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Discrete Models for Finance and Microlending

Teaching using Scilab and the Method of Exercices Leaflets

Part 2

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Hedging strategy and hedging price of an option

· Here the typical cell to set up the *hedging* price : $S_t = S$, $S_{t+\delta t} = S^+ = Su$ or $S_{t+\delta t} = S^- = Sd$, for option prices $C_t = C$, $C_{t+\delta t} = C^+$ or $C_{t+\delta t} = C^-$

$$\begin{cases} aS^+ + b &= C^+ \\ aS^- + b &= C^- \end{cases} \tag{1}$$

· Leads to $a = \frac{C^+ - C^-}{S^+ - S^-} = \frac{C^+ - C^-}{S(u-d)}$ and $b = \frac{C^- S^+ - C^+ S^-}{S^+ - S^-} = \frac{C^- u - C^+ d}{u-d}$

· Thus $C = aS + b = \frac{C^+ - C^-}{S^+ - S^-} S + \frac{S^+ C^- - C^+ S^-}{S^+ - S^-} = \frac{1-d}{u-d} C^+ + \frac{u-1}{u-d} C^-$

Hedging price of an option as a conditional expectation

- We found

$$C = aS + b = \frac{1-d}{u-d}C^+ + \frac{u-1}{u-d}C^- = pC^+ + (1-p)C^- = E_t^*(C_{t+\delta t})$$

We will come back on this.

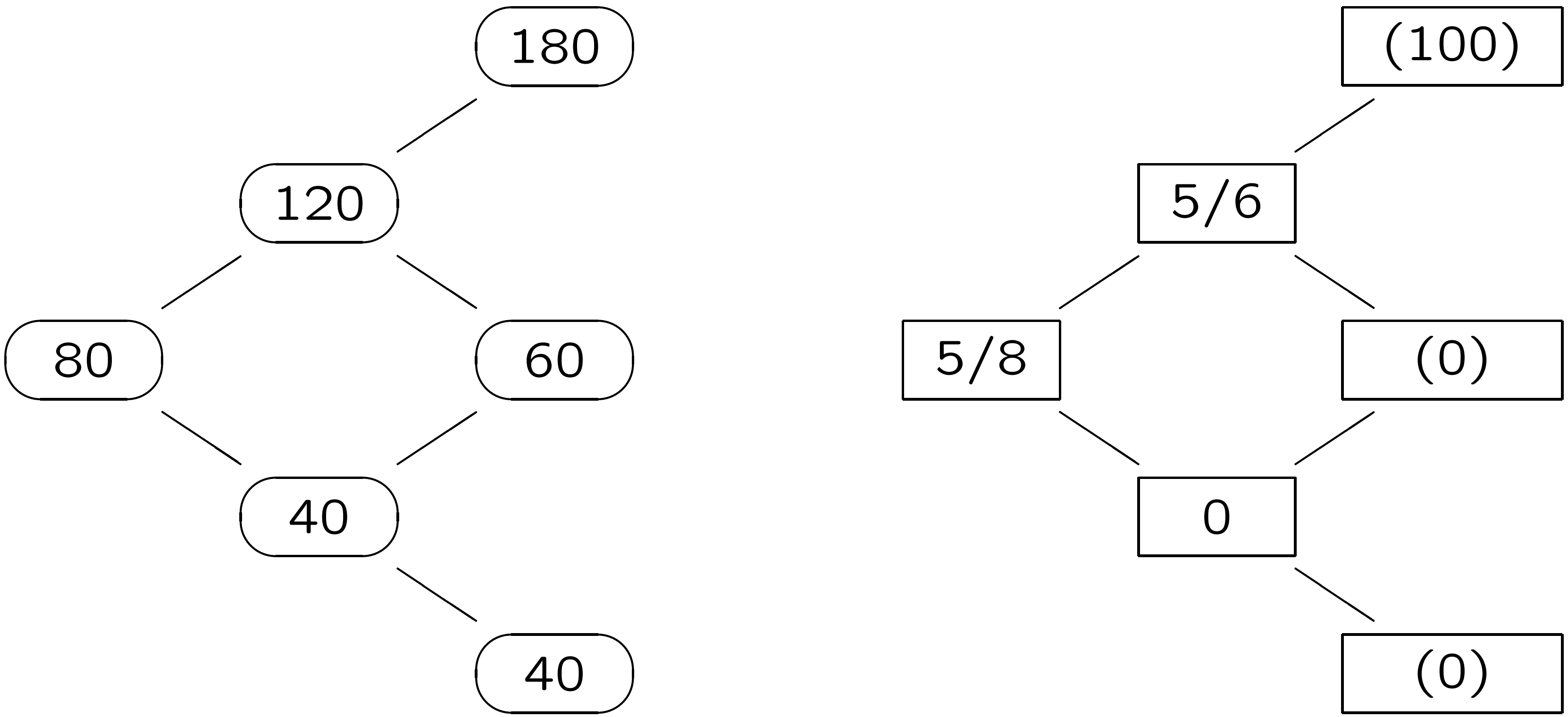
Hedging strategy of an option

· The typical cell

· We found $a = \frac{C^+ - C^-}{S^+ - S^-}$ (and $b = \frac{C^- S^+ - C^+ S^-}{S^+ - S^-} = \frac{C^- u - C^+ d}{u - d}$=C-aS)

· This means that, at each time step t, the trader who *hedges* the option knows that (s)he should have the quantity $a = a_t$ of stocks, where a_t can be computed from the observed price of S_t, so *with the information available at time* t. So this is a hedging *strategy*" : observe the prices and buy/sell stocks accordingly. If this is done rigorously (and the value of σ well chosen) the trader will end up with the right quantity of stocks to pay the agreed *payoff* $\varphi(S_T)$

Concrete example on the elementary model

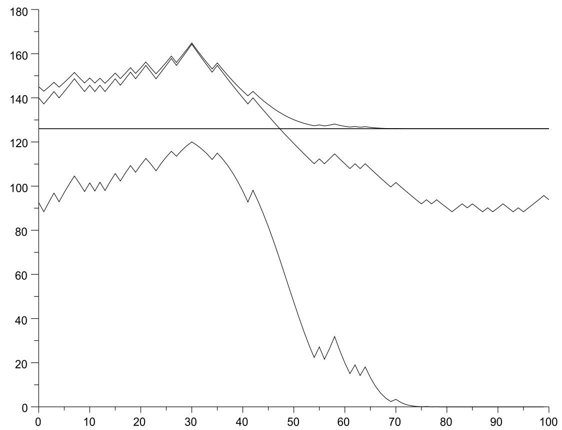


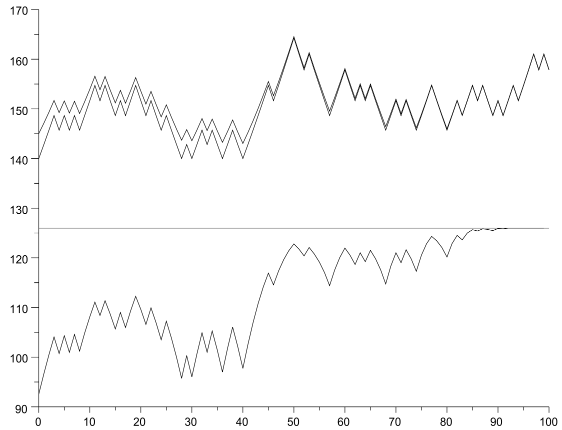
· So after one step, when the trader observes what is the outcome for S (120 or 60) (s)he realizes that (s)he needs to *buy* $\frac{5}{6} - \frac{5}{8} = \frac{1}{3}$ of extra stocks, or just cell all the $\frac{5}{8}$ stocks he had.

· This is called *dynamical hedging* following a *predictable* strategy.

Visualization on Scilab

· We adapt our Scilab program for plotting M trajectories to plot each time one trajectory $(S_t(\omega))_{t\in[0..T]_{\delta t}}$ (with different colours) so as the value of $(K + C_t(\omega))_{t\in[0..T]_{\delta t}}$ (that should end with K or S_T), and $(Ka_t(\omega))_{t\in[0..T]_{\delta t}}$ who ends up with 0 or $K = Ka_T$ with $a_T = 1$.





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The predictable hedging process and P&L

· The *Profit and Loss* $\mathsf{P\&L}_t^{t+\delta t}$ between t and $t+\delta t$ is

$$\mathsf{P\&L}_t^{t+\delta t} = a_t \delta S_{t+\delta t} \text{ , with } \delta S_{t+\delta t} := S_{t+\delta t} - S_t.$$

· By construction

$$C_0 + \mathsf{P\&L}_0^T = C_0 + \sum_{t \in [0..T)_{\delta t}} \mathsf{P\&L}_t^{t+\delta t} = \varphi(S_T) \text{ , with}$$

$$\mathsf{P\&L}_0^T = \sum_{t \in [0..T)_{\delta t}} a_t \delta S_{t+\delta t}, \text{ with } \delta S_{t+\delta t} := S_{t+\delta t} - S_t.$$

· The above sum, that gives the Profit and Loss of the predictable strategy $(a_t)_{t \in [0..T]_{\delta t}}$, is the so-called Itô stochastic integral of the discrete process $(S_t)_{t \in [0..T]_{\delta t}}$.

$(S_t)_{t\in[0..T]_{\delta t}}$ as a solution of a stochastic differential equation

· By definition

$$\delta S_{t+\delta t} := S_{t+\delta t} - S_t = S_t U_{t+\delta t} - S_t = S_t(U_{t+\delta t} - 1) = S_t \delta W_{t+\delta t}$$

$$, \text{ with } \delta W^{\sigma}_{t+\delta t} = U_{t+\delta t} - 1 = e^{\pm\sigma\sqrt{\delta t}} = \pm\sigma\sqrt{\delta t} + \frac{\sigma^2}{2}\delta t \pm \ldots$$

· So $S_t = S_0 + \sum_{s\in[0..t)_{\delta t}} S_s \delta W_{s+\delta t}$ is the solution of (discrete time) stochastic differential equation that is usually written

$$dS_t = S_t dW^{\sigma}_t.$$

Back to Contionnal Expectation

· $p = \frac{1-d}{u-d}$ is the risk-less probability :

· $p = P^*(\{U_{t+\delta t} = \texttt{up}\} \mid S_t) = P^*(\{U_{t+\delta t} = \texttt{up}\} \mid S_0, \ldots, S_t) = E_t^*(I_{\{U_{t+\delta t}=u\}})$, the so-called conditional expectation at time t.

· It is the random variable such that

$$E_t^*(f(S_0, \ldots, S_t)) = f(S_0, \ldots, S_t),$$

and for any r.v. X and any s and t

$$\text{if } s \leq t \text{ then } E_s^*(E_t^*(X)) = E_s^*(X) \;;$$

· and finally, $E_0^*(X) = E^*(X)$.

Contionnal Expectation and Expectation

$$
\begin{aligned}
E^*(\varphi(S_T)) &= E_0^*(E_{T-\delta t}^*(C_T)) = E_0^*(C_{T-\delta t}) \\
&= E_0^*(E_{T-2\delta t}^*(C_{T-\delta t})) = E_0^*(C_{T-2\delta t}^*) \\
&= \dots \\
&= E_0^*(E_{\delta t}^*(C_{2\delta t})) = E_0^*(C_{\delta t}) = C_0.
\end{aligned}
$$

So $\boxed{C_0 = E^*(\varphi(S_T)).}$

Expectation and CRR premium (price at $t = 0$)

So $\boxed{C_0 = E^*(\varphi(S_T))}$ and $S_T = S_0 u^{J_n} d^{n-J_n}$ with $J_n \rightsquigarrow \mathcal{B}(n,p)$. Thus

$$C_0 = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} p^j (1-p)^{n-j} \varphi(S(n,j)).$$

Recall : $S(i,j) = S_0 u^j d^{i-j}$.

The Black-Scholes formula

It can be shown (using generatrix functions for instance) that the price of an option computed with a CRR model tends, when n tends to infinity (or the time step $\delta t = T/n$ tends to zero, to the price computed for a Black-Scholes (continuous time) model :

$$C = S\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2),$$

where $\mathcal{N}(d) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} dx$ is the Gaussian function, with

$$d_1 = \frac{1}{\sigma\sqrt{T}} \left[\ln \frac{S_0}{K} + T \left(r + \frac{\sigma^2}{2} \right) \right] \text{ and } d_2 = d_1 - \sigma\sqrt{T}.$$

So, in `Scilab`, one may wish to define a `function BlackScholes(S,K,r,T,`σ`)` giving the Black-Scholes price of a Call option.

Thank You for your attention

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