Comsats Institute for Information Technology
Maths Department

Dynamical systems 2 : Nodes, saddle, spirals, and centers

Type of equilibria : the Poincaré classification

Henri Poincaré has introduced a classification of linear vector fields of the plan which collects all these vector fields in a finite number of classes, according to their qualitative behaviour. This classification is important because it can be used in the study of the non-linear systems near their equilibria, as we will see below.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with A a real 2×2 matrix. Poincaré assumes that A is *non-degenerate* which means that 0 is not an eigenvalue. Let us denote by λ and μ the two eigenvalues of A ; when they are complex we will also write $\alpha \pm i\omega$. One knows that it exists a basis of $\mathbb{R}^2$ in which the associate linear map associate with A becomes one of the following :

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix}$$

If we denote by U and V the coordinates in this basis, it is easy to solve the differential system satisfied by U and V : in the first case one has $(U,V) = (e^{\lambda t}U_0, e^{\mu t}V_0)$, in the second $(U,V) = e^{\lambda t}(U_0 + tV_0, V_0)$ and finally in the last case

$$\begin{pmatrix} U \\ V \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} U_0 \\ V_0 \end{pmatrix}.$$

It is then easy to find the qualitative behaviour of solutions. Figure 1 gives this behaviour, according to the values of λ, μ and α, so has the usual translation of the names given by Poincaré for these various cases.

Linearization of a non-linear differential system near an equilibrium Let (x^*, y^*) be an equilibrium of the differential system

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \tag{1}$$

, so a common zero of f and g. Let $\varepsilon > 0$ be a small parameter. Performing the changes of unknown $X := \frac{x-x^*}{\varepsilon}$, $Y := \frac{y-y^*}{\varepsilon}$ corresponds to *look at the equilibrium*(x^*, y^*) *through a magnifying glass*, as limitted (X,Y) corresponds to small $x - x^*$ et $y - y^*$. Elementary computations show that the corresponding system in X and Y is of kind

$$\begin{cases} X' = aX + bY + o_1(\varepsilon) \\ Y' = cX + dY + o_2(\varepsilon) \end{cases} \tag{2}$$

where $o_1(\varepsilon)$ and $o_2(\varepsilon)$ are of order ε. So, up to ε, limited solutions (X,Y) are close to solutions of

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$$

where matrix A is called the *jacobian matrix* at the equilibrium (x^*, y^*) and can be easily computed through elementary calculus, as its coefficients are just partial derivatives of f and g computed at (x^*, y^*). Indeed, we have

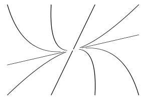
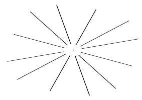
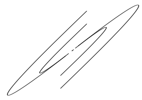
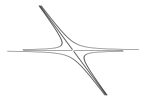
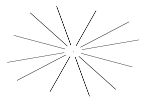
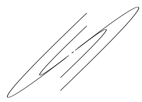
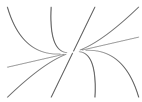
$$A = A(x^*, y^*) = \begin{pmatrix} \frac{\partial f}{\partial x}(x^*, y^*) & \frac{\partial f}{\partial y}(x^*, y^*) \\ \frac{\partial g}{\partial x}(x^*, y^*) & \frac{\partial g}{\partial y}(x^*, y^*) \end{pmatrix}$$

We saw that linear systems are (up to degenerate cases) of one of the four types : (stable or unstable) node and saddle, spiral or center. We use this same terminology for non-linear equilibria. Please observe that, whereas a linear center is build up with closed (periodic) solutions, in the non-linear case the solutions may be non-closed and could "spiral weakly" in or outwards. The type of the behaviour of the linearized system is sometimes called the *nature* of the equilibrium : it gives useful information on the qualitative behaviour of the system near the equilibrium which is easy to compute.

As an example, let us look at two following non-linear differential systems :

$$\begin{cases} x' = (2 - x - 2y/3)x \\ y' = (2 - 2x/3 - y)y \end{cases} \tag{3}$$

Real eigenvalues λ and μ

$0 < \lambda < \mu$		Unstable node
$0 < \lambda = \mu$, A diagonalizable		Degenerate unstable node
$0 < \lambda = \mu$, A non-diagonalizable		Unstable node
$\lambda < 0 < \mu$		Saddle
$\lambda = \mu < 0$, A diagonalizable		Degenerate stable node
$\lambda = \mu < 0$, A non-diagonalizable		Stable node
$\mu < \lambda < 0$		Stable node

Complex eigenvalues ($\alpha \pm i\omega$, $\omega \neq 0$)

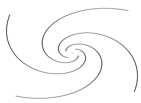
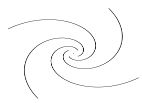
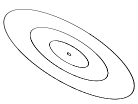
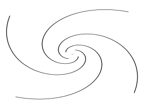
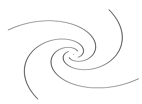
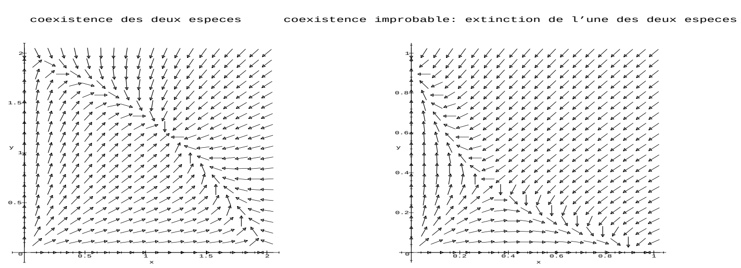
$\alpha > 0$	 ou 	Unstable spiral
$\alpha = 0$		Center
$\alpha < 0$	 ou 	Stable spiral

FIG. 1 – Poincaré classification of linear systems



$$
\begin{cases} x' = (1 - x - 2y)x \\ y' = (1 - 2x - y)y \end{cases} \qquad (4)
$$

Above the picture of the corresponding field of directions. In both of these examples there are three equilibria located at the coordinate axes, namely $(0, 0)$, $(0, 2)$, $(2, 0)$ and a fourth equilibirum with no-zero coordinates : $(\frac{6}{5}, \frac{6}{5})$ for the first example and $(\frac{1}{3}, \frac{1}{3})$ for the second one. One gets the nature of equilibria by linearizing the system at each equilibrium – actually, compute the jacobian matrix at these points– and compute the eigenvalues. Here we see that the equilibrium "with non-zero coordinates" is a stable node in the first case and a saddle point in the second one. Having a picture available, we can easily see this too.