# Mathematical models for individual and group lending in microfinance

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**Abstract**

In a previous paper [4], we have introduced a mathematical version of a microcredit model build on the pioneering work in 2006, of G.A. Tedeschi [10] based on dynamic incentives for borrowers to repay their loans. In this paper we extend the model to the case of a group of two borrowers with joint liability in order to compare individual and group lending efficiency. The extended model shows that group lending is preferable not only in increasing the borrowers expected return, when the joint liability coefficient is not too hight, but also in offering loans to more beneficiaries and in allowing the lender to charge lower interest rates.

*Keywords*: Microfinance; Individual lending; Group lending; Dynamic incentives; Joint liability.

# 1 Introduction

Microcredit consists of very small loans given by microfinance institutions (MFIs) to poor people to help them to develop micro entreprises. common borrowers belong to group of people who have no access to credits from banks because they can not put up any collateral for a loan. They are also mainly women.

Microcredit has been shown since its beginning and especially after the success of the Grameen Bank, to be really efficient in helping poor people all around the world. In addition microcredit appears to be a sustainable activity and even profitable for investors mainly due to a really low default rate.

There has been several important contributions that seek to explain the success of microcredit. Stiglitz [9] provide explanations based on peer monitoring. Besley and Coate [2] analyze a strategic repayment game with joint liability and demonstrate that successful group members may have an incentive to repay the loans of the less successful ones. They also highlight the effect of social collateral in ensuring repayment. Ghatak and Guinnane [5], on the other hand, analyze moral hazard problems in group-lending. But the first dynamical model who was not only a 1 ou 2 steps model[1] but takes into account all futurs steps is a model of Tedeschi [10]. In this model, she assumes as a rule that a borrower who reimbourses his loan will automatically have access to a new loan. The model also introduces an exclusion phase when reimboursement not occurs in order to enhance the incentive to repay. In this paper we first recall in the section 2 a mathematical version of the Tedeschi model, introduced as a Markov chain, and propose in section 3 a new mathematical model, also build on dynamic incentives and exclusion phase, but for group lending (group of two borrowers). This allows a comparaison between group and individual lending.

# 2 Model for individual lending

In the Tedeschi model, each borrower (usually woman) has a project that requires one unit of capital and bring an amount $w$ of profit if it is successfull. The project lasts for one period and will be successfull with probability $\alpha$ and fails (due to external shocks) with probability $1 - \alpha$. In case of success, the borrower will pay $1 + r$ to the lender, where $r$ is the interest rate for one period, and will be entitled to get (with certainty) a new loan of one unit.

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[1]For exemple, see Armendariz de Aghion [1], and Hulm an Mosley [7].

If not, she[2] will pay nothing but will not be allowed to get a new loan during the next $T$ periods. Then, when this credit-exclusion phase ends, she can apply again for a new loan but, depending on the number of eligible borrowers seeking a loan and the limit of the size of the lender's portfolio, she will become beneficiary only with probability $\gamma$ the first period after the exclusion phase, and with probability $(1-\gamma)$ she will not get a loan and have to wait one more period to apply again with the same chance to become beneficiary or not.

These rules of becoming a beneficiary or exclude can be summarized in a Markov chain $(X_t)_{t\in\mathbb{N}}$ with the set of states

$$E := \{B, E^T, E^{T-1}, \ldots, E^1\}, \tag{2.1}$$

where $B$ denotes the state of being a beneficiary of a loan, $E^1$ the state of being an applicant for a loan with the possibility to become a beneficiary for the next period, and $E^i$, $i = 2, \ldots, T$, the state of being in the exclusion phase for the next $i$ periods. The transition matrix of this Markov chain is given by

$$P_1 = \begin{pmatrix} \alpha & 1-\alpha & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ \gamma & 0 & 0 & 0 & \cdots & 0 & 1-\gamma \end{pmatrix} \tag{2.2}$$

The following diagramme summarizes the dynamic of this Markov chain (2.1,2.2):

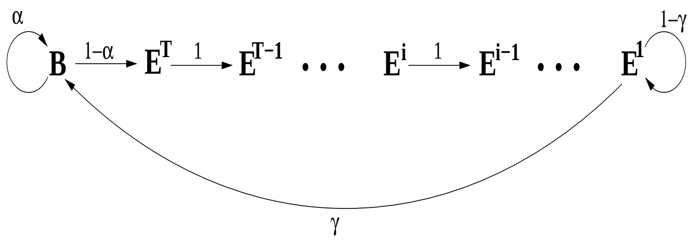


Figure 1: Diagramme of the Markov chain $X_t$

## 2.1 Principal hypothesis

The purpose of introducing this model of individual lending is to find an optimal contract $(r^*, T^*)$ which maximize the borrower return calculated in the next section, where $r^*$ is the optimal interest rate and $T^*$ is the optimal duration of the exclusion phase. In searching such contract, we take into account three kind of hypothesis as following :

1. **Participation hypothesis:** this constraint describes the fact that the borrower will only take a loan when she is able to make a profit on the venture:
$$w > 1 + r \tag{2.3}$$

2. **Sustainability hypothesis:** the interest rate should be set as low as possible in order to maximize the borrower's profit, but either it should be as long as possible to satisfy the sustainability constraint of the lender:
$$\alpha(1 + r) \geqslant 1 + z \tag{2.4}$$
where $z$ is the lending cost per unit, including operating costs.

[2]A borrower is always a women.

3. **Absence of strategic default hypothesis:** in order to induce the borrower to pay when she is able, the deprivation imposed by the punishment phase must be larger than the gain from non-repayment. If the profit from investing, repaying the loan, and having a new investment phase (with a new loan) is bigger than the one periode gain and going into an exclusion phase, than strategic default will not occur. This would be presented by
$$[w-(1+r)]+\delta V_s(B) \geqslant w+\delta V_s(E^T) \tag{2.5}$$
where $\delta \in (0,1)$ is the (fixed) *discount factor* for one period[3] and $V_s(x)$ is the total expected return for a borrower being at state $x$ in time $s$. This $V_s(x)$ is calculated in the next section for all $x \in E$.

## 2.2 Total expected return

In this model, when a borrower having $\alpha$ as probability of success of his project, knows that her expected return for one period is
$$\alpha[w-(1+r)]$$
But as she knows also that she will benefit automatically of a new loan if she reimbourses the current one, she can also evaluate the *expected discounted present value of the total return* for all futur periods, that we will simply call *total expected return* . To compute it, let us introduce the function $f : E \times E \to \mathbb{R}$ defined by
$$f(x,y)=\begin{cases} w-(1+r) & \text{if } (x,y)=(B,B) \\ 0 & \text{if not} \end{cases}$$
For each trajectory $(X_0, X_1, \dots)$ of the Markov chain, we define the total discounted return at time $s$ by
$$F(X_s, X_{s+1}, \dots.) := \Sigma_{t=1}^{\infty} \delta^{t-s-1} f(X_{t-1}, X_t)$$
Notice that the function $F$ is well defined, and bounded, because the serie converges since $0 \le \delta < 1$ and $f$ is bounded.

Finally let define the *total expected return* at time $s$, $V_s : E \to \mathbb{R}$, as a function of the state $x \in E$, by
$$V_s(x) = \mathbb{E}[F(X_s, X_{s+1}, \dots.) \mid X_s = x].$$
The following computation of the *total expected return* at time $s$, $V_s$, already given by Tedeschi, was proved in [4], in the case of the Markov Chain (2.1), (2.2).

**Theorem 2.1.** *In the model of individual lending defined by the Markov Chain (2.1), (2.2), the total expected return for an individual being in state $x$ at time $s$ is given by*
$$V_s(x)=\begin{cases} \alpha[w-(1+r)]\frac{1}{1-(\alpha\delta+(1-\alpha)\delta^T\Sigma)} & \text{if } x=B \\ \alpha[w-(1+r)]\frac{\delta^{i-1}\Sigma}{1-(\alpha\delta+(1-\alpha)\delta^T\Sigma)} & \text{if } x=E^i \ , \ i=1,\dots,T \end{cases} \tag{2.6}$$
*where* $\Sigma = \frac{\gamma\delta}{1-\delta(1-\gamma)}$.

Notice that as $\alpha[w-(1+r)]$ is just the expected return of the current loan, the *total expected return* $V_s(x)$ is the product of this immediat expected return by a factor that increases it to take into account the possible return of futur loans.

*Proof.* Notice first that, as $X_t$ is a Markov chain, for all $s \geq 0$ and all $x \in E$ we have
$$\begin{aligned} V_s(x) &= \mathbb{E}[F(X_s, X_{s+1}, \dots) \mid X_s = x] \\ &= \mathbb{E}[F(X_0, X_1, \dots) \mid X_0 = x] \\ &= V_0(x) \end{aligned}$$
Thus, instead of computing $V_s(x)$, it is enough to compute $V_0(x)$ at time $s = 0$. Now, as $F(X_0, X_1, \dots) = f(X_0, X_1) + \delta F(X_1, X_2, \dots)$, we have for all $x \in E$
$$V_0(x) = \mathbb{E}[f(X_0, X_1) + \delta F(X_1, X_2, \dots) \mid X_0 = x]$$

[3]Remark that this factor $\delta$ can be taken equal to $1/(1+r)$.

But, using first conditional expectation and then Markov property [4], we have

$$V_0(x) = \sum_{y\in E}[f(x,y) + \delta V_0(y)]p(X_1 = y \mid X_0 = x) \tag{2.7}$$

where $p(X_1 = y \mid X_0 = x)$ is the porbability of transition, in one step, from state $x$ to state $y$. So, all we have to do is to calculate $V_0(x)$ for all $x \in E$

- for $x = B$, then we have
$$V_0(B) = \alpha[w - (1+r)] + (1-\alpha)\delta V_0(E^T)$$
- for $x = E^i$, $i = 2, \ldots, T$, we have
$$V_0(E^i) = \delta V_0(E^{i-1})$$
- For $x = E^1$, some more computations are needed. Notice first that, when $X_0 = E^1$, using the definition of $f$, $F(X_0, X_1, \ldots) = \delta^\tau F(X_\tau, X_{\tau+1}, \ldots)$ where $\tau := Min\{t > 0 \mid X_t = B\}$ is a stopping time. Thus, we have

$$\begin{aligned}
V_0(E^1) &= \mathbb{E}[F(X_0, X_1, \ldots) \mid X_0 = E^1] \\
&= \mathbb{E}[\mathbb{E}[\delta^\tau F(X_\tau, X_{\tau+1}, \ldots) \mid X_0 = E^1, \ldots, X_{\tau-1} = E^1, X_\tau = B] \mid X_0 = E^1] \\
&= \mathbb{E}[\mathbb{E}[\delta^\tau F(X_\tau, X_{\tau+1}, \ldots) \mid X_\tau = B] \mid X_0 = E^1] \qquad \text{[strong Markov property]} \\
&= \mathbb{E}[\delta^\tau \mathbb{E}[F(X_\tau, X_{\tau+1}, \ldots) \mid X_\tau = B] \mid X_0 = E^1] \qquad [\delta^\tau \text{ is } \tau\text{-measurable}] \\
&= \mathbb{E}[\delta^\tau V_0(B) \mid X_0 = E^1] \\
&= \mathbb{E}[\delta^\tau V_0(B)] \\
&= V_0(B)\mathbb{E}[\delta^\tau]
\end{aligned}$$

Finally, as the stopping time $\tau$ follows a geometric law [11]). $\mathcal{G}(\gamma)$, then we have

$$\mathbb{E}(\delta^\tau) = \sum_{k\geq 1} \delta^k (1-\gamma)^{k-1}\gamma = \frac{\delta\gamma}{1 - \delta(1-\gamma)}$$

□

## 2.3 The optimal contract

The optimal contract is a contract that maximize the borrower profit in respecting the constraintes (2.3), (2.4) et (2.5). But a borrower will sign a contract only if it will make him a positif return, then I assume that the constraint (2.3) is usually satisfied. Then, the optimal contrat $(r^*, T^*)$ is a solution of the following optimization problem:

$$(\mathcal{P}) = \begin{cases} Max\ J(r,T) \\ (r,T) \in C \end{cases}$$

where

$$J(r,T) = \alpha[w - (1+r)]\frac{1}{1 - (\alpha\delta + (1-\alpha)\delta^T\Sigma)},$$
$$C = \{(r,T) \in K;\ h_1(r,T) \geq 0, h_2(r,T) \geq 0\}, K \text{ is a compact.}$$
$$h_1(r,T) = \alpha(1+r) - (1+z),$$

and

$$h_2(r,T) = (\delta - \Sigma\delta^T)J(r,T) - (1+r).$$

[4] For more informations about Markov properties, for exemple, see Çinlar [3].

The resolution[5] of this optimization problem gives:

$$r^* = \frac{1+z}{\alpha} - 1$$

and

$$T^* = \frac{1}{ln(\delta)} ln \left( \frac{[1-\delta(1-\gamma)][\alpha\delta w - (1+r^*)]}{\gamma[\alpha w - (1+r^*)]} \right) - 1$$

Notice that this optimal contract corresponds to the saturation of constraint (2.4). That means, the interest rate $r^*$ that both maximizes the borrower welfare and makes lending sustainable will exactly cover the total cost of lending.

### 2.4 Proportion of beneficiaries among the population

One interesting consequence of introducing a Markov chain to modelize this dynamic is that it eases the study of the evolution of the distribution of the population into the different states $B$, $E^1$, $E^2$, .... Indeed, we have the following result:

**Proposition 2.2.** *For any initial state distribution $\pi_0 = (\pi_0^0, \ldots, \pi_0^T)$ of the population among the different states, the Markov dynamic $\pi_0 P_1^t$ tends to the distribution $\pi_*$ when $t$ tends to infinity, with*

$$\pi_* = \frac{1}{\frac{1}{(1-\alpha)} + \frac{1}{\gamma} + (T-1)} \left( \frac{1}{1-\alpha}, 1, 1, \ldots, 1, \frac{1}{\gamma} \right). \tag{2.8}$$

*Proof.* The stochastic matrix $P_1$ is primitive, because $P_1^{2T} > 0$. Moreover the Perron-Frobenius theorem[6] shows that $\lim_{t\to+\infty} \pi_0 P_1^t = \pi_*$, thus the distribution of the population among the different states becomes closer and closer to the limiting distribution $\pi_*$ when $t$ tends to infinity. □

Notice that saying that the distribution $\pi_* = (\pi_*^0, \pi_*^1, \ldots, \pi_*^T)$ is a stationary distribution, means that if $N$ is the total (large and fixed) number of potential borrowers involved, then, at equilibrium for $t$ large enough, $\pi_*^0 N$ is the actual number of beneficiaries whereas $(1-\pi_*^0)N$ is the number of the people involved waiting for a loan. It is now possible to build up a prescribed *dynamic* increasing number $N(t)$ of involved potential borrowers or, similarly, a prescribed dynamic number $b(t)$ of actual beneficiaries of a loan. It suffices to add newcommers in each states in order to put the number of people in each states to $N(t)\pi_*$. This can be useful in order to meet some predetermined social-business plan of an increasing number of beneficiaries, taking advantage of the necessary waiting time to involve the candidates in some preparatory activity.

## 3 Model for group lending with joint liability

To study group lending, we consider for simplicity a group composed of only two identical borrowers. At the beginning of each lending period, the group has a loan of two units of capital, one for each borrower. Each borrower invests in a project and we assume that the returns of the two projects are independant. The duration of each investment is one period. At maturity of each period, the next situations describe the different possible states and consequences according to the ivestment results:

1. The two borrowers are successfull, this happens with probability $\alpha^2$ . They both will repay their loan[7] that is 2(1+$r$), $r$ is the interest rate for one period, and then they both will benefit of a new loan for the next period.

2. The two borrowers fail in their projects, this happens with probability $(1-\alpha)^2$. They are not able to reimbourse, then they both will be excluded for $T$ periods (to allow comparaison between individual and group lending, we choose the same length $T$ as in individual lending).

[5]The two funtions $h_1$ and $h_2$ satisfy the regularity conditions (or constraint qualifications), then by Karush-Kuhn-Tucker, we obtain the solution of the optimization problem $(\mathcal{P})$, see [6].

[6]For more details, see Serre [8].

[7]In this model of group lending, the interest rate $r$ is different from the interest rate in the individual lending model.

3. One borrower is successfull and the other fails, this happens with probability $\alpha(1-\alpha)$. Here, we assume that the first will repay not only his own loan (1+$r$) but also an amount $q$ that represents the joint liability component of contract. We assume that $0 \leq q \leq 1+r$. In this case where only one of the two borrower is successfull and repays $1+r+q$, she will get a new loan for the next period with another partner and the borrower who repays nothing will enter in an exclusion phase for $T_1$ periods, with $T_1 \leq T$.

4. As like as in the model of individual lending in section 2, we assume that after the exclusion phase ($T$ or $T_1$ periods), an excluded apply for a new loan and his chance to get it depends on the number of eligible applicants and the limited number of borrower in the lender portfolio. Then $\gamma$ represents the probability to have a new loan the first period following the exclusion phase, 1-$\gamma$ is the probability to still applicant, and $\gamma(1-\gamma)$ the probability to get it the second period following the exclusion phase, ...

The rules of a borrower participating to group lending can be summarized in a Markov chain $(X_t)_{t\in\mathbb{N}}$ with the set of states

$$H = \{B^0, B^1, E^T, E^{T-1}, \ldots, E^{T_1}, \ldots, E^1\} \tag{3.1}$$

where $B^0$ represents the state of being a beneficiary of a loan for a borrower who has the same partner as in the previous period, $B^1$ is the state of being a beneficiary of a loan for a borrower who changes his partner[8] and the $E^i$'s are the states of being in the exclusion phase for the next $i$ periods.
The transition matrix of this Markov chain is :

$$P_2 = \begin{pmatrix}
\alpha^2 & \alpha(1-\alpha) & (1-\alpha)^2 & 0 & 0 & 0\cdots0 & \alpha(1-\alpha) & 0\cdots0 & 0 \\
\alpha^2 & \alpha(1-\alpha) & (1-\alpha)^2 & 0 & 0 & 0\cdots0 & \alpha(1-\alpha) & 0\cdots0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0\cdots0 & 0 & 0\cdots0 & 0 \\
. & . & . & . & 1 & 0\cdots0 & 0 & 0\cdots0 & 0 \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & & . & . & 0\cdots1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0\cdots0 & 0 & 0\cdots0 & 1 \\
\gamma & 0 & 0 & 0 & 0 & 0\cdots0 & 0 & 0\cdots0 & 1-\gamma
\end{pmatrix} \tag{3.2}$$

The following diagram summarizes the dynamic of the Markov chain $(X_t)_{t\in\mathbb{N}}$ ((3.1), (3.2))

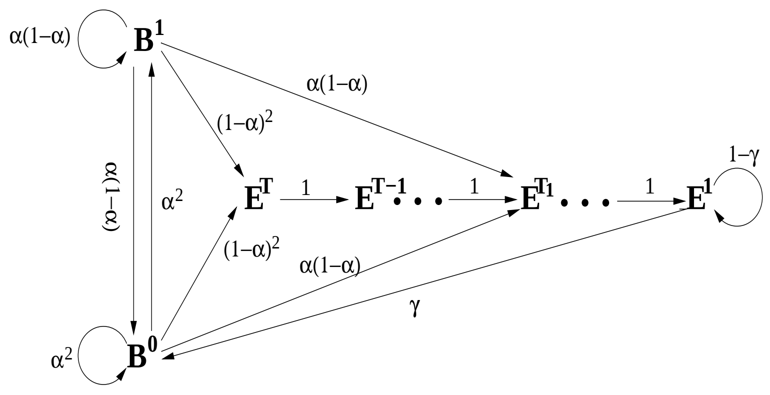


Figure 2: Various states for a participant to group lending with the corresponding transition probabilities

[8]In other words, $B^i$ presents the state of a success borrower in a groupe of $i$ defaulter, $i = 0, 1$.

## 3.1 Total Expected Return

In this model, the one period expected return, for a single borrower who has $\alpha$ as probability[9] of success of his project, is given by :

$$\alpha^2(w-(1+r))+\alpha(1-\alpha)(w-(1+r+q))=\alpha[w-(1+r)-(1-\alpha)q],$$

where, as before, $w$ is the profit for each borrower when she is successful and $q$ is the joint liability part of the contract. But each borrower knows also that she benefits automatically of a new loan if she reimbourses the current one, plus the joint part $q$ in case of having a partner in default. Thus she can also, as the previous model, evaluate the *expected discounted present value of the total return* for all futur periods, still called *total expected return* for simplicity.

In order to compute the *total expected return*, let us introduce the function $g : H \times H \longrightarrow \mathbb{R}$

$$g(x,y)=\begin{cases} w-(1+r) & \text{if } (x,y)=(B^i,B^0),\ \text{i=1,2} \\ w-(1+r+q) & \text{if } (x,y)=(B^i,B^1),\ \text{i=1,2} \\ 0 & \text{else} \end{cases}$$

For each trajectory $(X_0, X_1, \ldots)$ of the Markov chain, we define the total discounted return at time $s$ by

$$G(X_s, X_{s+1}, \ldots.) := \Sigma_{t=1}^{\infty}\delta^{t-s-1}g(X_{t-1},X_t)$$

Finally we define the total expected return at time $s$, $W_s : H \longrightarrow \mathbb{R}$, as a function of the state $x \in H$ by

$$W_s = \mathbb{E}[G(X_s, X_{s+1}, \ldots) \mid X_s = x].$$

In the next theorem, $\Sigma$ is the same quantity as in the previous theorem.

**Theorem 3.1.** *In the model of group lending defined by the Markov chain (3.1), (3.2), and under the rules defined in section 3, the total expected return for member who is at state $x$ at time $s$ is given by :*

$$W_s(x)=\begin{cases} \alpha[w-(1+r)-(1-\alpha)q]\frac{1}{1-[\alpha\delta+(1-\alpha)(\alpha\delta^{T_1}+(1-\alpha)\delta^T)\Sigma]} & \text{if } x\in\{B^0,B^1\} \\ \\ \alpha[w-(1+r)-(1-\alpha)q]\frac{\delta^{i-1}\Sigma}{1-[\alpha\delta+(1-\alpha)(\alpha\delta^{T_1}+(1-\alpha)\delta^T)\Sigma]} & \text{if } x=E^i,\ i=1,\cdots,T \end{cases} \tag{3.3}$$

*Proof.* The proof is very similar to the previous theorem's proof. As before, by the Markov properties we have $W_s(x) = W_0(x)$. we have only to compute $W_0(x)$ for all $x \in H$.
As $G(X_0, X_1, \ldots) = g(X_0, X_1) + \delta G(X_1, X_2, \ldots)$, the same arguments show that

$$W_0(x)=\sum_{y\in E}[g(x,y)+\delta W_0(y)]p(X_1=y\mid X_0=x)$$

where $p(X_1 = y \mid X_0 = x)$ is the porbability of transition, in one step, from state $x \in H$ to state $y \in H$.
Now let us compute $W_0(x)$ for each $x \in H$.

- For $x = B^0$

$$\begin{aligned} W_0(B^0) &= \alpha^2[w-(1+r)+\delta W_0(B^0)]+\alpha(1-\alpha)[w-(1+r+q)+\delta W_0(B^1)] \\ &\quad +\alpha(1-\alpha)\delta W_0(E^{T_1})+(1-\alpha)^2\delta W_0(E^T). \end{aligned}$$

- For $x = B^1$, we obtain that

$$W_0(B^0)=W_0(B^1).$$

- For $x = E^i$, $i = 2, \ldots, T$, as like as the previous theorem, we have

$$W_0(E^i)=\delta^{i-1}W_0(E^1)$$

[9] His partner has the same probability of success $\alpha$.

- For $x = E^1$, as like as the previous theorem, we obtain

$$W_0(E^1) = W_0(B^0)\Sigma$$

$\square$

From the theorem above and the theorem (2.1), we have the following corollary:

**Corollary 3.2.** *For two contracts at the same interest rate, individual contract $(r, T)$ and group contract with joint liability $(r, q, T_1, T)$ where $T_1 \leq T$, the group contract is more profitable for a borrower if and only if $q$ takes a small values, more precisely if $q \leq A$, where*

$$A = \alpha[w - (1 + r)] \frac{(\delta^{T_1} - \delta^T)\Sigma}{1 - (\alpha\delta + (1 - \alpha)\delta^T\Sigma)}$$

*Proof.* As we saw

$$V = \frac{\alpha[w - (1 + r)]}{1 - (\alpha\delta + (1 - \alpha)\delta^T\Sigma)}$$

and

$$W = \frac{\alpha[w - (1 + r) - (1 - \alpha)q]}{1 - [\alpha\delta + (1 - \alpha)(\alpha\delta^{T_1} + (1 - \alpha)\delta^T)\Sigma]}$$

$V$ is the total expected return for a beneficiary of an individual lending contract $(r, T)$, and $W$ is the total expected return for a beneficiary of a group lending contract with joint liability $(r, q, T_1, T)$ (we fixe the same interest rate $r$, the same $T$ exclusion periods for both contracts, and we assume that $T_1 \leq T$ and $q \leq 1 + r$). We find that $W \geq V$ is equivalent to $q \leq A$. $\square$

Notice that the maximal value $A$ for $q$ becomes zero when $T_1 = T$ which is easy to understand from the rules we have adapted for the liability constraint, thus it is natural to assume that $T_1 < T$ (and in that case $A > 0$).

## 3.2 Proportion of beneficiaries among the population

As in the model of individual lending, the evolution of the distribution of the population into the different states, $B^0$, $B^1$, $E^T$, ..., when $t$ tends to infinity, is easy to compute.

**Proposition 3.3.** *For any initial state distribution $\Pi_0 = (\Pi_0^0, \Pi_0^1, \cdots, \Pi_0^{T+1})$, the Markov dynamic, $(\Pi_0 P_2^t)_{t\in\mathbb{N}}$, tends to the distribution $\Pi_*$ when $t$ tends to infinity, with*

$$\Pi_* = \frac{1}{\frac{1}{1-\alpha} + \frac{1}{\gamma} + (1 - \alpha)(T - T_1) + (T_1 - 1)} \left( \frac{\alpha^2}{1 - \alpha}, 1 + \alpha, \underbrace{1 - \alpha, 1 - \alpha, \cdots, 1 - \alpha}_{(T - T_1)}, \underbrace{1, 1, \cdots, 1}_{(T_1 - 1)}, \frac{1}{\gamma} \right)$$

*Proof.* The stochastic matrix $P_2$ is primitive because that $P_2^{2T} > 0$. The dominant associeted left eigenvector, with positive coefficients adding up to 1, is equal to $\Pi_*$. Moreover the Perron-Frobenius theorem shows that $\lim_{t\to+\infty} \Pi_0 P_2^t = \Pi_*$. $\square$

From the proposition above and the proposition (2.2), the following corollory results:

**Corollary 3.4.** *Under assumption $T_1 < T$, the proportion of beneficiaries among all participant tends to be larger in the case of group lending then in individual lending.*

*Proof.* At equilibrium, the proportion of being a beneficiary in the case of group lending is the sum of the two first components of $\Pi_*$. Comparing this sum with the proportion of being a beneficiary in the case of individual lending (first component of $\pi_*$ in the expression (2.8) ), it is easy to show that if $T_1 < T$ then

$$\Pi_*^0 + \Pi_*^1 > \pi_*^0$$

$\square$

### 3.3 Optimal interest rate

In order to maintain a sustainable lending activity, the MFI has to charge an appropriate interest rate, usually chosen in such a way that the average repayment is larger than the costs for the MFI, assumed equal to $1+z$ for each unit of capital lent. In the case of group lending, the sustainabality constraint is :

$$\alpha^2(2(1+r)) + 2\alpha(1-\alpha)(1+r+q) \geqslant 2(1+z)$$

As in the model of individual lending, the profit is a decreasing function in $r$, then the interest rate that both maximizes the borrower welfare and makes lending sustainable will exactly cover the cost of lending. Then the optimal rate, $r*$ satisfies

$$1+r^* = \frac{1+z}{\alpha} - (1-\alpha)q$$

As result, we see that an MFI can achieve its equilibrium in charging , for the same cost $1+z$, a lower interest rate in case of group lending than in the case of individual lending.

## 4 Conclusion

This paper analyses two models of microlending using dynamic incentives, one for an individual lending contract and the other for a group lendign contract (group of two borrowers) with joint liability. In this analysis we have assumed that a borrower who repays his loan, plus the part of joint liability in group lending in cas of having a default partner, becomes a beneficiary of a new loan in the next lending period.

Under some circumstances, we show that a loan could be more profitable possibly for a borrower participating to group lending. Further more, the lender could offer more loan in case of group lending than in case of individuel lending.

A generalisation of these results to the case of $n$ borrowers group ($n > 2$) will be done in a further paper and we would try to find an optimal number of borrowers in a group (5 for Grameen bank [11]).

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