SPECTRUM OF HYPERSURFACES WITH SMALL EXTRINSIC RADIUS OR LARGE λ_1 IN EUCLIDEAN SPACES

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Abstract. In this paper, we prove that Euclidean hypersurfaces with almost extremal extrinsic radius or λ_1 have a spectrum that asymptotically contains the spectrum of the extremal sphere in the Reilly or Hasanis-Koutroufiotis Inequalities. We also consider almost extremal hypersurfaces which satisfy a supplementary bound on $v_M\|\mathrm{B}\|_\alpha^n$ and show that their spectral and topological properties depends on the position of α with respect to the critical value $\dim M$.

1. Introduction

Throughout the paper, $X\colon M^n \to \mathbb{R}^{n+1}$ is a closed, connected, immersed Euclidean hypersurface (with $n \geqslant 2$). We set v_M its volume, B its second fundamental form, $\mathrm{H} = \frac{1}{n}\mathrm{tr}\,\mathrm{B}$ its mean curvature, r_M its extrinsic radius (i.e. the least radius of the Euclidean balls containing M), $(\lambda_i^M)_{i\in\mathbb{N}}$ the non-decreasing sequence of its eigenvalues labelled with multiplicities and $\overline{X} := \frac{1}{v_M}\int_M X dv$ its center of mass. For any function $f: M \to \mathbb{R}$, we set $\|f\|_\alpha^\alpha = \frac{1}{v_M}\int_M |f|^\alpha dv$.

The Hasanis-Koutroufiotis inequality asserts that

$$r_M\|\mathrm{H}\|_2 \geqslant 1, \tag{1.1}$$

with equality if and only if M is the Euclidean sphere S_M with center $\overline{X}$ and radius $\frac{1}{\|\mathrm{H}\|_2}$.

The Reilly inequality asserts that

$$\lambda_1^M \leqslant n\|\mathrm{H}\|_2^2, \tag{1.2}$$

once again with equality if and only if M is the sphere S_M (we give some short proof of these inequalities in section 2).

Our aim is to study the spectral properties of the hypersurfaces that are almost extremal for each of this Inequalities. The results of this paper are used in [3] to study the metric shape of the almost extremal hypersurfaces.

We set $\mu_k^{S_M} = k(n+k-1)\|\mathrm{H}\|_2^2$ the k-th eigenvalue of S_M (labelled without multiplicities) and m_k its multiplicity. Throughout the paper we shall adopt the notation that $\tau(\varepsilon|n,\cdots)$ is a positive function which depends only on the variables $\varepsilon, n, \cdots$ and which converges to zero with ε when $n, \cdots$ are fixed.

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Theorem 1.1. *There exists a function $\tau(\varepsilon|n)$ such that for any immersed hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ (or with $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1^M} \leqslant 1+\varepsilon$) and for any $k \leqslant \frac{1}{\tau(\varepsilon|n)}$ the interval $[(1-\tau(\varepsilon|n))\mu_k^{S_M}, (1+\tau(\varepsilon|n))\mu_k^{S_M}]$ contains at least m_k eigenvalues of M counted with multiplicities.*

Note that by Theorem 1.1, almost extremal hypersurfaces for the Reilly inequality must have at least $n+1$ eigenvalues close to $\lambda_1^{S_M} = n\|\mathrm{H}\|_2^2$. However, they can have the topology of any immersed hypersurface of $\mathbb{R}^{n+1}$ (see below) and can be as close as possible in Hausdorff distance of any closed, connected subset of $\mathbb{R}^{n+1}$ containing S_M (see [3]). This is very different from the almost extremal manifolds for the Lichnerowicz Inequality in positive Ricci curvature (see for instance [1]).

The proof of Theorem 1.1 is based on estimates for the restrictions to M of homogeneous, harmonic polynomials of the ambient space $\mathbb{R}^{n+1}$. Such a polynomial of degree k satisfies the equality $\Delta^{S_M} P = n\|\mathrm{H}\|_2^2 dP(X) + \|\mathrm{H}\|_2^2 D^0 dP(X,X) = \mu_k^{S_M} P$ on S_M whereas it satisfies $\Delta^M P = n\mathrm{H} dP(\nu) + D^0 dP(\nu,\nu)$ on M, where $D^0 dP$ is the Euclidean Hessian and ν a local unit, normal vector to M. We prove that on almost extremal hypersurfaces, the quantities $\nu - \mathrm{H}X$ and $|\mathrm{H}| - \|\mathrm{H}\|_2$ are small in L^2-norms, which can be used to get the following estimates (see Lemmas 5.3 and 5.1)

$$\big|\|\varphi P\|^2_{L^2(M)} - \|\varphi P\|^2_{L^2(S_M)}\big| \leqslant \tau(\varepsilon|n,k)\|\varphi P\|_{L^2(S_M)}, \tag{1.3}$$

$$\|\Delta^M \varphi P - \mu_k^{S_M}\varphi P\|_{L^2(M)} \leqslant \tau(\varepsilon|n,k)\|\varphi P\|_{L^2(M)}, \tag{1.4}$$

where φ is a cut function localized near S_M from which we easily infer Theorem 1.1. Note that these estimates are not so easy to derive since there is no known good local control of the measure on M involving only the L^2-norm of the mean curvature.

Theorem 1.1 gives no information on the part of the spectrum of almost extremal hypersurfaces that is not close to the spectrum of the limit sphere S_M. Our next result shows that there is essentially no constraint on this part of the spectrum (in dimension larger than 2), even if we assume a supplementary bound on $\|B\|_p$ for $p < n$.

Theorem 1.2. *Let $M_1, M_2 \hookrightarrow \mathbb{R}^{n+1}$ be two immersed compact submanifolds of dimension $m \geqslant 3$, $M_1 \# M_2$ be their connected sum and F be any closed subset of $]0,+\infty[\setminus \mathrm{Sp}(M_1)$ (for the induced topology). Then there exists a sequence of immersions $i_k : M_1\#M_2 \hookrightarrow \mathbb{R}^{n+1}$ with induced metric g_k on $M_1\#M_2$ such that*

1) $i_k(M_1\#M_2)$ converges to M_1 in Hausdorff topology,
2) the curvatures of g_k satisfy

$$\frac{1}{\mathrm{Vol}\, g_k}\int_{M_1\#M_2} |\mathrm{H}|^\alpha \to \frac{1}{\mathrm{Vol}\, M_1}\int_{M_1} |\mathrm{H}|^\alpha \qquad \text{for any } \alpha \in [1,m),$$
$$\frac{1}{\mathrm{Vol}\, g_k}\int_{M_1\#M_2} |\mathrm{B}|^\alpha \to \frac{1}{\mathrm{Vol}\, M_1}\int_{M_1} |\mathrm{B}|^\alpha \qquad \text{for any } \alpha \in [1,m),$$

3) the limit spectrum $\cap_{k\in\mathbb{N}}\overline{\cup_{l\geqslant k}\mathrm{Sp}(g_l)}$ is equal to $F \cup \mathrm{Sp}(M_1)$,
4) $\mathrm{Vol}\,(g_k) \to \mathrm{Vol}\, M_1$.

To get almost extremal submanifolds from the previous result, we just have to consider the case where $M_1 = \mathbb{S}^n$ (and $F \subset [n,+\infty[$ in the Reilly case). It gives almost extremal hypersurfaces for the Reilly or Hasanis-Koutroufiotis Inequalities with the topology of any immersible Euclidean hypersurface, a spectrum arbitrarily close of any closed set containing $\mathrm{Sp}(\mathbb{S}^n)$ (and contained in $[n,+\infty[$ in the Reilly case), even if we assume a bound on $v_M\|\mathrm{B}\|_\alpha^n$ for any $\alpha<n$.

On the other hand, if we assume a bound on $\|\mathrm{B}\|_\alpha$ with $\alpha>n$, we prove in [3] that the almost extremal hypersrfaces converge to S_M in Hausdorff distance, which combined with the $\mathcal{C}^{1,\beta}$ pre-compactness theorem of [8] (or a Moser iteration as in the previous version of this paper [2]) imply the following stability in Lipschitz distance.

Proposition 1.3. *Let $n<\alpha\leqslant\infty$. Any immersed hypersurface $M\hookrightarrow\mathbb{R}^{n+1}$ with $v_M\|\mathrm{B}\|_\alpha^n\leqslant A$ and $r_M\|\mathrm{H}\|_2\leqslant 1+\varepsilon$ (or with $v_M\|\mathrm{B}\|_\alpha^n\leqslant A$ and $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1}\leqslant 1+\varepsilon$) is diffeomorphic to S_M and satisfies $d_L(M,S_M)\leqslant\tau(\varepsilon|n,\alpha,A)$. In particular, we have $|\lambda_k^M-\lambda_k^{S_M}|\leqslant\tau(\varepsilon|k,n,\alpha,A)$ for any $k\in\mathbb{N}$.*

The critical case, where we assume an upper bound on $v_M\|\mathrm{B}\|_n^n$ will be developed in a forthcoming paper. However, we construct in the present paper some examples of almost extremal hypersurfaces satisfying such a bound as a preliminary. First of all, considering the constructions of Theorem 1.2 in the case $\alpha=m$, we get a sequence of extremizing hypersurfaces for the two inequalities, with the topology of any immersible hypersurface, with $v_M\|B\|_n^n$ bounded and whose limit spectrum is equal to $\mathrm{Sp}(S_M)\cup F$, where F is any fixed, finite subset of $\mathbb{R}\setminus\mathrm{Sp}(S_M)$ (see section 6.1). Note however that the bound on $v_M\|\mathrm{B}\|_n^n$ for this sequence depends on the topology of the extremal hypersurfaces and on the subset F.

In section 6.2, we construct almost extremal hypersurfaces for the Hasanis-Koutroufiotis inequality, not diffeomorphic to $\mathbb{S}_M$, not Gromov-Hausdorff close to S_M, with limit spectrum larger than the spectrum of $\mathbb{S}^n$ and with $\|\mathrm{H}\|_\infty$ bounded. We set $E(x)$ the integral part of x.

Example 1.4. *For any couple (l,p) of integers there exists a sequence of embedded hypersurfaces $M_j\hookrightarrow\mathbb{R}^{n+1}$ diffeomorphic to p spheres $\mathbb{S}^n$ glued by connected sum along l points, such that $\|\mathrm{H}_j\|_\infty\leqslant C(n)$, $\|\mathrm{H}\|_2=1$, $\|\mathrm{B}_j\|_n\leqslant C(n)$, $\big\||X_j|-1\big\|_\infty\to 0$, $\big\||\mathrm{H}_j|-1\big\|_1\to 0$, and for any $\sigma\in\mathbb{N}$ we have $\lambda_\sigma^{M_j}\to\lambda_{E(\frac{\sigma}{p})}^{\mathbb{S}^n}$. In particular, the M_j have at least p eigenvalues close to 0 whereas its extrinsic radius is close to 1.*

Example 1.5. *There exists sequence of immersed hypersurfaces $M_j\hookrightarrow\mathbb{R}^{n+1}$ diffeomorphic to 2 spheres $\mathbb{S}^n$ glued by connected sum along 1 great subsphere $\mathbb{S}^{n-2}$, such that $\|\mathrm{H}_j\|_\infty\leqslant C(n)$, $\|\mathrm{H}_j\|_2=1$, $\|\mathrm{B}_j\|_2\leqslant C(n)$, $\big\||X_j|-1\big\|_\infty\to 0$, $\big\||\mathrm{H}_j|-1\big\|_1\to 0$, and for any $\sigma\in\mathbb{N}$ we have $\lambda_\sigma^{M_j}\to\lambda_{E(\frac{\sigma}{2})}^{\mathbb{S}^{n,d}}$, where $\mathbb{S}^{n,d}$ is the sphere $\mathbb{S}^n$ endowed with the singular metric, pulled-back of the canonical metric of $\mathbb{S}^n$ by the map $\pi:(y,z,r)\in\mathbb{S}^1\times\mathbb{S}^{n-2}\times[0,\frac{\pi}{2}]\mapsto(y^d,z,r)\in\mathbb{S}^1\times\mathbb{S}^{n-2}\times[0,\frac{\pi}{2}]$, where $\mathbb{S}^1\times\mathbb{S}^{n-2}\times[0,\frac{\pi}{2}]$ is identified with $\mathbb{S}^n\subset\mathbb{R}^2\times\mathbb{R}^{n-1}$ via the map $\Phi(y,z,r)=\big((\sin r)y,(\cos r)z\big)$. Note that $\mathbb{S}^{n,d}$ has infinitely many eigenvalues that are not eigenvalues of $\mathbb{S}^n$.*

The structure of the paper is as follows: after a preliminary section 2, where we give short proofs of the Reilly and Hasanis-Koutroufiotis inequalities, we prove some concentration properties for the volume, mean curvature and position vector X of almost extremal hypersurfaces in Section 3. Section 4 is devoted to estimates on the restriction on hypersurfaces of the homogeneous, harmonic polynomials of $\mathbb{R}^{n+1}$. These estimates are used in Section 5 to prove Theorem 1.1. We end the paper in section 6 by the constructions of Theorem 1.2 and of Examples 1.4 and 1.5.

Throughout the paper we adopt the notation that $C(n,k,p,\cdots)$ is function greater than 1 which depends on p, q, n, $\cdots$. It eases the exposition to disregard the explicit nature of these functions. The convenience of this notation is that even though C might change from line to line in a calculation it still maintains these basic features.

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2. Some geometric optimal inequalities

Any function F on $\mathbb{R}^{n+1}$ gives rise to a function $F\circ X$ on M which, for more convenience, will be also denoted F subsequently. An easy computation gives the formula

$$\Delta F = n\mathrm{H}dF(\nu)+\Delta^0 F+\nabla^0 dF(\nu,\nu), \tag{2.1}$$

where ν denotes a local normal vector field of M in $\mathbb{R}^{n+1}$, ∇^0 is the Euclidean connection, Δ denotes the Laplace operator of (M,g) and Δ^0 is the Laplace operator of $\mathbb{R}^{n+1}$. This formula is fundamental to control the geometry of a hypersurface by its mean curvature. Applied to $F(x)=\langle x,x\rangle$, where $\langle\cdot,\cdot\rangle$ is the canonical product on $\mathbb{R}^{n+1}$, Formula 2.1 gives the Hsiung formulae,

$$\frac{1}{2}\Delta|X|^2 = n\mathrm{H}\,\langle\nu,X\rangle - n,\qquad \int_M \mathrm{H}\langle\nu,X\rangle dv = v_M \tag{2.2}$$

2.1. **A rough geometrical bound.** The integrated Hsiung formula (2.2) and the Cauchy-Schwarz inequality give the following

$$1=\int_M\frac{\mathrm{H}\langle\nu,X\rangle dv}{v_M}\leqslant \|\mathrm{H}\|_2\big\|X-\overline{X}\big\|_2 \tag{2.3}$$

This inequality $\|\mathrm{H}\|_2\|X-\overline{X}\|_2\geqslant 1$ is optimal since M satisfies $\|\mathrm{H}\|_2\big\|X-\overline{X}\big\|_2=1$ if and only if M is a sphere of radius $\frac{1}{\|\mathrm{H}\|_2}$ and center $\overline{X}$. Indeed, in this case $X-\overline{X}$ and ν are collinear on $M\setminus\{H=0\}$, hence $|X-\overline{X}|^2$ is locally constant on $M\setminus\{H=0\}$. This implies that $\{H=0\}=\emptyset$ and that X is an isometric-cover of M on the sphere S of center $\overline{X}$ and radius $\|X-\bar{X}\|_2=\frac{1}{\|\mathrm{H}\|_2}$, hence an isometry.

2.2. **Hasanis-Koutroufiotis inequality on extrinsic radius.** We set R the extrinsic Radius of M, i.e. the least radius of the balls of $\mathbb{R}^{n+1}$ which contain M. Then Inequality (2.3) gives $\|\mathrm{H}\|_2 r_M=\|\mathrm{H}\|_2\inf_{u\in\mathbb{R}^{n+1}}\|X-u\|_\infty\geqslant\|\mathrm{H}\|_2\inf_{u\in\mathbb{R}^{n+1}}\|X-u\|_2=\|\mathrm{H}\|_2\|X-\overline{X}\|_2\geqslant 1$ and $r_M=\frac{1}{\|\mathrm{H}\|_2}$ if and only if we have equality in (2.3).

2.3. **Reilly inequality on λ_1^M.** Since we have $\frac{1}{v_M}\int_M(X_i-\bar{X}_i)\,dv=0$ for any component function of $X-\bar{X}$, by the min-max principle and Inequality (2.3), we have $\lambda_1^M\frac{1}{\|\mathrm{H}\|_2^2}\leqslant\lambda_1^M\|X-\bar{X}\|_2^2=\lambda_1^M\sum_i\|X_i-\bar{X}_i\|_2^2\leqslant\sum_i\|\nabla X_i\|_2^2=n$ where λ_1^M is the first non-zero eigenvalue of M and where the last equality comes from the fact that $\sum_i|\nabla X_i|^2$ is the trace of the quadratic form $Q(u)=|p(u)|^2$ with respect to the canonical scalar product, where p is the orthogonal projector from $\mathbb{R}^{n+1}$ to T_xM. This gives the Reilly inequality (1.2).

Here also, equality in the Reilly inequality gives equality in 2.3 and so it characterizes the sphere of radius $\frac{1}{\|\mathrm{H}\|_2} = \|X\|_2 = \sqrt{\frac{n}{\lambda_1^M}}$.

3. Concentration estimates

We say that M satisfies the pinching $P_{p,\varepsilon}$ when $\|\mathrm{H}\|_p\|X - \overline{X}\|_2 \leqslant 1+\varepsilon$. From the proofs of Inequalities (1.1) and (1.2) above, it appears that pinchings $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ or $n\|\mathrm{H}\|_2^2/\lambda_1 \leqslant 1+\varepsilon$ imply the pinching $P_{2,\varepsilon}$.

From now on, we assume, without loss of generality, that $\bar{X} = 0$. Let $X^T(x)$ denote the orthogonal projection of $X(x)$ on the tangent space T_xM.

Lemma 3.1. *If $(P_{2,\varepsilon})$ holds, then we have $\|X^T\|_2 \leqslant \sqrt{3\varepsilon}\|X\|_2$ and $\|X - \frac{\mathrm{H}}{\|\mathrm{H}\|_2^2}\nu\|_2 \leqslant \sqrt{3\varepsilon}\|X\|_2$.*

Proof. Since we have $1 = \frac{1}{v_M}\int_M \mathrm{H}\langle X,\nu\rangle dv \leqslant \|\mathrm{H}\|_2\|\langle X,\nu\rangle\|_2$, Inequality $(P_{2,\varepsilon})$ gives us $\|X\|_2 \leqslant (1+\varepsilon)\|\langle X,\nu\rangle\|_2$ and $1 \leqslant \|\mathrm{H}\|_2\|X\|_2 \leqslant 1+\varepsilon$. Hence $\|X - \langle X,\nu\rangle\nu\|_2 \leqslant \sqrt{3\varepsilon}\,\|X\|_2$ and $\|X - \frac{\mathrm{H}\nu}{\|\mathrm{H}\|_2^2}\|_2^2 = \|X\|_2^2 - \|\mathrm{H}\|_2^{-2} \leqslant 3\varepsilon\,\|X\|_2^2$. □

We set $A_\eta = B_0(\frac{1+\eta}{\|\mathrm{H}\|_2}) \setminus B_0(\frac{1-\eta}{\|\mathrm{H}\|_2})$.

Lemma 3.2. *If $(P_{p,\varepsilon})$ (for $p > 2$), or $n\|\mathrm{H}\|_2^2/\lambda_1^M \leqslant 1+\varepsilon$, or $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ holds (with $\varepsilon \leqslant \frac{1}{100}$), then we have $\big\|\|X\| - \frac{1}{\|\mathrm{H}\|_2}\big\|_2 \leqslant \frac{C}{\|\mathrm{H}\|_2}\sqrt[8]{\varepsilon}$, $\||\mathrm{H}| - \|\mathrm{H}\|_2\|_2 \leqslant C\sqrt[8]{\varepsilon}\|\mathrm{H}\|_2$ and $\mathrm{Vol}\,(M \setminus A_{\sqrt[8]{\varepsilon}}) \leqslant C\sqrt[8]{\varepsilon}v_M$, where $C = 6 \times 2^{\frac{2p}{p-2}}$ in the case $(P_{p,\varepsilon})$ and $C = 100$ in the other cases.*

Proof. When $(P_{p,\varepsilon})$ holds, we have

$$\|\mathrm{H}\|_p\|X\|_2 \leqslant (1+\varepsilon) \leqslant (1+\varepsilon)\|\mathrm{H}\|_p\|X\|_{\frac{p}{p-1}} \leqslant (1+\varepsilon)\|\mathrm{H}\|_p\|X\|_1^{1-\frac{2}{p}}\|X\|_2^{\frac{2}{p}},$$

hence we get $\big\||X| - \frac{1}{\|\mathrm{H}\|_2}\big\|_2^2 = \|X\|_2^2 - 2\frac{\|X\|_1}{\|\mathrm{H}\|_2} + \frac{1}{\|\mathrm{H}\|_2^2} \leqslant 2^{\frac{2p}{p-2}}\frac{1}{\|\mathrm{H}\|_2^2}\varepsilon$. Combined with the second inequality of Lemma 3.1, it gives

$$\big\||\mathrm{H}| - \|\mathrm{H}\|_2\big\|_2 \leqslant \|\mathrm{H}\|_2^2\big\||X| - \frac{|\mathrm{H}|}{\|\mathrm{H}\|_2^2}\big\|_2 + \|\mathrm{H}\|_2^2\big\||X| - \frac{1}{\|\mathrm{H}\|_2}\big\|_2 \leqslant C\sqrt[4]{\varepsilon}\|\mathrm{H}\|_2$$

Now, by the Chebyshev inequality and Lemma 3.1, we get

$$\begin{aligned}\mathrm{Vol}\left(M \setminus A_{\sqrt[4]{\varepsilon}}\right) &= \mathrm{Vol}\left\{x \in M / \big||X(x)| - \frac{1}{\|\mathrm{H}\|_2}\big| \geqslant \frac{\sqrt[4]{\varepsilon}}{\|\mathrm{H}\|_2}\right\}\\ &\leqslant \frac{\|\mathrm{H}\|_2^2}{\sqrt{\varepsilon}}\int_M \big||X| - \frac{1}{\|\mathrm{H}\|_2}\big|^2 \leqslant C(p)\sqrt{\varepsilon}v_M\end{aligned}$$

When $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ holds. We set X_0 the center of the circumsphere to M of radius r_M. We have $\|X - X_0\|_2^2 = \|X\|_2^2 + |X_0|^2 = r_M^2 \leqslant \frac{(1+\varepsilon)^2}{\|\mathrm{H}\|_2^2}$ and then we have $|X_0| \leqslant \frac{\sqrt{3\varepsilon}}{\|\mathrm{H}\|_2}$ and $|X| \leqslant |X_0| + r_M \leqslant \frac{1+3\sqrt{\varepsilon}}{\|\mathrm{H}\|_2}$. So we have $\frac{1}{\|\mathrm{H}\|_2^2} - |X|^2 \in [\frac{\sqrt[4]{\varepsilon}}{\|\mathrm{H}\|_2^2}, \frac{1}{\|\mathrm{H}\|_2}]$ on $M \setminus A_{\sqrt[4]{\varepsilon}}$. Chebyshev inequality and (2.3) give us

$$\frac{\mathrm{Vol}\,(M \setminus A_{\sqrt[4]{\varepsilon}})}{v_M}\frac{\sqrt[4]{\varepsilon}}{\|\mathrm{H}\|_2^2} \leqslant \frac{1}{v_M}\int_{M\setminus A_{\sqrt[4]{\varepsilon}}}\frac{1}{\|\mathrm{H}\|_2^2} - |X|^2 \leqslant \frac{1}{v_M}\int_{M\cap A_{\sqrt[4]{\varepsilon}}}|X|^2 - \frac{1}{\|\mathrm{H}\|_2^2} \leqslant \frac{9\sqrt{\varepsilon}}{\|\mathrm{H}\|_2^2}$$

where in the last inequality we have used $|X| \leqslant \frac{1+3\sqrt{\varepsilon}}{\|\mathrm{H}\|_2}$ and, so we get

$$\big\||X| - \frac{1}{\|\mathrm{H}\|_2}\big\|_2^2 = \frac{1}{v_M}\int_{M\cap A_{\sqrt[4]{\varepsilon}}} \big||X| - \frac{1}{\|\mathrm{H}\|_2}\big|^2 + \frac{1}{v_M}\int_{M\setminus A_{\sqrt[4]{\varepsilon}}} \big||X| - \frac{1}{\|\mathrm{H}\|_2}\big|^2$$
$$\leqslant \frac{\sqrt{\varepsilon}}{\|\mathrm{H}\|_2^2} + \frac{\mathrm{Vol}\,(M\setminus A_{\sqrt[4]{\varepsilon}})}{v_M}\frac{1}{\|\mathrm{H}\|_2^2} \leqslant \frac{10\sqrt[4]{\varepsilon}}{\|\mathrm{H}\|_2^2}$$

Combined with the second inequality of Lemma 3.1, we get $\|\frac{1}{\|\mathrm{H}\|_2} - \frac{|\mathrm{H}|}{\|\mathrm{H}\|_2^2}\|_2 \leqslant \frac{C\sqrt[8]{\varepsilon}}{\|\mathrm{H}\|_2}$.

When $n\|\mathrm{H}\|_2^2/\lambda_1^M \leqslant 1+\varepsilon$ holds, we have $\int_M (|X|^2 - \|X\|_2^2)dv = 0$ and so by the Poincare inequality we get $\big\||X|^2 - \|X\|_2^2\big\|_2^2 \leqslant \frac{4\|X^T\|_2^2}{\lambda_1^M} \leqslant \frac{12(1+\varepsilon)^2\varepsilon\|X\|_2^2}{n\|\mathrm{H}\|_2^2} \leqslant \frac{200\varepsilon}{n\|\mathrm{H}\|_2^4}$, which gives $\frac{1}{\|\mathrm{H}\|_2}\big\||X| - \frac{1}{\|\mathrm{H}\|_2}\big\|_2 \leqslant \big\||X|^2 - \frac{1}{\|\mathrm{H}\|_2^2}\big\|_2 \leqslant \big\||X|^2 - \|X\|_2^2\big\|_2 + \big|\|X\|_2^2 - \frac{1}{\|\mathrm{H}\|_2^2}\big| \leqslant \frac{12\sqrt{\varepsilon}}{\|\mathrm{H}\|_2^2}$ and then we get the estimate on the volume of $A_{\sqrt[4]{\varepsilon}}$ by the same Chebyshev procedure as for $P_{p,\varepsilon}$ and the estimate on the mean curvature by the same procedure as for $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$. □

Let ψ:$[0,\infty) \to [0,1]$ be a smooth function with ψ=0 outside $[\frac{(1-2\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}, \frac{(1+2\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}]$ and ψ=1 on $[\frac{(1-\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}, \frac{(1+\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}]$. Let us consider the function φ on M defined by $\varphi(x) = \psi(|X_x|^2)$ and the vector field Z on M defined by $Z = \nu - \mathrm{H}X$. The previous estimates then imply the following.

Lemma 3.3. $(P_{p,\varepsilon})$ *(for $p > 2$) or $n\|\mathrm{H}\|_2^2/\lambda_1 \leqslant 1+\varepsilon$ or $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ implies $\|\varphi^2(\mathrm{H}^2 - \|\mathrm{H}\|_2^2)\|_1 \leqslant C\sqrt[8]{\varepsilon}\|\mathrm{H}\|_2^2$, $\|\varphi Z\|_2 \leqslant C\varepsilon^{\frac{3}{32}}$ and $|\|\varphi\|_2^2 - 1| \leqslant C\sqrt[8]{\varepsilon}$, where C is a constant which depends on p in the case $(P_{p,\varepsilon})$.*

Proof. We have $\|\varphi^2(\mathrm{H}^2 - \|\mathrm{H}\|_2^2)\|_1 \leqslant \big\||\mathrm{H}| - \|\mathrm{H}\|_2\big\|_2 2\|\mathrm{H}\|_2 \leqslant C\sqrt[8]{\varepsilon}\|\mathrm{H}\|_2^2$ and

$$\begin{aligned}\|\varphi Z\|_2^2 =& \frac{1}{v_M}\int_M \varphi^2|Z|^2 dv = \frac{1}{v_M}\int_M \varphi^2(1 - 2\mathrm{H}\langle\nu, X\rangle + \mathrm{H}^2|X|^2)dv\\ =&\frac{\|\mathrm{H}\|_2^2}{v_M}\int_M \varphi^2\big|X - \frac{\mathrm{H}}{\|\mathrm{H}\|_2^2}\nu\big|^2 dv + \frac{1}{\|\mathrm{H}\|_2^2 v_M}\int_M (\|\mathrm{H}\|_2^2 - \mathrm{H}^2)\varphi^2(1 - |X|^2\|\mathrm{H}\|_2^2)dv\\ \leqslant&\|\mathrm{H}\|_2^2\big\|X - \frac{\mathrm{H}}{\|\mathrm{H}\|_2^2}\nu\big\|_2^2 + 8\sqrt[16]{\varepsilon}\frac{\big\|\varphi^2(\mathrm{H}^2 - \|\mathrm{H}\|_2^2)\big\|_1}{\|\mathrm{H}\|_2^2},\end{aligned}$$

which gives the result by Lemma 3.1. Finally, we have $1 - \frac{\mathrm{Vol}\,(M\setminus A_{\sqrt[8]{\varepsilon}})}{v_M} \leqslant \frac{\mathrm{Vol}\,(A_{\sqrt[8]{\varepsilon}}\cap M)}{v_M} \leqslant \|\varphi\|_2^2$ and $\|\varphi\|_2^2 \leqslant 1$. □

4. Homogeneous, harmonic polynomials of degree k

In this section, we give some estimates on harmonic homogeneous polynomials restricted to almost extremal hypersurfaces. They will be used subsequently to derive our result on the spectrum and on the volume of almost extremal manifolds. Let us begin by general estimates on harmonic, homogeneous polynomials.

4.1. **General estimates.** Let $\mathcal{H}^k(\mathbb{R}^{n+1})$ be the space of homogeneous, harmonic polynomials of degree k on $\mathbb{R}^{n+1}$. Note that $\mathcal{H}^k(\mathbb{R}^{n+1})$ induces on $\mathbb{S}^n$ the spaces of eigenfunctions of $\Delta^{\mathbb{S}^n}$ associated to the eigenvalues $\mu_k := k(n+k-1)$ with multiplicity $m_k := \binom{n+k-1}{k}\dfrac{n+2k-1}{n+k-1}$.

On the space $\mathcal{H}^k(\mathbb{R}^{n+1})$, we set $(P,Q)_{\mathbb{S}^n} := \frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n} PQdv_{\text{can}}$, where dv_{can} denotes the element volume of the sphere with its standard metric.

Remind that for any $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$ and any $Y \in \mathbb{R}^{n+1}$, we have $dP(X) = kP(X)$ and $\nabla^0 dP(X,Y) = (k-1)dP(Y)$.

Lemma 4.1. *For any $x \in \mathbb{R}^{n+1}$ and $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$, we have $|P(x)|^2 \leqslant \|P\|^2_{\mathbb{S}^n} m_k |x|^{2k}$.*

Proof. Let $(P_i)_{1\leqslant i\leqslant m_k}$ be an orthonormal basis of $\mathcal{H}^k(\mathbb{R}^{n+1})$. For any $x \in \mathbb{S}^n$, $Q_x(P) = P^2(x)$ is a quadratic form on $\mathcal{H}^k(\mathbb{R}^{n+1})$ whose trace is given by $\sum_{i=1}^{m_k} P_i^2(x)$. Since for any $x' \in \mathbb{S}^n$ and any $O \in O_{n+1}$ such that $x' = Ox$ we have $Q_{x'}(P) = Q_x(P\circ O)$ and since $P \mapsto P\circ O$ is an isometry of $\mathcal{H}^k(\mathbb{R}^{n+1})$, we have $\sum_{i=1}^{m_k} P_i^2(x) = \operatorname{tr}(Q_x) = \sum_{i=1}^{m_k} P_i^2(x') = \operatorname{tr}(Q_{x'})$. We infer that $\sum_{i=1}^{m_k} \frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n} P_i^2(x)dv = m_k = \frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n}\left(\sum_{i=1}^{m_k} P_i^2(x)\right)dv$ and so $\sum_{i=1}^{m_k} P_i^2(x) = m_k$. By homogeneity of the P_i we get

$$\sum_{i=1}^{m_k} P_i^2(x) = m_k|x|^{2k}, \tag{4.1}$$

and by the Cauchy-Schwarz inequality applied to $P(x) = \sum_i (P,P_i)_{\mathbb{S}^n} P_i(x)$, we get the result. $\square$

As an immediate consequence, we have the following lemma.

Lemma 4.2. *For any $x, u \in \mathbb{R}^{n+1}$ and $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$, we have*

$$|d_xP(u)|^2 \leqslant \|P\|^2_{\mathbb{S}^n} m_k\Big(\frac{\mu_k}{n}|x|^{2(k-1)}|u|^2 + \big(k^2 - \frac{\mu_k}{n}\big)\langle u,x\rangle^2|x|^{2(k-2)}\Big).$$

Proof. Let $x \in \mathbb{S}^n$ and $u \in \mathbb{S}^n$ so that $\langle u,x\rangle = 0$. Once again the quadratic forms $Q_{x,u}(P) = \big(d_xP(u)\big)^2$ are conjugate (since O_{n+1} acts transitively on orthonormal couples) and so $\displaystyle\sum_{i=1}^{m_k}\big(d_xP_i(u)\big)^2$ does not depend on $u \in x^\perp$ nor on $x \in \mathbb{S}^n$. By choosing an orthonormal basis $(u_j)_{1\leqslant j\leqslant n}$ of $x^\perp$, we obtain that

$$\begin{aligned}\sum_{i=1}^{m_k}\big(d_xP_i(u)\big)^2 &= \frac{1}{n}\sum_{i=1}^{m_k}\sum_{j=1}^{n}\big(d_xP_i(u_j)\big)^2 = \frac{1}{n\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n}\sum_{i=1}^{m_k}|\nabla^{\mathbb{S}^n}P_i|^2\\ &= \frac{1}{n\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n}\sum_{i=1}^{m_k}P_i\Delta^{\mathbb{S}^n}P_i = \frac{m_k\mu_k}{n}\end{aligned}$$

Now suppose that $u \in \mathbb{R}^{n+1}$. Then $u = v + \langle u, x\rangle x$, where $v = u - \langle u, x\rangle x$, and we have

$$\begin{aligned}\sum_{i=1}^{m_k}\big(d_xP_i(u)\big)^2 &= \sum_{i=1}^{m_k}\big(d_xP_i(v) + k\langle u,x\rangle P_i(x)\big)^2\\ &= \sum_{i=1}^{m_k}\big(d_xP_i(v)\big)^2 + 2k\langle u,x\rangle\sum_{i=1}^{m_k} d_xP_i(v)P_i(x) + m_k\langle u,x\rangle^2k^2\\ &= \frac{m_k\mu_k}{n}|v|^2 + m_k\langle u,x\rangle^2k^2 = m_k\left(\frac{\mu_k}{n}|u|^2 + \left(k^2 - \frac{\mu_k}{n}\right)\langle u,x\rangle^2\right),\end{aligned}$$

where we have taken the derivative the equality (4.1) to compute $\sum_{i=1}^{m_k} d_xP_i(v)P_i(x)$. By homogeneity of P_i we get $\sum_{i=1}^{m_k}\big(d_xP_i(u)\big)^2 = m_k\big(\frac{\mu_k}{n}|x|^{2(k-1)}|u|^2 + (k^2 - \frac{\mu_k}{n})\langle u,x\rangle^2|x|^{2(k-2)}\big)$ and conclude once again by the Cauchy-Schwarz inequality. □

Lemma 4.3. *For any $x \in \mathbb{R}^{n+1}$ and $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$, we have*

$$|\nabla^0 dP(x)|^2 \leqslant \|P\|^2_{\mathbb{S}^n} m_k\alpha_{n,k}|x|^{2(k-2)},$$

where $\alpha_{n,k} = (k-1)(k^2+\mu_k)(n+2k-3) \leqslant C(n)k^4$.

Proof. The Bochner equality gives

$$\begin{aligned}\sum_{i=1}^{m_k}|\nabla^0 dP_i(x)|^2 &= \sum_{i=1}^{m_k}\left(\langle d\Delta^0P_i, dP_i\rangle - \frac{1}{2}\Delta^0\big|dP_i\big|^2\right)\\ &= -\frac{1}{2}m_k\big(k^2+\mu_k\big)\Delta^0|X|^{2k-2} = m_k\alpha_{n,k}|X|^{2k-4} \qquad (4.2)\end{aligned}$$

□

4.2. **Estimates on hypersurfaces.** Let $\mathcal{H}^k(M) = \{P \circ X \ ,\ P \in \mathcal{H}^k(\mathbb{R}^{n+1})\}$ be the space of functions induced on M by $\mathcal{H}^k(\mathbb{R}^{n+1})$. We will identify P and $P \circ X$ subsequently. There is no ambiguity since we have

Lemma 4.4. *Let M^n be a compact manifold immersed by X in $\mathbb{R}^{n+1}$ and let $(P_1,\dots,P_m)$ be a linearly independent set of homogeneous polynomials of degree k on $\mathbb{R}^{n+1}$. Then the set $(P_1\circ X,\dots,P_m\circ X)$ is also linearly independent.*

Proof. Any homogeneous polynomial P which is zero on M is zero on the cone $\mathbb{R}^+\cdot M$. Since M is compact there exists a point $x \in M$ so that $X_x \notin T_xM$ and so $\mathbb{R}^+\cdot M$ has non empty interior. Hence $P \circ X = 0$ implies $P = 0$. □

We now compare the L^2-norm of P on M with L^2-norm of P on the sphere $S_M = \frac{1}{\|\mathrm{H}\|_2}\mathbb{S}^n$. We still denote $\psi : [0,\infty) \longrightarrow [0,1]$ a smooth function which is 0 outside $[\frac{(1-\eta)^2}{\|\mathrm{H}\|_2^2}, \frac{(1+\eta)^2}{\|\mathrm{H}\|_2^2}]$, is 1 on $[\frac{(1-\eta/2)^2}{\|\mathrm{H}\|_2^2}, \frac{(1+\eta/2)^2}{\|\mathrm{H}\|_2^2}]$ and satisfies the upper bounds $|\psi'| \leqslant \frac{4\|\mathrm{H}\|_2^2}{\eta}$ and $|\psi''| \leqslant \frac{8\|\mathrm{H}\|_2^4}{\eta^2}$. We set $\varphi(x) = \psi(|X_x|^2)$ on M.

Lemma 4.5. *With the above restrictions on ψ we have*

$$\|\Delta\varphi^2\|_1 \leqslant \frac{192\|\mathrm{H}\|_2^4}{\eta^2}\|X^T\|_2^2 + \frac{16n\|\mathrm{H}\|_2^2}{\eta}\|\varphi Z\|_1$$

Proof. An easy computation yields that

$$\begin{aligned}\Delta(\varphi^2) &= -(\psi^2)''(|X|^2)|d|X|^2|^2 + (\psi^2)'(|X|^2)\Delta|X|^2\\ &= -4(\psi^2)''(|X|^2)|X^T|^2 - 2n(\psi^2)'(|X|^2)\langle \nu, Z\rangle\end{aligned}$$

But the bound on the derivatives of ψ gives us $|(\psi^2)'| \leqslant \frac{8\|\mathrm{H}\|_2^2}{\eta}\psi$ and $|(\psi^2)''| \leqslant \frac{48\|\mathrm{H}\|_2^4}{\eta^2}$. Hence we get $\|\Delta\varphi^2\|_1 \leqslant \frac{192\|\mathrm{H}\|_2^4}{\eta^2}\|X^T\|_2^2 + \frac{16n\|\mathrm{H}\|_2^2}{\eta}\|\varphi Z\|_1$. □

Lemma 4.6. *Let $\varphi : M \to [0,1]$ be as above. There exists a constant $C = C(n)$ such that for any isometrically immersed hypersurface M of $\mathbb{R}^{n+1}$ and any $P \in \mathcal{H}^k(M)$, we have* $\left|\|\mathrm{H}\|_2^{2k}\|\varphi P\|_2^2 - \|P\|_{\mathbb{S}^n}^2\right| \leqslant \left(1 - \|\varphi\|_2^2 + DC(n)\sum_{i=1}^k m_i(1+\eta)^{2k}\right)\|P\|_{\mathbb{S}^n}^2$, *where* $D = \|\varphi Z\|_2 + \|\varphi Z\|_2^2 + \frac{200\|\mathrm{H}\|_2^2}{\eta^2}\|X^\perp\|_2^2 + \frac{16n}{\eta}\|\varphi Z\|_1 + \frac{\|\varphi^2(\mathrm{H}^2 - \|\mathrm{H}\|_2^2)\|_1}{\|\mathrm{H}\|_2^2}$.

Proof. For any $P \in \mathcal{H}^k(M)$ we have

$$\begin{aligned}&\|\varphi\nabla^0 P\|_2^2 = \|\varphi dP(\nu)\|_2^2 + \|\varphi dP\|_2^2\\ &= \|\varphi dP(Z)\|_2^2 + k^2\|\varphi \mathrm{H} P\|_2^2 + \frac{1}{v_M}\int_M \big(2k\mathrm{H} dP(\varphi Z)\varphi P + \varphi^2 P\Delta P - \frac{P^2\Delta(\varphi^2)}{2}\big)dv\end{aligned}$$

Now, Formula (2.1) applied to $P \in \mathcal{H}^k(\mathbb{R}^{n+1})$ gives

$$\Delta P = \mu_k \mathrm{H}^2 P + (n+2k-2)\mathrm{H} dP(Z) + \nabla^0 dP(Z,Z) \tag{4.3}$$

hence, we get

$$\begin{aligned}\|\varphi\nabla^0 P\|_2^2 =& \|dP(\varphi Z)\|_2^2 + (\mu_k + k^2)\|\mathrm{H}\varphi P\|_2^2\\ &+ \frac{1}{v_M}\int_M \big(\varphi^2 P\nabla^0 dP(Z,Z) + (n+4k-2)\varphi \mathrm{H} dP(\varphi Z)P - \frac{P^2\Delta(\varphi^2)}{2}\big)dv\\ =& \frac{1}{v_M}\int_M \Big((\mu_k + k^2)\big(\mathrm{H}^2 - \|\mathrm{H}\|_2^2\big)\varphi^2 P^2 + (n+4k-2)\mathrm{H} dP(\varphi Z)\varphi P\Big)dv\\ &+ \frac{1}{v_M}\int_M \big(P\nabla^0 dP(\varphi Z, \varphi Z) - \frac{P^2\Delta(\varphi^2)}{2}\big)dv\\ &+ (\mu_k + k^2)\|\mathrm{H}\|_2^2\|\varphi P\|_2^2 + \|dP(\varphi Z)\|_2^2\end{aligned}$$

Now we have

$$\left\|\nabla^0 P\right\|_{\mathbb{S}^n}^2 = \left\|\nabla^{\mathbb{S}^n} P\right\|_{\mathbb{S}^n}^2 + k^2\|P\|_{\mathbb{S}^n}^2 = (\mu_k + k^2)\|P\|_{\mathbb{S}^n}^2 \tag{4.4}$$

Hence

$$\begin{aligned}&\|\mathrm{H}\|_2^{2k-2}\|\varphi\nabla^0 P\|_2^2 - \left\|\nabla^0 P\right\|_{\mathbb{S}^n}^2 = (\mu_k + k^2)\big(\|\mathrm{H}\|_2^{2k}\|\varphi P\|_2^2 - \|P\|_{\mathbb{S}^n}^2\big) + \|\mathrm{H}\|_2^{2k-2}\|dP(\varphi Z)\|_2^2\\ &+ \frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M \varphi^2 P\Big((\mu_k + k^2)\big(\mathrm{H}^2 - \|\mathrm{H}\|_2^2\big)P + \mathrm{H}(n+4k-2)dP(Z) + \nabla^0 dP(Z,Z)\Big)dv\\ &- \frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M \frac{P^2\Delta(\varphi^2)}{2}\end{aligned}$$

Which gives

(4.5)

$$
\begin{aligned}
&\Big|\|\mathrm{H}\|_2^{2k}\|\varphi P\|_2^2-\|P\|_{\mathbb{S}^n}^2\Big|\\
&\leqslant\frac{1}{\mu_k+k^2}\Big|\|\mathrm{H}\|_2^{2k-2}\|\varphi\nabla^0P\|_2^2-\big\|\nabla^0P\big\|_{\mathbb{S}^n}^2\Big|\\
&\quad+\frac{\|\mathrm{H}\|_2^{2k-2}}{\mu_k+k^2}\int_M\Big((n+4k-2)|\mathrm{H}|\varphi|P||dP(\varphi Z)|+|dP(\varphi Z)|^2+|P||\nabla^0dP||\varphi Z|^2\Big)\\
&\quad+\frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M\big(\varphi^2\big|\mathrm{H}^2-\|\mathrm{H}\|_2^2\big|P^2+\frac{P^2|\Delta(\varphi^2)|}{2}\big)dv
\end{aligned}
$$

By Lemma 4.1, we have

$$
\begin{aligned}
\frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M\big|\mathrm{H}^2-\|\mathrm{H}\|_2^2\big|(\varphi P)^2dv&\leqslant\frac{m_k\,\|P\|_{\mathbb{S}^n}^2\,\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M\big|\varphi^2(\mathrm{H}^2-\|\mathrm{H}\|_2^2)\big||X|^{2k}dv\\
&\leqslant\|P\|_{\mathbb{S}^n}^2\,m_k(1+\eta)^{2k}\frac{\|\varphi^2(\mathrm{H}^2-\|\mathrm{H}\|_2^2)\|_1}{\|\mathrm{H}\|_2^2}
\end{aligned}
$$

In the same way, we have

$$
\frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M\frac{P^2|\Delta(\varphi^2)|}{2}dv\leqslant\|P\|_{\mathbb{S}^n}^2\,m_k(1+\eta)^{2k}\frac{\|\Delta(\varphi^2)\|_1}{\|\mathrm{H}\|_2^2}
$$

and using Lemma 4.2, we get

$$
\begin{aligned}
\frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M\varphi^2|PdP(Z)\mathrm{H}|dv&\leqslant\frac{m_kk\,\|P\|_{\mathbb{S}^n}^2\,\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M\varphi^2|X|^{2k-1}|\mathrm{H}Z|\,dv\\
&\leqslant\|P\|_{\mathbb{S}^n}^2\,m_kk(1+\eta)^{2k}\|\varphi Z\|_2
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M|dP(\varphi Z)|^2&\leqslant\|P\|_{\mathbb{S}^n}^2m_kk^2\frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M|\varphi Z|^2|X|^{2(k-1)}\,dv\\
&\leqslant\|P\|_{\mathbb{S}^n}^2\,m_kk^2(1+\eta)^{2k}\|\varphi Z\|_2^2
\end{aligned}
$$

Finally, using Lemma 4.3, we get

$$
\begin{aligned}
\frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M|P||\nabla^0dP||\varphi Z|^2&\leqslant\|P\|_{\mathbb{S}^n}^2m_k\sqrt{\alpha_{n,k}}\frac{\|\mathrm{H}\|_2^{2k-2}}{v_M}\int_M|X|^{2(k-1)}|\varphi Z|^2\,dv\\
&\leqslant\|P\|_{\mathbb{S}^n}^2\,m_k\sqrt{\alpha_{n,k}}(1+\eta)^{2k}\|\varphi Z\|_2^2
\end{aligned}
$$

which, combined with (4.5) and equation (4.4), gives

$$
\begin{aligned}
\frac{\big|\|\mathrm{H}\|_2^{2k}\|\varphi P\|_2^2-\|P\|_{\mathbb{S}^n}^2\big|}{\|P\|_{\mathbb{S}^n}^2}&\leqslant\frac{\Big|\|\mathrm{H}\|_2^{2k-2}\|\varphi\nabla^0P\|_2^2-\big\|\nabla^0P\big\|_{\mathbb{S}^n}^2\Big|}{\|\nabla^0P\|_{\mathbb{S}^n}^2}\\
&\quad+C(n)m_k(1+\eta)^{2k}\Big(\|\varphi Z\|_2+\|\varphi Z\|_2^2+\frac{\|\Delta(\varphi^2)\|_1}{\|\mathrm{H}\|_2^2}+\frac{\|\varphi^2(\mathrm{H}^2-\|\mathrm{H}\|_2^2)\|_1}{\|\mathrm{H}\|_2^2}\Big)\\
&\leqslant\frac{\Big|\|\mathrm{H}\|_2^{2k-2}\|\varphi\nabla^0P\|_2^2-\big\|\nabla^0P\big\|_{\mathbb{S}^n}^2\Big|}{\|\nabla^0P\|_{\mathbb{S}^n}^2}+C(n)m_k(1+\eta)^{2k}D
\end{aligned}
$$

In particular for $k=1$, we have $|\nabla^0 P|$ constant equal to $(1+n)\|P\|^2_{\mathbb{S}^n}$ and so

$$\big|\|\mathrm{H}\|_2^2\|\varphi P\|_2^2-\|P\|^2_{\mathbb{S}^n}\big|\leqslant\big(1-\|\varphi\|_2^2+C(n)m_1(1+\eta)^2D\big)\,\|P\|^2_{\mathbb{S}^n}$$

Now, let $B_k=\sup\Big\{\frac{\left|\|\mathrm{H}\|_2^{2k}\|\varphi P\|_2^2-\|P\|^2_{\mathbb{S}^n}\right|}{\|P\|^2_{\mathbb{S}^n}}\quad\big|\quad P\in\mathcal{H}^k(\mathbb{R}^{n+1})\setminus\{0\}\Big\}$. Then using that $\nabla^0 P\in\mathcal{H}^{k-1}(\mathbb{R}^{n+1})$ and (4.4), we get

$$B_k\leqslant B_{k-1}+C(n)m_k(1+\eta)^{2k}D\leqslant 1-\|\varphi\|_2^2+C(n)D\sum_{i=1}^k m_i(1+\eta)^{2k}$$

□

5. Proof of Theorem 1.1

Under the assumption of Theorem 1.1 we can use Lemmas 3.1 and 3.3 to improve the estimate in Lemma 4.6 in the case $\eta=2\sqrt[16]{\varepsilon}$.

Lemma 5.1. *For any isometrically immersed hypersurface $M\hookrightarrow\mathbb{R}^{n+1}$ with $r_M\|\mathrm{H}\|_2\leqslant 1+\varepsilon$ (or $\lambda_1(1+\varepsilon)^2\geqslant n\|\mathrm{H}\|_2^2$ or $(P_{p,\varepsilon})$ for $p>2$) and for any $P\in\mathcal{H}^k(M)$, we have*

$$\big|\|\mathrm{H}\|_2^{2k}\|\varphi P\|_2^2-\|P\|^2_{\mathbb{S}^n}\big|\leqslant C\sqrt[32]{\varepsilon}\,\|P\|^2_{\mathbb{S}^n}\,,$$

where $C=C(n,k)$ in the first two cases and $C=C(p,k,n)$ in the latter case.

As a consequence, the map $P\mapsto\varphi P$ is injective on $\mathcal{H}^k(M)$ for ε small enough.

Lemma 5.2. *Under the assumption of Lemma 5.1, if $\varepsilon\leqslant\frac{1}{(2C)^{32}}$ then* $\dim(\varphi\mathcal{H}^k(M))=m_k$.

Lemma 5.1 allows us to prove the following estimate on ΔP.

Lemma 5.3. *Under the assumptions of Lemma 5.1, if $\varepsilon\leqslant\frac{1}{(2C)^{32}}$, then for any $P\in\mathcal{H}^k(M)$, we have $\big\|\Delta(\varphi P)-\mu_k^{S_M}\varphi P\big\|_2\leqslant C\sqrt[16]{\varepsilon}\mu_k^{S_M}\|\varphi P\|_2$ where $C=C(n,k)$ ($C=C(n,k,p)$ under the pinching $(P_{p,\varepsilon})$).*

Proof. Let $P\in\mathcal{H}^k(M)$. Using (2.1) we have

$$\begin{aligned}\Delta(\varphi P)=&P\Delta\varphi-2\langle dP,d\varphi\rangle+\varphi\Delta P=P\Delta\varphi-2\langle dP,d\varphi\rangle+\varphi n\mathrm{H}dP(\nu)+\varphi\nabla^0dP(\nu,\nu)\\=&P\Delta\varphi-2\langle dP,d\varphi\rangle+\varphi\mu_k|H|\|\mathrm{H}\|_2P+\varphi(n+k-1)\frac{\mathrm{H}}{|\mathrm{H}|}\|\mathrm{H}\|_2dP(Z)\\&+\varphi(n+k-1)\frac{\mathrm{H}}{|\mathrm{H}|}(|\mathrm{H}|-\|\mathrm{H}\|_2)dP(\nu)+\varphi\nabla^0dP(\nu,Z)\end{aligned}$$

hence, we get

$$\begin{aligned}\|\Delta(\varphi P)-\mu_k\|H\|_2^2\varphi P\|_2\leqslant&\|(\Delta\varphi)P\|_2+2\|\langle d\varphi,dP\rangle\|_2+\mu_k\|(|H|-\|H\|_2)\varphi P\|_2\|\mathrm{H}\|_2\\&+(n+k-1)\|\mathrm{H}\|_2\|\varphi|dP||Z|\|_2+(n+k-1)\big\|\varphi(|\mathrm{H}|-\|\mathrm{H}\|_2)dP(\nu)\big\|_2+\|\varphi|\nabla^0dP||Z|\|_2\end{aligned}\tag{5.1}$$

Let us estimate $\|(\Delta\varphi)P\|_2$.

$$\begin{aligned}\|(\Delta\varphi)P\|_2^2 &\leqslant \frac{1}{v_M}\int_M (4|\psi''(|X|^2)||X^T|^2+2n|\psi'(|X|^2)||Z|)^2P^2dv\\ &\leqslant \frac{m_k}{v_M}\Big(\int_M |X|^{2k}\big(4|\psi''(|X|^2)||X^T|^2+2n|\psi'(|X|^2)||Z|\big)^2dv\Big)\,\|P\|_{\mathbb{S}^n}^2\\ &\leqslant \frac{m_k}{v_M}\frac{(1+2\sqrt[16]{\varepsilon})^{2k}}{\|H\|_2^{2k}}\Big(\int_{A_{2\sqrt[16]{\varepsilon}}}\big(\frac{8\|H\|_2^4}{\sqrt[8]{\varepsilon}}|X^T|^2+2n\frac{2\|H\|_2^2}{\sqrt[16]{\varepsilon}}|Z|\big)^2dv\Big)\,\|P\|_{\mathbb{S}^n}^2\\ &\leqslant \frac{m_k}{v_M}\frac{(1+2\sqrt[16]{\varepsilon})^{2k}}{\|H\|_2^{2k}}\Big(\int_{A_{2\sqrt[16]{\varepsilon}}}\frac{128\|H\|_2^8}{\sqrt[4]{\varepsilon}}|X^T|^4+32n^2\frac{\|H\|_2^4}{\sqrt[8]{\varepsilon}}|Z|^2dv\Big)\,\|P\|_{\mathbb{S}^n}^2\end{aligned}$$

Since we have $|X^T|\leqslant|X|$ and since Lemma 3.3 is valid with $\|\varphi Z\|_2^2$ replaced by $\frac{1}{v_M}\int_{A_{2\sqrt[16]{\varepsilon}}}|Z|^2$, we get

$$\begin{aligned}\|(\Delta\varphi)P\|_2^2 &\leqslant \frac{C(n,k)\mu_k}{v_M}\frac{\|P\|_{\mathbb{S}^n}^2}{\|H\|_2^{2k}}\int_{A_{2\sqrt[16]{\varepsilon}}}\big(\frac{\|H\|_2^6}{\sqrt[4]{\varepsilon}}|X^T|^2+\frac{\|H\|_2^4}{\sqrt[8]{\varepsilon}}|Z|^2\big)dv\\ &\leqslant \frac{C(n,k)\mu_k}{\|H\|_2^{2k}}\|H\|_2^4\sqrt[16]{\varepsilon}\,\|P\|_{\mathbb{S}^n}^2\end{aligned}$$

From the lemma 5.1, $\varepsilon\leqslant\frac{1}{(2C)^{32}}$ implies that

$$\|P\|_{\mathbb{S}^n}^2\leqslant 2\|H\|_2^{2k}\|\varphi P\|_2^2 \tag{5.2}$$

which gives

$$\|(\Delta\varphi)P\|_2^2\leqslant C(n,k)\mu_k\|H\|_2^4\sqrt[16]{\varepsilon}\|\varphi P\|_2^2 \tag{5.3}$$

Now

$$\begin{aligned}\|\langle d\varphi,dP\rangle\|_2^2\leqslant 4\|\psi'(|X|^2)|X^T||dP|\|_2^2 &\leqslant \frac{16\|\mathrm{H}\|_2^4}{\sqrt[16]{\varepsilon}v_M}\int_{A_{2\sqrt[16]{\varepsilon}}}|X^T|^2|dP|^2dv\\ &\leqslant \frac{16\|\mathrm{H}\|_2^4}{\sqrt[16]{\varepsilon}v_M}\|P\|_{\mathbb{S}^n}^2\int_{A_{2\sqrt[16]{\varepsilon}}}|X^T|^2m_knk^2|X|^{2(k-1)}dv\\ &\leqslant C(n,k)\mu_k\sqrt[16]{\varepsilon}\|\mathrm{H}\|_2^{4-2k}\|P\|_{\mathbb{S}^n}^2\leqslant C(n,k)\|\mathrm{H}\|_2^4\sqrt[16]{\varepsilon}\|\varphi P\|_2^2\end{aligned} \tag{5.4}$$

By the same way, we get

$$\|\varphi|dP|Z\|_2^2\leqslant C(n,k)\mu_k\|\mathrm{H}\|_2^2\sqrt[16]{\varepsilon}\|\varphi P\|_2^2 \tag{5.5}$$

Now, by Lemma 3.2, we have

$$\begin{aligned}\|(|\mathrm{H}|-\|\mathrm{H}\|_2)\varphi P\|_2^2 &\leqslant \frac{m_k}{v_M}\|P\|_{\mathbb{S}^n}^2\int_M||H|-\|H\|_2|^2|X|^{2k}\varphi^2dv\\ &\leqslant \frac{C(n,k)}{\|H\|_2^{2k}}\|P\|_{\mathbb{S}^n}^2\,\|\varphi(|H|-\|H\|_2)\|_2^2\\ &\leqslant C(n,k)\mu_k\|H\|_2^2\sqrt[16]{\varepsilon}\|\varphi P\|_2^2\end{aligned} \tag{5.6}$$

By the same way, we get

$$\|\varphi(|\mathrm{H}|-\|\mathrm{H}\|_2)dP(\nu)\|_2^2\leqslant C(n,k)\mu_k\sqrt[16]{\varepsilon}\|\mathrm{H}\|_2^4\|\varphi P\|_2^2 \tag{5.7}$$

Now let us estimate the last terms of (5.1)

$$\|\varphi|\nabla^0 dP||Z|\|_2^2 \leqslant \frac{C(n,k)\mu_k}{v_M}\|P\|^2_{\mathbb{S}^n}\int_M \varphi^2|X|^{2k-4}|Z|^2dv$$

$$\leqslant C(n,k)\mu_k\|H\|_2^4\sqrt[16]{\varepsilon}\|\varphi P\|_2^2 \tag{5.8}$$

Reporting (5.3), (5.4), (5.5), (5.6), (5.7) and (5.8) in (5.1) we get

$$\|\Delta(\varphi P)-\mu_k\|H\|_2^2\varphi P\|_2 \leqslant C(n,k)\sqrt[16]{\varepsilon}\mu_k\|H\|_2^2\|\varphi P\|_2$$

□

Let E_k^ε be the space spanned by the eigenfunctions of M associated to an eigenvalue in the interval $\left[(1-\sqrt[16]{\varepsilon}2C(n,k))\mu_k^{S_M},(1+\sqrt[16]{\varepsilon}2C(k,n))\mu_k^{S_M}\right]$. If $\dim E_k^\varepsilon < m_k$, then there exists $\varphi P \in (\varphi\mathcal{H}^k(M))\setminus\{0\}$ which is L^2-orthogonal to E_k^ν. Let $\varphi P=\sum_i f_i$ be the decomposition of φP in the Hilbert basis given by the eigenfunctions f_i of M associated respectively to λ_i. Putting $N:=\{i/\, f_i\notin E_k^\varepsilon\}$, by assumption on P we have

$$4C(n,k)^2\sqrt[8]{\varepsilon}(\mu_k^{S_M})^2\|\varphi P\|_2^2 \leqslant \sum_{i\in N}\left(\lambda_i-\mu_k^{S_M}\right)^2\|f_i\|_2^2=\|\Delta(\varphi P)-\mu_k^{S_M}\varphi P\|_2^2$$

$$\leqslant (\mu_k^{S_M})^2C(n,k)^2\sqrt[8]{\varepsilon}\|\varphi P\|_2^2$$

which gives a contradiction. We then have $\dim E_k^\varepsilon \geqslant m_k$.

6. Some examples

6.1. **Proof of Theorem 1.2.** We adapt the constructions made in [4, 12, 3]. We first consider submanifolds obtained by connected sum of a small submanifold εM_2 with a fixed submanifold M_1 along a small, adequately pinched cylinder $\varepsilon T'_\varepsilon$ (this is actually a 2 scales collapsing sequence of submanifolds). It gives Theorem 1.2 in the case where F is a singleton.

In the case where F is finite, it will suffice to iterate the construction (i.e. to glue several such cylinders) to add any finite set of eigenvalues to the spectrum of M_1. Since for a general F, $F\setminus \mathrm{Sp}(M_1)$ is the Hausdorff limit of a sequence of finite sets, we get Theorem 1.2 in the general case by a diagonal procedure.

6.1.1. *Flattening of submanifolds.* For any submanifold M of $\mathbb{R}^{n+1}$, we set $\tilde{M}^\varepsilon$ a submanifold of $\mathbb{R}^{n+1}$ obtained by smooth deformation of M at the neighbourhood of a point $x_0\in M$ such that $B_{x_0}(4\varepsilon)$ is flat in $\tilde{M}^\varepsilon$ and $M^\varepsilon\setminus B_{x_0}(10\varepsilon)$ is a subset of M. We also set $M^\varepsilon=\tilde{M}^\varepsilon\setminus B_{x_0}(3\varepsilon)$, whose boundary has a neighbourhood isometric to the flat annulus $B_0(4\varepsilon)\setminus B_0(3\varepsilon)$ in $\mathbb{R}^m$.

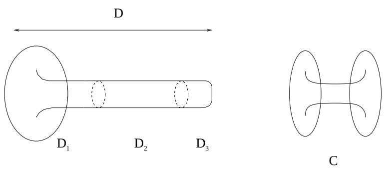
We describe precisely how to construct such a flattening $\tilde{M}^\varepsilon$ in [3] so that it also satisfies the following curvature estimates for any $\alpha\geqslant 1$.

$$\lim_{\varepsilon\to 0}\int_{\tilde{M}^\varepsilon}|\mathrm{H}_\varepsilon|^\alpha dv=\lim_{\varepsilon\to 0}\int_{M^\varepsilon}|\mathrm{H}_\varepsilon|^\alpha dv=\int_M|\mathrm{H}|^\alpha dv$$

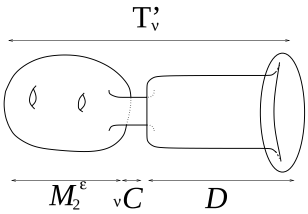
$$\lim_{\varepsilon\to 0}\int_{\tilde{M}^\varepsilon}|\mathrm{B}_\varepsilon|^\alpha dv=\lim_{\varepsilon\to 0}\int_{M^\varepsilon}|\mathrm{B}_\varepsilon|^\alpha dv=\int_M|\mathrm{B}|^\alpha dv$$

Note also that by construction, any function on M can be seen as a function on $\tilde{M}^\varepsilon$ and this identification of $H^1(M)$ with $H^1(\tilde{M}^\varepsilon)$ tends to an isometry as ε tends to 0.

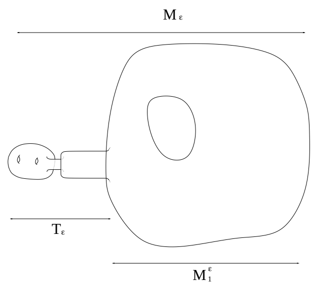
6.1.2. *Control of the curvature of the gluing.* Let M_1, M_2 be 2 manifolds of dimension m isometrically immersed in $\mathbb{R}^{n+1}$ and λ, L be some fixed, positive real numbers, with $\lambda \notin \mathrm{Sp}(M_1)$ and $L > \max\big(\frac{C(M_1)(1+\lambda)^2}{d^2}, 1\big)$, where d is the distance between λ and $\mathrm{Sp}(M_1)$ in $\mathbb{R}$. We consider the flattenings $\tilde{M}_2^\varepsilon$ of M_2 around the point x_2 and M_1^ε of M_1 around x_1. Let D be a smooth hypersurface of revolution of $\mathbb{R}^{m+1}$, composed of three parts, D_1, D_2, D_3, where D_1 is a cylinder of revolution isometric to $B_0(3) \setminus B_0(2) \subset \mathbb{R}^{m+1}$ at the neighbourhood of one of its boundary component and isometric to $[0,1] \times \mathbb{S}^{m-1}$ at the neighbourhood of its other boundary component, where $D_2 = [0, L] \times \mathbb{S}^{m-1}$ and where D_3 is a disc of revolution with pole x_3 and isometric to $[0,1] \times \mathbb{S}^{m-1}$ at its boundary and to a flat disc at the neighbourhood of x_3. Let C be a cylinder of revolution of dimension m isometric to $B_0(2) \setminus B_0(1) \subset \mathbb{R}^m$ at the neighbourhood of its 2 boundary components.



There exists $\nu_0 > 0$ such that for any $\nu \in]0, \nu_0[$ the gluing of $\tilde{M}_2^\varepsilon \setminus B_{x_2}(2\nu)$, of νC and of $D \setminus B_{x_3}(2\nu)$ along their isometric boundary components is a smoothly immersed submanifold T'_ν of dimension m. By standard arguments (see for instance [4] or what is done in section 6.1.3 in a more complicate case), when ν tends to 0, the Dirichlet spectrum of T'_ν converges to the disjoint union of the Dirichlet spectrum of D and of the spectrum of M_2. Moreover, for ν small enough, $\lambda_1^D(T'_\nu)$ depends continuously on ν. We infer that for any $\varepsilon \in]0, \varepsilon_0(M_2, \lambda, L, D_1, D_3)[$ there exists a $\nu_\varepsilon \in]0, \nu_0(M_2, \lambda, L, D_1, D_3)[$ such that $\lambda_1^D(T'_{\nu_\varepsilon}) = \varepsilon^2 \lambda$ and $\lambda_2^D(T'_{\nu_\varepsilon}) \geqslant \Lambda_2(L, M_2, \lambda, D_1, D_3) > 0$. We set $T_\varepsilon = \varepsilon T'_{\nu_\varepsilon}$. Note that we have $\int_{T_\varepsilon} |\mathrm{B}|^p \leqslant \varepsilon^{m-p} C_2(M_2, \lambda, L, D_1, D_3)$ for any $p < m$, $\lim_{\varepsilon \to 0} \int_{T_\varepsilon} |\mathrm{B}|^m = \int_{M_2} |\mathrm{B}|^m + \int_{D_1} |\mathrm{B}|^m + \int_{D_3} |\mathrm{B}|^m + LC(m)$, $\lambda_1^D(T_\varepsilon) = \lambda$ and $\lambda_2^D(T_\varepsilon) \geqslant \frac{\Lambda_2}{\varepsilon^2}$ for any $\varepsilon \leqslant \varepsilon_0$.



We set M_ε the m-submanifold of $\mathbb{R}^{n+1}$ obtained by gluing M_1^ε and T_ε along their boundaries in a fixed direction $\nu \in N_{x_1} M_1$. Note that M_ε is a smooth immersion of $M_1 \# M_2$ (resp. an embedding when M_1 and M_2 are embedded).



By the computations above, the sequence $i_k(M_1\#M_2)=M_{\frac{1}{k}}$ converges to M_1 in Hausdorff distance and we have

$$\lim_{\varepsilon\to 0}\int_{M_\varepsilon}|\mathrm{H}_\varepsilon|^\alpha dv=\int_{M_1}|\mathrm{H}|^\alpha dv \qquad \lim_{\varepsilon\to 0}\int_{M_\varepsilon}|\mathrm{B}_\varepsilon|^\alpha dv=\int_{M_1}|\mathrm{B}|^\alpha dv$$

for any $\alpha<m$ and

$$\lim_{\varepsilon\to 0}\int_{M_\varepsilon}|\mathrm{H}_\varepsilon|^m=\int_{M_1}|\mathrm{H}|^m+\int_{D_1\cup D_3}|\mathrm{H}|^m+C(m)L+\int_{M_2}|\mathrm{H}|^m$$
$$\lim_{\varepsilon\to 0}\int_{M_\varepsilon}|\mathrm{B}_\varepsilon|^m=\int_{M_1}|\mathrm{B}|^m+\int_{D_1\cup D_3}|\mathrm{B}|^m+C(m)L+\int_{M_2}|\mathrm{B}|^m$$

6.1.3. *Computation of the spectrum of M_ε.* We will prove that there exists a sequence $(\varepsilon_p)_{p\in\mathbb{N}}$ such that $\varepsilon_p\to 0$ and the spectrum of M_{ε_p} converges to the disjoint union of $\mathrm{Sp}(M_1)$ and of $\{\tilde\lambda\}$, where $\tilde\lambda$ satisfies $\lambda-\frac{C(M_1)(1+\lambda)}{\sqrt{L}}\leqslant\tilde\lambda\leqslant\lambda$. Since the collapsing of M_ε is multiscale, the cutting and rescaling technique of [4, 12] has to be adapted. Indeed, after rescaling of T_ε we get another collapsing sequence of submanifolds with no uniform control of the trace and Sobolev Inequalities.

We denote by $(\lambda_k)_{k\in\mathbb{N}}$ the union with multiplicities of the spectrum of M_1 and of $\{\lambda\}$, by $(\lambda_k^\varepsilon)_{k\in\mathbb{N}}$ the spectrum of M_ε and by $(\mu_k^\varepsilon)_{k\in\mathbb{N}}$ the Dirichlet spectrum of the disjoint union $M'_\varepsilon=T_\varepsilon\cup\Big(M_1^\varepsilon\setminus B_{x_1}(10\varepsilon)\Big)$. By the Dirichlet principle, we have $\lambda_k^\varepsilon\leqslant\mu_k^\varepsilon$ for any $k\in\mathbb{N}$. It is well known (see for instance [6]) that the Dirichlet spectrum of $M_1^\varepsilon\setminus B_{x_1}(10\varepsilon)$ converges to the spectrum of M_1. We infer that $\mu_k^\varepsilon\to\lambda_k$ as $\varepsilon\to 0$ and so $\limsup\lambda_k^\varepsilon\leqslant\lambda_k$ for any $k\in\mathbb{N}$.

We set $\alpha_k=\liminf_{\varepsilon\to 0}\lambda_k^\varepsilon$. To get some lower bound on the α_k, we need some local trace inequalities. We set $S_t=\{x\in T_\varepsilon/\, d(x,\partial T_\varepsilon)=-t\}$ for any $t\leqslant 0$ and $S_t=\{x\in M_1^\varepsilon/\, d(x,\partial M_1^\varepsilon)=t\}$ for any $t\geqslant 0$. We also set $B_{t,r}=\cup_{\{s/\,|s-t|\leqslant r\}}S_s$, $N_r=M_1^\varepsilon\cup B_{\frac{r}{2},\frac{-r}{2}}$ for any $r\leqslant 0$ and $N_r=M_1^\varepsilon\setminus B_{\frac{r}{2},\frac{r}{2}}$ for any $r\geqslant 0$. Let a_{M_1} be a constant such that the volume density θ_ε of M_ε in normal coordinates to $S_{-2\varepsilon}$ satisfies $\frac{1}{a_{M_1}}(3+\frac{t}{\varepsilon})^{m-1}\geqslant\theta_\varepsilon(t,u)\geqslant a_{M_1}(3+\frac{t}{\varepsilon})^{m-1}$ for any $t\in[-2\varepsilon,a_{M_1}]$ and any $u\in S_{-2\varepsilon}$. Let εd be the distance in M_ε between M_1^ε and εD_2 and $C(D_1)$ be a constant such that for any $t\in[-(L+d+2)\varepsilon,-2\varepsilon]$ and any $u\in S_{-2\varepsilon}$ we have $\frac{\theta_\varepsilon(t,u)}{\theta_\varepsilon(-2\varepsilon,u)}\in[\frac{1}{C(D_1)},C(D_1)]$. Let $\eta:[-2\varepsilon,a_{M_1}]\to[0,1]$ be a smooth function such that $\eta(t)=1$ for any $t\leqslant\frac{a_{M_1}}{2}$, $\eta(a_{M_1})=0$ and $|\eta'|\leqslant\frac{4}{a_{M_1}}$. For any $r\in[-2\varepsilon,a_{M_1}/2]$ and any $f\in H^1(M_\varepsilon)$, we have

$$\begin{aligned}\int_{S_r}f^2&=\int_{S_{-2\varepsilon}}\Big(\int_r^{a_{M_1}}\frac{\partial}{\partial s}[\eta(\cdot)f(\cdot,u)]ds\Big)^2\theta_\varepsilon(r,u)du\\&\leqslant\int_r^{a_{M_1}}\frac{\sup_{u\in S_{-2\varepsilon}}\theta_\varepsilon(r,u)}{\inf_{u\in S_{-2\varepsilon}}\theta_\varepsilon(s,u)}ds\int_{S_{-2\varepsilon}}\int_r^{a_{M_1}}\big(\frac{\partial}{\partial s}[\eta(\cdot)f(\cdot,\frac{\varepsilon}{10}u)]\big)^2\theta_\varepsilon(s,u)\\&\leqslant c(M_1)\int_r^{a_{M_1}}\frac{(3+r/\varepsilon)^{m-1}}{(3+s/\varepsilon)^{m-1}}ds\|f\|^2_{H^1(M_\varepsilon)}\end{aligned}$$

which gives

$$\int_{S_r}f^2\leqslant c(M_1)(3\varepsilon+r)\|f\|^2_{H^1(M_\varepsilon)} \tag{6.1}$$

when $m \geqslant 3$. By the same way, for any $r \in [-(L+d+2)\varepsilon, -2\varepsilon]$, we have

$$\int_{S_r} f^2 \leqslant -c(M_1, D_1)(\varepsilon + r)\|f\|^2_{H^1(M_\varepsilon)} \tag{6.2}$$

We now use this local trace inequality to get some estimates on the eigenfunctions of M_ε. We set $\varphi : M_\varepsilon \to [0,1]$ be a smooth function equal to 1 on $N_{21\varepsilon/2} \cup \big(M_\varepsilon \setminus N_{-\varepsilon/2}\big)$, equal to 0 outside M'_ε and such that $|\varphi'| \leqslant \frac{4}{\varepsilon}$. For any $f_1, f_2 \in H^1(M_\varepsilon)$, integration of Inequalities (6.1) and (6.2) gives us

$$\Big|\int_{M_\varepsilon} f_1 f_2 - \int_{M_\varepsilon} \varphi f_1 \varphi f_2\Big| \leqslant \int_{M_\varepsilon} |\varphi^2 - 1||f_1||f_2| \leqslant c(M_1)\varepsilon^2 \|f_1\|_{H^1(M_\varepsilon)}\|f_2\|_{H^1(M_\varepsilon)} \tag{6.3}$$

and

$$\begin{aligned}\int_{M_\varepsilon} |d\varphi f_1|^2 &\leqslant \int_{M_\varepsilon} |d\varphi|^2 f_1^2 + 2\varphi f_1 (df_1, d\varphi) + \varphi^2 |df_1|^2\\ &\leqslant \frac{16}{\varepsilon^2}\|f_1\|^2_{L^2(\mathrm{Supp}(d\varphi))} + \frac{8}{\varepsilon}\|f_1\|_{L^2(\mathrm{Supp}(d\varphi))}\|df_1\|_2 + \|df_1\|_2^2\\ &\leqslant c(M_1)\|f_1\|^2_{H^1(M_\varepsilon)} \end{aligned}\tag{6.4}$$

Let (f_k^ε) be a L^2-orthonormal, complete set of eigenfunctions of M_ε. For any k, we set $\tilde{f}_k^\varepsilon$ the function on M_1 equal to φf_k^ε on $N_{10\varepsilon}$ and extended by 0. By Inequality (6.4), we have $\|\tilde{f}_k^\varepsilon\|^2_{H^1(M_1)} \leqslant c(M_1)(1+\lambda_k)$ for ε small enough. We infer by diagonal extraction that there exists some sequences $(\varepsilon_p)_{p\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}} \in H^1(M_1)^{\mathbb{N}}$ such that $\lambda_k^{\varepsilon_p} \to \alpha_k$ and $(\tilde{f}_k^{\varepsilon_p})_p$ converges weakly in $H^1(M_1)$ and strongly in $L^2(M_1)$ to h_k, for any k. It is easy to prove that h_k is a weak solution of $\Delta h_k = \alpha_k h_k$ on $H^1(M_1 \setminus \{x_1\}) = H^1(M_1)$. By elliptic regularity, either $h_k = 0$ or α_k is an eigenvalue of M_1.

Let $k_0 \in \mathbb{N}$ such that $\lambda_{k_0} = \lambda$. Since D_2 isometric to $[0,L] \times \mathbb{S}^{m-1}$, any $f_k^{\varepsilon_p}$ can be seen as a function on $[0, \varepsilon_p L] \times \varepsilon_p \mathbb{S}^{m-1}$. For any $f = \sum_{i \leqslant k_0} \beta_i f_i^{\varepsilon_p} \in \mathrm{Vect}\{f_i^{\varepsilon_p} / \, i \leqslant k_0\}$, we define the rescaling F_p on $c = [0,1] \times \mathbb{S}^{m-1}$ by $F_p(t,x) = \varepsilon_p^{\frac{m}{2}-1} L^{-\frac{1}{2}} f(\varepsilon_p L t, \varepsilon_p x)$. By Inequality (6.2), we have

$$\begin{aligned}\int_c F_p^2 &= \frac{1}{\varepsilon_p^2 L^2}\int_{\varepsilon_p D_2} f^2 \leqslant c(M_1, D_1)(1 + \frac{2}{L})(1+\lambda)\|f\|_2^2,\\ \int_{\{0\}\times\mathbb{S}^{m-1}} F_p^2 &= \frac{1}{L\varepsilon_p}\int_{\varepsilon_p(D_1\cap D_2)} f^2 \leqslant \frac{c(M_1,D_1)(1+\lambda)\|f\|_2^2}{L},\\ \text{and}\quad \int_{\{1\}\times\mathbb{S}^{m-1}} F_p^2 &= \frac{1}{L\varepsilon_p}\int_{\varepsilon_p(D_3\cap D_2)} f^2 \leqslant (1+\lambda)c(M_1,D_1)(1+\frac{d}{L})\|f\|_2^2,\end{aligned}$$

for p large enough (note that we have $d \geqslant 2$ by construction). Moreover, we have $\int_c |dF_p|^2 \leqslant \int_{\varepsilon_p D_2} |df|^2 \leqslant \lambda \|f\|_2^2$. So we can assume that there exists $F_\infty \in H^1(c)$ such that the sequence (F_p) converges to F_∞ weakly in $H^1(c)$ and strongly in $L^2(c)$. We set $j_p(t) = \int_{\mathbb{S}^{m-1}} F_p(t,x)dx$ and $j_\infty(t) = \int_{\mathbb{S}^{m-1}} F_\infty(t,x)dx$, we have $j_p, j_\infty \in H^1([0,1])$ (with $j_p'(t) = \int_{\mathbb{S}^{m-1}} \frac{\partial F_p}{\partial t}(t,x)dx$), $j_p \to j_\infty$ strongly in $L^2([0,1])$ and weakly in $H^1([0,1])$. By the estimates above and the compactness of the trace operator on c, we have $|j_\infty(0)| \leqslant \frac{C(M_1)\sqrt{1+\lambda}\|f\|_2}{\sqrt{L}}$ and $|j_\infty(1)| \leqslant \sqrt{1+\lambda}\|f\|_2 C(M_1)$. Hence $l(t) = j_\infty(t) -$

$\big(j_\infty(0)+(j_\infty(1)-j_\infty(0))t\big)$ is in $H^1_0([0,1])$. For any $\psi\in\mathcal{C}^\infty_c([0,1])$, we set $\psi_p(t,x)=\varepsilon_pL\psi(\frac{t}{\varepsilon_pL})$ seen as a function in $H^1_0(\varepsilon_pD_2)$. We have

$$\begin{aligned}
&\int_0^1 l'\psi'\,dt=\int_0^1 j'_\infty\psi'\,dt\\
&=\lim_p\int_0^1 j'_p(t)\psi'(t)\,dt=\lim_p\int_c\frac{\partial F_p}{\partial t}\psi'=\lim_p\frac{1}{\varepsilon_p^{\frac{m}{2}}\sqrt{L}}\int_{\varepsilon_pD_2}\langle df,d\psi_p\rangle\,dt\,dx\\
&=\lim_p\sum_i\frac{\beta_i\lambda_i^{\varepsilon_p}}{\varepsilon_p^{\frac{m}{2}}\sqrt{L}}\int_{\varepsilon_pD_2}f_i^{\varepsilon_p}\psi_p\,dt\,dx=\sum_i\alpha_i\beta_iL^2\lim_p\varepsilon_p^2\int_cF_{i,p}\psi\,dt\,dx\\
&=0,
\end{aligned}$$

where $F_{i,p}(t,x)=\varepsilon_p^{\frac{m}{2}-1}L^{-\frac12}f_i^{\varepsilon_p}(\varepsilon_pLt,\varepsilon_px)$. We infer l is harmonic and in $H^1_0([0,1])$, i.e. $l=0$ and $j_\infty(t)=j_\infty(0)+(j_\infty(1)-j_\infty(0))t$ on $[0,1]$. Since the Poincare inequality on $\mathbb{S}^{m-1}$ gives us

$$\begin{aligned}
\int_{\mathbb{S}^{m-1}}F_p(t,x)^2\,dx&\leqslant\frac{1}{\operatorname{Vol}\mathbb{S}^{m-1}}\Big(\int_{\mathbb{S}^{m-1}}F_p(t,x)\,dx\Big)^2+\frac{1}{m-1}\int_{\mathbb{S}^{m-1}}|d_{\mathbb{S}^{m-1}}F_p|^2\\
&\leqslant\frac{1}{\operatorname{Vol}\mathbb{S}^{m-1}}j_p^2(t)+\frac{\varepsilon_p}{(m-1)L}\int_{\varepsilon_p\mathbb{S}^{m-1}}|d_{\varepsilon_p\mathbb{S}^{m-1}}f|^2(\varepsilon_pLt,x)\,dx,
\end{aligned}$$

we get that

$$\begin{aligned}
&\frac{1}{L\varepsilon_p^2}\int_{[0,\varepsilon_p\sqrt{L}]\times\varepsilon_p\mathbb{S}^{m-1}}f^2=L\int_{[0,\frac{1}{\sqrt{L}}]\times\mathbb{S}^{m-1}}F_p^2\\
&\leqslant\frac{L}{\operatorname{Vol}\mathbb{S}^{m-1}}\int_0^{\frac{1}{\sqrt{L}}}j_p^2(t)\,dt+\frac{1}{(m-1)L}\int_{[0,\varepsilon_p\sqrt{L}]\times\varepsilon_p\mathbb{S}^{m-1}}|d_{\varepsilon_p\mathbb{S}^{m-1}}f|^2\\
&\leqslant\frac{L}{\operatorname{Vol}\mathbb{S}^{m-1}}\int_0^{\frac{1}{\sqrt{L}}}j_p^2(t)\,dt+\frac{\lambda}{(m-1)L}\|f\|^2\\
&\to\frac{L}{\operatorname{Vol}\mathbb{S}^{m-1}}\int_0^{\frac{1}{\sqrt{L}}}j_\infty(t)^2\,dt+\frac{\lambda}{(m-1)L}\|f\|^2\leqslant\frac{C(M_1)(1+\lambda)\|f\|^2}{\sqrt{L}}
\end{aligned}$$

If the family $(h_i)_{i<k_0}$ is not free in $L^2(M_1)$, then either one h_i is null or they are all eigenfunctions of M_1. Since the eigenspaces are in direct sum, we infer that there exists $\mu\leqslant\lambda_{k_0-1}$ and $(\beta_i)\in\mathbb{R}^{k_0}\setminus\{0\}$ such that $\sum_i\beta_i^2=1$, $\sum_i\beta_ih_i=0$ and $\alpha_i=\mu$ for any i such that $\beta_i\neq0$. We set $f=\sum_i\beta_if_i^{\varepsilon_p}$ and $\eta:M_{\varepsilon_p}\to[0,1]$ a smooth function equal to 1 on $M_{\varepsilon_p}\setminus N_{-(2+d+\sqrt{L})\varepsilon_p}$, equal to 0 on $N_{-(2+d)\varepsilon_p}$ and such that $|\varphi'|\leqslant\frac{2}{\varepsilon_p\sqrt{L}}$. We then have

$$\begin{aligned}
&\Big|\int_{M_{\varepsilon_p}}|d(\eta f)|^2-\mu\int_{M_{\varepsilon_p}}(\eta f)^2\Big|=\Big|\int_{M_{\varepsilon_p}}|d\eta|^2f^2+\langle df,d(\eta^2f)\rangle-\mu\int_{M_{\varepsilon_p}}(\eta f)^2\Big|\\
&\leqslant\frac{C(M_1)(1+\lambda)}{\sqrt{L}}\|f\|_2^2+\int_{M_{\varepsilon_p}}\sum_{i,j}(\lambda_i^{\varepsilon_p}-\mu)\beta_i\eta f_i^{\varepsilon_p}\beta_j\eta f_j^{\varepsilon_p}
\end{aligned}\tag{6.5}$$

Inequalities (6.3) and (6.2) imply that $\int_{M_{\varepsilon_p}}(\eta f)^2\to1$. Since $\eta f\in H^1_0(T_{\varepsilon_p})$ and since by construction of T_{ε_p}, we have $\lambda_1^D(T_{\varepsilon_p})=\lambda$, we then have $\int_{M_{\varepsilon_p}}|d(\eta f)|^2\geqslant\lambda\int_{M_{\varepsilon_p}}(\eta f)^2$.

Letting p tend to ∞ in Inequality (6.5) we get that $\lambda-\lambda_{k_0-1}\leqslant\frac{C(M_1)(1+\lambda)}{\sqrt{L}}$, which contradicts the choice made on L at the beginning of this subsection.

We infer that $(h_i)_{i<k_0}$ is free in $L^2(M_1)$. This implies that α_i is an eigenvalue of M_1 and h_i is an eigenfunction of M_1 for any $i<k_0$. Since $\alpha_i=\lim\lambda_i^{\varepsilon_p}\leqslant\lambda_i=\lambda_i(M_1)$ for any $i<k_0$, we infer that $\alpha_i=\lambda_i$ for any $i<k_0$ and that the $(h_i)_{i<k_0}$ is a basis of the eigenspaces of M_1 associated to the first k_0 eigenvalues. By the same way, if $h_{k_0}\neq 0$, then $\alpha_{k_0}=\lambda_{k_0-1}$ (since it is an eigenvalue of M_1 less than λ) and so the family $(h_i)_{i\leqslant k_0}$ is not free. The same argument as above gives a contradiction. So we have that $h_{k_0}=0$.

Assume that there exists another index $l\neq k_0$ such that $h_l=0$. Then, Inequality (6.3) gives that $\int_{T_{\varepsilon_p}}\varphi f_{k_0}^{\varepsilon_p}\varphi f_l^{\varepsilon_p}\to 0$, $\int_{T_{\varepsilon_p}}(\varphi f_{k_0}^{\varepsilon_p})^2\to 1$ and $\int_{T_{\varepsilon_p}}(\varphi f_l^{\varepsilon_p})^2\to 1$ and Inequality (6.4) gives that $\int_{T_{\varepsilon_p}}|d\varphi f_{k_0}^{\varepsilon_p}|^2$ and $\int_{T_{\varepsilon_p}}|d\varphi f_l^{\varepsilon_p}|^2$ remain bounded as $\varepsilon_p\to 0$. We set g_p a unitary eigenfunction of T_{ε_p} for the Dirichlet problem associated to the eigenvalue λ. If we set $(\varphi f_{k_0}^{\varepsilon_p})_{|T_{\varepsilon_p}}=\beta_{k_0}^p g_p+\delta_{k_0}^p$ and $(\varphi f_l^{\varepsilon_p})_{|T_{\varepsilon_p}}=\beta_l^p g_p+\delta_l^p$, with $\beta_{k_0}^p,\beta_l^p\in\mathbb{R}$ and $\delta_{k_0}^p,\delta_l^p$ orthogonal to g_p in $H_0^1(T_{\varepsilon_p})$. The previous relations and the lower bound on $\lambda_2^D(T_{\varepsilon_p})$ imply that

$$\int_{T_{\varepsilon_p}}|d(\varepsilon f_{k_0}^{\varepsilon_p})|^2\geqslant\lambda(\beta_{k_0}^p)^2+\lambda_2^D(T_{\varepsilon_p})\|\delta_{k_0}^p\|_{L^2(T_{\varepsilon_p})}^2\geqslant(\beta_{k_0}^p)^2\lambda+\frac{\Lambda_2}{\varepsilon_p^2}\|\delta_{k_0}^p\|_{L^2(T_{\varepsilon_p})}^2.$$

By the same way, $(\beta_l^p)^2\lambda+\frac{\Lambda_2}{\varepsilon_p^2}\|\delta_l^p\|_{L^2(T_{\varepsilon_p})}^2$ is bounded, and so $\|\delta_{k_0}^p\|_{L^2(T_{\varepsilon_p})}^2$ and $\|\delta_l^p\|_{L^2(T_{\varepsilon_p})}^2$ tend to 0 with ε_p. Now, we have $(\beta_{k_0}^p)^2+\|\delta_{k_0}^p\|_{L^2(T_{\varepsilon_p})}^2\to 1$ and so $|\beta_{k_0}^p|\to 1$. By the same way, we have $|\beta_l^p|\to 1$, which contradicts the fact that $\int_{T_{\varepsilon_p}}\varphi f_{k_0}^{\varepsilon_p}\varphi f_l^{\varepsilon_p}\to 0$. We infer that for any $k\in\mathbb{N}\setminus\{k_0\}$ we have that α_k is an eigenvalue of M_1. Moreover, if we decompose $(\varphi f_k^{\varepsilon_p})_{|T_{\varepsilon_p}}=\beta_k^p g_p+\delta_k^p$ as above, Inequality (6.4) implies that $(\beta_k^p)^2+\frac{\Lambda_2}{\varepsilon_p^2}\|\delta_k^p\|_{L^2(T_{\varepsilon_p})}^2$ remains bounded and so we have $\lim\|\delta_k^p\|_{L^2(T_{\varepsilon_p})}^2=0$ and Inequality (6.3) gives

$$0=\lim\int_{M_\varepsilon}f_{k_0}^{\varepsilon_p}f_k^{\varepsilon_p}=\lim\beta_k^p\beta_{k_0}^p=\lim\beta_k^p$$

and so $(\varphi f_k^{\varepsilon_p})_{|T_{\varepsilon_p}}\to 0$ in $L^2(T_{\varepsilon_p})$ for any $k\neq k_0$. Once again, Inequality (6.3) gives us that for any $k,l\in\mathbb{N}\setminus\{k_0\}$, we have

$$\int_{M_1}h_k h_l=\delta_{kl}.$$

From the min-max principle, it gives that we have $\alpha_k\geqslant\lambda_k$ for any $k\neq k_0$. Since we have $\alpha_k\leqslant\lambda_k$ for any $k\in\mathbb{N}$, we infer that for any $k\in\mathbb{N}\setminus\{k_0\}$ we have $\alpha_k=\lambda_k$. Finally, Inequality (6.5), applied to $f=f_{k_0}^{\varepsilon_p}$ and $\mu=\alpha_{k_0}$ gives that $\alpha_{k_0}\in[\lambda-\frac{C(M_1)(1+\lambda)}{\sqrt{L}},\lambda]$.

6.1.4. *End of the proof of Theorem 1.2 and case $\alpha=m$.* Since we can take L as large as needed while keeping $\int_{M_\varepsilon}|\mathrm{B}|^\alpha\to\int_{M_1}|B|^\alpha$ for any $\alpha<n$, we get Theorem 1.2 for $F=\mathrm{Sp}(M_1)\cup\{\lambda\}$ by diagonal extraction. Iterating the construction (with M_2 replaced by $\mathbb{S}^m$ for any supplementary gluing) we get the result for any disjoint union $F=\mathrm{Sp}(M_1)\cup\{\text{finite set}\}$ and then for any F, since any closed set F is the limit in pointed-Hausdorff topology of a sequence of finite sets.

In the case $\alpha = m$, the limit $\int_{M_\varepsilon} |\mathrm{B}|^m$ depend on L and so we are only able to get a weak version of Theorem 1.2 with $F = \mathrm{Sp}(M_1) \cup G$, where G is a finite set whose elements are known up to an error term and where the point 2) is replaced by $\int_{i_k(M_1 \# M_2)} |\mathrm{B}|^m$ is bounded by a constant that depend on M_1, M_2, D_1, D_3, G and on the error term.

6.2. **Example 1.4.** We set $I_\varepsilon = [\varepsilon, \frac{\pi}{2}]$ for $\varepsilon > 0$ and let $\varphi : I_\varepsilon \longrightarrow (-1, +\infty)$ be a function continuous on I_ε and smooth on $(\varepsilon, \frac{\pi}{2}]$. For any $0 \leqslant k \leqslant n-2$, we consider the map

$$\begin{array}{rcl}\Phi_\varphi : \mathbb{S}^{n-k-1} \times \mathbb{S}^k \times I_\varepsilon & \longrightarrow & \mathbb{R}^{n+1} = \mathbb{R}^{n-k} \oplus \mathbb{R}^{k+1} \\ x = (y, z, r) & \longmapsto & (1 + \varphi(r))(y \sin r + z \cos r)\end{array}$$

whose image X_φ is a smooth embedded submanifold (with boundary) diffeomorphic to $\mathbb{S}^n \setminus B(\mathbb{S}^k, \varepsilon)$. We denote respectively by $\mathrm{B}_q(\varphi)$ and $\mathrm{H}_q(\varphi)$ the second fundamental form and the mean curvature of X_φ at the point q. They are given by the following formulae.

Lemma 6.1. *Let $x = (y, z, r) \in \mathbb{S}^{n-k-1} \times \mathbb{S}^k \times I_\varepsilon$, $q = \Phi_\varphi(x)$ and $(u, v, h) \in T_x X_\varepsilon$. Then we have*

$$n\mathrm{H}_q(\varphi) = \left(\varphi'^2 + (1+\varphi)^2\right)^{-3/2} \Big[-(1+\varphi(r))\varphi''(r) + (1+\varphi(r))^2 + 2\varphi'^2(r)\Big]$$
$$+ \frac{\left(\varphi'^2 + (1+\varphi)^2\right)^{-1/2}}{1+\varphi(r)} \Big[-(n-k-1)\varphi'(r)\cot r + (n-1)(1+\varphi(r)) + k\varphi'(r)\tan r\Big]$$

$$|\mathrm{B}_q(\varphi)| = \frac{(1+\varphi(r))^{-1}}{\left(1 + (\frac{\varphi'(r)}{1+\varphi(r)})^2\right)^{1/2}} \max\Big(\big|1 - \frac{\varphi'}{1+\varphi}\cot r\big|, \big|1 + \frac{\varphi'}{1+\varphi}\tan r\big|, \big|1 + \frac{(\varphi')^2 - (1+\varphi)\varphi''}{\varphi'^2 + (1+\varphi)^2}\big|\Big)$$

To prove Theorem 1.4, we set $a < \frac{\pi}{10}$ and define the function φ_ε on I_ε by

$$\varphi_\varepsilon(r) = \begin{cases} f_\varepsilon(r) = \varepsilon \displaystyle\int_1^{\frac{r}{\varepsilon}} \frac{dt}{\sqrt{t^{2(n-k-1)} - 1}} & \text{if } \varepsilon \leqslant r \leqslant a + \varepsilon, \\ u_\varepsilon(r) & \text{if } r \geqslant a + \varepsilon, \\ b_\varepsilon & \text{if } r \geqslant 2a + \varepsilon, \end{cases}$$

where b_ε is a constant and u_ε is chosen so that φ_ε is smooth on $(\varepsilon, \frac{\pi}{2}]$ and strictly concave on $(\varepsilon, 2a+\varepsilon]$. Since we have $f_\varepsilon(x) \to 0$, $f'_\varepsilon(x) \to 0$, $f''_\varepsilon(x) \to 0$ for any fixed $x \in (\varepsilon, a+\varepsilon]$, the concavity implies that $b_\varepsilon \to 0$ as $\varepsilon \to 0$ (hence b_ε can be chosen less than $\frac{1}{2}$), that $\varphi_\varepsilon \to 0$ uniformly on I_ε and that φ'_ε converges uniformly to 0 on any compact subset of $(\varepsilon, \frac{\pi}{2}]$. Moreover, u_ε can be chosen such that φ''_ε converges to 0 uniformly on any compact subset of $(\varepsilon, \frac{\pi}{2}]$.

On $(\varepsilon, a+\varepsilon]$, φ_ε satisfies

$$\varphi''_\varepsilon = -\frac{(n-k-1)(1+\varphi'^2_\varepsilon)}{r}\varphi'_\varepsilon, \tag{6.6}$$

$\varphi_\varepsilon(\varepsilon) = 0$ and $\lim\limits_{t\to\varepsilon} \varphi'_\varepsilon(t) = +\infty = -\lim\limits_{t\to\varepsilon} \varphi''_\varepsilon(t)$. On $(-b_\varepsilon, b_\varepsilon)$, we define $\tilde{\varphi}_\varepsilon$ by $\tilde{\varphi}_\varepsilon(t) = \varphi_\varepsilon^{-1}(|t|)$. Since $\tilde{\varphi}_\varepsilon$ satisfies the equation $yy'' = (n-k-1)\big(1 + (y')^2\big)$ with initial data $\tilde{\varphi}_\varepsilon(0) = \varepsilon$ and $\tilde{\varphi}'_\varepsilon(0) = 0$, it is smooth at 0, hence on $(-b_\varepsilon, b_\varepsilon)$.

Now we consider the two applications $\Phi_{\varphi_\varepsilon}$ and $\Phi_{-\varphi_\varepsilon}$ defined as above, and we set $M_\varepsilon^+ = X_{\varphi_\varepsilon}$, $M_\varepsilon^- = X_{-\varphi_\varepsilon}$ and $M_\varepsilon^k = M_\varepsilon^+ \cup M_\varepsilon^-$. M_ε^k is a smooth submanifold of $\mathbb{R}^{n+1}$ since the function $F_\varepsilon(p_1,p_2) = |p_1|^2 - |p|^2 \sin^2\big(\tilde{\varphi}_\varepsilon(|p|-1)\big)$, defined on

$$U = \{p = (p_1,p_2) \in \mathbb{R}^{n-k} \oplus \mathbb{R}^{k+1} / \, p_1 \neq 0,\, p_2 \neq 0,\, -b_\varepsilon + 1 < |p| < b_\varepsilon + 1\}$$

gives a smooth, local equation of M_ε^k at the neighborhood of $M_\varepsilon^+ \cap M_\varepsilon^-$ that satisfies

$$\nabla F_\varepsilon(p_1,p_2) = 2p_1 \cos^2\varepsilon - 2p_2 \sin^2\varepsilon \neq 0$$

on $M_\varepsilon^+ \cap M_\varepsilon^-$.

We denote respectively by H_ε and B_ε, the mean curvature and the second fundamental form of M_ε^k.

Theorem 6.2. *$\|H_\varepsilon\|_\infty$ and $\|\mathrm{B}_\varepsilon\|_{n-k}$ remain bounded whereas $\|H_\varepsilon - 1\|_1 \to 0$ and $\big\||X|-1\big\|_\infty \to 0$ when $\varepsilon \to 0$.*

Remark 6.3. *We have $\|\mathrm{B}_\varepsilon\|_q \to \infty$ when $\varepsilon \to 0$, for any $q > n-k$.*

Proof. From the lemma 6.1 and the definition of φ_ε, H_ε and $|\mathrm{B}_\varepsilon|$ converge uniformly to 1 on any compact of $M_\varepsilon^k \setminus M_\varepsilon^+ \cap M_\varepsilon^-$. On the neighborhood of $M_\varepsilon^+ \cap M_\varepsilon^-$, we have $n(H_\varepsilon)_x = nh_\varepsilon^\pm(r)$ and $|nh_\varepsilon^\pm| \leqslant h_{1,\varepsilon}^\pm + h_{2,\varepsilon}^\pm + h_{3,\varepsilon}^\pm$, where

$$h_{2,\varepsilon}^\pm(r) = \frac{k|\tan(r)|}{1\pm\varphi_\varepsilon}\frac{\varphi_\varepsilon'}{(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{1/2}} \leqslant \frac{k}{1-b_\varepsilon}\tan\frac{\pi}{5}$$

$$h_{3,\varepsilon}^\pm(r) = \frac{1}{(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{1/2}}\Big(n-1++\frac{2\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2}{\varphi_\varepsilon'^2+(1\pm\varphi_\varepsilon)^2}\Big) \leqslant \frac{n+1}{1-b_\varepsilon}$$

and by differential Equation (6.6) we have

$$\begin{aligned}
h_{1,\varepsilon}^\pm(r) &= \Big|(n-k-1)\frac{(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-1/2}}{1\pm\varphi_\varepsilon}\varphi_\varepsilon'\cot(r) + (\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-3/2}(1\pm\varphi_\varepsilon)\varphi_\varepsilon''\Big| \\
&\leqslant (n-k-1)\frac{(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-1/2}}{1\pm\varphi_\varepsilon}\varphi_\varepsilon'\Big|\cot(r) - \frac{1}{r}\Big| \\
&\quad + \frac{n-k-1}{r}\Big|\frac{(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-1/2}}{1\pm\varphi_\varepsilon}\varphi_\varepsilon' - (\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2)^{-3/2}(1\pm\varphi_\varepsilon)(1+\varphi_\varepsilon'^2)\varphi_\varepsilon'\Big| \\
&\leqslant \frac{n}{1-b_\varepsilon}\Big(\frac{1}{r} - \cot(r)\Big) \\
&\qquad + \frac{n\left(\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2\right)^{-3/2}}{r(1\pm\varphi_\varepsilon)}\varphi_\varepsilon'\Big|\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2 - (1\pm\varphi_\varepsilon)^2(1+\varphi_\varepsilon'^2)\Big| \\
&\leqslant \frac{n}{1-b_\varepsilon}\Big(\frac{1}{r} - \cot(r)\Big) + \frac{n}{r}\varphi_\varepsilon\frac{2\pm\varphi_\varepsilon}{1\pm\varphi_\varepsilon}\frac{\varphi_\varepsilon'^3}{[\varphi_\varepsilon'^2 + (1\pm\varphi_\varepsilon)^2]^{3/2}} \\
&\leqslant \frac{n}{1-b_\varepsilon}\Big(\frac{1}{r} - \cot(r)\Big) + \frac{\varphi_\varepsilon}{r}n\frac{2+b_\varepsilon}{1-b_\varepsilon}
\end{aligned}$$

Since $\dfrac{\varphi_\varepsilon}{r} = \dfrac{\varepsilon}{r}\displaystyle\int_1^{r/\varepsilon}\frac{dt}{\sqrt{t^{2(n-k-1)}-1}} \leqslant \dfrac{\varepsilon}{r}\displaystyle\int_1^{r/\varepsilon}\frac{dt}{\sqrt{t^2-1}}$ and $\frac{1}{x}\int_1^x \frac{dt}{\sqrt{t^2-1}} \sim_{+\infty} \frac{\ln x}{x}$, we get that $h_{1,\varepsilon}^\pm$ is bounded on M_ε^k, hence H_ε is bounded on M_ε. By the Lebesgue theorem we have $\|\mathrm{H}_\varepsilon - 1\|_1 \to 0$.

We now bound $\|\mathrm{B}_\varepsilon\|_q$ with $q = n-k$. The volume element at the neighbourhood of $M_\varepsilon^+ \cap M_\varepsilon^-$ is

$$dv_{g_\varepsilon} = (1 \pm \varphi_\varepsilon)^n (1 + (\frac{\varphi'_\varepsilon}{1 \pm \varphi_\varepsilon})^2)^{1/2} \sin^{n-k-1}(r) \cos^k(r) dv_{n-k-1} dv_k dr \tag{6.7}$$

where dv_{n-k-1} and dv_k are the canonical volume element of $\mathbb{S}^{n-k-1}$ and $\mathbb{S}^k$ respectively. By Lemma 6.1 and Equation (6.6), we have

$$|\mathrm{B}_\varepsilon|^q dv_{g_\varepsilon} = \frac{1}{(\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2)^{\frac{q}{2}}} \max\Big(\big|1 - \frac{\varphi'_\varepsilon}{1 \pm \varphi_\varepsilon} \cot r\big|, \big|1 + \frac{\varphi'_\varepsilon}{1 \pm \varphi_\varepsilon} \tan r\big|,$$
$$\big|1 + \frac{\varphi_\varepsilon'^2 + (n-k-1)(1 \pm \varphi_\varepsilon)(1 + \varphi_\varepsilon'^2)\varphi'_\varepsilon / r}{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}\big|\Big)\Big]^q dv_{g_\varepsilon}$$

Noting that $\frac{x}{\sqrt{1+x^2}} \leqslant \min(1, x)$, it is easy to see that, if we set $h_\varepsilon = \min(1, |\varphi'_\varepsilon|)$

$$\frac{\big|1 - \frac{\varphi'_\varepsilon}{1 \pm \varphi_\varepsilon} \cot r\big|}{\sqrt{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}} \leqslant \frac{1}{\sqrt{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}} + \frac{\frac{\varphi'_\varepsilon}{1 \pm \varphi_\varepsilon}}{\sqrt{\frac{\varphi_\varepsilon'^2}{(1 \pm \varphi_\varepsilon)^2} + 1}} \frac{\cot r}{1 \pm \varphi_\varepsilon}$$
$$\leqslant \frac{1}{1 - \varphi_\varepsilon} + \frac{h_\varepsilon \cot r}{(1 - \varphi_\varepsilon)^2} \leqslant 4\left(1 + \frac{h_\varepsilon}{r}\right)$$

Similarly for $r \in [\varepsilon, \pi/5 + \varepsilon]$ and ε small enough, we have

$$\frac{\big|1 + \frac{\varphi'_\varepsilon}{1 \pm \varphi_\varepsilon} \tan r\big|}{\sqrt{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}} \leqslant 4(1 + h_\varepsilon \tan r) \leqslant 8(1 + h_\varepsilon r) \leqslant 8\big(1 + \frac{h_\varepsilon}{r}\big)$$

And since $\varphi'_\varepsilon = 0$ for $r \geqslant \pi/5 + \varepsilon$, this inequality is also true for $r \in (\varepsilon, \pi/2]$. Moreover

$$\frac{1}{\sqrt{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}} \big|1 + \frac{\varphi_\varepsilon'^2 + (n-k-1)(1 \pm \varphi_\varepsilon)(1 + \varphi_\varepsilon'^2)\varphi'_\varepsilon / r}{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2}\big|$$
$$\leqslant \frac{1}{1 \pm \varphi_\varepsilon} + \frac{\varphi_\varepsilon'^2}{(\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2)^{3/2}} + \frac{n}{r} \frac{(1 \pm \varphi_\varepsilon)(1 + \varphi_\varepsilon'^2)}{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2} \frac{|\varphi'_\varepsilon|}{(\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2)^{1/2}}$$
$$\leqslant \frac{2}{1 \pm \varphi_\varepsilon} + \frac{n h_\varepsilon}{r(1 - \varphi_\varepsilon)} \frac{(1 \pm \varphi_\varepsilon)(1 + \varphi_\varepsilon'^2)}{\varphi_\varepsilon'^2 + (1 \pm \varphi_\varepsilon)^2} \leqslant \frac{2}{1 \pm \varphi_\varepsilon} + 2\frac{n h_\varepsilon}{r} \frac{(1 + \varphi_\varepsilon)^2}{(1 - \varphi_\varepsilon)^2}$$
$$\leqslant 2\big(2 + 9\frac{n h_\varepsilon}{r}\big)$$

It follows that

$$|\mathrm{B}_\varepsilon|^q dv_{g_\varepsilon} \leqslant C(n,k)\big(1 + \frac{h_\varepsilon}{r}\big)^q dv_{g_\varepsilon} \leqslant C(n,k)(r + h_\varepsilon)^q r^{-1}\big(1 + \frac{\varphi'_\varepsilon}{1 \pm \varphi_\varepsilon}\big) dv_{n-k-1} dv_k dr$$
$$\leqslant C(n,k) r^{-1} (r + h_\varepsilon)^q \left(1 + \frac{1}{\sqrt{(r/\varepsilon)^{2(n-k-1)} - 1}}\right) dv_{n-k-1} dv_k dr$$

Now

$$\int_{M_\varepsilon^k} |\mathrm{B}_\varepsilon|^q dv_{g_\varepsilon} \leqslant C(n,k)\Big(\int_\varepsilon^{2^{\frac{1}{2(n-k-1)}}\varepsilon} r^{-1}\Big(1+\frac{1}{\sqrt{(r/\varepsilon)^{2(n-k-1)}-1}}\Big)dr + \int_{2^{\frac{1}{2(n-k-1)}}\varepsilon}^{2a+\varepsilon} r^{n-k-1}\Big(1+\frac{1}{r\sqrt{(r/\varepsilon)^{2(n-k-1)}-1}}\Big)^q dr\Big)$$

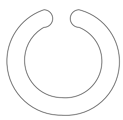
$$\leqslant C(n,k)\Big(\int_1^{2^{\frac{1}{2(n-k-1)}}} s^{-1}\Big(1+\frac{1}{\sqrt{s^{2(n-k-1)}-1}}\Big)ds + \int_{2^{\frac{1}{2(n-k-1)}}}^{2a/\varepsilon+1} s^{n-k-1}\big(\varepsilon+\frac{1}{s^q}\big)^q ds\Big)$$

Since $\varepsilon^{\frac{-1}{q}} \leqslant \frac{2a}{\varepsilon}+1$ for ε small enough we have

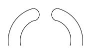
$$\int_{M_\varepsilon^k} |\mathrm{B}_\varepsilon|^q dv_{g_\varepsilon} \leqslant C(n,k)\Big(1+\int_{2^{\frac{1}{2(n-k-1)}}}^{\varepsilon^{\frac{-1}{q}}} \frac{2s^{n-k-1}}{s^{q^2}}ds + \int_{\varepsilon^{\frac{-1}{q}}}^{2a/\varepsilon+1} 2s^{n-k-1}\varepsilon^q ds\Big)$$
$$\leqslant C(n,k)\big(1+\varepsilon^{n-k-1}\big)$$

which remains bounded when $\varepsilon \to 0$. □

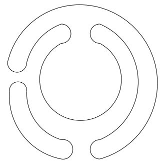
Since φ_ε is constant outside a neighborhood of $M_\varepsilon^+ \cap M_\varepsilon^-$ (given by a), M_ε^k is a smooth submanifold diffeomorphic to the sum of two spheres $\mathbb{S}^n$ along a (great) subsphere $\mathbb{S}^k \subset \mathbb{S}^n$.



If we denote $\tilde{M}_\varepsilon^k$ one connected component of the points of M_ε^k corresponding to $r \leqslant 3a$, we get some pieces of hypersurfaces



that can be glued together along pieces of spheres of constant curvature to get a smooth submanifold M_ε, diffeomorphic to p spheres $\mathbb{S}^n$ glued each other along l subspheres S_i, and with curvature satisfying the bounds of Theorem 1.4 (when all the subspheres have dimension 0) or of the remark before Theorem 1.3.



Since the surgeries are performed along subsets of capacity zero, the manifold constructed have a spectrum close to the spectrum of p disjoints spheres of radius close to 1 (i.e. close to the spectrum of the standard $\mathbb{S}^n$ with all multiplicities multiplied by p). More precisely, we set $\eta \in [2\varepsilon, \frac{\pi}{20}]$, and for any subsphere S_i, we set $N_{i,\eta,\varepsilon}$ the tubular neighborhood of radius η of the submanifold $\tilde{S}_i = M_{\varepsilon,i}^+ \cap M_{\varepsilon,i}^-$ in the local parametrization of M_ε given by the map $\Phi_{\varphi_{\varepsilon,i}}$ associated to the subsphere S_i. We have $M_\varepsilon = \Omega_{1,\eta,\varepsilon} \cup \cdots \cup \Omega_{p,\eta,\varepsilon} \cup N_{1,\eta,\varepsilon} \cup \cdots \cup N_{l,\eta,\varepsilon}$ where $\Omega_{i,\eta,\varepsilon}$ are the connected component of $M \setminus \cup_i N_{i,\eta,\varepsilon}$. The $\Omega_{i,\eta,\varepsilon}$ are diffeomorphic to some $S_{i,\eta}$ (which does not depend on ε

and η) open set of $\mathbb{S}^n$ which are complements of neighborhoods of subspheres of dimension less than $n-2$ and radius η, endowed with metrics which converge in $\mathcal{C}^1$ topology to standard metrics of curvature 1 on $S_{i,\eta}$. Indeed, φ_ε converge to 0 in topology $\mathcal{C}^2$ on $[r_{\varepsilon,\eta}^{i,\pm},\frac{\pi}{2}]$, where $\int_\varepsilon^{r_{\varepsilon,\eta}^{i,\pm}}\sqrt{(1\pm\varphi_{\varepsilon,i})^2+(\varphi_{\varepsilon,i}')^2}=\eta$ since it converges in $\mathcal{C}^1$ topology on any compact of $[\varepsilon,\frac{\pi}{2}]$ and since we have

$$\eta\geqslant\int_\varepsilon^{r_{\varepsilon,\eta}^{i,\pm}}(1-b_{i,\varepsilon})\,dt=(r_{\varepsilon,\eta}^{i,\pm}-\varepsilon)(1-b_{i,\varepsilon})$$

$$\eta\leqslant\int_\varepsilon^{r_{\varepsilon,\eta}^{i,\pm}}(1+b_{i,\varepsilon})\,dt+\int_\varepsilon^{r_{\varepsilon,\eta}^{i,\pm}}\frac{dt}{\sqrt{(\frac{t}{\varepsilon})^{2(n-k-1)}-1}}=(r_{\varepsilon,\eta}^{i,\pm}-\varepsilon)(1+b_{i,\varepsilon})$$
$$+\varepsilon\int_1^{+\infty}\frac{dt}{\sqrt{t^{2(n-k-1)}-1}}$$

so $r_{\varepsilon,\eta}^{\pm}\to\eta$ when $\varepsilon\to0$. So the spectrum of $\cup_i\Omega_{i,\eta,\varepsilon}\subset M_\varepsilon$ for the Dirichlet problem converges to the spectrum of $\amalg_iS_{i,\eta}\subset\amalg_i\mathbb{S}^n$ for the Dirichlet problem as ε tends to 0 (by the min-max principle). Since any subsphere of codimension at least 2 has zero capacity in $\mathbb{S}^n$, we have that the spectrum of $\amalg_iS_{i,\eta}\subset\amalg_i\mathbb{S}^n$ for the Dirichlet problem converges to the spectrum of $\amalg_i\mathbb{S}^n$ when η tends to 0 (see for instance [6] or adapt what follows). Since the spectrum of $\amalg_i\mathbb{S}^n$ is the spectrum of $\mathbb{S}^n$ with all multiplicities multiplied by p, by diagonal extraction we infer the existence of two sequences (ε_m) and (η_m) such that $\varepsilon_m\to0$, $\eta_m\to0$ and the spectrum of $\cup_i\Omega_{i,\eta_m,\varepsilon_m}\subset M_{\varepsilon_m}$ for the Dirichlet problem converges to the spectrum of $\mathbb{S}^n$ with all multiplicities multiplied by p. Finally, note that $\lambda_l(M_\varepsilon)\leqslant\lambda_l(\cup_i\Omega_{i,2\eta,\varepsilon})$ for any l by the Dirichlet principle.

On the other hand, by using functions of the distance to the $\tilde{S}_i$ we can easily construct on M_ε a function ψ_ε with value in $[0,1]$, support in $\cup_i\Omega_{i,\eta,\varepsilon}$, equal to 1 on $\cup_i\Omega_{i,2\eta,\varepsilon}$ and whose gradient satisfies $|d\psi_\varepsilon|_{g_\varepsilon}\leqslant\frac{2}{\eta}$. It readily follows that

$$\|1-\psi_\varepsilon^2\|_1+\|d\psi_\varepsilon\|_2^2\leqslant(1+\frac{4}{\eta^2})\sum_i\frac{\operatorname{Vol}N_{i,2\eta,\varepsilon}}{\operatorname{Vol}M_\varepsilon}$$

To estimate $\sum_i\operatorname{Vol}N_{i,2\eta,\varepsilon}$, note that $N_{i,2\eta,\varepsilon}$ corresponds to the set of points with $r^{i,\pm}\leqslant r_{\varepsilon,2\eta}^{i,\pm}$ in the parametrization of M_ε given by $\Phi_{\varphi_{\varepsilon,i}}$ at the neighborhood of $\tilde{S}_i$, where, as above, $r_{\varepsilon,2\eta}^{i,\pm}$ is given by

$$\int_\varepsilon^{r_{\varepsilon,2\eta}^{i,\pm}}\sqrt{(1\pm\varphi_{\epsilon,i})^2+(\varphi_{\epsilon,i}')^2}=2\eta$$

hence satisfies $\frac{1}{2}(r_{\varepsilon,2\eta}^{i,\pm}-\varepsilon)\leqslant2\eta$ (since we have $1-\varphi_{\varepsilon,i}\geqslant\frac{1}{2}$). By formula 6.7, we have

$$\operatorname{Vol}N_{i,2\eta,\varepsilon}\leqslant C(n)\int_\varepsilon^{r_\eta^-}(1-\varphi_{\varepsilon,i})^{n-1}\sqrt{(1-\varphi_{\varepsilon,i})^2+(\varphi_{\varepsilon,i}')^2}t^{n-k-1}dt$$
$$+C(n)\int_\varepsilon^{r_\eta^+}(1+\varphi_{\varepsilon,i})^{n-1}\sqrt{(1+\varphi_{\varepsilon,i})^2+(\varphi_{\varepsilon,i}')^2}t^{n-k-1}dt$$
$$\leqslant C(n)(4\eta+\varepsilon)^{n-k-1}\eta\leqslant C(n,k)\eta^{n-k}$$

where we have used that $\varphi_{\varepsilon,i} \leqslant 2$ and $2\varepsilon \leqslant \eta$. We then have

$$\|1-\psi_\varepsilon^2\|_1 + \|d\psi_\varepsilon\|_2^2 \leqslant C(n,k,l,p)\eta^{n-k}$$

To end the proof of the fact that M_{ε_m} has a spectrum close to that of $\cup_i \Omega_{i,\eta_m,\varepsilon_m}$ we need the following proposition, whose proof is a classical Moser iteration (we use the Simon and Michael Sobolev Inequality).

Proposition 6.4. *For any $q>n$ there exists a constant $C(q,n)$ so that if (M^n,g) is any Riemannian manifold isometrically immersed in $\mathbb{R}^{n+1}$ and $E_N = \langle f_0,\cdots,f_N\rangle$ is the space spanned by the eigenfunctions associated to $\lambda_0 \leqslant \cdots \leqslant \lambda_N$, then for any $f\in E_N$ we have*

$$\|f\|_\infty \leqslant C(q,n)\Big((v_M)^{1/n}(\lambda_N^{1/2} + \|\mathrm{H}\|_q)\Big)^\gamma \|f\|_2$$

where $\gamma = \frac{1}{2}\frac{qn}{q-n}$.

Since we already know that $\lambda_\sigma(M_{\varepsilon_m}) \leqslant \lambda_\sigma(\cup_i\Omega_{i,\eta_m,\varepsilon_m}) \to \lambda_{E(\sigma/p)}(\mathbb{S}^n)$ for any σ when $m\to\infty$, we infer that for any N there exists $m=m(N)$ large enough such that on M_{ε_m} and for any $f\in E_N$, we have (with $q=2n$ and since $\|\mathrm{H}\|_\infty \leqslant C(n)$)

$$\|f\|_\infty \leqslant C(p,N,n)\|f\|_2$$

By the previous estimates, if we set

$$L_{\varepsilon_m} : f\in E_N \mapsto \psi_{\varepsilon_m} f \in \mathrm{H}_0^1(\cup_i\Omega_{i,\eta_m,\varepsilon_m})$$

then we have

$$\|f\|_2^2 \geqslant \|L_{\varepsilon_m}(f)\|_2^2 \geqslant \|f\|_2^2 - \|f\|_\infty^2\|1-\psi_{\varepsilon_m}^2\|_1 \geqslant \|f\|_2^2\big(1 - C(k,l,p,N,n)\eta_m^{n-k}\big)$$

and

$$\begin{aligned}\|dL_{\varepsilon_m}(f)\|_2^2 &= \frac{1}{\operatorname{Vol} M_{\varepsilon_m}}\int_{M_{\varepsilon_m}} |f d\psi_{\varepsilon_m} + \psi_{\varepsilon_m} df|^2 \\ &\leqslant (1+h)\|df\|_2^2 + (1+\frac{1}{h})\frac{1}{\operatorname{Vol} M_{\varepsilon_m}}\int_{M_{\varepsilon_m}} f^2|d\psi_{\varepsilon_m}|^2 \\ &\leqslant (1+h)\|df\|_2^2 + (1+\frac{1}{h})C(k,l,p,N,n)\|f\|_2^2\eta_m^{n-k}\end{aligned}$$

for any $h>0$. We set $h=\eta_m^{\frac{n-k}{2}}$. For $m=m(k,l,p,N,n)$ large enough, $L_{\varepsilon_m} : E_N \to \mathrm{H}_0^1(\cup_i\Omega_{i,\eta_m,\varepsilon_m})$ is injective and for any $f\in E_N$, we have

$$\frac{\|dL_{\varepsilon_m}(f)\|_2^2}{\|L_{\varepsilon_m}(f)\|_2^2} \leqslant (1 + C(k,l,p,N,n)\eta_m^{\frac{n-k}{2}})\frac{\|df\|_2^2}{\|f\|_2^2} + C(k,l,p,N,n)\eta_m^{\frac{n-k}{2}}$$

By the min-max principle, we infer that for any $\sigma \leqslant N$, we have

$$\lambda_\sigma(M_{\varepsilon_m}) \leqslant \lambda_\sigma(\cup_i\Omega_{i,\eta_m,\varepsilon_m}) \leqslant (1 + C(k,l,p,N,n)\eta_m^{\frac{n-k}{2}})\lambda_\sigma(M_{\varepsilon_m}) + C(k,l,p,N,n)\eta_m^{\frac{n-k}{2}}$$

Since $\lambda_\sigma(\cup_i\Omega_{i,\eta_M,\varepsilon_m}) \to \lambda_{E(\sigma/p)}(\mathbb{S}^n)$, this gives that $\lambda_\sigma(M_{\varepsilon_m}) \to \lambda_{E(\sigma/p)}(\mathbb{S}^n)$ for any $\sigma \leqslant N$. By diagonal extraction we get the sequence of manifolds (M_j) of Theorem 1.4.

To construct the sequence of Theorem 1.5, we consider the sequence of embedded submanifolds (M_j) of Theorem 1.4 for $p=2$, $k=n-2$ and $l=1$. Each element of the sequence admits a covering of degree d given by $y\mapsto y^d$ in the local charts associated to the maps Φ. We endow these covering with the pulled back metrics. Arguing as above, we get that the spectrum of the new sequence converge to the spectrum of two disjoint copies of

$$\big(\mathbb{S}^1 \times \mathbb{S}^{n-2} \times [0, \frac{\pi}{2}], dr^2 + d^2 \sin^2 r g_{\mathbb{S}^1} + \cos^2 r g_{\mathbb{S}^{n-2}}\big).$$

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Appendix A. Proof of Lemma 6.1

Let $(u, v, h) \in T_x S_\varepsilon$ and put $w = d(\Phi_\varphi)_x(u, v, h) \in T_q X_\varphi$ where $S_\varepsilon = \mathbb{S}^{n-k-1} \times \mathbb{S}^k \times I_\varepsilon$. An easy computation shows that

$$\begin{aligned} w &= (1+\varphi(r))((\sin r)u + (\cos r)v) \\ &\quad + \varphi'(r)((\sin r)y + (\cos r)z)h + (1+\varphi(r))((\cos r)y - (\sin r)z)h \end{aligned} \tag{A.1}$$

We set

$$\tilde{N}_q = -\varphi'(r)((\cos r)y - (\sin r)z) + (1+\varphi(r))((\sin r)y + (\cos r)z)$$

and $N_q = \dfrac{\tilde{N}_q}{(\varphi'^2 + (1+\varphi)^2)^{1/2}}$ is a unit normal vector field on X_φ. Then we have

$$\begin{aligned} \mathrm{B}_q(\varphi)(w,w) &= \langle \nabla^0_w N, w \rangle = \big(\varphi'^2 + (1+\varphi)^2\big)^{-1/2} \langle \nabla^0_w \tilde{N}, w \rangle \\ &= \big(\varphi'^2 + (1+\varphi)^2\big)^{-1/2} \Big\langle \sum_{i=1}^{n+1} w(\tilde{N}^i)\partial_i, w \Big\rangle \end{aligned} \tag{A.2}$$

where $(\partial_i)_{1\leqslant i\leqslant n+1}$ is the canonical basis of $\mathbb{R}^{n+1}$. A straightforward computation shows that

$$\sum_{i=1}^{n+1} w(\tilde{N}^i)\partial_i = -\varphi'(r)((\cos r)u-(\sin r)v)+(1+\varphi(r))((\sin r)u+(\cos r)v)$$
$$-\varphi''(r)((\cos r)y-(\sin r)z)h+2\varphi'(r)((\sin r)y+(\cos r)z)h$$
$$+(1+\varphi(r))((\cos r)y-(\sin r)z)h$$

Reporting this in (A.2) and using (A.1) we get

$$\mathrm{B}_q(\varphi)((u,v,h),(u,v,h))=\frac{1}{\sqrt{\varphi'^2+(1+\varphi)^2}}\Big[-\varphi'(r)\big(1+\varphi(r)\big)\sin r\cos r(|u|^2-|v|^2)$$
$$+(1+\varphi(r))^2(\sin^2 r|u|^2+\cos^2 r|v|^2)-\big(1+\varphi(r)\big)\varphi''(r)h^2+2\varphi'^2(r)h^2+(1+\varphi(r))^2h^2\Big]$$

Now let $(u_i)_{1\leqslant i\leqslant n-k-1}$ and $(v_i)_{1\leqslant i\leqslant k}$ be orthonormal bases of respectively $\mathbb{S}^{n-k-1}$ at y and $\mathbb{S}^k$ at z. We set $g=\Phi_\varphi^\star can$ and $\xi=(0,0,1)$, then we have

$$g(u_i,u_j)=(1+\varphi(r))^2\sin^2 r\delta_{ij},\quad g(v_i,v_j)=(1+\varphi(r))^2\cos^2 r\delta_{ij},\quad g(u_i,v_j)=0,$$
$$g(\xi,\xi)=\varphi'^2+(1+\varphi)^2,\qquad g(u_i,\xi)=g(v_j,\xi)=0.$$

Now setting $\tilde{u}_i=d(\Phi_\varphi)_x(u_i)$, $\tilde{v}_i=d(\Phi_\varphi)_x(u_i)$ and $\tilde{\xi}=d(\Phi_\varphi)_x(\xi)$, the relation above allows us to compute the trace and norm

$$|\mathrm{B}_q(\varphi)|=\max\Big(\max_i\frac{|\mathrm{B}_q(\varphi)(\tilde{u}_i,\tilde{u}_i)|}{g(u_i,u_i)},\max_j\frac{|\mathrm{B}_q(\varphi)(\tilde{v}_j,\tilde{v}_j)|}{g(v_j,v_j)},\frac{|\mathrm{B}_q(\varphi)(\tilde{\xi},\tilde{\xi})|}{g(\xi,\xi)}\Big)$$
$$=\frac{1}{\sqrt{\varphi'^2+(1+\varphi)^2}}\max\Big(\big|1-\frac{\varphi'}{1+\varphi}\cot r\big|,\big|1+\frac{\varphi'}{1+\varphi}\tan r\big|,\big|1+\frac{(\varphi')^2-(1+\varphi)\varphi''}{\varphi'^2+(1+\varphi)^2}\big|\Big)$$

of the second fundamental form.

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