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Randomness of interest rates in microcredit

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Abstract

Microcredit, as described by M. Yunus, allows efficient lending without collateral. One of its characteristics is that it is based on a large number of (frequent) settlements leading to little default risk but possible delay in settlements, creating randomness of actual interest rate. We take the example of the Yunus equation to examine the probability characteristics (law) of this interest rate, as a measure of the risks in microcredit.

In strong contrast to finance of stock derivatives or credit risk, the probabilistic approach to microcredit has not benefited much of the tools of stochastic calculus. In [2] we considered a Markov-chain model inspired by G. Tedeschi's seminal work [5]. We want to introduce another stochastic model to take into account the fact that, in case of microcredit, the main risk is in the delays that can occur in foreseen settlements. Indeed, in our mind what characterizes microcredit is that the borrower can't provide any collateral to the loan she[1] receives and thus has no access to regular credit. Anyway, and despite this apparently more risky feature, the microcredit activity, when correctly dealt with by the lender (a "microfinance institution" (IMF) usually), has turned out to be sustainable, leading to a modeling challenge that we want to tackle here. The feature we want to address is that the risk of default is mainly replaced by the risk of delay, and the consequent randomness of interest rate. Indeed, as the loan is small, the borrower will do her utmost efforts to avoid to lose the advantage related to have access to microcredit if she fails to reimburse her loan. As the number of settlements is large their amount is very small but, even so, as the borrower is very poor, she may face anyway difficulties to pay in time a settlement and postpone by one (or possibly more) time periods all the subsequent settlements, and this affects the interest rate received by the IMF.

1 The generalized Yunus equation

In his book [6], Yunus explains how he could lend, with his students, an amount of 1000 BDT[2], that were repaied in 50 settlements of 22BDT paied in week k, so at time $t_k = k$.

If we denote by r^Y the yearly continuous interest rate, we see that this "Yunus" interest rate r^Y is implicitly defined by the equation

$$1000 = \sum_{k=1}^{50} 22 e^{-t_k \frac{r^Y}{52}} = 22 \sum_{k=1}^{50} q^k = 22 \frac{q - q^{51}}{1-q} \text{ as } t_k = k, \text{ with } q = e^{-\frac{r^Y}{52}}.$$

This equations can easily be solved numerically, leading to $q = 0.9962107\ldots$ and $r^Y = -52\log(q) = 19.74\ldots\% \simeq 20\%$, as given by Yunus.

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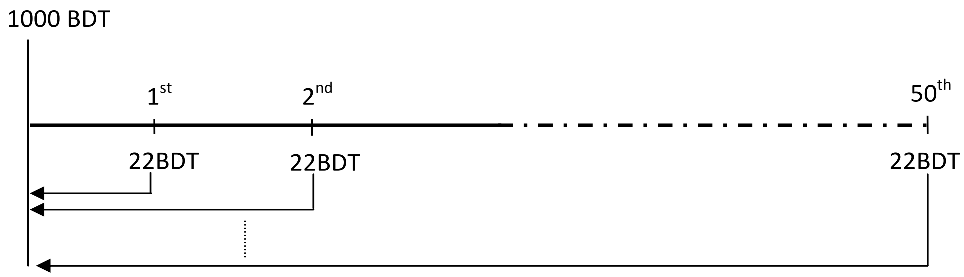
[1] As most of the Grameen Bank's borrowers are women, we have choosen here to use a feminine pronoun for the borrowers.

[2] One EUR is a little less then hundred BDT

Figure 1: Weekly Installments



The problem with this computation is that if the borrower faces a problem in week i, she will not be able to pay, and will postpone the payment to the next week, thus introducing a delay in all the subsequent settlements. To take this into account we now introduce a stochastic process in such a way that the t_ks become random variables T_k.

We will also take into account that the number (50) is large, so we introduce a new large parameter N and end up with the following *generalized Yunus equation*

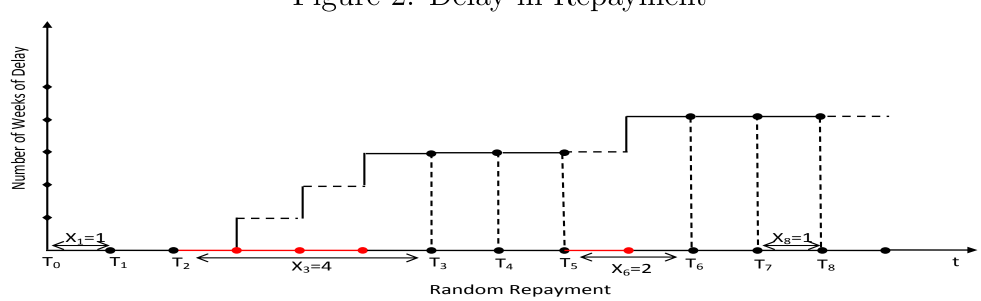
$$N = (1 + r_f) \sum_{k=1}^{N} Q^{T_k} \text{ , with } Q = e^{-\frac{R_N}{N}} \tag{1}$$

where the original equation corresponds to $N = 50$, $r_f = 10\%$ is the *flat rate* such that $22 = (1 + r_f)1000/N$, and $r^Y = \frac{N+2}{N} R_N$. We use here a capital letter R_N for the implicit interest rate, as, the T_ks being random, this interest rate becomes random too. We claim that this randomness is one of the main risk factors and we wish to better understand its probabilisty distribution.

2 The distribution of the implicit interest rate

It still remains to provide the stochastic model for the random settlement dates T_k. To that purpose, let $\mathcal{B} = (B_m)_{m \geq 1}$ be a Bernoulli process, i.e. a sequence of independent Bernoulli random variables $B_m \rightsquigarrow \mathcal{B}(1, p)$. In our model $B_m = 1$ stands for "the borrower can pay at week m", and $B_m = 0$ models that she can't pay at this week. So the k-th settlement takes place at (the $\mathcal{F}^{\mathcal{B}}$ stopping-time) $T_k = \min\{m | B_1 + \ldots + B_m = k\}$, $k = 1 \ldots N$. It is convenient to define the sequence of *time to k-th settlement* $X_k = T_k - T_{k-1}$, with $T_0 = 0$. From the strong-Markov property of process $\mathcal{B}$ we see that $\mathbf{P}(X = x) = p(1-p)^{x-1}$ so $X_k \rightsquigarrow \mathcal{G}(p)$, the geometric distribution. This gives a way to choose a reasonably realistic value of p. Indeed, if we take the often cited 3% of default-rate and if we consider[3] that default means that some $X_m > 4$, we get $p = 0.84$, value that we will use in our examples. Now

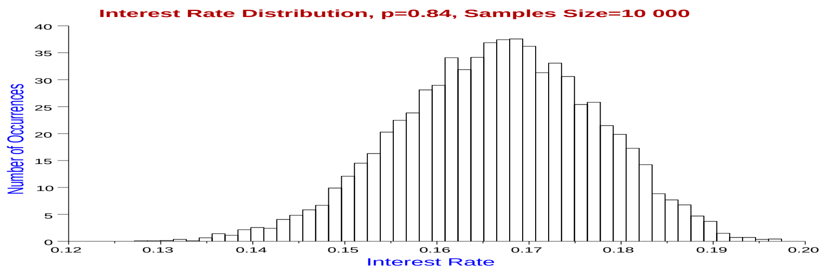
Figure 2: Delay in Repayment



the model is fully settled, and this defines, implicitly from equation (1), the random variable Q and $R_N = -N \log(Q)$. Using `Scilab` we built a sample of size $10,000$ of random settlements of lending. In figure 3 the histogram of the consequent actuarial interest rate received by the microcredit institution. The maximum is of course $r^Y = 19.74$, the actuarial rate when there is no delay in installments; but we can compute the weighted average and we can observe that it is more then 3% below this "no-delay"

[3]private communication of some BRAC credit-officer

Figure 3: Distribution of actuarial interest rate R_N computed from equation (1) for $N = 50$ and $r_f = 10\%$ on a Monte-Carlo sample of 10,000 borrowers.



value r^Y, just as the actuarial expected rate (see [3]). This explains why, for microcredit, we think that the "delay-risk" is, at least, as important as the default-risk.

3 The asymptotics of the interest rate for a single delay

Looking at the histogram suggests, at first, a normal distribution. But it is obviously bounded from above, and the skewness is not zero (the hump is lopsided). We introduced the generalized Yunus equation because we expect to get an asymptotic expansion converging towards the distribution of R_N, i.e.

$$R_N = \sum_{l=0}^{n} \frac{\alpha_l}{N^l} + \frac{1}{N^n}\epsilon_n(N)$$

with $\lim_{N\to\infty} \epsilon_n(N) = 0$ for all $n \geq 0$. This program is far from being reached but we succeeded to get an asymptotic expansion for $n = 3$ of R^1_N, the implicit interest rate assuming a single delay, i.e. $T_N = N + 1$, or, equivalently, there is one and only one k such that $X_k = 2$, all other X_is being equal to 1. Let $\Omega_1 = \{T_N = N + 1\}$. On Ω_1 all events $\{X_k = 2\}$ have same probability, so $\mathbf{P}_{\Omega_1}(\{X_k = 2\}) = \frac{1}{N}$. Moreover, still on Ω_1, if $X_k = 2$, R^1_N is determined and is the unique positive solution $r = r_k$ of

$$N = (1 + r_f)\left(\sum_{m=1}^{k-1} q^m + \sum_{m=k}^{N} q^{m+1}\right) \text{ , where } q = e^{-\frac{r}{N}} \tag{2}$$

which can be rewritten $N = (1 + r_f)(\sum_{m=1}^{N+1} q^m - q^k)$ $\left(= (1 + r_f)q[\frac{1-q^{N+1}}{1-q} - q^{k-1}]\right)$.

Theorem 1 *The unique positive solution q_k of (2) has the following asymptotic expansion*

$$q_k = 1 - \frac{\beta_1}{N} + \frac{\beta_2}{N^2} + \frac{\beta_3(k)}{N^3} + \frac{\epsilon(N)}{N^3} \text{ , with } \lim_{N\to\infty} \epsilon(N) = 0,$$

where β_1 is the unique positive solution of $\beta = (1+r_f)(1-e^{-\beta})$; $\beta_2 = \frac{\beta_1^2(3+\beta_1-r_f)}{2(\beta_1-r_f)}$; $\beta_3 = \beta_3(k) = \lambda k+\mu$, with $\lambda = -\frac{\beta_1^2(1+r_f)}{\beta_1-r_f}$, and $\mu = -\frac{\beta_1(1+r_f)}{\beta_1-r_f}\left[\frac{\beta_2^2}{\beta_1^2(1+r_f)} + \frac{1+r_f-\beta_1}{1+r_f}\left[\beta_2\left(\frac{3}{2} - \frac{\beta_2}{\beta_1^2} - \frac{\beta_2}{2\beta_1}\right) - \beta_1\left(1 + \frac{\beta_1}{2} - \frac{\beta_2}{2} + \frac{\beta_1^2}{8}\right)\right]\right]$.

When $r_f = 10\%$ $\beta_1 = 0.193748$. The striking fact here is that the three first terms of the expansion of q_k do not depend on k, and only the fourth, β_3 does, and is an affine function of k, $\beta_3(k) = \lambda k + \mu$. This is again so for the values of $r_k = -N\log(q_k)$; indeed expanding $\log(1 - x)$ for small x we get

Corollary 2 *On Ω_1, when only one delay occurs in week k, the implicit interest rate r_k has an order two expansion*

$$r_k = \alpha_0 + \frac{\alpha_1}{N} + \frac{\alpha_2(k)}{N^2} + \frac{\epsilon(N)}{N^2} \text{ , with } \lim_{N\to\infty} \epsilon(N) = 0.$$

with $\alpha_0 = \beta_1$, $\alpha_1 = \frac{1}{2}\beta_1^2 - \beta_2$, and $\alpha_2 = \alpha_2(k) = \frac{1}{3}\beta_1^3 - \frac{1}{2}\beta_1\beta_2 - \beta_3(k)$.

So, up to terms small with respect to $\frac{1}{N^2}$, the interest rate depends linearly on the number of the week when the single delay occurs. See figure 4.

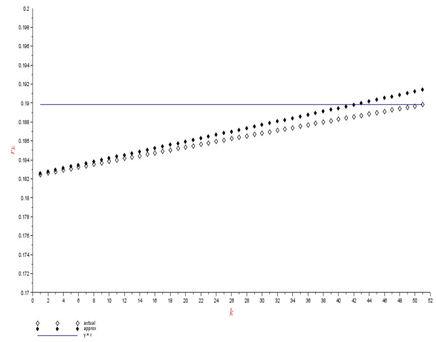


Figure 4: Actual interest rate in case of one single delay at week k and its asymptotic approximation. The horizontal line corresponds to the interest rate $r = r_{N+1}$, when there is no delay ($N = 50$, $r_f = 10\%$, $r^Y = \frac{52}{50}r$))

4 Conclusion

In this paper we have examined the risk of microcredit related to delay in payment of one or more timely foreseen settlements. Having in mind the 50 payments of the example given in Yunus' book we have set up a model with a large parameter N of payment, each subject to some delay leading to delay to all subsequent foreseen payments. We wish to use the fact that 50 is rather large to get an asymptotic value of the distribution of the implicit (here deterministic) interest rate. In the case of the example of Yunus, the flat rate r_f is 10%, the implicit (deterministic, actuarial) interest rate is nearly 20%, but this is a maximum: any delay reduces the received interest rate, leading to an average rate of less then 17% when $N = 50$ and the probability of a single delay is 16%, a value extrapolated, within the model, for the often claimed default risk of 3%, when default means a sequence of four delayed weekly settlements. Finally we have given a second-order asymptotic expansion of the implicit interest rate when restricting the risk to a single, equally distributed, delay.

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