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Teaching ODE with Scilab

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Why we use Scilab to teach ODE to 3rd year students ?

· We wanted to reniew a course that was too abstract and too technical for the audiance we have now

· We wanted to emphasize the use of ODE as models of concrete situations

· We had students with different backgrounds

· We wanted to reach nevertheless a deep understanding of the subject

ODE : three possible approaches

- Algebraic computation of explicit solutions
- Qualitative (or geometric) approach
- Numeriacal computation of approximate solutions

Models of population dynamics

· The model of Malthus (1798), or **exponential model** ,

$$\frac{dy(t)}{dt} = ry(t)$$

It is unrealistic because the population does not tend to infinity

· The logistic model (Verhulst, 1905)

$$\frac{dy(t)}{dt} = ry(t)\left(1 - \frac{y(t)}{K}\right)$$

It shows a damped growth, called **logistic growth**, toward the carrying capacity K of the population.

· The Allee model (Allee, 1930)

$$\frac{dy(t)}{dt} = ry(t)\left(\frac{y(t)}{M} - 1\right)\left(1 - \frac{y(t)}{N}\right)$$

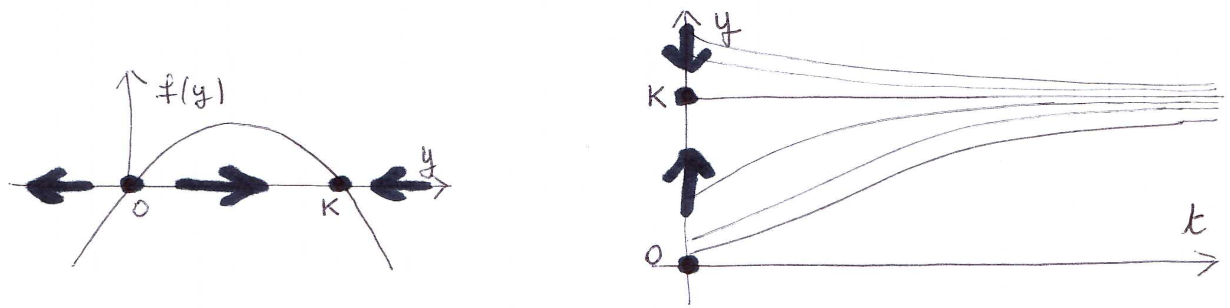
It shows possible **extinction** when the density of the population is too low.

Qualitative study

Définition : For a differential equation

$$\frac{dy(t)}{dt} = f(y(t))$$

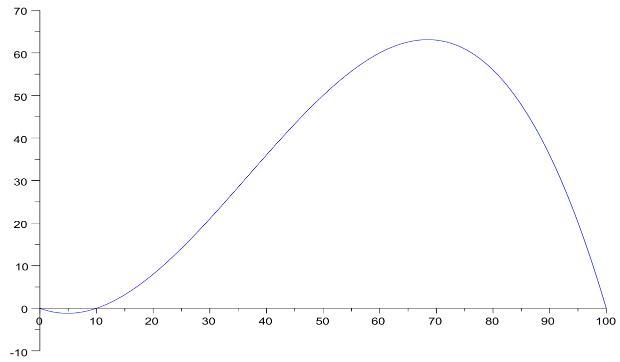
we call *equilibrium* or *stationnary point* a constant y^* s.t. if $y(0) = y^*$ then $y(t) = y^*$ for all t. Thus it is a solution of $f(y^*) = 0$.

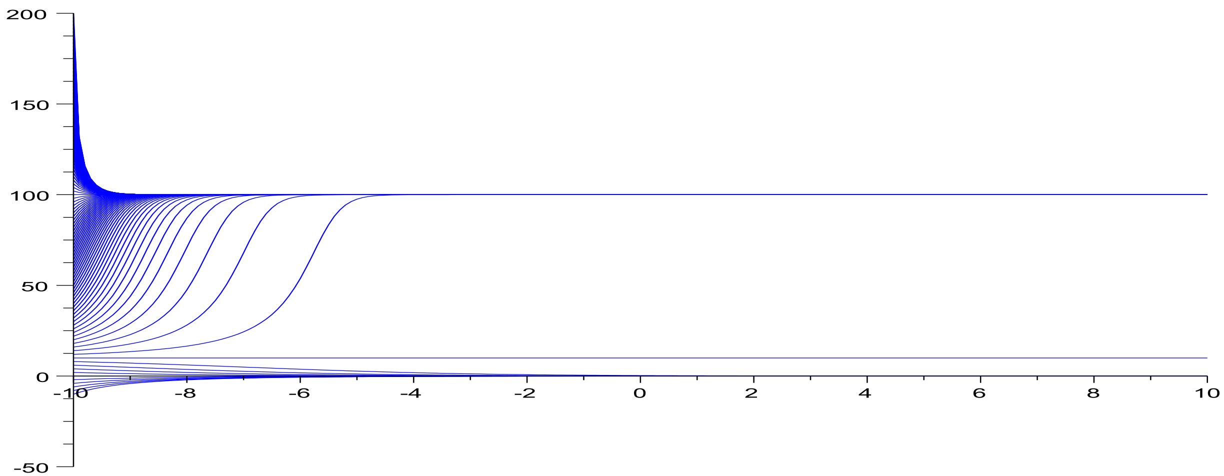


Graph of the function $f(y) = ry(1 - \frac{y}{K})$ and outline of the solutions of $y'(t) = ry(t)(1 - \frac{y(t)}{K})$ in the plan (t, y).

An equilibrium y^* is *stable* if $f'(y^*) < 0$ and *unstable* if $f'(y^*) > 0$.

Understanding the Allee model





Now, it is time to do exercices !

Approximate solutions

The easiest method to compute numerical solutions of an equation as $y' = f(t, y)$, is due to a Swiss mathematician, Leonard Euler (published in 1768).

Euler's method has been improved by two German mathematicians Runge and Kutta in 1901 and it is today the order 4 Runge-Kutta method which is the most used.

But there exist many other methods, including the so-called *predictor-corrector* method of Adams which is the one used by default by Scilab.

Euler's Method

The approximate value of the solution $y(t)$ s.t.$y(t_0) = y_0$ is given as a sequence of (t_i, y_i) recursively defined by

$$\begin{cases} t_{i+1} &= t_i + h \\ y_{i+1} &= y_i + hf(t_i, y_i) \end{cases} \tag{1}$$

where $h > 0$ est a (small) timestep.

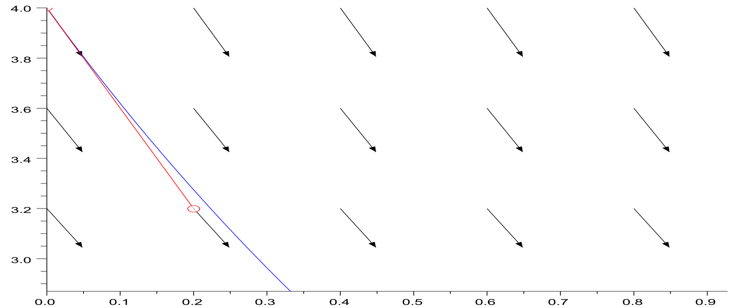
One can see that the error, after one step, computed using a Taylor formula

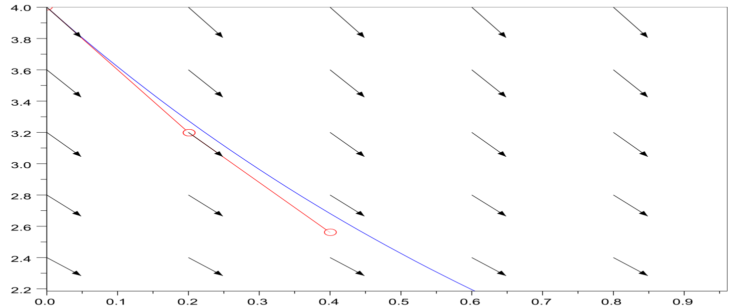
$$y(t_1) = y(t_0) + hy'(t_0) + \frac{h^2}{2}y''(t_0) + o(h^2) = y_1 + \frac{h^2}{2}y''(t_0) + o(h^2)$$

tends to zero as h^2. Thus, at time $t = t_0 + 1$, i.e. after $n = 1/h$ steps, this error tends to zero as h (heuristic.., proof more subtle).

This is why Euler's method is *of order 1* (and local order 2).

Geometrical idea of Euler Method





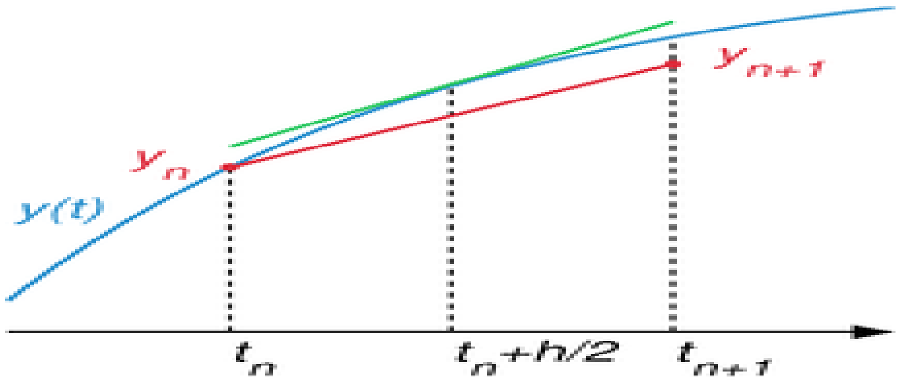
RK2 (or Midpoint) Method

To get a method of (global) order 2, Runge and Kutta introduced the following algorithm :

$$\begin{cases} t_{i+1} = t_i + h \\ y_{i+1} = y_i + hf(t_i + \frac{h}{2}, y_i + \frac{k_1}{2}) \end{cases} \quad (2)$$

où, à chaque pas, k_1 désigne $k_1 = hf(t_i, y_i)$.

Geometrical idea



Time to do exercices again !