cosh(\mathscr{D}h) = \cosh(d\mathscr{D})[1 + \tfrac{1}{2}\mathscr{D}^2\eta^2 + \cdots] + \sinh(d\mathscr{D})[\mathscr{D}\eta + \tfrac{1}{6}\mathscr{D}^3\eta^3 + \cdots], \tag{A9}$$

hence, with $\mathscr{G}_0 \stackrel{\text{def}}{=} \mathscr{D}\tanh(d\mathscr{D})$,

$$(\cosh(\mathscr{D}h))^{-1} = [1 + \mathscr{G}_0\eta + \tfrac{1}{2}\,\mathscr{D}^2\eta^2 + \tfrac{1}{6}\,\mathscr{G}_0\,\mathscr{D}^2\eta^3 + \cdots]^{-1}\operatorname{sech}(d\mathscr{D}) = [1 - \mathscr{G}_0\eta - \tfrac{1}{2}\,\mathscr{D}^2\eta^2 + \mathscr{G}_0\eta\,\mathscr{G}_0\eta + \cdots]\operatorname{sech}(d\mathscr{D}), \tag{A10}$$

$$\nabla\cdot\mathscr{D}^{-1}\sinh(\mathscr{D}h)\,\nabla = -\cosh(d\mathscr{D})[\mathscr{G}_0 - \nabla\cdot\eta\nabla - \tfrac{1}{2}\,\mathscr{G}_0\nabla\cdot\eta^2\nabla + \cdots]. \tag{A11}$$

Thus, with $\mathscr{G}$ defined in (5.12), one gets

$$\mathscr{G} = [1 + \mathscr{G}_0\eta + \tfrac{1}{2}\,\mathscr{D}^2\eta^2 + \cdots]^{-1}[\mathscr{G}_0 - \nabla\cdot\eta\nabla - \tfrac{1}{2}\,\mathscr{G}_0\nabla\cdot\eta^2\nabla + \cdots]. \tag{A12}$$

Expanding the DNO as $\mathscr{G} = \mathscr{G}_0 + \mathscr{G}_1 + \mathscr{G}_2 + \cdots$, with $\mathscr{G}$ defined in (5.11), one obtains

$$\mathscr{G}_1 = -\mathscr{G}_0\eta\,\mathscr{G}_0 - \nabla\cdot\eta\nabla, \quad \mathscr{G}_2 = -\tfrac{1}{2}\,\mathscr{D}^2\eta^2\,\mathscr{G}_0 - \mathscr{G}_0\eta\,\mathscr{G}_1 - \tfrac{1}{2}\,\mathscr{G}_0\nabla\cdot\eta^2\nabla, \quad \text{etc.,} \tag{A13a,b}$$

so the expansion of Craig & Sulem (1993) is recovered.

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

Substituting $h + \delta h$ for h, for some small δh, we have the first-order Taylor expansions

$$\begin{aligned}\cosh(\mathscr{D}(h+\delta h)) &\stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{\mathscr{D}^{2n}\,(h+\delta h)^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{\mathscr{D}^{2n} h^{2n}}{(2n)!} + \sum_{n=1}^{\infty} \frac{\mathscr{D}^{2n} h^{2n-1}\delta h}{(2n-1)!} + O(\delta h^2) \\ &= \cosh(\mathscr{D}h) + \mathscr{D}\sinh(\mathscr{D}h)\delta h + O(\delta h^2)\end{aligned} \tag{A14}$$

and

$$\begin{aligned}\sinh(\mathscr{D}(h+\delta h)) &= \sum_{n=0}^{\infty} \frac{\mathscr{D}^{2n+1} h^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty} \frac{\mathscr{D}^{2n+1} h^{2n}\delta h}{(2n)!} + O(\delta h^2) \\ &= \sinh(\mathscr{D}h) + \mathscr{D}\cosh(\mathscr{D}h)\delta h + O(\delta h^2),\end{aligned} \tag{A15}$$

hence

$$\cosh(\mathscr{D}(h+\delta h))^{-1} = [1 - \cosh(\mathscr{D}h)^{-1}\mathscr{D}\sinh(\mathscr{D}h)\delta h]\cosh(\mathscr{D}h)^{-1} + O(\delta h^2). \tag{A16}$$

We are then in position to compute explicitly the functional variations of the DNO.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I.A. 1965 *Handbook of Mathematical Functions*. Dover.

ALAZARD, T. & BALDI, P. 2015 Gravity capillary standing water waves. *Arch. Rat. Mech. Anal.* **217**, 741–830.

ALAZARD, T., BURQ, N. & ZUILY, C. 2012 On the Cauchy problem for gravity water waves. *Invent. Math.* **198**, 71–163.

ANDRADE, D. & NACHBIN, A. 2018 A three-dimensional Dirichlet–Neumann operator for water waves over topography. *J. Fluid Mech.* **845**, 321–345.

CLAMOND, D. & GRUE, J. 2001 A fast method for fully nonlinear water-wave computations. *J. Fluid Mech.* **447**, 337–355.

CONSTANTIN, A. & IVANOV, R.I. 2019 Equatorial wave-current interactions. *Commun. Math. Phys.* **370**, 1–48.

CONSTANTIN, A., IVANOV, R.I. & MARTIN, C.-I. 2016 Hamiltonian formulation for wave-current interactions in stratified rotational flows. *Arch. Rat. Mech. Anal.* **221**, 1417–1447.

CRAIG, W. & GROVES, M.D. 1994 Hamiltonian long-wave approximations to the water-wave problem. *Wave Motion* **19**, 367–389.

CRAIG, W. & SULEM, C. 1993 Numerical simulation of gravity waves. *J. Comput. Phys.* **108**, 73–83.

CRAIG, W., GUYENNE, P. & KALISCH, H. 2005 Hamiltonian log-wave expansions for free surfaces and interfaces. *Commun. Pure Appl. Maths* **58**, 1587–1641.

CRAIG, W., GUYENNE, P., NICHOLLS, D.P. & SULEM, C. 2005 Hamiltonian long-wave expansions for water waves over a rough bottom. *Proc. R. Soc. Lond.* A **461**, 839–873.

DOMMERMUTH, D.G. & YUE, D.K.P. 1987 A high-order spectral method for the study of nonlinear gravity waves. *J. Fluid. Mech.* **184**, 267–288.

FAZIOLI, C. & NICHOLLS, D.P. 2010 Stable computation of the functional variation of the Dirichlet–Neumann operator. *J. Comput. Phys.* **229**, 906–920.

FONTELOS, M.A., LECAROS, R., LÓPEZ-RÍOS, J.C. & ORTEGA, J.H. 2017 Bottom detection through surface measurements on water waves. *SIAM J. Control Optim.* **55** (6), 3890–3907.

FRUCTUS, D. & GRUE, J. 2007 An explicit method for the nonlinear interaction between water waves and variable and moving bottom topography. *J. Comput. Phys.* **222** (2), 720–739.

FRUCTUS, D., CLAMOND, D., GRUE, J. & KRISTIANSEN, Ø. 2005 Efficient numerical model for three-dimensional gravity waves simulations. Part I: periodic domains. *J. Comput. Phys.* **205**, 665–685.

GROVES, M.D. & HORN, J. 2020 A variational formulation for steady surface water waves on a Beltrami flow. *Proc. R. Soc. Lond.* A **476** (2234), 20190495.

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press

IGUCHI, T. 2011 A mathematical analysis of tsunami generation in shallow water due to seabed deformation. *Proc. R. Soc. Edin.* **141**, 551–608.

ISAACSON, E. & KELLER, H.B. 1994 *Analysis of Numerical Methods*. Dover.

LANNES, D. 2013 *The Water Waves Problem*. Math. Surveys and Monographs, vol. 188. American Mathematical Society.

MILNE-THOMSON, L.M. 2011 *Theoretical Hydrodynamics*, 5th edn. Dover Books on Physics. Dover.

NICHOLLS, D.P. & REITICH, F. 2001 A new approach to analyticty of Dirichlet–Neumann operators. *Proc. R. Soc. Edin.* **131**, 1411–1433.

SCHÄFFER, H.A. 2008 Comparison of Dirichlet–Neumann operator expansions for nonlinear surface gravity waves. *Coast. Engng.* **55** (4), 288–294.

WEHAUSEN, J.V. & LAITONE, E.V. 1960 Surface waves. In *Fluid Dynamics III*, (ed. S. Flugge & C. Truesdell), Encyclopaedia of Physics, vol. IX, pp. 446–778. Springer

WEST, B.J., BRUECKNER, K.A., JANDA, R.S., MILDER, D.M. & MILTON, R.L. 1987 A new numerical method for surface hydrodynamics. *J. Geophys. Res.* **92**, 11803–11824.

WILKENING, J. & VASAN, V. 2015 Comparison of five methods of computing the Dirichlet–Neumann operator for the water wave problem. In *AMS Special Session on Nonlinear Wave and Integrable Systems* (ed. C.W. Curtis, A. Dzhamay, & W.A. Hereman), Contemporary Mathematics, vol. 635, pp. 175–210. American Mathematical Society.

ZAKHAROV, V.E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **9**, 1990–1994.

https://doi.org/10.1017/jfm.2022.830 Published online by Cambridge University Press