Comsats Institute for Information Technology
Maths Department
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Dynamical Systems

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**Dynamical Systems : Answer-sheet 3**
**Système dynamiques linéaires du plan et linéarisé d'un système non-linéaire**

# 1 First a linear system

Let's consider the following system of *linear differential equations*, with unknow functions $t \mapsto x(t)$ and $t \mapsto y(t)$ :

$$\left\{ \begin{array}{rcl} x' & = & -\frac{1}{3}x - \frac{2}{3}y \\ y' & = & -\frac{2}{3}x - \frac{1}{3}y \end{array} \right. \tag{1}$$

1. Show that $(x_1(t), y_1(t)) = (e^{-t}, e^{-t})$ is a solution of system (1). Hint : begin with the right hand side of each equation. Give the equation of the straight line $D_1$ to which belong all $(x_1(t), y_1(t))$ ; show what are the points $D_1^+$ of that line which are reached for $t \geq 0$ and what are those $D_1^-$ which are reached for $t \leq 0$.

2. Similarly, show that $(x_2(t), y_2(t)) = (-e^{\frac{t}{3}}, e^{\frac{t}{3}})$ is a solution of system (1). Give the equation of the straight line $D_2$ to which belong all $(x_2(t), y_2(t))$ ; show what are the points $D_2^+$ of that line which are reached for $t \geq 0$ and what are those points $D_2^-$ which are reached for $t \leq 0$.

3. Following `Scilab` commands provide a geometric view on system (1) as a field of directions.

```
A=[-1/3,-2/3;-2/3,-1/3];
function www=WWW(t,V); www=A*V; endfunction;
xset("window",0);
xMin=-1;xMax=+1;yMin=-1;yMax=+1;
fchamp(WWW,0,xMin :0.1 :xMax,yMin :0.1 :yMax);
```

Give here a sketch of this field of directions, drawing the lines $D_1$ and $D_2$, indication in what direction evolve the solutions $(x_1, y_1)$ and $(x_2, y_2)$ and showing what are the subsets $D_1^{\pm}$ and $D_2^{\pm}$.

4. Write system (1) as an equation involving a matrix $A$ that you will provide.

5. Following command allows to get the eigenvectors and that eigenvalues of matrix $A$

```
[R,diagevals]=spec(A);
disp(R,"eigenvectors",diagevals,"eigenvalues");
```

What are the eigenvalues $\lambda$ of $A$. What can you say about each vector $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$, for any $t$ ?

Following commands provide a plot in red or blue of solutions $(x_1, y_1)$ and $(x_2, y_2)$ so as $(-x_1, -y_1)$ and $(-x_2, -y_2)$ which are also solutions.

```
function x=x1(t); x=exp(-t) endfunction;
function x=y1(t); x=exp(-t) endfunction;
function x=x2(t); x=-exp(+t/3) endfunction;
function x=y2(t); x=exp(+t/3) endfunction;
tplus=0 :0.01 :2;
plot(x1(tplus),y1(tplus),'r-'); plot(-x1(tplus),-y1(tplus),'r-');
tmoins=0 :-0.01 :-4;
plot(x2(tmoins),y2(tmoins),'b-'); plot(-x2(tmoins),-y2(tmoins),'b-');
```

6. Show that for any $a \in \mathbb{R}$ and any $b \in \mathbb{R}$ $(x, y) := (ax_1 + bx_2, ay_1 + by_2)$ is asolution of (1)

7. Following commands use this remark to plot (in `black-`) 100 *segments of trajectories* chosen randomly

```
tt=-4 :0.01 :2 ;
for traject=1 :100 ;
       a=-0.5+rand() ; b=-0.5+rand() ;
       plot(a*x1(tt)+b*x2(tt),a*y1(tt)+b*y2(tt),'k-') ;
end ;
a=gca() ;a.data_bounds=[xMin,yMin ;xMax,yMax] ;
```

Where are (randomly) chosen the *initial conditions*$(x(0), y(0))$ ? What is the purpose of the last code line ? Give a nice sketch of what you obtain ; draw the lines $D_1$ a d $D_2$ ; indicate in what direction evolve the solutions.

8. Let $M_1 = (x_1(0), y_1(0))$ and $M_2 = (x_2(0), y_2(0))$ ; show that $(x(t), y(t)) = U(t)M_1 + V(t)M_2$ for some choice of function $U$ and $V$ that you will find and providde and which satisfy $cU^{\lambda_1} + dV^{\lambda_2} = 0$ for some constants $c$ and $d$ (to be found and provided). How can this be understood on the picture ?

## 2 A non-linear differential system exhibiting a saddle-point

Let us now consider the following (*non-linear*) system

$$\left\{\begin{array}{rcl} x' &=& x(1-x-2y) \\ y' &=& y(1-2x-y) \end{array}\right. \tag{2}$$

1. Show that the point $(x_0, y_0) = (\frac{1}{3}, \frac{1}{3})$ is an *equilibrium point* and that the matrix $A$ above is the *jacobian matrix* of this system at this point $(x_0, y_0)$.

2. The following code allows to plot the field of directions associated with system(2) and to plot some trajectories with initial point chosen randomly

```
xset("window",1);
function xprim=f(x,y); xprim=x*(1-x-2*y); endfunction;
function yprim=g(x,y); yprim=y*(1-y-2*x); endfunction;
function vprim=www(t,v);
vprim=[f(v(1),v(2)),g(v(1),v(2))]';
endfunction;
fchamp(www,0,0 :0.05 :1.1,0 :0.05 :1.1);
Tmax=10; N=100;
smallstep=Tmax/N;t=0 :smallstep :Tmax;
for trajnumber=1 :100
      M0=[rand(),rand()]';
      M=ode(M0,0,t,www);
      x=M(1, :);y=M(2, :);
      plot(x,y);
end;
```

   Run this code and zoom-in around the equilibrium point $(x_0, y_0)$. What can you observe?

3. Run following commands

```
M1=[1/3;1/3];eps=0.0000001;
JAC=[www(0,M1+eps*[1;0])-www(0,M1),www(0,M1+eps*[0;1])-www(0,M1)]/eps;
disp(JAC;'system under a looking-glass');
```

   What can you observe? Please explain.