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MATHEMATICAL MODELS FOR MICROLENDING

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Abstract

Microlending has not yet been placed on a firm mathematical foundation, in contrast to the highly developed theory of Mathematical Finance. Here we propose a first step, modeling a slightly simplified procedure than the one actually used, as a Markov chain. Using this model we compute the expected benefit each borrower gains from his or her activity. We compute the distribution of the beneficiaries among the population involved, and discuss the resultant equilibrium, as well as the issue of strategic defaults. Our proposed model builds on the pioneering work of 2006, by G.A. Tedeschi.

1 Introduction

The mathematical formulation of microcredit is in its infancy, in stark contrast to the highly developed theory of Asset Pricing, and even of Credit Risk. We take a first step here, by proposing a Markov chain model to formalize the dynamic model of microcredit lending. In doing so, we recover some of the basic results and formulas of G.A. Tedeschi [3] , who obtained them through economic reasoning alone. Our model leads naturally to an optimization problem which is for the lender to choose the optimal time of exclusion with regards to a given borrower. This arises to avoid abuse by the borrower, which stems ultimately from a complete lack of credit ratings and the possibility of posted collateral. (This lack of credit ratings is integral to microcredit, which tries to extend beneficial lending practices in a financially primitive setting.) Microcredit has been shown to be sustainable in practice. Indeed its creator and main promoter, Muhammad Yunus & Grameen Bank of Bangladesh received the Nobel Peace Prize. Microfinance has received significant attention in the Economics literature. See for example [2] and the references therein. Therefore it is a bit surprising that we know of no previous attempts to model microcredit with modern mathematical tools.

2 Description of the Model

Our simplified model is as follows : a potential borrow can borrow one unit over a period of time t. At time t the borrower is obligated to repay the one unit, plus interest. Thus we have the scheme $1 \to 1+r$. The amount r will depend on the interest rate charged, and the time duration t. For the currency we could take the Bangladeshi Taka, but for our model the actual currency used (or numéraire in financial parlance) is irrelevant. The borrower is expected to invest the 1 (Taka[1]) in a business proposition, resulting in an amount w at time t. If $w > 1 + r$ then the borrower can repay the loan, and we call this a *success*. A borrower is assumed to be successful with a fixed probability α. We call these two states A (for applicant) and B (for Beneficiary). Not all applicants receive loans, thus one does not move from state A to state B with certainty.

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[1] or, more realistically, one *thousand* Taka, as in the original loan described by M. Yunus in [4]

If she[2] is succesful, she is entitled to get (with certainty) a new loan of 1, so she is again in the state B. If not, she enters a *credit-exclusion period* of length T at least. A count down of *application states* proceeds, A^T, A^{T-1}, $\ldots A^1$, and as long she is in state A^i, with $i > 1$, she cannot get any loan. After $T-1$ steps she obtains (for sure) $A^1 =: A$ when she can apply again, with probability γ to become beneficiary B of a loan at the next step. With probability $(1-\gamma)$ she will stay applicant for the next step and can apply again with the same chances to become beneficiary. Please observe that the outcome (B stays B or becomes A^T, A^1 becomes B or stays A^1) is known at the end of the time period, so becoming A^T implies a wait of T time steps to obtain the possibility of a new loan.

These rules can be summarized in a Markov chain $(X_t)_{t\in\mathbb{N}}$ with $X_t \in \mathcal{S} := \{B, A^1, A^2, \ldots, A^T\} = \{A^0, A^1, A^2, \ldots, A^T\}$, letting $A^0 := B$. The transition matrix of this Markov chain is given by

$$\mathcal{P} = \begin{pmatrix} \alpha & 0 & 0 & \cdots & 0 & 1-\alpha \\ \gamma & 1-\gamma & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{where}$$

$\mathbb{P}(X_{t+1} = B \mid X_t = B) = \alpha$ (succesful beneficiary)
$\mathbb{P}(X_{t+1} = A^T \mid X_t = B) = 1-\alpha$ (unsuccesful beneficiary, leads to credit exclusion for at least T)
$\mathbb{P}(X_{t+1} = A^{i-1} \mid X_t = A^i) = 1$, $i = 2\ldots, T$ (countdown of credit exclusion period)
$\mathbb{P}(X_{t+1} = B \mid X_t = A^1) = \gamma$ (applicant gets a loan)
$\mathbb{P}(X_{t+1} = A^1 \mid X_t = A^1) = 1-\gamma$ (applicant fails to get a loan and stays an applicant)

As soon as a Markov chain modelizes some dynamic, it is natural to check if it admits a limit stationary distribution. In our case, there is one and it gives the limit distribution of the total population into the different states B, A^1, $\ldots$, A^T.

Proposition 1 *For any initial state distribution $\pi_0 = (\pi_0^0, \ldots, \pi_0^T)$ the Markov dynamic tends to the distribution π_*, with*

$$\pi_* = \frac{1}{(1-\alpha)(1+\gamma(T-1))+\gamma}\left(\gamma, 1-\alpha, \gamma(1-\alpha), \ldots, \gamma(1-\alpha)\right). \tag{1}$$

Proof: The matrix $\mathcal{P}$ is stochastic and 1 is its largest (dominant) eigenvalue. It is easy to compute an associated (dominant) left eigenvector π_* with positive coefficients adding up to 1. As $\mathcal{P}$ is primitive, the Perron-Frobenius theorem shows that $\lim_{n\to+\infty} \pi_0 \mathcal{P}^n = \pi_*$. □

Notice that the distribution $\pi_* = (\pi_*^0, \pi_*^1, \ldots, \pi_*^T)$ is a stationary distribution. This means that, at the equilibrium, if N is the total (large and fixed) number of potential borrowers involved (i.e. in one of the states B, A^1, $\ldots$, A^T), then, in view of the law of large numbers, $\pi_*^0 N$ is the actual number of beneficiaries whereas $(1-\pi_*^0)N$ is the number of the people involved waiting for a loan. It is now possible to build up a prescribed *dynamic* increasing number $N(t)$ of involved potential borrowers or, similarly, a prescribed dynamic number $b(t)$ of actual beneficiaries of a loan. It suffice to add newcommers in each states in order to put the number of people in each states to $N(t)\pi^*$. This can be useful in order to meet some predetermined social-business plan of an increasing number of beneficiaries, taking advantage of the necessary waiting time to involve the candidates in some preparatory activity.

3 Computing the expected total discounted return

Let us now describe the (linear) model for the activity related to a loan. When lent 1, the borrower can enter a production activity that will produce, if nothing bad happens, an income of w during time 1, for which she will have to pay $1+r$, principal plus interest : this is what takes place with probability α. But if she is unlucky, her income is 0 and she has nothing to reimburse for the lent 1 (this is the specific feature of microcredit) and is excluded of any credit for a time period T at least. So the net income available at time t for the activity related to the loan 1 lent at time $t-1$ is a function $f(X_{t-1}, X_t)$ with $f(B,B) = w-(1+r)$, and $f(x,y) = 0$ for all other (x,y) than $(x,y) = (B,B)$.

[2] As most of the Grameen Bank's borrowers are women, we have chosen here to use a feminin pronoun for the borrowers.

Let us consider, for any $s \geq 0$, the *expected total discounted future income* W_s

$$W_s = \mathbb{E}\left(\sum_{t=s+1}^{\infty} \delta^{t-s} f(X_{t-1}, X_t) \;\mid\; \mathcal{F}_s\right),$$

where $\mathcal{F}_t = \sigma(X_0, \ldots, X_t)$ is the filtration associated with X_t and $\delta \in (0,1)$ the (fixed) *discount factor* for a time period 1. The discount factor δ could be taken equal to $1/(1+r)$ but more likely it will be chosen such that $\delta = \frac{1}{1+r^+} < \frac{1}{1+r}$, with $r^+ > r$ being the interest rate the borrower would have to pay when she does *not* have access to the MFI loan and has to use an alternative. Notice that the series converges since $0 \leq \delta < 1$ and f is bounded. The quantity W_s is the expected present value of the intertemporal benefit that a borrower could expect from being in a state B (getting a loan), or in a state A^i (waiting i periods before applying for a loan) at time s. Following Tedeschi, we can compute this quantity as follows.

Theorem 2 *The* expected total discounted future income W_s *of a borrower at time s is given by* $W_s = h(X_s)$, *with*

$$h(x) = \begin{cases} (1-\delta(1-\gamma))W^0 & \text{if } x = B \\ \delta^i \gamma W^0 & \text{if } x = A^i \text{ for } i = 1 \ldots T \end{cases}$$

where $W^0 := \dfrac{\alpha\delta(w-(1+r))}{(1-\alpha\delta)(1-\delta(1-\gamma)) - \gamma\delta^{T+1}(1-\alpha)}$.

Proof: First using the Markov property, we have

$$\mathbb{E}\left(\sum_{t=s+1}^{\infty} \delta^{t-s} f(X_{t-1}, X_t) \;\mid\; \mathcal{F}_s\right) = \mathbb{E}\left(\sum_{t=s+1}^{\infty} \delta^{t-s} f(X_{t-1}, X_t) \;\mid\; X_s\right)$$

thus the value of W_s is a function of X_s equal to $\mathbb{E}_{\{X_s=x\}}\left(\sum_{t=s+1}^{\infty} \delta^{t-s} f(X_{t-1}, X_t)\right)$ for $x \in \{B, A^1, \ldots, A^T\}$. Define the operator Φ that associates to any Markov chain $Y = (Y_t)_{t\geq 0}$ with state-space $\mathcal{S}$ the random variable $\Phi(Y) := \sum_{t=1}^{\infty} \delta^t f(Y_{t-1}, Y_t)$. Introducing $\hat{X}_t := X_{t+s}$, we can rewrite the previous expectation :

$$\mathbb{E}_{\{X_s=x\}}\left(\sum_{t=s+1}^{\infty} \delta^{t-s} f(X_{t-1}, X_t)\right) = \mathbb{E}_{\{\hat{X}_0=x\}}\left(\sum_{t=1}^{\infty} \delta^t f(\hat{X}_{t-1}, \hat{X}_t)\right) = \mathbb{E}_{\{\hat{X}_0=x\}}(\Phi(\hat{X})).$$

Now, by the Markov property applied to X, if $\hat{X}_0 = x$, we see that $\hat{X}$ is a Markov chain beginning at x with transition probability matrix $\mathcal{P}$ (we write $\hat{X} \rightsquigarrow \mathcal{M}(\delta_B, \mathcal{P})$), just like X if $X_0 = x$, so $\mathbb{E}_{\{\hat{X}_0=x\}}(\Phi(\hat{X})) = \mathbb{E}_{\{X_0=x\}}(\Phi(X))$.

Finally, if $X_s = x$, W_s is simply equal to $h(x)$, with $h(x) := \mathbb{E}_{\{X_0=x\}}(\Phi(X))$. It remains to compute $h(x)$ for all $x \in \{B, A^1, \ldots, A^T\}$. We have

$$\begin{aligned} h(x) &= \mathbb{E}_{\{X_0=x\}}(\Phi(X)) = \mathbb{E}_{\{X_0=x\}}\left(\mathbb{E}\left(\delta f(X_0, X_1) + \delta \sum_{t=1+1}^{\infty} \delta^{t-1} f(X_{t-1}, X_t) | \mathcal{F}_1\right)\right) \\ &= \delta \mathbb{E}_{\{X_0=x\}}(f(x, X_1) + W_1) = \delta \mathbb{E}_{\{X_0=x\}}(f(x, X_1) + h(X_1)) \\ &= \delta \sum_{a \in \{A^0, \ldots, A^T\}} (f(x,a) + h(a)) p_{xa}, \text{ where } p_{xa} = P(X_t = a | X_{t-1} = x). \end{aligned}$$

- For $x = A^i$ with $i > 1$, $f(x,a) = 0$ and only $p_{A^i A^{i-1}}$ is not 0 (and actually equal to 1) thus

$$h(A^i) = \delta h(A^{i-1}) = \delta^2 h(A^{i-2}) = \ldots = \delta^{i-1} h(A^1) = \delta^{i-1} h(A).$$

- For $x = B(= A^0)$, $h(B) = \delta[(f(B,B) + h(B))\alpha + (f(B, A^T) + h(A^T))(1-\alpha)]$, thus

$$h(B) = \delta[(w - (1+r) + h(B))\alpha + \delta^{T-1} h(A)(1-\alpha)]. \tag{2}$$

- For $x = A(= A^1)$, we introduce the stopping time τ defined by $\tau := \text{Min}\{t > 0 \;\mid\; X_t = B\}$ if this set is not empty, and else $\tau = +\infty$. This stopping time follows a geometric law $\mathcal{G}(\gamma)$, with $\mathbb{P}\{\tau = k\} = (1-\gamma)^{k-1}\gamma$, so $\mathbb{P}(\{\tau < +\infty\}) = 1$ and thus $\tau < +\infty$ a.s.. This allows one to apply the strong Markov property to $\hat{X}^\tau := (X_{t+\tau})_{t\geq 0}$, with $\hat{X}_0^\tau = X_\tau = B$, thus $\hat{X}^\tau \rightsquigarrow \mathcal{M}(\delta_B, \mathcal{P})$, and $\hat{X}^\tau$ and τ are independent. As $f(X_{t-1}, X_t) = 0$ for $t \leq \tau$ we have

$$\begin{aligned}\Phi(X) = \sum_{t=1}^{\infty} \delta^t f(X_{t-1}, X_t) &= \sum_{t=1}^{\tau-1} \delta^t f(X_{t-1}, X_t) + \sum_{t=\tau}^{\infty} \delta^t f(X_{t-1}, X_t) \\ &= 0 + \delta^\tau \sum_{t=0}^{\infty} \delta^t f(\hat{X}_{t-1}^\tau, \hat{X}_t^\tau) = \delta^\tau \Phi(\hat{X}^\tau). \qquad (3)\end{aligned}$$

Let $G_\tau(\delta)$ be the value at $u = \delta$ of the probability generatrix function $u \mapsto G_\tau(u) := \mathbb{E}(u^\tau)$ of the distribution $\mathcal{G}(\gamma)$. So $G_\tau(\delta) = \mathbb{E}(\delta^\tau) = \frac{\delta\gamma}{1-\delta(1-\gamma)}$. Now

$$\begin{aligned}h(A) &= \mathbb{E}_{\{X_0=A\}}(\Phi(X)) = \mathbb{E}_{\{X_0=A\}}(\delta^\tau \Phi(\hat{X}^\tau)) \text{ by (3)} \\ &= \mathbb{E}_{\{X_0=A\}}(\delta^\tau)\mathbb{E}_{\{X_0=A\}}(\Phi(\hat{X}^\tau)) \text{ as } \hat{X}^\tau \text{ and } \tau \text{ are independent} \\ &= G_\tau(\delta)\mathbb{E}_{\{X_0=B\}}(\Phi(X)) \text{ as } \hat{X}^\tau \text{ and } X \text{ are both } \mathcal{M}(\delta_B, \mathcal{P}) \\ &= G_\tau(\delta)h(B) = \frac{\delta\gamma}{1-\delta(1-\gamma)} h(B), \qquad (4)\end{aligned}$$

Now, introducing these results in (2) we get

$$\begin{aligned}h(A^0) = h(B) &= \frac{\alpha\delta(w-(1+r))(1-\delta(1-\gamma))}{(1-\alpha\delta)(1-\delta(1-\gamma)) - \gamma\delta^{T+1}(1-\alpha)} =: (1-\delta(1-\gamma))W^0 \qquad (5) \\ h(A^1) = h(A) &= \frac{\delta\gamma}{1-\delta(1-\gamma)} h(B) = \delta\gamma W^0 \\ h(A^i) &= \delta^{i-1} h(A) = \delta^i \gamma W^0 \text{for } 1 < i \leq T, \text{ and actually } 1 \leq i \leq T\ . \qquad (6)\end{aligned}$$

□

4 Condition of Absence of Strategic Default (ASD)

The purpose of introducing a period of loan-exclusion is to avoid any incentive to default, by creating some fear of being excluded of the right to get a new loan. This goal will be achieved as soon as the borrower is better off when paying $1+r$ as agreed and have full access to microcredit as Beneficiary than not paying and being excluded of any microcredit loan for a period at least equal to T. For a rational risk-neutral borrower this can be expressed by

$$h(B) - (1+r) \geq h(A^T), \qquad (7)$$

that we call the *Absence of Strategic Default* condition, or ASD condition[3].

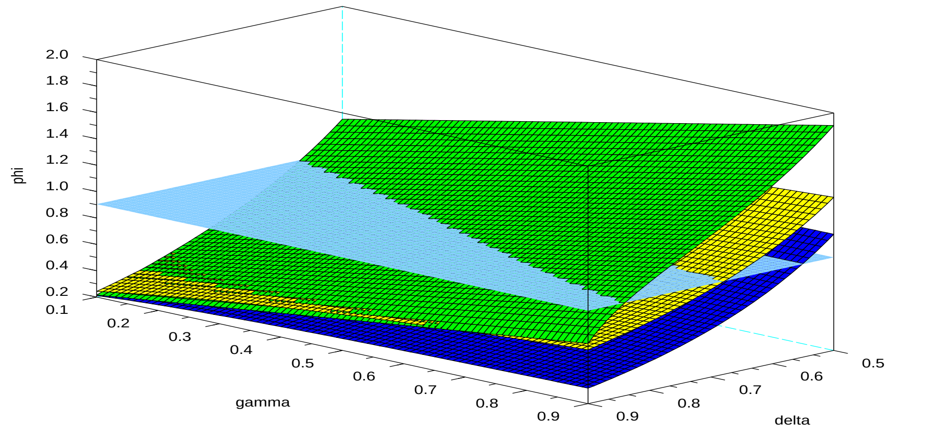
Corollary 3 *The ASD condition (7) is achieved if and only if*

$$\frac{w}{1+r} - 1 \geq \frac{1}{\alpha\delta} \frac{(1-\alpha\delta)(1-\delta(1-\gamma)) - \delta^{T+1}\gamma(1-\alpha)}{1-(\delta(1-\gamma)+\delta^T\gamma)} (=: \varphi(\alpha,\delta,\gamma,T)). \qquad (8)$$

Let $\Delta := \alpha\delta w - (1+r)$; T can be chosen such that the ASD condition holds if and only if $\Delta > 0$, and, if so, ASD holds if and only if $T \geq T^*$, where

$$T^* = \frac{\ln \frac{(1-\delta(1-\gamma))(\alpha\delta w-(1+r))}{\delta\gamma(\alpha w-(1+r))}}{\ln(\delta)}.$$

[3]Strategic Default is the (morally dishonnest) behaviour of claming not being able to reimburse (Default) when it is not the case, just because this is individually more profitable.



low high patient impatient

chances to get a loan preference for the present

Figure 1: Condition (8) for $T = 1, 2, 5$.

Proof: In view of theorem (2) the ASD condition (7) is equivalent to $(1-\delta(1-\gamma))W_0 - (1+r) \geq \delta^T W_0$, that by a straight forward computation is equivalent to (8). As the denominators in (8) are positive, this inequality is equivalent to

$$\delta^T \Delta' \delta\gamma \leq (1-\delta(1-\gamma))\Delta \text{ , with } \Delta' := \alpha w - (1+r) \text{ and } \Delta := \alpha\delta w - (1+r). \tag{9}$$

If $\Delta > 0$, as $\Delta' > \Delta$, (9) is equivalent to $T \geq T^*$, and if $\Delta \leq 0$, (9) can never hold, as $\delta = \frac{1}{1+r} \in (0, 1)$. $\square$

Remark: The hypothesis $\Delta > 0$ is just $w > \frac{1+r}{\alpha\delta}$, which means that the return for the borrower for each borrowed unit has to be large enough to make sure that she is interested in keeping the possibility to borrow again. If not she is better off keeping the $1+r$ she owes and have never more a chance to borrow again. A second remark is that $T^* \leq 0$ if and only if

$$w \geq w^* \text{ with } w^* := \frac{1}{\alpha\delta}\left(1 + r + \gamma\left(\alpha + \frac{1-\alpha}{1-\delta(1-\gamma)}\right)\right), \tag{10}$$

in which case the interest for the borrower of having acces to credit is so high that there is no point in imposing a minimum time of exclusion and the fear of loosing this acces to credit suffices to push her to pay what she owes.

5 Conclusion

We have presented a simplified mathematical model of microcredit. Our model extends the approach given by G.A. Tedeschi. We note that our model is too simple for immediate applicability : For example it does not take into account the possibility of frequent small partial reimbursements, or joint liability factors arising from group borrowing. While this has been broached excellent in the Economics literature (see, for example, [1]), it still awaits a proper mathematical treatment. We hope however it is a start which will inspire future, more comprehensive studies.

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