Asymptotics of the binomial formula for option pricing*

Francine Diener and Marc Diener
Laboratoire C.N.R.S. Jean-Alexandre Dieudonné
Université de Nice Sophia-Antipolis
Parc Valrose
F06108 Nice cedex 2

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1 Introduction

There are mainly three methods to compute the price of derivatives of financial assets : the tree methods, the numerical methods for solving partial differential equations, and the Monte Carlo methods. Although they are the most rudimentary, the tree methods are of very frequent use, mostly because they are really easy to understand, and thus to use safely. Moreover one knows that they converge as the number of steps tends to infinity, and thus, it suffice to choose a large enough number of time steps to get a satisfactory value. Nevertheless there are few theoretical results about the convergence (how many time steps are necessary to have a given precision ?), its nature, monotone or oscillatory (is the result overestimated or underestimated ?), or what is the effect of the position of the strike price K or the barrier H with respect to the nodes of the tree (is it advisable to choose the tree such that its node are close to these values or not ?)

In what follows we propose a new method to compute an asymptotic expansion of the price of an option as the number n of time steps becomes large. It consists in replacing the binomial sums involved in the tree method by eulerian fir integrals and expand these integrals using the Laplace method. In applying this method to the computation of a call option at the money we will be able to explain the observed oscillations in the price, as n tends to infinity. We will show that the distance to the limit, in this case, is equal to $\frac{A}{n} + o(\frac{1}{n})$, with a number A that depends on the *parity* of n. This approach, when generalised, will give usefull answers to the previous questions and a better control of the quality of algorithms using tree methods.

2 Tree model expectation's estimate

Let $\delta t > 0$, $]0..T]_{\delta t} := \{\delta t, 2\delta t, \ldots, n\delta t = T\}$ be a subdivision with equal steps δt of the interval $[0, T]$. Similarly, define $]0..t]_{\delta t} := \{\delta t, 2\delta t, \ldots, i\delta t = t\}$. Consider a familly $X_0, X_{\delta t}, \ldots, X_{n\delta t}$ of identically distributed independent random variable (r.v.) and, for all $t \in]0..T]_{\delta t}$, denote by Y_t their sum up to t :

$$Y_t := \sum_{s \in]0..t]_{\delta t}} X_s.$$

In other words one has $Y_t = Y_{t-\delta t} + \delta Y_t$, with $\delta Y_t := X_t$. The central limit theorem shows that, if $\sigma = \mathbb{E}(X_1^2) < \infty$,

$$\frac{Y_T}{\sigma\sqrt{n}} - n\mathbb{E}(X) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

Moreover, the Berry-Eseen inequality [6] shows that the error is bounded above in the following way

$$\operatorname{Sup}_x \left| P\left(\frac{Y_T}{\sigma\sqrt{n}} < x\right) - N(x)\right| \le c\frac{\mathbb{E}(X^3)}{\sigma^3\sqrt{n}}$$

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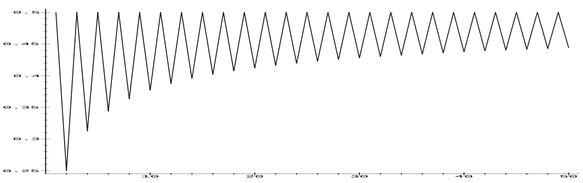

Figure 1: Oscillatory convergence of $P(Y_{n\delta t} < 0)$ when n tends to infinity.

provided that $\mathbb{E}(X^3) < \infty$, where $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy = \frac{1}{2} + \frac{1}{2}\text{erf}\,(\frac{x}{\sqrt{2}})$, and with c a constant independent of the law of the X_t. This inequality suggests that the rate of convergence of the sum Y_T towards the normal law is usually of order $\frac{1}{\sqrt{n}}$ and the following classical example of a binomial sum shows that, indeed, one can not expect, in general, a more rapid rate of convergence, of order $\frac{1}{n}$ for example :

Example: [6] Let X_t be centered independent Bernouilli r.v. $P(X_t = -1) = P(X_t = 1) = \frac{1}{2}$. Then, $P(Y_T > 0) = 1 - P(Y_T < 0) - P(Y_T = 0)$, and thus, by symetry, $2P(Y_T > 0) = 1 - P(Y_T = 0)$. Now,

$$P(Y_T = 0) = \begin{cases} \begin{pmatrix} 2m \\ m \end{pmatrix} \frac{1}{2^{2m}} & \text{si } n = 2m \\ 0 & \text{si } n = 2m+1 \end{cases}$$

and thus, using Stirling's formula, $m\,! = m^m e^{-m}\sqrt{2\pi}\sqrt{m}(1 + o(1))$ and as $N(0) = \frac{1}{2}$:

$$P(Y_T > 0) - \Phi(0) = \begin{cases} -\frac{1}{\sqrt{2\pi}\sqrt{n}} + o(\frac{1}{\sqrt{n}}) & \text{si } n = 2m \\ 0 & \text{si } n = 2m+1. \end{cases} \tag{1}$$

This first example shows a $\frac{1}{\sqrt{n}}-$ convergence, and exhibits oscillations when n tends to infinity (figure 1) of the same kind as the oscillations we will observe below in the tree model computations of a call option price.

In the martingale approach of option pricing (brownian model of Black and Scholes, tree model of Cox, Ross and Rubinstein), the price is the discounted expected pay-off for the so-called risk-neutral probability. Thus, computing such a price consists in computing an expectation like $\mathbb{E}(f(S_T))$. With the previous example in mind, where $f(y) = \mathbb{I}_{\{y<0\}}$ and S_T is a binomial r.v., $\mathcal{B}(n, \frac{1}{2})$, one would suspect, for such an expectation, an asymptotic expansion beginning with a multiple of $\frac{1}{\sqrt{n}}$. Nevertheless, under smoothness asumption for the function f and for specific r.v. S_T, better results have been obtained :

- In [7], Talay and Tubaro show that if f is $\mathcal{C}^4$ and if S_t is a solution of the stochastic differential equation (s.d.e.) $dS_t = \sigma S_t dW_t$, $\mathbb{E}(f(S_T))$ the error in compution this expectation using an Euler scheem is of type $\frac{C}{n} + o(\frac{1}{n})$, where C is some explicit constant. This estimation can nevertheless not be applied in the case of a call option, as, in this case, f is continuous but not smooth.
- In [4], Lamberton has also estimates in $\frac{1}{n}$ for a function f that is lipschitz, for a discrete valued model for the underlying asset of type $\frac{\sum X_s}{\sqrt{n}}$ for a familly of i.i.d. X_s such that $\mathbb{E}(X) = \mathbb{E}(X^3) = 0$, $\mathbb{E}(X^2) = 1$ and $\mathbb{E}(X^4) < \infty$, conditions that are also not satisfied for the Cox-Ross-Rubinstein model considered here.

3 Experimental observations

The fact that the price of an option, computed on a tree model with n time steps, converges to the Black-Scholes price in an oscillatory way is known by the users of this method (see for example [3]). In order to explain these oscillations, we have studied the easiest case : a call option (C_t) *at the money*, that is $C_T = (S_T - K)^+$ with $K = S_0$, on an underlying asset S_t defined by $S_0 = 1$ and, for all $t \in]0..T = n\delta t]_{\delta t}$,

$$S_{t+\delta t} = \begin{cases} S_t e^{\sigma/\sqrt{n}} & \text{(with probability} \quad p = 1/(1 + e^{\sigma/\sqrt{n}}) \\ S_t e^{-\sigma/\sqrt{n}} & \text{(with probability} \quad 1 - p = e^{\sigma/\sqrt{n}}/(1 + e^{\sigma/\sqrt{n}}) \end{cases} \tag{2}$$

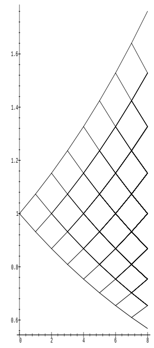

Figure 2: Binomial tree of the values of the underlying asset S_t for $n = 8$.

This discrete model, denoted by CRR, has been chosen by Cox, Ross and Rubinstein [2] in order to fit to the usual continuous model, in the following sense : as n tends to infinity, the two first moments of the increments $\delta S_t := S_t - S_{t-\delta t}$ of the continuous and the tree model are asymptotic to each other up to $\delta t = (\sqrt{\delta t})^2$. It should be noted that, as $e^{\sigma/\sqrt{n}} e^{-\sigma/\sqrt{n}} = 1$, the binary tree (figure 2) of the values of the random walk S_t has its nodes lined up on horizontal lines, and thus contains the point $(T, K = S_0) = (1, 1)$ for even n, but not for odd n. Figure 3 shows a succession of values of the call option at time $t = 0$ for n between 10 and 200. One observes that the values corresponding to even n increase towards the limit (wich is the price given by the Black-Scholes formula) whereas the values obtained for odd n decrease towards this limit. A careful look at the order of magnitude of the difference between the values C_0 and their limit that this order is more likely to be $\frac{1}{n}$ than $\frac{1}{\sqrt{n}}$. This is the subject of the next section.

4 Asymptotic of the call option price

Let us consider the CRR model and denote by u(=up) and d(=down) the quantities $u = e^{\sigma/\sqrt{n}}$ and $d = e^{-\sigma/\sqrt{n}}$. The value of S_T defined by (2) at a node of the tree reached after j up (and $n - j$ down) is simply

$$S_T = S_0 u^j d^{n-j}$$

and $\Pr\{S_T = S_0 u^j d^{n-j}\} = \left(\begin{array}{c} n \\ j \end{array} \right) p^j (1-p)^{n-j}$, where p is the so-called *risk-neutral probability* given by

$$p := \frac{1}{1+u} = \frac{1}{1+e^{\sigma/\sqrt{n}}},$$

for which the random walk $(S_t)_{t \in [0..T]}$ is a martingale (here we asume for simplicity that the riskless interest rate is zero). So the price of the call option with pay-off $C_T = (S_T - K)^+$ is :

$$C_0 = S_0 \sum_{j=a}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) p^j (1-p)^{n-j} u^j d^{n-j} - K \sum_{j=a}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) p^j (1-p)^{n-j} \tag{3}$$

with $a = \mathrm{Min}\{j \;\; S_0 u^j d^{n-j} > K\}$. Remark that both the probability p and the integer a depend on n, more precisely $p = \frac{1}{2} - \frac{\sigma}{4\sqrt{n}} + o(\frac{1}{\sqrt{n}}$ and $a = [\frac{n}{2} + \frac{\sqrt{n}}{2\sigma} \ln K] + 1$, where [.] denote the integer part. Using the function $\Phi(a, n, p)$, called *complementary binomiale distribution* in [2] :

$$\Phi(a, n, p) = \sum_{j=a}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) p^j (1-p)^{n-j},$$

and the martingale relation $pu + (1-p)d$, the value of the call option (3) can be rewritten as the difference of two terms, in a form very similar to the Black Scholes formula :

$$C_0 = S_0 \Phi(a, n, q) - K \Phi(a, n, p) \tag{4}$$

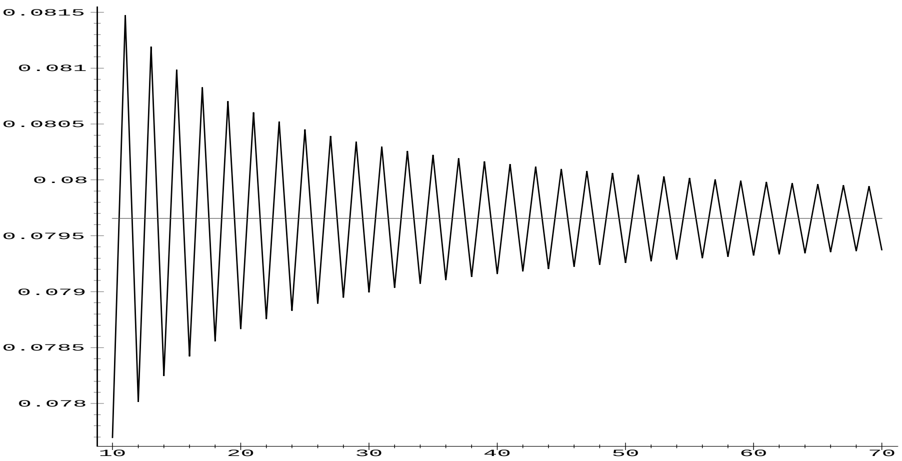

Figure 3: Call option price at time $t = 0$ as a function of the number of time steps ; S_t is defined by (2), with $\sigma = 0.2$ and $C_T = (S_T - K)^+$, with $K = S_0 = 1$.

with $q := pu = 1 - p$. The following result[1] shows that the function $\Phi(a, n, p)$ can be written as an integral. This will make it possible to compute its asymptotic expansion.

Proposition 1 *For all $n \in \mathbb{N}$, for all a, $0 < a \leq n$, one has :*

$$\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} = a \binom{n}{a} \int_0^p t^{a-1}(1-t)^{n-a} dt.$$

This identity follows immediately from an iterated integration by parts of the integral.

The following theorem generalizes to the case of the non symetric Bernouilli low the example of the first section. In the special case $K = S_0$ (at the money call), as $a = \frac{n}{2} + 1$ if n is even and $a = \frac{n+1}{2} + 1$ if n is odd, this result is exactly what we need to compute the asymptotic expansion of the call (4).

Theorem 2 *Let (X_t), $t \in]0..T]_{\delta t}$, be a familly of independent Bernouilli r.v., with $Pr(X_t = 1) = p$ and $Pr(X_t = -1) = 1 - p$, for $0 < p < 1$, and let Y_t be their sum up to t. If the quantity $\tilde{p} := (p - \frac{1}{2})\sqrt{n}$ is a $O(1)$ then*

$$Pr(Y_T > 0) = \Phi(m+1, n, p) = \begin{cases} \frac{1}{2} + \frac{1}{2} erf(\sqrt{2}\tilde{p}) - \frac{e^{-2\tilde{p}^2}}{\sqrt{2\pi}} \frac{1}{\sqrt{n}} - \sqrt{\frac{2}{\pi}} \frac{e^{-2\tilde{p}^2}}{n}(-\tilde{p}/4 + \tilde{p}^3) + o(\frac{1}{n}) & si\ n = 2m \\ \frac{1}{2} + \frac{1}{2} erf(\sqrt{2}\tilde{p}) + \sqrt{\frac{2}{\pi}} \frac{e^{-2\tilde{p}^2}}{n}(\tilde{p}/4 + \tilde{p}^3) + o(\frac{1}{n}) & si\ n = 2m+1 \end{cases}$$

where erf is defined by $\frac{1}{2} + \frac{1}{2} erf(x/\sqrt{2}) = N(x)$.

Proof: We give here just the sketch of the proof explaining how to compute the first terms of the asymptotic expansion we are looking for. We do not show here that the error is actually of order $o(\frac{1}{n})$. In order to do this, one has to generalize standard results on Laplace's method to our case, as the intervalle $[0, p]$ depends on n.

To compute the asymptotic expansion of the function Φ, we first apply the formula of proposition 1 (and the fact that $a = m + 1$), in order to transform the binomial sum in an integral :

$$\Phi(m+1, n, p) = \begin{cases} (m+1) \binom{2m}{m+1} \int_0^p t^m (1-t)^m \frac{dt}{1-t} & \text{if } n = 2m \\ (m+1) \binom{2m+1}{m+1} \int_0^p t^m (1-t)^m dt & \text{if } n = 2m+1 \end{cases}$$

[1] We are deeply grateful to Adri Olde Daalhuis for drawing our attention to the properties of the incomplete Beta function ([1], p 263 and 944).

```
> restart;
> n:=2*m:a:=m+1:coef:=a*binomial(n,a):asymptcoef:=asympt(coef,m):
> integrand:=subs(t=1/2+tau/sqrt(n),t^(a-1)*(1-t)^(n-a)):
> asymptintegrand:=asympt(integrand,m,3):
> asymptintegrale:=int(eval(subs(O=0,asymptintegrand)),tau=b..-b):
> ptau:=eval(subs(O=0,asympt(sqrt(n)*(1/(1+exp(sigma/sqrt(n)))-1/2),m,2))):
> C:=asympt(asymptcoef*subs(b=ptau,asymptintegrale)/sqrt(n),m,3):
> map(simplify,asympt(C,m,2));
```

$$\operatorname{erf}(\frac{1}{4}\,\sigma\,\sqrt{2}) - \frac{1}{192}\,\frac{\sqrt{2}\,e^{(-1/8\,\sigma^2)}\,\sigma\,(\sigma^2+12)}{\sqrt{\pi}\,m} + \mathrm{O}(\frac{1}{m^2})$$

Figure 4: Computation of the asymptotic expansion of C_0 for $n = 2m$, using Maple.

These integrals being Laplace integrals, and the function $t \to t(1-t)$ having a maximum at $t = 1/2$, it is natural to perform the change of variable $\tau = (t - \frac{1}{2})\sqrt{n}$. Then one computes the asymptotic expansion of the coefficient $(m+1)\begin{pmatrix} 2m \\ m+1 \end{pmatrix}$ (resp. $(m+1)\begin{pmatrix} 2m+1 \\ m+1 \end{pmatrix}$) using Stirling's formula (of order 2), and then the asymptotic expansion of the function to be integrated which is equal, in the new variable,

$$\left(\frac{1}{2} + \frac{\tau}{\sqrt{2m}}\right)^m \left(\frac{1}{2} - \frac{\tau}{\sqrt{2m}}\right)^m \frac{1}{\sqrt{2m}\left(\frac{1}{2} - \frac{\tau}{\sqrt{2m}}\right)}$$

when n is even (and an analogous expression for odd n) and finally one computes an asymptotic expansion of the upper end of the integration interval $p = \frac{1}{1+e^{\sigma/\sqrt{n}}}$. Finally, one performs the integration.

These somewhat boring computations can profitably be performed with help of *Maple* (see figure 4) □

Corollary 3 *Let C_0 be the price at time $t = 0$ of an at the money call option, with expiry time $T = n\delta t = 1$ on the underlying asset S_t defined by (2). One has :*

$$C_0 = \begin{cases} \sqrt{\frac{2}{\pi}} \int_0^{\sigma/2} e^{-y^2/2} dy - \frac{1}{n} e^{-\sigma^2/8} \sqrt{\frac{2}{\pi}} \frac{\sigma}{8}(1 + \frac{\sigma^2}{12}) + o(\frac{1}{n}) & si\ n = 2m \\ \sqrt{\frac{2}{\pi}} \int_0^{\sigma/2} e^{-y^2/2} dy + \frac{1}{n} e^{-\sigma^2/8} \sqrt{\frac{2}{\pi}} \frac{\sigma}{8}(1 - \frac{\sigma^2}{12}) + o(\frac{1}{n}) & si\ n = 2m+1 \end{cases} \qquad (5)$$

Proof: One has $S_{n\delta t} = \exp(\frac{\sigma \sum X_t}{\sqrt{n}}) = \exp(\frac{\sigma Y_{n\delta t}}{\sqrt{n}})$. Thus

$$\begin{aligned} C_0 &= \mathbb{E}^p(S_{n\delta t} - 1)^+ \\ &= \mathbb{E}^p \mathbb{I}_{\{Y_{n\delta t} > 0\}} S_{n\delta t} - \mathbb{E}^p \mathbb{I}_{\{Y_{n\delta t} > 0\}} \\ &= \mathbb{E}^q \mathbb{I}_{\{Y_{n\delta t} > 0\}} - \mathbb{E}^p \mathbb{I}_{\{Y_{n\delta t} > 0\}} = \Phi(m+1, n, q) - \Phi(m+1, n, p). \end{aligned}$$

One applies the theorem to the two terms of this difference. One observes that, as $q = pu = 1 - p$, $\tilde{q} := (q - \frac{1}{2})\sqrt{n} = -\tilde{p}$. Now, in the asymptotic expansion of the theorem, the coefficients of the terms in $\frac{1}{\sqrt{n}}$ are even functions of $\tilde{p}$ whereas those of the constant terms and of the terms in $\frac{1}{n}$ are odd functions. Taking the difference, the first ones will cancel out whereas the others will add up. The result follows. □

Figure 5 shows that formula (5) of this corollary gives, even when neglecting all the terms of order larger then $1/n$, an excellent approximation of the two "curves" corresponding to the even and the odd values of n respectively.

5 So, what now ?

When one leaves the special case of an option at the money ($K = S_0$) for the general case, the plottings still exhibit oscillations, but these become much more irregular (see figure 6). This new problem, which

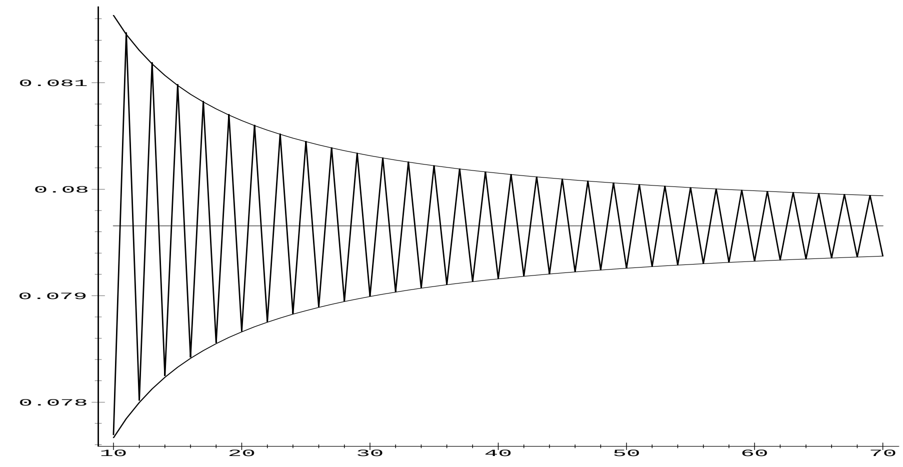

Figure 5: This figure shows the values of an at the money call option as a function of n, with line segments interconnecting them in order to visualize the oscillations, together with the two curves given by formula (5) discarding the term $o(\frac{1}{n})$. The two curves fit very well to the call option values, even for rather small n.

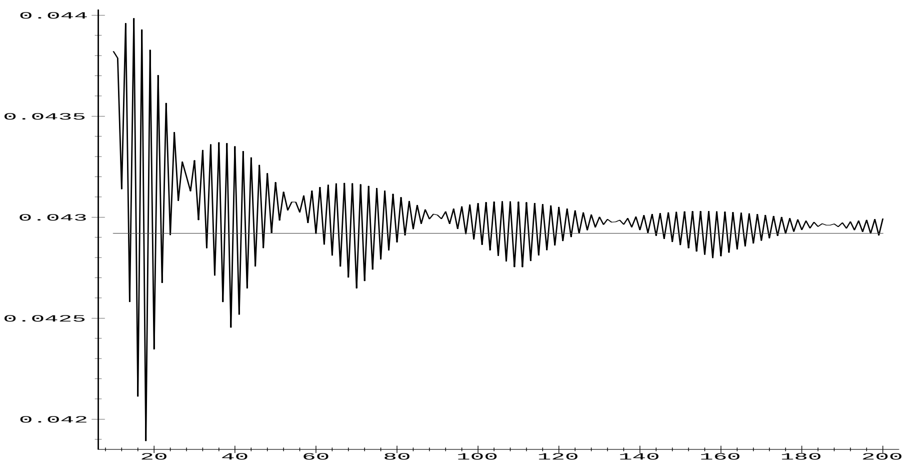

Figure 6: Price, as a function of the number n of steps, at time $t = 0$ of a call option of final value $C_T = (S_T - K)^+$, $T = n\delta t = 1$, on an underlying asset S_T with volatility $\sigma = 0.2$. Here $K = S_0(1+0.10)$.

is described in [5], lies in the geometry of the tree (the nodes of which line up on horizontals) : when for $K = S_0$ the point $(t = n\delta t, K)$ is a node of the tree for any even $n = 2m$ (and it stays precisely in the midle between two nodes for any odd $n = 2m + 1$), to the contrary, when $K \neq S_0$, it moves gradually between the nodes of the tree (who move as n changes). As in the previous case

$$C_0(n) = S_0\Phi(a, n, q) - K\Phi(a, n, p)$$

but now the binomial sums Φ in each of the two terms of this difference do no longer have a number of terms equal to $\frac{n}{2} = m$ (or to $\frac{n-1}{2}$ for odd n), but actually the number of terms is equal to $\frac{n}{2}$ only as long as $n < \nu_1$, then it becoms $\frac{n}{2} - 1$ as long as $n < \nu_2$, and so on, with integers $\nu_1, \nu_2, \ldots$ given by

$$\nu_k^{\text{even}} = 2\left[2\frac{(\sigma k)^2}{\ln^2 K}\right] \text{ , or } \nu_k^{\text{odd}} = 2\left[2\frac{\sigma k(\sigma k - 1)}{\ln^2 K}\right] + 1,$$

according if n is even or odd. Forexample, the first values are $\nu_1^{\text{even}} = 16$, $\nu_2^{\text{even}} = 70$, $\nu_3^{\text{even}} = 158$, and $\nu_1^{\text{odd}} = 3$, $\nu_2^{\text{odd}} = 39$, $\nu_3^{\text{odd}} = 109$ (see also figure 6). The sequence of integers ν_k correspond to the angular points on the picture (actually one tip of two for even n, and the analogous sequence for odd n) as can be seen on figure 6. Then the two curves we are looking for will not be define by the same formula for any n. Nevertheless the same method, using proposition 1 and Laplace integral gives the asymptotic expansion of this more complicated curves.

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