ON THE BOUNDARY OF ALMOST ISOPERIMETRIC DOMAINS

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Abstract. We prove that finite perimeter subsets of $\mathbb{R}^{n+1}$ with small isoperimetric deficit have boundary Hausdorff-close to a sphere up to a subset of small measure. We also refine this closeness under some additional a priori integral curvature bounds. As an application, we answer a question raised by B. Colbois concerning the almost extremal hypersurfaces for Chavel's inequality.

1. Introduction

In all the paper, $B_x(r)$ and $S_x(r)$ denote respectively the Euclidean ball and sphere with center x and radius r in $\mathbb{R}^{n+1}$. We also set $\mathbb{B}^k$ the unit ball centred at 0 in $\mathbb{R}^k$ and $\mathbb{S}^{k-1}$ the unit sphere centred at 0 in $\mathbb{R}^k$.

For any Borel set Ω of $\mathbb{R}^{n+1}$, we denote $|\Omega|$ its Lebesgue measure, $P(\Omega)$ its perimeter (see definition in section 2) and $I(\Omega) = \frac{P(\Omega)}{|\Omega|^{\frac{n}{n+1}}}$ its isoperimetric ratio. Then it satisfies the isoperimetric inequality

$$I(\Omega) \geqslant I\big(\mathbb{B}^{n+1}\big) \tag{1.1}$$

with equality if and only if Ω is a Euclidean ball up to set of Lebesgue measure 0. To study the stability of the isoperimetric inequality, we denote by

$$\delta(\Omega) := \frac{I(\Omega)}{I\big(\mathbb{B}^{n+1}\big)} - 1$$

the isoperimetric deficit of a Borel set Ω of finite perimeter and address the following question:

"How far from a ball are almost isoperimetric domains?(i.e. with small $\delta(\Omega)$)"

More quantitatively, by stability of the isoperimetric inequality, we understand the validity of an inequality of the form

$$\text{"distance" from } \Omega \text{ to some ball } \leqslant C\delta(\Omega)^{1/\alpha} \text{ for a given category of } \Omega \subset \mathbb{R}^{n+1}$$

where the "distance" need to be defined and where C and α are some positive universal constants. Many authors have studied this stability problem with the Fraenkel asymmetry $\mathcal{A}(\Omega)$ as distance function. We recall that

$$\mathcal{A}(\Omega) := \inf_{x \in \mathbb{R}^{n+1}} \frac{|\Omega \Delta B_x(R_\Omega)|}{|\Omega|} \text{ for } \Omega \subset \mathbb{R}^{n+1}$$

Date: 6th September 2018.
2000 *Mathematics Subject Classification.* 53A07, 53C21.
Key words and phrases. Isoperimetric inequality.

where R_Ω is given by $R_\Omega^{n+1}|\mathbb{B}^{n+1}| = |\Omega|$ and $U\Delta V = (U\setminus V)\cup(V\setminus U)$. So the isoperimetric inequality is said stable with respect to the Fraenkel asymmetry if there exists $C(n), \alpha(n) > 0$ such that

$$\mathcal{A}(\Omega) \leqslant C(n)\delta(\Omega)^{1/\alpha(n)} \tag{1.2}$$

holds for a given category of domains $\Omega \subset \mathbb{R}^{n+1}$.

Such inequalities were first obtained for domains of $\mathbb{R}^2$ by Bernstein ([4]) and Bonnesen ([5]). The first result in higher dimension was due to Fuglede ([11]) for convex domains. Without convexity assumption, the main contributions are due to Hall, Haymann, Weitsman (see [17] and [18]) who established this inequality with $\alpha(n) = 4$, and later to Fusco, Maggi and Pratelli who proved this inequality with the sharp exponent $\alpha(n) = 2$ in [15] (see also the paper of Figalli, Magelli and Pratelli ([10]) or [8] and [14] for other proofs of this last result).

To get more precise informations on the geometry of almost isoperimetric domains than a small Fraenkel asymmetry, we can take as "distance" function the Hausdorff distance. The first result in that direction was the following inequality proved by Bonnesen ([5]) for convex curves and by Fuglede ([12]) in the general case: if $\partial\Omega$ is a C^1-piecewise closed curve there exists a Euclidean circle $\mathcal{C}$ such that

$$16\pi d_H^2(\mathcal{C}, \partial\Omega) \leqslant P(\Omega)^2 - 4\pi|\Omega| \leqslant 4\pi|\Omega|\delta(\Omega)\big(2 + \delta(\Omega)\big) \tag{1.3}$$

where d_H denotes the Hausdorff distance. Note that assuming $\delta(\Omega) \leqslant 1$ and using the isodiametric inequality $|\Omega| \leqslant \frac{\pi}{4}(\mathrm{d}iam\,\Omega)^2$, we infer the following inequality

$$\frac{d_H(\mathcal{C}, \partial\Omega)}{\mathrm{d}iam\,\Omega} \leqslant \frac{\sqrt{3\pi}}{4}\delta(\Omega)^{\frac{1}{2}} \tag{1.4}$$

However, this result is false for more general domains in $\mathbb{R}^2$, especially non connected one (consider for instance the disjoint union of a large ball and a tiny one far from each other). Moreover, in higher dimension $n \geqslant 2$, even for connected smooth domains, we cannot expect to control the Hausdorff distance from $\partial\Omega$ to a sphere by the isoperimetric deficit alone, as proves the sets obtained by adding or subtracting to a ball a thin tubular neighbourhood of a Euclidean subset of dimension not larger than $n-1$ (see for instance [6]). So to generalize this kind of stability result in higher dimension, it is necessary to assume additional informations on the geometry of the domains we consider. In [11] Fuglede proved that if $n \geqslant 3$, Ω is a convex set and $\delta(\Omega)$ small enough then

$$\inf_{x\in\mathbb{R}^{n+1}} \frac{d_H(\Omega, B_x(R_\Omega))}{R_\Omega} \leqslant C(n)\delta(\Omega)^{\frac{2}{n+2}}. \tag{1.5}$$

($\delta(\Omega)^{\frac{2}{n+2}}$ is replaced by $\sqrt{\delta(\Omega)}$ for $n = 1$ and by $(\delta(\Omega)\log[1/\delta(\Omega)])^{1/2}$ for $n = 2$). Note that since Ω is convex, $\partial\Omega$ is also close to a sphere of radius R_Ω. Actually, Fuglede deals with more general sets called nearly spherical domains and this Fuglede's result has been generalized by Fusco, Gelli and Pisante ([13]) for any set of finite perimeter satisfying an interior cone condition.

In this paper, we prove generalizations of inequalities (1.4) and (1.5) to any smooth domain (even nonconvex) with integral control on the mean curvature of the boundary. We even get a weak Hausdorff control for almost isoperimetric domains that need no additional assumption on their boundary.

1.1. **No assumption on the boundary.** Let $\mathcal{F}(\Omega)$ be the reduced boundary of Ω (see the section 2 for the definition). When Ω is a smooth domain, we have $\mathcal{F}(\Omega) = \partial\Omega$.

Theorem 1. *Let Ω be a set of $\mathbb{R}^{n+1}$ with finite perimeter with $\delta(\Omega) \leqslant \frac{1}{C(n)}$. There exists $x_\Omega \in \mathbb{R}^{n+1}$ and $A(\Omega) \subset \mathcal{F}(\Omega)$ such that*

(1) $\dfrac{\mathcal{H}^n(\mathcal{F}(\Omega) \setminus A(\Omega))}{P(\Omega)} \leqslant C(n)\delta(\Omega)^{\frac{1}{4}}$,

(2) $\dfrac{d_H\big(A(\Omega), S_{x_\Omega}(R_\Omega)\big)}{R_\Omega} \leqslant C(n)\delta(\Omega)^{\beta(n)}$.

Here $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure and $\beta(n) = \min(\frac{1}{4n}, \frac{1}{8})$.

In other words, the boundary $\mathcal{F}(\Omega)$ is Hausdorff close to a sphere up to a set of small measure. Note that we have

$$A(\Omega) = \mathcal{F}(\Omega) \cap A_{\delta(\Omega)^{\frac{1}{4}}}$$

where for any $\eta > 0$ we set $A_\eta = \big\{x \in \mathbb{R}^{n+1} / \big||x - x_\Omega| - R_\Omega\big| \leqslant R_\Omega \eta\big\}$.

Remark 1. *Note that the sets of the previous theorem also satisfy*

(1) $\dfrac{|\Omega \Delta B_{x_\Omega}(R_\Omega)|}{|\Omega|} \leqslant C(n)\delta(\Omega)^{1/2}$,

(2) $\dfrac{d_H\big(\Omega \cap B_{x_\Omega}(R_\Omega), B_{x_\Omega}(R_\Omega)\big)}{R_\Omega} \leqslant C(n)\delta(\Omega)^{\frac{1}{2(n+1)}}$ *(see the end of the section 2).*

In other words $\Omega \cap B_{x_\Omega}(R_\Omega)$ is Hausdorff close to the ball $B_{x_\Omega}(R_\Omega)$ up to a set of small measure, which is a weak generalization of inequality (1.5).

Remark 2. *When $n = 1$ or Ω convex Theorem 1 easily implies earlier results à la Bonnesen* [5] *and Fuglede* [11] *but with non optimal power $\beta(n)$.*

Remark 3. *See also Theorem 8 in Section 4.4 that is a reformulation of Theorem 1 in term of Preiss distance between the normalized measures associated to $\mathcal{F}(\Omega)$ and $S_{x_\Omega}(R_\Omega)$.*

To get informations on the smooth domain Ω itself, and not up to a set of small measure, additional assumptions are required. A reasonable assumption is an integral control on the mean curvature H. In the sequel, for any $p \geqslant 1$, we define

$$\|f\|_p = \left(\frac{1}{P(\Omega)} \int_{\partial\Omega} |f|^p d\mathcal{H}^n\right)^{1/p} \text{ for any measurable } f : \partial\Omega \to \mathbb{R}.$$

Note that a upper bound on $\|\mathrm{H}\|_p$ with $p < n-1$ is not sufficient. Indeed, we can refer to examples constructed by the authors in [2, 3]: by adding small tubular neighbourhood of well chosen trees to $B_0(1)$, we get a set almost isoperimetric domains on which $\|\mathrm{H}\|_p$ is uniformly bounded for any $p < n-1$ and that is dense for the Hausdorff distance among all the closed set of $\mathbb{R}^{n+1}$ that contain $B_0(1)$.

1.2. **Upper bound on $\|\mathrm{H}\|_{n-1}$.**

Theorem 2. *Let Ω be an open set with a smooth boundary $\partial\Omega$, finite perimeter and $\delta(\Omega) \leqslant \frac{1}{C(n)}$. There exists a subset T of $\mathbb{R}^{n+1}$ which satisfies whose* 1*-dimensional Hausdorff measure satisfies*

(1) $\mathcal{H}^1(T) \leqslant C(n)R_\Omega \int_{\partial\Omega\setminus A_{\delta(\Omega)^{1/4}}} |\mathrm{H}|^{n-1} d\mathcal{H}^n$,

(2) $d_H\big(\partial\Omega, S_{x_\Omega}(R_\Omega)\cup T\big) \leqslant C(n)R_\Omega\delta(\Omega)^{\beta(n)}$,

(3) the set $A_{\delta(\Omega)^{1/4}}\cup T$ *has at most* $N+1$ *connected components,*

where $\mathcal{H}^1(T)$ *denotes the 1-dimensional Hausdorff measure of* T *and* N *is the number of the connected components of* $\partial\Omega$ *that do not intercept* $A_{\delta(\Omega)^{\frac14}}$.

Note that by Theorem 1 at least one connected component of $\partial\Omega$ intercepts $A_{\delta(\Omega)^{\frac14}}$ and so if $\partial\Omega$ is connected then we have $N=0$ and $A_{\delta(\Omega)^{1/4}}\cup T$ is connected. Moreover note that for $n=1$ we recover Fuglede's result (1.3) for C^2-piecewise closed curves.

The case $N=\infty$ in Theorem 2 is trivial since the sets obtained by the union of a sphere and infinitely numebrable many points are dense for the Hausdorff distance among all the closed sets containing $S_{x_\Omega}(R_\Omega)$.

Similarly to the case of curves, Theorem 2 is quite optimal as prove examples given by a domain $\Omega_\varepsilon = \big[B_0(R)\setminus\bigcup_i T_{i,\varepsilon}\big]\cup\bigcup_j T_{j,\varepsilon}$, where (T_i) and (T_j) are some families of Euclidean trees and the $T_{i,\varepsilon}$ denotes the ε-tubular neighbourhood of T_i. In these examples, the integral of $|\mathrm{H}|^{n-1}$ on $\partial\Omega_\varepsilon\setminus A_{\delta(\Omega)^{\frac14}}$ will converge, up to a multiplicative constant $C(n)$, to the sum of the length of the trees as ε tends to 0.

We refer to Theorem 10 of Section 5.2 for a generalization of inequality (1.5) similar to Theorem 2.

1.3. **Bound on $\|\mathrm{H}\|_p$ with $p>n-1$.** If we assume some upper bound on the L^p norm of $|\mathrm{H}|$ with $p>n-1$, then combining Theorem 2 and Lemma 2 with Hölder inequality readily gives the following improved result.

Theorem 3. *Let* $p\geqslant n-1$ *and* Ω *be an open set with a smooth boundary* $\partial\Omega$*, finite perimeter and* $\delta(\Omega)\leqslant\frac{1}{C(n)}$*. Let* $(\partial\Omega_i)_{i\in I}$ *be the connected components of* $\partial\Omega$ *that do not intercept* $A_{\delta(\Omega)^{\frac14}}$*. For any* $i\in I$*, there exists* $x_i\in\partial\Omega_i$ *such that*

$$d_H\big(\partial\Omega, S_{x_\Omega}(R_\Omega)\cup\bigcup_{i\in I}\{x_i\}\big)\leqslant C(n,p)R_\Omega\Big[\delta(\Omega)^{\beta(n)}+\delta(\Omega)^{\frac{p-n+1}{4p}}\big(P(\Omega)^{\frac1n}\|\mathrm{H}\|_p\big)^{n-1}\Big]$$

Moreover if $p\geqslant n$ *and if* H *is* L^p*-integrable then* I *is finite and we have*

$$\mathrm{Card}(I)\leqslant C(n,p)P(\Omega)\|\mathrm{H}\|_p^n\delta(\Omega)^{\frac{p-n}{4p}}. \tag{1.6}$$

Remark 4. *We will see in the proof that the above estimates are more precise since as in Theorem 2, we can replaced* $\|\mathrm{H}\|_p$ *by* $\Big(\frac{1}{P(\Omega)}\int_{\partial\Omega\setminus A_{\delta(\Omega)^{1/4}}}|\mathrm{H}|^p d\mathcal{H}^n\Big)^{\frac1p}$.

Remark 5. *If we assume that* $\partial\Omega$ *is connected, then Theorem 3 implies that* $\partial\Omega$ *is Hausdorff close to a sphere. If* $\partial\Omega$ *has* N *connected component, the it asserts that* $\partial\Omega$ *is Hausdorff close to a sphere union a finite set with at most* $N-1$ *points.*

Note that in the case $p<n$ we can not control the cardinal of I in terms of $\|\mathrm{H}\|_p$. Indeed, consider the sequence of domains Ω_k obtained by the union of $\mathbb{B}^{n+1}$ and k balls $B_{x_i}(r_i/k)$ where x_i are some points satisfying for instance $\mathrm{d}ist(0,x_i)=2i$. If $\sum_{i\geqslant0} r_i^{n-p}$ is

convergent then $\lim_{k\longrightarrow\infty}\delta(\Omega_k)=0$ and $P(\Omega_k)\|\mathrm{H}_k\|_p$ (where H_k denotes the mean curvature of $\partial\Omega_k$) remains bounded when $Card(I)$ tends to infinity.

Here also we refer to Theorem 11 of Section 5.2 for a version of Theorem 3 generalizing inequality 1.5.

1.4. **Bound on $\|\mathrm{H}\|_p$ with $p>n$.** When $p>n$, it follows from 1.6 that if $\delta(\Omega)$ is small enough then $I=\emptyset$ and $\partial\Omega$ is Hausdorff close to $S_{x_\Omega}(R_\Omega)$. More precisely we have that

Theorem 4. *Let $p>n$. There exists a constant $C(n,p)>0$ such that if Ω is an open set with smooth boundary $\partial\Omega$ such that $\mathcal{H}^n(\partial\Omega)\|\mathrm{H}\|_p^n\leqslant K$ and $\delta(\Omega)\leqslant\frac{1}{C(n,p,K)}$ then $\partial\Omega$ is diffeomorphic and quasi-isometric to $S_{x_\Omega}(R_\Omega)$. Moreover the Lipschitz distance d_L satisfies*

$$d_L(\partial\Omega,\mathbb{S}_{x_\Omega}(R_\Omega))\leqslant C(n,p)\delta(\Omega)^{\frac{2(p-n)}{p(n+2)-2n}}$$

for any $n\geqslant 2$ and the Hausdorff distance

$$d_H(\partial\Omega,S_{x_\Omega}(R_\Omega))\leqslant C(n,p,K)R_\Omega\delta(\Omega)^{\frac{2p-n}{2p-2n+np}}$$

when $n\geqslant 3$ and

$$d_H(\partial\Omega,S_{x_\Omega}(R_\Omega))\leqslant C(p,K)R_\Omega(-\delta(\Omega)\ln\delta(\Omega))^{\frac{1}{2}}$$

when $n=2$.

Remark 6. *Actually, under the assumption of the previous theorem, we show that $\partial\Omega=\{\varphi(w)w, w\in S_{x_\Omega}(R_\Omega)\}$, where $\varphi\in W^{1,\infty}(S_{x_\Omega}(R_\Omega))\cap W^{2,p}(S_{x_\Omega}(R_\Omega))$, with $\|d\varphi\|_\infty\leqslant\frac{C(n,p,K)}{R_\Omega}\delta(\Omega)^{\frac{p-n}{2p-2n+pn}}$ and $\|\nabla d\varphi\|_p\leqslant C(n,p,K)/R_\Omega^2$. So Ω is a nearly spherical domain in the sense of Fuglede and is the graph over $S_{x_\Omega}(R_\Omega)$ of a $C^{1,1-\frac{n}{p}}(S_{x_\Omega}(R_\Omega))$ function. It implies that any sequence of domain $(\Omega_k)_k$ with $\delta(\Omega_k)\to 0$ and $\mathcal{H}^n(\partial\Omega_k)\|\mathrm{H}_k\|_p^n\leqslant K$ converges to $S_{x_\Omega}(R_\Omega)$ in $C^{1,q}$ topology for any $q<1-\frac{p}{n}$.*

Remark 7. *The estimates on d_L and d_H in Theorem 4 are sharp with respect of the exponent of $\delta(\Omega)$ involved, but not for what concern the constant $C(n,p,K)$. We show it by constructing example at the end of section 6. Note moreover that in the case $p=\infty$ we recover the same exponent as in the convex case.*

1.5. **Stability of the Chavel Inequality.** In the last part of this paper we answer a question asked by Bruno Colbois concerning the almost extremal hypersurfaces for the Chavel's inequality: if we set λ_1^Σ the first nonzero eigenvalue of a compact hypersurface Σ that bounds a domain Ω, Chavel's inequality says that

$$\lambda_1^\Sigma\leqslant\frac{n}{(n+1)^2}\Big(\frac{\mathcal{H}^n(\Sigma)}{|\Omega|}\Big)^2 \tag{1.7}$$

Moreover equality holds if and only if Σ is a geodesic sphere. Now if we denote by $\gamma(\Omega)$ the deficit of Chavel's inequality (i.e. $\gamma(\Omega)=\frac{n}{\lambda_1^\Sigma(n+1)^2}\Big(\frac{\mathcal{H}^n(\Sigma)}{|\Omega|}\Big)^2-1)$, we have

Theorem 5. *Let Σ be an embedded compact hypersurface bounding a domain Ω in $\mathbb{R}^{n+1}$. If $\gamma(\Omega)\leqslant\frac{1}{C(n)}$ then we have*

$$\delta(\Omega)\leqslant C(n)\gamma(\Omega)^{1/2}$$

Consequently, $\delta(\Omega)$ can be replaced by $\gamma(\Omega)^{\frac{1}{2}}$ in all the previous theorems, which gives the stability of the Chavel's inequality. Note moreover that γ small implies readily that $\Sigma=\partial\Omega$ is connected and so we have $N=0$ and $I=\emptyset$ is this case.

2. Preliminaries

2.1. **Definitions.** First let us introduce some notations and recall some definitions used in the paper. Throughout the paper we adopt the notation that $C(n,k,p,\cdots)$ is function which depends on p, q, n, $\cdots$. It eases the exposition to disregard the explicit nature of these functions. The convenience of this notation is that even though C might change from line to line in a calculation it still maintains these basic features.

Given two bounded sets A and B the Hausdorff distance between A and B is defined by

$$d_H(A,B)=\inf\{\varepsilon\mid A\subset B_\varepsilon \text{ and } B\subset A_\varepsilon\}$$

where for any subset E, $E_\varepsilon=\{x\in\mathbb{R}^{n+1}\mid \mathrm{d}ist(x,E)\leqslant\varepsilon\}$.

Let μ be a $\mathbb{R}^{n+1}$-valued Borel measure on $\mathbb{R}^{n+1}$. Its total variation is the nonnegative measure $|\mu|$ defined on any Borel set Ω by

$$|\mu|(\Omega):=\sup\left\{\sum_{k\in\mathbb{N}}\|\mu(\Omega_k)\| \ \mid\ \Omega_i\cap\Omega_j=\emptyset\ ,\ \bigcup_{k\in\mathbb{N}}\Omega_k\subset\Omega\right\}$$

Given a Borel set Ω of $\mathbb{R}^{n+1}$, we say that Ω is of finite perimeter if the distributional gradient $D\chi_\Omega$ of its characteristic function is a $\mathbb{R}^{n+1}$-valued Borel measure such that $|D\chi_\Omega|(\mathbb{R}^{n+1})<\infty$. The perimeter of Ω is then $P(\Omega):=|D\chi_\Omega|(\mathbb{R}^{n+1})$. Of course if Ω is a bounded domain with a smooth boundary we have $P(\Omega)=\mathcal{H}^n(\partial\Omega)$. For any set Ω with finite perimeter, we have $P(\Omega)=\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)$ where $\mathcal{F}(\Omega)$ is the reduced boundary defined by

$$\mathcal{F}(\Omega):=\left\{x\in\mathbb{R}^{n+1}\ \mid\ \forall r>0\ ,\ |D\chi_\Omega|(B_x(r))>0 \text{ and } \lim_{r\longrightarrow 0^+}\frac{D\chi_\Omega(B_x(r))}{|D\chi_\Omega|(B_x(r))}\in\mathbb{S}^n\right\}$$

Moreover Federer (see [1]) proved that $\mathcal{F}(\Omega)\subset\partial^\star\Omega$ where $\partial^\star\Omega$ is the essential boundary of Ω defined by

$$\partial^\star\Omega:=\mathbb{R}^{n+1}\setminus(\Omega^0\cup\Omega^1)$$

where $\Omega^t:=\left\{x\in\mathbb{R}^{n+1}\ \mid\ \lim\limits_{r\longrightarrow 0}\dfrac{|\Omega\cap B_x(r)|}{|B_x(r)|}=t\right\}$.

2.2. **Some results proved in** [10]**.** Now we gather some results proved in [10] about almost isoperimetric sets, that will be used in this paper.

Theorem 6. *(*A. Figalli, F. Maggi, A. Pratelli, [10]*) Let Ω be a set of $\mathbb{R}^{n+1}$ of finite perimeter, with $0<|\Omega|<\infty$ and $\delta(\Omega)\leqslant\min\left(1,\dfrac{k(n)^2}{8}\right)$ where $k(n):=\dfrac{2-2^{\frac{n}{n+1}}}{3}$. Then there exists a domain $G\subset\Omega$ such that*

(1) $0\leqslant|\Omega|-|G|\leqslant|\Omega\setminus G|\leqslant\frac{\delta(\Omega)}{k(n)}|\Omega|$,
(2) $P(G)\leqslant P(\Omega)$,
(3) $\delta(G)\leqslant\frac{3}{k(n)}\delta(\Omega)$,
(4) There exists a point $x_\Omega\in\mathbb{R}^{n+1}$ such that

$$\int_{\mathcal{F}(G)}\big||x-x_\Omega|-R_G\big|d\mathcal{H}^n\leqslant\frac{10(n+1)^3}{k(n)}|G|\delta(\Omega)^{1/2}$$

where X is the vector position of $\mathbb{R}^{n+1}$,
(5) $|G\Delta B_{x_\Omega}(R_G)|\leqslant\frac{20(n+1)^3}{k(n)}|G|\delta(\Omega)^{1/2}$.

The following property is important for our purpose and derive easily from [10], but since it is not proved nor stated in [10], we give a proof of it for sake of completeness.

Lemma 1. *There exists a constant $C(n) > 0$ such that under the assumptions and notations of the previous theorem, we have*

$$\big(1 - C(n)\delta(\Omega)\big)\mathcal{H}^n\big(\mathcal{F}(\Omega)\big) \leqslant \mathcal{H}^n\big(\mathcal{F}(\Omega) \cap \mathcal{F}(G)\big)$$

Proof. We reuse the notations of [10]. First of all, by the previous theorem, we have

$$\mathcal{H}^n\big(\mathcal{F}(G)\big) \geqslant I(\mathbb{B}^{n+1})|G|^{\frac{n}{n+1}} \geqslant (1 - \frac{\delta(\Omega)}{k(n)})^{\frac{n}{n+1}} I(\mathbb{B}^{n+1})|\Omega|^{\frac{n}{n+1}} \geqslant \frac{(1 - \frac{\delta(\Omega)}{k(n)})^{\frac{n}{n+1}}}{1 + \delta(\Omega)} \mathcal{H}^n\big(\mathcal{F}(\Omega)\big)$$

$$\geqslant \big(1 - C(n)\delta(\Omega)\big)\mathcal{H}^n\big(\mathcal{F}(\Omega)\big) \tag{2.1}$$

and by the construction made in [10], Ω is the disjoint union of G and a set F_∞ which satisfy

$$\mathcal{H}^n\big(\mathcal{F}(F_\infty)\big) \leqslant \big(1 + k(n)\big)\mathcal{H}^n\big(\mathcal{F}(\Omega) \cap \mathcal{F}(F_\infty)\big).$$

Then we have

$$\mathcal{H}^n(\mathcal{F}(\Omega)) = \mathcal{H}^n(\mathcal{F}(\Omega) \cap \mathcal{F}(G)) + \mathcal{H}^n(\mathcal{F}(\Omega) \cap \mathcal{F}(F_\infty))$$

and

$$\begin{aligned}
(1 + k(n))\mathcal{H}^n\big(\mathcal{F}(\Omega)\big) &+ (1 - k(n))\mathcal{H}^n\big(\mathcal{F}(G) \cap \mathcal{F}(\Omega)\big) \\
&= 2\mathcal{H}^n\big(\mathcal{F}(G) \cap \mathcal{F}(\Omega)\big) + (1 + k(n))\mathcal{H}^n\big(\mathcal{F}(\Omega) \cap \mathcal{F}(F_\infty)\big) \\
&\geqslant 2\mathcal{H}^n\big(\mathcal{F}(G) \cap \mathcal{F}(\Omega)\big) + \mathcal{H}^n\big(\mathcal{F}(F_\infty)\big) \\
&= \mathcal{H}^n\big(\mathcal{F}(G)\big) + \mathcal{H}^n\big(\mathcal{F}(\Omega)\big) \\
&\geqslant \big(2 - C(n)\delta(\Omega)\big)\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)
\end{aligned}$$

where we have used Inequality (2.1). We infer that

$$\mathcal{H}^n\big(\mathcal{F}(G) \cap \mathcal{F}(\Omega)\big) \geqslant \big(1 - C(n)\delta(\Omega)\big)\mathcal{H}^n\big(\mathcal{F}(\Omega)\big).$$

□

2.3. **Proof of remark 1.** Up to a translation we can assume that $x_\Omega = 0$ and from the Theorem 6 we have :

$$|G\Delta B_0(R_G)| \leqslant C(n)|G|\delta(\Omega)^{1/2}$$

Since

$$\Omega\Delta B_0(R_\Omega) \subset (\Omega\Delta G) \cup (G\Delta B_0(R_G)) \cup (B_0(R_G)\Delta B_0(R_\Omega))$$

we deduce immediately that $|\Omega\Delta B_0(R_\Omega)| \leqslant C(n)|\Omega|\delta(\Omega)^{1/2}$ which proves the point (1) of the remark.

On the other hand let $x \in B_0(R_\Omega)$ and $R_\Omega \geqslant \varepsilon > 0$ such that

$$B_x(\varepsilon) \cap (\Omega \cap B_0(R_\Omega)) = \emptyset.$$

We then have $B_x(\varepsilon) \cap B_0(R_\Omega) \subset \Omega\Delta B_0(R_\Omega)$ and since $B_x(\varepsilon) \cap B_0(R_\Omega)$ contains the ball with diameter $\mathbb{R}x \cap B_x(\varepsilon) \cap B_0(R_\Omega)$ whose length is larger than ε, we get

$$\frac{1}{C(n)}\varepsilon^{n+1} \leqslant |\Omega\Delta B_0(R_\Omega)| \leqslant C(n)|\Omega|\delta(\Omega)^{1/2}$$

Since $\Omega \cap B_0(R_\Omega) \subset B_0(R_\Omega)$, it suffices to get the point (2) that is

$$d_H(\Omega \cap B_0(R_\Omega), B_0(R_\Omega)) \leqslant C(n)|\Omega|^{\frac{1}{n+1}}\delta(\Omega)^{\frac{1}{2(n+1)}}$$

3. Concentration in a tubular neighborhood of a sphere

The main result of this section is the following theorem :

Lemma 2. *Let Ω be a set of $\mathbb{R}^{n+1}$ with finite perimeter and let*

$$A_\eta := \Big\{x \in \mathbb{R}^{n+1} / \big||x - x_\Omega| - R_\Omega\big| \leqslant R_\Omega \eta\Big\}.$$

If $\delta(\Omega) \leqslant \frac{1}{C(n)}$ then for any $\alpha \in (0, \frac{1}{2})$, we have

$$\mathcal{H}^n\big(\mathcal{F}(\Omega) \setminus A_{\delta(\Omega)^\alpha}\big) \leqslant C(n) P(\Omega) \delta(\Omega)^{\frac{1}{2}-\alpha}$$

Proof. By inequalities (4) and (1) of Theorem 6, we get

$$\begin{aligned}
\mathcal{H}^n\big(\mathcal{F}(G) \setminus A_\eta\big) &\leqslant \frac{1}{R_\Omega \eta} \int_{\mathcal{F}(G)\setminus A_\eta} \big||x - x_\Omega| - R_\Omega\big| d\mathcal{H}^n \\
&\leqslant \frac{1}{R_\Omega \eta} \int_{\mathcal{F}(G)\setminus A_\eta} \big||x - x_\Omega| - R_G\big| d\mathcal{H}^n + \frac{|R_\Omega - R_G|}{\eta R_\Omega} \mathcal{H}^n\big(\mathcal{F}(G)\big) \\
&\leqslant \frac{1}{R_\Omega \eta} \frac{10(n+1)^3}{k(n)} |G| \delta(\Omega)^{1/2} + \frac{|R_\Omega - R_G|}{\eta R_\Omega} \mathcal{H}^n\big(\mathcal{F}(G)\big) \\
&\leqslant C(n) \frac{|\Omega|^{\frac{n}{n+1}}}{\eta} \sqrt{\delta(\Omega)}
\end{aligned}$$

where we have used that $\mathcal{H}^n\big(\mathcal{F}(G)\big) = P(G) = C(n)\big(1 + \delta(G)\big)|G|^{\frac{n}{n+1}} \leqslant C(n)|\Omega|^{\frac{n}{n+1}}$ (by Theorem 6 (1) and (3)).

Now by Lemma 1 and Inequality (2) of Theorem 6, we have

$$\mathcal{H}^n\Big(\mathcal{F}(G) \setminus \big(\mathcal{F}(\Omega) \cap \mathcal{F}(G)\big)\Big) \leqslant \mathcal{H}^n\Big(\mathcal{F}(\Omega) \setminus \big(\mathcal{F}(\Omega) \cap \mathcal{F}(G)\big)\Big) \leqslant C(n)\delta(\Omega)\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)$$

And so

$$\begin{aligned}
\mathcal{H}^n\big(\mathcal{F}(\Omega) \setminus A_\eta\big) &\leqslant \mathcal{H}^n\Big(\big(\mathcal{F}(\Omega) \cap \mathcal{F}(G)\big) \setminus A_\eta\Big) + \mathcal{H}^n\Big(\mathcal{F}(\Omega) \setminus \big(\mathcal{F}(\Omega) \cap \mathcal{F}(G)\big)\Big) \\
&\leqslant \mathcal{H}^n\big(\mathcal{F}(G) \setminus A_\eta\big) + C(n)\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)\delta(\Omega) \\
&\leqslant \frac{C(n)}{\eta} |\Omega|^{\frac{n}{n+1}} \delta(\Omega)^{1/2} + C(n)|\Omega|^{\frac{n}{n+1}} \delta(\Omega) \\
&\leqslant C(n) \left(\frac{1}{\eta}\delta(\Omega)^{1/2} + \delta(\Omega)\right) |\Omega|^{\frac{n}{n+1}}
\end{aligned}$$

Then choosing $\eta := \delta(\Omega)^\alpha$ and $\delta(\Omega) \leqslant 1$ we get the desired result. □

4. Domains with small deficit without assumption on the boundary

In this section, we gather the proofs of several geometric-measure properties of the boundary of almost isoperimetric domains.

4.1. **Proof of Theorem 1.** By Lemma 2, we have Inequality (1) with $A(\Omega) = \mathcal{F}(\Omega) \cap A_{\delta(\Omega)^{1/4}}$. Inequality (2) will be a consequence of the following density theorem.

Theorem 7. *Let Ω be a set of $\mathbb{R}^{n+1}$ with finite perimeter and $\rho \in \big[C(n)\delta(\Omega)^{\frac{1}{8}} R_\Omega, R_\Omega\big]$. Then for any $x \in S_{x_\Omega}(R_\Omega)$ we have*

$$\Big|\frac{\mathcal{H}^n\big(B_x(\rho) \cap S_{x_\Omega}(R_\Omega)\big)}{R_\Omega^n \mathrm{Vol}\, \mathbb{S}^n} - \frac{\mathcal{H}^n\big(\mathcal{F}(\Omega) \cap B_x(\rho)\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)}\Big| \leqslant C(n)\delta(\Omega)^{\frac{1}{4}}$$

Let $x \in S_{x_\Omega}(R_\Omega)$ and $\rho = C_1(n)R_\Omega\delta(\Omega)^{\beta(n)}$ with $\beta(n) := \min(\frac{1}{8}, \frac{1}{4n})$. Then for $C_1(n)$ large enough and $\delta(\Omega) \leqslant (1/C_1(n))^{1/\beta(n)}$, $\rho \in \big[C(n)\delta(\Omega)^{\frac{1}{8}}R_\Omega, R_\Omega\big]$ and the estimate of Theorem 7 combined to the fact that there exists a constant $C_2(n)$ such that

$$\frac{\mathcal{H}^n\big(B_x(\rho) \cap S_{x_\Omega}(R_\Omega)\big)}{R_\Omega^n \mathrm{Vol}\,\mathbb{S}^n} \geqslant C_2(n)\left(\frac{\rho}{R_\Omega}\right)^n$$

gives for $C_1(n)$ great enough

$$\begin{aligned}\frac{\mathcal{H}^n\big(\mathcal{F}(\Omega) \cap B_x(\rho)\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)} &\geqslant -C(n)\delta(\Omega)^{1/4} + C_2(n)\left(\frac{\rho}{R_\Omega}\right)^n \\ &\geqslant (-C(n) + C_2(n)C_1(n)^n)\delta(\Omega)^{\min(\frac{n}{8},\frac{1}{4})} \\ &\geqslant C_3(n)\delta(\Omega)^{\min(\frac{n}{8},\frac{1}{4})}\end{aligned}$$

Moreover from the lemma 2 we have

$$\begin{aligned}\mathcal{H}^n(\mathcal{F}(\Omega) \cap A_{\delta(\Omega)^{1/4}} \cap B_x(\rho)) &\geqslant \mathcal{H}^n\big(\mathcal{F}(\Omega) \cap B_x(\rho)\big) - C(n)P(\Omega)\delta(\Omega)^{1/4} \\ &\geqslant C_3(n)P(\Omega)\delta(\Omega)^{\min(\frac{n}{8},\frac{1}{4})} - C(n)P(\Omega)\delta(\Omega)^{1/4} \\ &\geqslant (C_3(n) - C(n))P(\Omega)\delta(\Omega)^{\min(\frac{n}{8},\frac{1}{4})} \\ &\geqslant C_4(n)P(\Omega)\delta(\Omega)^{\min(\frac{n}{8},\frac{1}{4})}\end{aligned}$$

If $C_1(n)$ is large enough. This implies that $\mathcal{F}(\Omega) \cap A_{\delta(\Omega)^{1/4}} \cap B_x(\rho)$ has non-zero measure, hence is non-empty for any $x \in S_{x_\Omega}(R_\Omega)$. Putting $A(\Omega) = \mathcal{F}(\Omega) \cap A_{\delta(\Omega)^{1/4}}$, we obtain that $d_H(A(\Omega), S_{x_\Omega}(R_\Omega)) \leqslant \rho$ for $C_1(n)$ large enough which gives the fact (2) of Theorem 1. □

Note that Theorem 7 implies that density of $\mathcal{F}(\Omega)$ near each point of $\mathbb{S}^n(R_\Omega)$ converges to 1 at any fixed scale. It will be combined with Allard's regularity theorem in Section 6 to prove Theorem 4.

4.2. **Proof of Theorem 7.** It will be a consequence of the following fundamental proposition.

Proposition 1. *Let Ω be a set of $\mathbb{R}^{n+1}$ of finite perimeter, with $\delta(\Omega) \leqslant \frac{1}{C(n)}$. For any $f \in C_c^1(\mathbb{R}^{n+1})$, we have*

$$\Big|\frac{1}{P(\Omega)}\int_{\mathcal{F}(\Omega)} f d\mathcal{H}^n - \frac{1}{R_\Omega^n \mathrm{Vol}\,\mathbb{S}^n}\int_{S_{x_\Omega}(R_\Omega)} f d\mathcal{H}^n\Big| \leqslant C(n)\big(\|f\|_\infty + \|df\|_\infty\big)\delta(\Omega)^{\frac{1}{2}},$$

where we denote $\|df\|_\infty = \sup_y |d_y f(y)|$.

Proof. Up to translation, we can assume that $x_\Omega = 0$ subsequently. Let $G \subset \Omega$ be the subset associated to Ω in Theorem 6. We note X the field $X_x = x$ for any $x \in \Omega$. We have $\mathrm{d}iv_x(fX) = df_x(X_x) + (n+1)f(x)$ and so we get

$$\begin{aligned}
&\Big|\int_{\mathcal{F}(G)} f\langle X,\nu_G\rangle d\mathcal{H}^n - R_G\int_{S_0(R_G)} f d\mathcal{H}^n\Big| \\
&\qquad = \Big|\int_{\mathcal{F}(G)} f\langle X,\nu_G\rangle d\mathcal{H}^n - \int_{S_0(R_G)} f\langle X,\nu_{S_0(R_G)}\rangle d\mathcal{H}^n\Big| \\
&\qquad = \Big|\int_G \mathrm{d}iv(fX) d\mathcal{H}^{n+1} - \int_{B_0(R_G)} \mathrm{d}iv(fX) d\mathcal{H}^{n+1}\Big| \\
&\qquad \leqslant (n+1)\Big|\int_G f d\mathcal{H}^{n+1} - \int_{B_0(R_G)} f d\mathcal{H}^{n+1}\Big| \\
&\qquad\quad + \Big|\int_G df(X) d\mathcal{H}^{n+1} - \int_{B_0(R_G)} df(X) d\mathcal{H}^{n+1}\Big| \\
&\qquad \leqslant (n+1)\big(\|f\|_\infty + \|df\|_\infty\big)|G\Delta B_0(R_G)| \\
&\qquad \leqslant \frac{20(n+1)^4}{k(n)}|G|\big(\|f\|_\infty + \|df\|_\infty\big)\delta(\Omega)^{1/2} \qquad (4.1)
\end{aligned}$$

Where we have used Inequality (5) of Theorem 6. Now we have

$$\begin{aligned}
&\int_{\mathcal{F}(G)} \big|f(R_G - \langle X,\nu_G\rangle)\big| d\mathcal{H}^n \\
&\leqslant \|f\|_\infty \int_{\mathcal{F}(G)} \big|R_G - |X|\big| d\mathcal{H}^n + \|f\|_\infty \int_{\mathcal{F}(G)} \big||X| - \langle X,\nu_G\rangle\big| d\mathcal{H}^n \\
&= \|f\|_\infty \int_{\mathcal{F}(G)} \big|R_G - |X|\big| d\mathcal{H}^n + \|f\|_\infty \int_{\mathcal{F}(G)} |X| d\mathcal{H}^n - \|f\|_\infty \int_{\mathcal{F}(G)} \langle X,\nu_G\rangle d\mathcal{H}^n \\
&\leqslant 2\|f\|_\infty \int_{\mathcal{F}(G)} \big|R_G - |X|\big| d\mathcal{H}^n + R_G\|f\|_\infty \mathcal{H}^n\big(\mathcal{F}(G)\big) - \|f\|_\infty \int_G \mathrm{d}iv(X) d\mathcal{H}^{n+1} \\
&\leqslant \frac{20(n+1)^3}{k(n)}|G|\|f\|_\infty \delta(\Omega)^{1/2} + \|f\|_\infty R_G \mathcal{H}^n\big(\mathcal{F}(G)\big) - \|f\|_\infty (n+1)|G|
\end{aligned}$$

Now a straightforward computation shows that $\delta(G) = \frac{R_G\mathcal{H}^n(\mathcal{F}(G))}{(n+1)|G|} - 1$. Consequently

$$\int_{\mathcal{F}(G)} \big|f(R_G - \langle X,\nu_G\rangle)\big| d\mathcal{H}^n \leqslant \|f\|_\infty C(n)|G|\delta(\Omega)^{1/2} \qquad (4.2)$$

Combining Inequalities (4.1) and (4.2) gives

$$\begin{aligned}
\frac{1}{P(\Omega)}\Big|\int_{\mathcal{F}(G)} f d\mathcal{H}^n - \int_{\mathbb{S}^n(R_G)} f d\mathcal{H}^n\Big| &\leqslant C(n)\frac{|G|}{P(\Omega)R_G}\big(\|f\|_\infty + \|df\|_\infty\big)\delta(\Omega)^{\frac{1}{2}} \\
&\leqslant C(n)\big(\|f\|_\infty + \|df\|_\infty\big)\delta(\Omega)^{\frac{1}{2}} \qquad (4.3)
\end{aligned}$$

Where we have used Inequality (1) of Theorem 6 to get

$$\frac{|G|}{P(\Omega)R_G} \leqslant C(n)\frac{|G|^{\frac{n}{n+1}}}{P(\Omega)} \leqslant C(n)\frac{|\Omega|^{\frac{n}{n+1}}}{P(\Omega)} \leqslant C(n). \qquad (4.4)$$

We have

$$\left|\frac{1}{P(\Omega)}\int_{\mathcal{F}(\Omega)} f d\mathcal{H}^n - \frac{1}{R_\Omega^n \mathrm{V}ol\,\mathbb{S}^n}\int_{\mathbb{S}^n(R_\Omega)} f d\mathcal{H}^n\right| \leqslant A_1 + A_2 + A_3 + A_4$$

with

$$A_1 = \frac{1}{P(\Omega)}\left|\int_{\mathcal{F}(\Omega)} f d\mathcal{H}^n - \int_{\mathcal{F}(G)} f d\mathcal{H}^n\right|$$
$$A_2 = \frac{1}{P(\Omega)}\left|\int_{\mathcal{F}(G)} f d\mathcal{H}^n - \int_{\mathbb{S}^n(R_G)} f d\mathcal{H}^n\right|$$
$$A_3 = \frac{1}{P(\Omega)}\left|\int_{\mathbb{S}^n(R_G)} f d\mathcal{H}^n - \int_{\mathbb{S}^n(R_\Omega)} f d\mathcal{H}^n\right|$$
$$A_4 = \Big|\frac{R_\Omega^n \mathrm{V}ol\,\mathbb{S}^n}{P(\Omega)} - 1\Big|\|f\|_\infty \leqslant C(n)\delta(\Omega)\|f\|_\infty$$

Note that A_2 is controlled by Inequality (4.3). Let us now estimate A_1. By Lemma 1 we have

$$\begin{aligned}
A_1 &= \frac{1}{P(\Omega)}\Big|\int_{\mathcal{F}(\Omega)\setminus(\mathcal{F}(\Omega)\cap\mathcal{F}(G))} f d\mathcal{H}^n - \int_{\mathcal{F}(G)\setminus(\mathcal{F}(G)\cap\mathcal{F}(\Omega))} f d\mathcal{H}^n\Big| \\
&\leqslant \|f\|_\infty C(n)\frac{|\Omega|^{\frac{n}{n+1}}}{P(\Omega)}\delta(\Omega) \leqslant \|f\|_\infty C(n)\delta(\Omega) \\
A_3 &\leqslant \frac{1}{P(\Omega)}\int_{\mathbb{S}^n}\big|R_G^n f(R_G u) - R_\Omega^n f(R_\Omega u)\big| d\mathcal{H}^n \\
&\leqslant \frac{1}{P(\Omega)}\int_{\mathbb{S}^n}\big|(R_G^n - R_\Omega^n) f(R_G u)\big| + R_\Omega^n\big|f(R_\Omega u) - f(R_G u)\big| d\mathcal{H}^n \\
&\leqslant C(n)\frac{|\Omega|^{\frac{n}{n+1}} - |G|^{\frac{n}{n+1}}}{P(\Omega)}\|f\|_\infty + \frac{R_\Omega^n}{P(\Omega)R_G}\int_{\mathbb{S}^n}\int_{R_G}^{R_\Omega}|d_{tu}f(tu)|\,dt\,du \\
&\leqslant C(n)(\|f\|_\infty + \|df\|_\infty)\delta(\Omega)
\end{aligned}$$

where once again we have used the estimates of Theorem 6. □

Proof of the theorem 7: Up to translation, we can assume that $x_\Omega = 0$. Let $\rho \leqslant R_\Omega$. By Lemma 2, we have

$$\Big|\frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap B_x(\rho)\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)} - \frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap B_x(\rho)\cap A_{\delta(\Omega)^{\frac{1}{4}}}\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)}\Big| \leqslant C(n)\delta(\Omega)^{\frac{1}{4}} \tag{4.5}$$

We set $\eta = \delta(\Omega)^{\frac{1}{4}} \leqslant \frac{1}{2}$ and $\varphi : [0, +\infty) \to [0, 1]$ be a C^1 function with compact support in $(0, 2R_\Omega)$, $\frac{2}{R_\Omega}$-Lipschitz and such that $\varphi(t) = 1$ on $\big[R_\Omega(1-\eta), R_\Omega(1+\eta)\big]$. For any function $v \in C^1\big(S_0(R_\Omega)\big)$, we set $f(x) = \varphi(|x|)v(\frac{R_\Omega x}{|x|})$. Then $|df_x(x)| \leqslant 4\|v\|_\infty$ and applying Proposition 1 to f, we get

$$\Big|\frac{1}{P(\Omega)}\int_{\mathcal{F}(\Omega)} f d\mathcal{H}^n - \frac{1}{R_\Omega^n \mathrm{V}ol\,\mathbb{S}^n}\int_{S_0(R_\Omega)} v d\mathcal{H}^n\Big| \leqslant C(n)\|v\|_\infty\delta(\Omega)^{\frac{1}{2}} \tag{4.6}$$

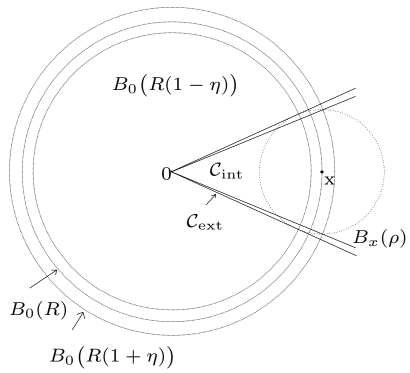

FIGURE 1

Let $x\in S_0(R_\Omega)$ and v_r be the characteristic function of the geodesic ball of center x and radius r in $S_0(R_\Omega)$. By convolution, we can approximate v_r in $L^1(S_0(R_\Omega))$ by C^1 functions u_k such that $\|u_k\|_\infty\leqslant 1$. Applying Inequality (4.6) to $v=u_k$ and letting k tends to ∞, we get

$$\Big|\frac{1}{P(\Omega)}\int_{\mathcal{F}(\Omega)} f_r d\mathcal{H}^n-\frac{\mathcal{H}^n(\mathcal{C}_{r/R_\Omega}\cap S_0(R_\Omega))}{R_\Omega^n \mathrm{V}ol\,\mathbb{S}^n}\Big|\leqslant C(n)\delta(\Omega)^{\frac12} \tag{4.7}$$

where $f_r=\varphi_{R_\Omega}(\|x\|)v_r(\frac{R_\Omega x}{\|x\|})$ and where $\mathcal{C}_\alpha=\big\{y\in\mathbb{R}^{n+1}\setminus\{0\}/\,\langle\frac{y}{\|y\|},\frac{x}{\|x\|}\rangle\geqslant\cos\alpha\big\}$. Now, since $\|f_r\|_\infty\leqslant 1$, Lemma 2 gives us

$$\frac{1}{P(\Omega)}\Big|\int_{\mathcal{F}(\Omega)} f_r d\mathcal{H}^n-\int_{\mathcal{F}(\Omega)\cap A_{\delta(\Omega)^{\frac14}}} f_r d\mathcal{H}^n\Big|\leqslant C(n)\delta(\Omega)^{\frac14} \tag{4.8}$$

By construction of f_r, we have

$$\int_{\mathcal{F}(\Omega)\cap A_{\delta(\Omega)^{\frac14}}} f_r d\mathcal{H}^n=\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap A_{\delta(\Omega)^{\frac14}}\cap\mathcal{C}_{r/R_\Omega}\big) \tag{4.9}$$

Combining Inequalities (4.7), (4.8) and (4.9), we get

$$\Big|\frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap A_{\delta(\Omega)^{\frac14}}\cap\mathcal{C}_{r/R_\Omega}\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)}-\frac{\mathcal{H}^n\big(\mathcal{C}_{r/R_\Omega}\cap\mathbb{S}_0(R_\Omega)\big)}{R_\Omega^n\mathrm{V}ol\,\mathbb{S}^n}\Big|\leqslant C(n)\delta(\Omega)^{\frac14} \tag{4.10}$$

We now assume that $\delta(\Omega)^{\frac14}\leqslant\frac{\rho^2}{2R_\Omega^2}$. The following angles

$$\alpha_{\rm ext}=\arccos\Big(\frac{1+(1-\delta(\Omega)^{\frac14})^2-\frac{\rho^2}{R_\Omega^2}}{2(1-\delta(\Omega)^{\frac14})}\Big)\quad\text{and}\quad\alpha_{\rm int}=\arccos\Big(\frac{1+(1+\delta(\Omega)^{\frac14})^2-\frac{\rho^2}{R_\Omega^2}}{2(1+\delta(\Omega)^{\frac14})}\Big)$$

satisfy the following property (see figure 4.2)

$$\mathcal{C}_{int}\cap A_{\delta(\Omega)^{\frac14}}\subset B_x(\rho)\cap A_{\delta(\Omega)^{\frac14}}\subset\mathcal{C}_{\rm ext}\cap A_{\delta(\Omega)^{\frac14}},$$

where we have set $C_{\text{int}} = C_{\alpha_{int}}$ and $C_{\text{ext}} = C_{\alpha_{\text{ext}}}$, so we get the following inequalities

$$\begin{aligned}
\frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap B_x(\rho)\cap A_{\delta(\Omega)^{\frac14}}\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)} &\geqslant \frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap \mathcal{C}_{int}\cap A_{\delta(\Omega)^{\frac14}}\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)}\\
&\geqslant \frac{\mathcal{H}^n\big(\mathcal{C}_{\text{int}}\cap \mathbb{S}_0(R_\Omega)\big)}{R_\Omega^n Vol\,\mathbb{S}^n} - C(n)\delta(\Omega)^{\frac14}\\
\frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap B_x(\rho)\cap A_{\delta(\Omega)^{\frac14}}\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)} &\leqslant \frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap \mathcal{C}_{\text{ext}}\cap A_{\delta(\Omega)^{\frac14}}\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)}\\
&\leqslant \frac{\mathcal{H}^n\big(\mathcal{C}_{\text{ext}}\cap \mathbb{S}_0(R_\Omega)\big)}{R_\Omega^n Vol\,\mathbb{S}^n} + C(n)\delta(\Omega)^{\frac14}
\end{aligned}$$

Since we have $B_x(\rho)\cap S_0(R_\Omega) = \mathcal{C}_{\alpha_\rho}\cap S_0(R_\Omega)$ for $\alpha_\rho = \arccos(1-\frac{\rho^2}{2R_\Omega^2})$, we infer the estimate

$$\begin{aligned}
D = \Big|\frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap B_x(\rho)\cap A_{\delta(\Omega)^{\frac14}}\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)} - \frac{B_x(\rho)\cap S_0(R_\Omega)}{R_\Omega^n Vol\,\mathbb{S}^n}\Big|\\
\leqslant \frac{\mathcal{H}^n\big(\mathcal{C}_{\text{ext}}\cap S_0(R_\Omega)\big) - \mathcal{H}^n\big(\mathcal{C}_{\text{int}}\cap S_0(R_\Omega)\big)}{R_\Omega^n Vol\,\mathbb{S}^n} + C(n)\delta(\Omega)^{\frac14}
\end{aligned}$$

Now, by the Bishop's and Bishop-Gromov's theorems, we have

$$\mathcal{H}^n\big(\mathcal{C}_{\text{ext}}\cap S_0(R_\Omega)\big) = \mathcal{H}^n(B_x^{S_0(R_\Omega)}(R_\Omega\alpha_{\text{ext}})) \leqslant R_\Omega^n\alpha_{\text{ext}}^n Vol\,(\mathbb{B}^n) = \frac{Vol\,(\mathbb{S}^{n-1})}{n}R_\Omega^n\alpha_{\text{ext}}^n$$

and

$$\frac{\mathcal{H}^n(B_x^{S_0(R_\Omega)}(R_\Omega\alpha_{\text{int}}))}{\mathcal{H}^n(B_0(R_\Omega\alpha_{\text{int}}))} \geqslant \frac{\mathcal{H}^n(B_x^{S_0(R_\Omega)}(R_\Omega\alpha_{\text{ext}}))}{\mathcal{H}^n(B_0(R_\Omega\alpha_{\text{ext}}))}$$

that is

$$\frac{\mathcal{H}^n\big(\mathcal{C}_{\text{int}}\cap S_0(R_\Omega)\big)}{\mathcal{H}^n\big(\mathcal{C}_{\text{ext}}\cap S_0(R_\Omega)\big)} \geqslant \frac{\alpha_{\text{int}}^n}{\alpha_{\text{ext}}^n},$$

where $B_x^{S_0(R_\Omega)}(r)$ denotes the ball of center x and radius r in $S_0(R_\Omega)$. These inequalities give

$$\begin{aligned}
D &\leqslant \frac{1}{R_\Omega^n Vol\,(\mathbb{S}^n)}\left(1-\frac{\alpha_{\text{int}}^n}{\alpha_{\text{ext}}^n}\right)\frac{Vol\,(\mathbb{S}^{n-1})}{n}R_\Omega^n\alpha_{\text{ext}}^n + C(n)\delta(\Omega)^{\frac14}\\
&\leqslant \frac{Vol\,(\mathbb{S}^{n-1})}{Vol\,(\mathbb{S}^n)}\pi^{n-1}|\alpha_{\text{ext}}-\alpha_{\text{int}}| + C(n)\delta(\Omega)^{\frac14}
\end{aligned}$$

Since by assumption $\delta(\Omega)^{\frac14} \leqslant \frac{\rho^2}{2R_\Omega^2}$, we get $|\alpha_{\text{ext}}-\alpha_{\text{int}}| \leqslant C(n)\delta(\Omega)^{\frac14}$ which gives

$$D \leqslant C(n)\delta(\Omega)^{\frac14}$$

Finally, by Lemma 2, we have

$$\Big|\frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap B_x(\rho)\cap A_{\delta(\Omega)^{\frac14}}\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)} - \frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap B_x(\rho)\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)}\Big| \leqslant C(n)\delta(\Omega)^{\frac14}$$

which gives

$$\Big|\frac{\mathcal{H}^n\big(B_x(\rho)\cap S_0(R_\Omega)\big)}{R_\Omega^n \mathrm{Vol}\,\mathbb{S}^n}-\frac{\mathcal{H}^n\big(\mathcal{F}(\Omega)\cap B_x(\rho)\big)}{\mathcal{H}^n\big(\mathcal{F}(\Omega)\big)}\Big|\leqslant C(n)\delta(\Omega)^{\frac14}.$$

□

4.3. **A control of the unit normal to $\mathcal{F}(\Omega)$.** In this subsection, we prove a result that we will use latter. It gives a weak control of the oscillation of the tangent planes of $\mathcal{F}(\Omega)$. Note that another proof of this result is proposed in [14].

Lemma 3. *Let Ω be a set of finite perimeter such that $\delta(\Omega)\leqslant\frac{1}{C(n)}$. Then we have*

$$\int_{\mathcal{F}(\Omega)}\Big|\nu_\Omega-\frac{x}{|x|}\Big|^2 d\mathcal{H}^n\leqslant C(n)P(\Omega)\delta(\Omega)^{\frac12}$$

Proof. By Lemma 1 and the fact that $\nu_\Omega=\nu_G$ $\mathcal{H}^n$-almost everywhere in $\mathcal{F}(G)\cap\mathcal{F}(\Omega)$, we have

$$\begin{aligned}&\Big|\int_{\mathcal{F}(\Omega)}\Big|\nu_\Omega-\frac{x}{|x|}\Big|^2 d\mathcal{H}^n-\int_{\mathcal{F}(G)}\Big|\nu_G-\frac{x}{|x|}\Big|^2 d\mathcal{H}^n\Big|\\&\quad=\Big|\int_{\mathcal{F}(\Omega)\setminus\mathcal{F}(G)}\Big|\nu_\Omega-\frac{x}{|x|}\Big|^2 d\mathcal{H}^n-\int_{\mathcal{F}(G)\setminus\mathcal{F}(\Omega)}\Big|\nu_G-\frac{x}{|x|}\Big|^2 d\mathcal{H}^n\Big|\leqslant 4C(n)\delta(\Omega)P(\Omega)\end{aligned}$$

Now, we have

$$\int_{\mathcal{F}(G)}\Big|\nu_G-\frac{x}{|x|}\Big|^2 d\mathcal{H}^n=2\int_{\mathcal{F}(G)}\Big(1-\Big\langle\nu_G,\frac{x}{|x|}\Big\rangle\Big)d\mathcal{H}^n$$

and by inequality (4.2), we have that

$$\begin{aligned}\Big|\int_{\mathcal{F}(G)}\Big(1-\Big\langle\nu_G,\frac{x}{|x|}\Big\rangle\Big)d\mathcal{H}^n\Big|&\leqslant\Big|\int_{\mathcal{F}(G)}\Big(1-\Big\langle\nu_G,\frac{x}{R_\Omega}\Big\rangle\Big)d\mathcal{H}^n\Big|\\&\quad+\Big|\int_{\mathcal{F}(G)}\Big\langle\nu_G,\frac{x}{|x|}-\frac{x}{R_\Omega}\Big\rangle d\mathcal{H}^n\Big|\\&\leqslant C(n)R_G^n\delta(\Omega)^{\frac12}+\frac{1}{R_\Omega}\int_{\mathcal{F}(G)}\big||x|-R_\Omega\big|d\mathcal{H}^n\\&\leqslant C(n)R_G^n\delta(\Omega)^{\frac12}\leqslant C(n)P(G)\delta(\Omega)^{\frac12}\leqslant C(n)P(\Omega)\delta(\Omega)^{\frac12}\end{aligned}$$

where the last inequality comes from fact (4) of Theorem 6. □

4.4. **A stability result involving the Preiss distance.** First we recall the definition of the Preiss distance on Radon measures of $\mathbb{R}^{n+1}$.

Definition 1. *Let μ and ν be two Radon measures on $\mathbb{R}^{n+1}$, for any $i\in\mathbb{N}$, we set*

$$F_i(\mu,\nu)=\sup\Big\{\Big|\int f d\mu-\int f d\nu\Big|,\ \mathrm{spt}\,f\subset B_0(i),\ f\geqslant 0,\ \mathrm{Lip}\,f\leqslant 1\Big\}$$

and

$$d_P(\mu,\nu)=\sum_{i\in\mathbb{N}}\frac{1}{2^i}\min\big(1,F_i(\nu,\mu)\big)$$

it gives a distance on the Radon measure of $\mathbb{R}^{n+1}$ whose converging sequences are the weakly* converging sequences.

For almost isoperimetric domains we have a control on the boundary in term of Preiss distance

Theorem 8. *Let* Ω *be a set of* $\mathbb{R}^{n+1}$ *with finite perimeter. Then there exists* $x_\Omega \in \mathbb{R}^{n+1}$ *such that*

$$d_P\left(\frac{|D\chi_{B_{x_\Omega}(R_\Omega)}|}{R_\Omega^n \mathrm{Vol}\,\mathbb{S}^n}, \frac{|D\chi_\Omega|}{P(\Omega)}\right) \leqslant C(n)\sqrt{\delta(\Omega)} \tag{4.11}$$

where d_P *is the Preiss distance on Radon measures of* $\mathbb{R}^{n+1}$*.*

Proof. Note that if f has support in $B_0(i)$ and is 1-Lipschitz, then by convolution, it can be uniformly approximated by a sequence of 1-Lipschitz, C^1 and compactly supported functions (f_k). We then have $\lim\limits_k \|f_k\|_\infty = \|f\|_\infty \leqslant i$ and $\lim\limits_k |d_X f_k(X)| \leqslant \lim\limits_k \|df_k\|_\infty i \leqslant i$ and applying Proposition 1 to f_k and letting k tends to ∞ gives us

$$\Big|\frac{1}{P(\Omega)}\int_{\mathcal{F}(\Omega)} f d\mathcal{H}^n - \frac{1}{R_\Omega^n \mathrm{Vol}\,\mathbb{S}^n}\int_{S_{x_\Omega}(R_\Omega)} f d\mathcal{H}^n\Big| \leqslant 2iC(n)\delta(\Omega)^{\frac{1}{2}}$$

and so

$$F_i\Big(\frac{|D\chi_\Omega|}{P(\Omega)}, \frac{|D\chi_{B_{x_\Omega}(R_G)}|}{P(B_{x_\Omega}(R_G))}\Big) \leqslant 2iC(n)\sqrt{\delta(\Omega)}$$

Hence we get that if $\delta(\Omega) \leqslant \frac{1}{C(n)}$, then we have

$$d_P\left(\frac{|D\chi_\Omega|}{P(\Omega)}, \frac{|D\chi_{B_{x_\Omega}(R_G)}|}{P(B_{x_\Omega}(R_G))}\right) \leqslant C(n)\sqrt{\delta(\Omega)}$$

Since for any couple of measures μ, ν we have $d_P(\mu,\nu) \leqslant 2$, we infer that we can leave the condition $\delta(\Omega) \leqslant \frac{1}{C(n)}$ as soon as we consider a larger $C(n)$. $\square$

5. Domains with small deficit and $\|\mathrm{H}\|_p$ bounded in the case $p \leqslant n$

5.1. **Proof of Theorems 2 and 3.** These theorems are consequence of the following.

Theorem 9. *(*E. Aubry, J.-F. Grosjean, [3]*) There exists a (computable) constant* $C = C(m)$ *such that, for any compact submanifold* M^m *of* $\mathbb{R}^{n+1}$ *and any closed subset* $A \subset M$ *that intercepts any connected component of* M*, there exists a finite family* $(T_i)_{\in I}$ *of geodesic trees in* M *with* $A \cap T_i \neq \emptyset$ *for any* $i \in I$*,* $d_H\big(A \cup \bigcup\limits_{i\in I} T_i, M\big) \leqslant C\big(\mathrm{Vol}\,(M\setminus A)\big)^{\frac{1}{m}}$ *and* $\sum\limits_{i\in I}\mathcal{H}^1(T_i) \leqslant C^{m(m-1)}\int_{M\setminus A}|\mathrm{H}|^{m-1}$*.*

Remark 8. *Note that by construction the* $A \cup \bigcup\limits_{i\in I} T_i$ *has the same number of connected components than* A*.*

Proof Theorems 2 and 3 : We set $\partial_r\Omega$ the union of the connected components of $\partial\Omega$ that intercept $A_{\delta(\Omega)^{\frac{1}{4}}}$ and we apply Theorem 9 to the hypersurface $\partial_r\Omega$ and the set $A_0 = \partial\Omega \cap A_{\delta(\Omega)^{1/4}} = \partial_r\Omega \cap A_{\delta(\Omega)^{1/4}}$. We set T_0 the union of the trees given by the theorem. Then we get $\mathcal{H}^1(T_0)\leqslant C(n)\int_{\partial_r\Omega\setminus A_0}|\mathrm{H}|^{n-1}$, the set $A_{\delta(\Omega)^{1/4}}\cup T_0$ is connected and by the first point of Theorem 1 (or Lemma 2) and Theorem 9, we have

$$\begin{aligned}d_H(A_0\cup T_0,\partial_r\Omega)&\leqslant C(n)\mathcal{H}^n(\partial_r\Omega\setminus A_0)^{\frac{1}{n}}\leqslant C(n)\mathcal{H}^n(\partial\Omega\setminus A_{\delta(\Omega)^{1/4}})^{\frac{1}{n}}\\&\leqslant C(n)P(\Omega)^{1/n}\delta(\Omega)^{\frac{1}{4n}}\leqslant C(n)R_\Omega\delta(\Omega)^{\frac{1}{4n}}\end{aligned}$$

If we now apply Theorems 9 and Theorem 1 to each connected component C_i of $\partial\Omega\setminus\partial_r\Omega$ with $A_i=\{x_i\}\subset C_i$, we get a connected union of trees T_i such $d_H(T_i,C_i)\leqslant C(n)R_\Omega\delta(\Omega)^{\frac{1}{4n}}$ and $\mathcal{H}^1(T_i)\leqslant C(n)\int_{C_i}|\mathrm{H}|^{n-1}d\mathcal{H}^n$. If we set $T=T_0\cup\bigcup\limits_{i\in I}T_i$, then we have $\mathcal{H}^1(T)\leqslant C(n)\int_{\partial\Omega\setminus A_{\delta(\Omega)^{\frac14}}}|\mathrm{H}|^{n-1}d\mathcal{H}^n$ and

$$\begin{aligned}d_H\big(\partial\Omega,S_{x_\Omega}(R_\Omega)\cup T\big)&\leqslant\max\Big(d_H\big(\partial_r\Omega,S_{x_\Omega}(R_\Omega)\cup T_0\big),(d_H(C_i,T_i))_{i\in I}\Big)\\&\leqslant\max\Big(d_H(\partial_r\Omega,A_0\cup T_0)+d_H\big(A_0\cup T_0,S_{x_\Omega}(R_\Omega)\cup T_0\big),C(n)R_\Omega\delta(\Omega)^{\frac{1}{4n}}\Big)\\&\leqslant C(n)R_\Omega\delta(\Omega)^{\frac{1}{4n}}+d_H\big(A_0,S_{x_\Omega}(R_\Omega)\big)\\&\leqslant C(n)R_\Omega\delta(\Omega)^{\beta(n)}\end{aligned}$$

the last inequality comes from Theorem 1. This completes the proof of Theorem 2.

Now to prove Theorem 3, we have

$$\begin{aligned}&d_H\left(\partial\Omega,S_{x_\Omega}(R_\Omega)\cup\left(\bigcup_{i\in I}\{x_i\}\right)\right)\\&\qquad\leqslant d_H\big(\partial\Omega,S_{x_\Omega}(R_\Omega)\cup T\big)+d_H\left(S_{x_\Omega}(R_\Omega)\cup T,S_{x_\Omega}(R_\Omega)\cup\left(\bigcup_{i\in I}\{x_i\}\right)\right)\\&\qquad\leqslant C(n)R_\Omega\delta(\Omega)^{\beta(n)}+d_H\left(T,(T_0\cap S_{x_\Omega}(R_\Omega))\cup\left(\bigcup_{i\in I}\{x_i\}\right)\right)\\&\qquad\leqslant C(n)R_\Omega\delta(\Omega)^{\beta(n)}+\max\big(d_H\big(T_0,T_0\cap S_{x_\Omega}(R_\Omega)\big),(d_H(T_i,\{x_i\}))_{i\in I}\big)\\&\qquad\leqslant C(n)R_\Omega\delta(\Omega)^{\beta(n)}+C(n)\int_{\partial\Omega\setminus A_{\delta(\Omega)^{\frac14}}}|\mathrm{H}|^{n-1}d\mathcal{H}^n\end{aligned}$$

To finish the proof of Theorem 3 we just have to use Hölder's inequality and Lemma 2. For what concerns cardinality of I, remark that the Michael-Simon Inequality applied to the function $f=1$ and to any connected component C of $\partial\Omega\setminus\partial_r\Omega$ gives us

$$\mathcal{H}^n(C)^{\frac{n-1}{n}}\leqslant C(n)\int_C|\mathrm{H}|d\mathcal{H}^n\leqslant C(n)\Big(\int_C|\mathrm{H}|^nd\mathcal{H}^n\Big)^{\frac{1}{n}}(\mathcal{H}^n(C)^{\frac{n-1}{n}}$$

and so $\int_C|\mathrm{H}|^nd\mathcal{H}^n\geqslant\frac{1}{C(n)}$ for any connected component of $\partial\Omega\setminus\partial_r\Omega$. We infer that

$$\frac{Card(I)}{C(n)}\leqslant\sum_C\int_C|\mathrm{H}|^nd\mathcal{H}^n\leqslant\int_{\partial\Omega\setminus A_{\delta(\Omega)^{\frac14}}}|\mathrm{H}|^nd\mathcal{H}^n$$

we conclude for any $p\geqslant n$ by Hölder inequality and Lemma 2. □

5.2. **Variants of Theorems 2 and 3 that generalize inequality** (1.5)**.**

Theorem 10. *Let Ω be an open set with smooth boundary, finite perimeter and $\delta(\Omega) \leqslant \frac{1}{C(n)}$. There exists a subset $T \subset \mathbb{R}^{n+1}$ with*

(1) $\mathcal{H}^1(T) \leqslant C(n)R_\Omega \int_{\partial\Omega\setminus B_{x_\Omega}\left(R_\Omega(1+\delta(\Omega)^{\frac{1}{4}})\right)} |\mathrm{H}|^{n-1}d\mathcal{H}^n$,

(2) $d_H\big(\Omega, B_{x_\Omega}(R_\Omega) \cup T\big) \leqslant C(n)R_\Omega\delta(\Omega)^{\beta(n)}$,

(3) the set $B_{x_\Omega}\big(R_\Omega(1+\delta(\Omega)^{\frac{1}{4}})\big) \cup T$ has at most $N+1$ connected components,

where N is the number of connected components of $\partial\Omega$ that do not intercept the ball $B_{x_\Omega}\big(R_\Omega(1+\delta(\Omega)^{\frac{1}{4}})\big)$.

Theorem 11. *Let $p \geqslant n-1$ and Ω be an open set with smooth boundary $\partial\Omega$, finite perimeter and $\delta(\Omega) \leqslant \frac{1}{C(n)}$. Let $(\partial\Omega_i)_{i\in I}$ be the connected components of $\partial\Omega$ that do not intercept the ball $B_{x_\Omega}\big(R_\Omega(1+\delta(\Omega)^{\frac{1}{4}})\big)$. For any $i \in I$, there exists $x_i \in \partial\Omega_i$ such that*

$$d_H\Big(\Omega, B_{x_\Omega}(R_\Omega) \cup \bigcup_{i\in I}\{x_i\}\Big) \leqslant C(n,p)R_\Omega\left[\delta(\Omega)^{\beta(n)} + \delta(\Omega)^{\frac{p-n+1}{4p}}\big(P(\Omega)^{\frac{1}{n}}\|\mathrm{H}\|_p\big)^{n-1}\right]$$

Moreover if $p \geqslant n$ and H *is L^p-integrable then I is of finite cardinal N and we have*

$$N \leqslant C(n,p)P(\Omega)\|\mathrm{H}\|_p^n\delta(\Omega)^{\frac{p-n}{4p}}.$$

Remark 9. *The norm $\|\mathrm{H}\|_p$ can be replaced by* $\left(\frac{1}{P(\Omega)}\int_{\partial\Omega\setminus B_{x_\Omega}\left(R_\Omega(1+\delta(\Omega)^{\frac{1}{4}})\right)} |\mathrm{H}|^p d\mathcal{H}^n\right)^{\frac{1}{p}}$.

Proof of Theorems 10 and 11 : We set $\partial_r\Omega$ the union of the connected components of $\partial\Omega$ that intercept $B_{x_\Omega}\big(R_\Omega(1+\delta(\Omega)^{\frac{1}{4}}\big)$ and then we construct T as in the previous section. Arguing as in the previous subsection, we get that the $C(n)R_\Omega\delta(\Omega)^{\beta(n)}$-tubular neighbourhood of $B_{x_\Omega}(R_\Omega) \cup T$ contains $\partial\Omega \setminus B_{x_\Omega}\big(R_\Omega(1+\delta(\Omega)^{\frac{1}{4}})\big)$. We set with $R'_\Omega = R_\Omega(1+2C(n)\delta(\Omega)^{\frac{1}{2(n+1)}})$ (where $C(n)$ is the constant of Remark 1 (2)). Then for any $x \in \Omega$, either we have $x \in B_{x_\Omega}(R'_\Omega)$ and then $d\big(x, B_{x_\Omega}(R_\Omega) \cup T\big) \leqslant 2C(n)R_\Omega\delta(\Omega)^{\frac{1}{2(n+1)}}$, either we have $x \in \Omega \setminus B_{x_\Omega}(R'_\Omega)$, and then $x \in \Omega\Delta B_{x_\Omega}(R_\Omega)$. From the Remark 1 (1), we infer (as in the proof of Remark 1 (2)) that

$$d(x,\partial\Omega) \leqslant C(n)R_\Omega\delta(\Omega)^{\frac{1}{2(n+1)}}$$

and even more precisely, $d\big(x, \partial\Omega \setminus B_{x_\Omega}\big(R_\Omega(1+\delta(\Omega)^{\frac{1}{4}})\big)\big) \leqslant C(n)R_\Omega\delta(\Omega)^{\frac{1}{2(n+1)}}$. We infer that we have

$$d\big(x, B_{x_\Omega}(R_\Omega) \cup T\big) \leqslant C(n)R_\Omega\delta(\Omega)^{\min(\frac{1}{8},\frac{1}{4n})}$$

On the other hand, for any $x \in B_{x_\Omega}(R_\Omega) \cup T$, either $x \in Bx_\Omega(R_\Omega)$ and then $d(x,\Omega) \leqslant C(n)R_\Omega\delta(\Omega)^{\frac{1}{2(n+1)}}$ by Remark 1 (2), either $x \in T$ and then $d(x,\Omega) = 0$. We then get

$$d_H\big(B_{x_\Omega}(R_\Omega) \cup T, \Omega\big) \leqslant C(n)R_\Omega\delta(\Omega)^{\min(\frac{1}{8},\frac{1}{4n})}$$

which gives the result as in the proofs of Theorems 2 and 3. □

6. A quasi-isometry result : proof of Theorem 4

Let us first remind Duggan's version of Allard's local regularity theorem about hypersurface of suitably bounded mean curvature.

Theorem 12 (J.P. Duggan [9]). *If $p > n$ is arbitrary, then there are $\eta = \eta(n,p)$, $\gamma = \gamma(n,p) \in (0,1)$ and $c = c(n,p)$ such that if $M \subset \mathbb{R}^{n+1}$ is a hypersurface, $x \in M$ and $\rho > 0$ satisfy the hypotheses*

(1) $\mathcal{H}^n(B_x(\rho) \cap M) \leqslant (1+\eta)\rho^n |\mathbb{B}^n|$

(2) $\rho^{p-n} \int_{B_x(\rho)\cap M} |\mathrm{H}|^p d\mathcal{H}^n \leqslant \eta^p$

then there exists a linear isometry q of $\mathbb{R}^{n+1}$ and $u \in W^{2,p}(B_0^{\mathbb{R}^n}(\gamma\rho))$ with $u(0) = 0$, $M \cap B_x(\gamma\rho) = \big(x + q(graph\ u)\big) \cap B_x(\gamma\rho)$ and

$$\frac{\sup|u|}{\rho} + \sup|du| + \rho^{1-\frac{n}{p}} \left(\int_{B_0^{\mathbb{R}^n}(\gamma\rho)} |\nabla du|^p d\mathcal{H}^n \right)^{1/p} \leqslant c\eta^{\frac{1}{4n}}. \tag{6.1}$$

So the Morrey-Campanato says that for any $v \in W^{1,p}(B_0^{\mathbb{R}^n}(1))$ we have

$$\sup_{x\neq y \in B_0^{\mathbb{R}^n}(1)} \frac{|v_x - v_y|}{|x-y|^{1-\frac{n}{p}}} \leqslant C(n,p) \left(\int_{B_0^{\mathbb{R}^n}(1)} |v|^p d\mathcal{H}^n + \int_{B_0^{\mathbb{R}^n}(1)} |dv|^p d\mathcal{H}^n \right)$$

Up to a normalization and under the assumptions of Theorem 12, the Morrey-Campanato theorem gives us that

$$\rho^{1-\frac{n}{p}} \sup_{x\neq y\in B_0^{\mathbb{R}^n}(\gamma\rho)} \frac{\big||du_x| - |du_y|\big|}{|x-y|^{1-\frac{n}{p}}} \leqslant C(n,p)\eta^{\frac{1}{4n}}$$

Now let $\Phi : B_0^{\mathbb{R}^n}(\gamma\rho) \to \mathbb{R}^{n+1}$, $a \mapsto q(a, u(a))$. Then $d\Phi_a(h) = q(h, du_a(h))$. Since q is an isometry, a unit normal is given by $\nu_{\Phi(a)} = q\big(\frac{((\nabla u)|_a, -1)}{\sqrt{1+|\nabla u|_a|^2}}\big)$ which gives for any $x \in \partial\Omega$

$$\rho^{1-\frac{n}{p}} \sup_{y,z\in B_x(\gamma\rho)\cap\partial\Omega,\ y\neq z} \frac{\big|\nu_y - \nu_z\big|}{\big|y-z\big|^{1-\frac{n}{p}}} \leqslant C(n,p)\eta^{\frac{1}{4n}} \tag{6.2}$$

Lemma 4. *Let $p > n$. There exist 3 positive constants $C_1(n,p)$, $C_2(n,p)$ and $C_3(n,p)$ such that for any domain Ω with smooth boundary $\partial\Omega$ satisfying $P(\Omega)\,\|\mathrm{H}\|_p^n \leqslant K$, and $\delta(\Omega) \leqslant \frac{1}{C_1(n,p)K^{\alpha(n,p)}}$, we have*

$$\sup_{x\in\partial\Omega} \big||x - x_\Omega| - R_\Omega\big| \leqslant C_2(n,p) R_\Omega \delta(\Omega)^{\beta(n)}, \tag{6.3}$$

and the assumptions of Theorem 12 are satisfied for $\bar\rho = \frac{R_\Omega}{C_3(n,p)K^{\frac{p}{n(p-n)}}}$. Moreover we have

$$\bar\rho^{1-\frac{n}{p}} \sup_{y,z\in B_x(\gamma\bar\rho)\cap\partial\Omega,\ y\neq z} \frac{\big|Z_y - Z_z\big|}{\big|y-z\big|^{1-\frac{n}{p}}} \leqslant C(n,p)\eta^{\frac{1}{4n}} \tag{6.4}$$

Where $Z_x = \frac{x - x_\Omega}{|x-x_\Omega|} - \nu_x$. Here we have set $\alpha(n,p) = \frac{8p}{p-n}$.

Proof. Since the computations are a bit messy, we organize them in several steps:

(1) For what concern the point (2) of Theorem 12, we have for any $\rho > 0$

$$\rho^{p-n} \int_{B_x(\rho)\cap\partial\Omega} |\mathrm{H}|^p d\mathcal{H}^n \leqslant \rho^{p-n} P(\Omega) \|\mathrm{H}\|_p^p \leqslant \left(\frac{\rho}{P(\Omega)^{\frac{1}{n}}} \right)^{p-n} K^{\frac{p}{n}}$$

From (1.1) and the definition of R_Ω, we have $R_\Omega^n \leqslant C(n)P(\Omega)$ and so

$$\rho^{p-n} \int_{B_x(\rho)\cap\partial\Omega} |\mathrm{H}|^p d\mathcal{H}^n \leqslant C(n,p) \left(\frac{\rho}{R_\Omega} \right)^{p-n} K^{\frac{p}{n}}$$

From this we deduce that there exists a constant $C_3(n,p)$ large enough such that $\partial\Omega$ satisfies assumption (2) of Theorem 12 for $\rho = \bar{\rho} = \frac{R_\Omega}{C_3(n,p) K^{\frac{p}{n(p-n)}}}$.

(2) Let $x \in \mathbb{S}^n$ then there exists a $r(n,p) \in]0,1]$ such that we have $\frac{\mathcal{H}^n(B_x(r)\cap\mathbb{S}^n)}{|\mathbb{B}^n| r^n} \in [1/2, 1+\eta(n,p)/4]$, for any $r < r(n,p)$, where $\eta(n,p)$ is the constant of Theorem 12. By Michael-Simon Sobolev inequality, we have $K \geqslant P(\Omega)\|\mathrm{H}\|_p^n \geqslant k(n)$, and so we can assume $C_3(n,p)$ large enough to have $\bar{\rho}/R_\Omega \leqslant r(n,p) \leqslant 1$.

From now on $C_3(n,p)$ is fixed so that it satisfies both the two previous conditions.

(3) Since $K \geqslant k(n)$, we can assume $C_1(n,p)$ large enough for $\delta(\Omega) \leqslant \frac{1}{C_1(n,p) K^{\alpha(n,p)}}$ to imply that $\delta(\Omega) \leqslant \min\big(\frac{\eta}{\eta+4}, (\frac{|\mathbb{B}^n|(C(n))^{n-1}\eta}{8\mathcal{H}^n(\mathbb{S}^n)})^8\big) \leqslant 1$ in what follows, where $C(n)$ is the constant of Theorem 7.

(4) From Theorem 3, the number N of connected components of $\partial\Omega$ that do not intercept $A_{\delta(\Omega)^{1/4}}$ satisfies

$$N \leqslant C(n,p) P(\Omega) \|\mathrm{H}\|_p^n \delta(\Omega)^{\frac{p-n}{4p}} \leqslant C(n,p) K \delta(\Omega)^{\frac{p-n}{4p}} \leqslant C(n,p) K \delta(\Omega)^{\frac{1}{\alpha(n,p)}}$$

So, when $\delta(\Omega) \leqslant \frac{1}{(2C(n,p)K)^{\alpha(n,p)}}$, we have $N = 0$. We infer by Theorem 3

$$\begin{aligned} d_H\big(\partial\Omega, S_{x_\Omega}(R_\Omega)\big) &\leqslant C(n,p) R_\Omega \left[\delta(\Omega)^{\beta(n)} + \delta(\Omega)^{\frac{p-n+1}{4p}} (P(\Omega)^{\frac{1}{n}} \|\mathrm{H}\|_p)^{n-1} \right] \\ &\leqslant C(n,p) R_\Omega \left(1 + \delta(\Omega)^{\left(\frac{n-1}{n}\right)\left(\frac{p-n}{4p}\right)} K^{\frac{n-1}{n}} \right) \delta(\Omega)^{\beta(n)} \\ &\leqslant C(n,p) R_\Omega \Big(1 + \big(\delta(\Omega)^{\frac{1}{\alpha(n,p)}} K\big)^{\frac{n-1}{n}} \Big) \delta(\Omega)^{\beta(n)} \\ &\leqslant C_2(n,p) R_\Omega \delta(\Omega)^{\beta(n)} \end{aligned}$$

which gives inequality (6.3) for any $C_1(n,p) \geqslant (2C(n,p))^{\alpha(n,p)}$ such that the previous condition (3) also holds. Note that we have used $\delta(\Omega) \leqslant 1$. At this stage, $C_2(n,p)$ is fixed, and does not depends on $C_1(n,p)$.

(5) Similarly for $C_1(n,p)$ large enough and $\delta(\Omega) \leqslant \frac{1}{C_1 K^\alpha}$, we have from the previous point that

$$|x| \geqslant R_\Omega(1 - C_2\delta(\Omega)^\beta) \geqslant R_\Omega \left(1 - \frac{C_2}{C_1^\beta K^{\beta\alpha}} \right) \geqslant R_\Omega \left(1 - \frac{C_2}{C_1^\beta k(n)^{\beta\alpha}} \right) \geqslant \frac{1}{2} R_\Omega \tag{6.5}$$

From this we deduce that

$$\frac{\left| \frac{x-x_\Omega}{|x-x_\Omega|} - \frac{y-x_\Omega}{|y-x_\Omega|} \right|}{|x-y|^{1-\frac{n}{p}}} \leqslant \frac{4}{R_\Omega} |x-y|^{\frac{n}{p}} \leqslant C(n,p) R_\Omega^{\frac{n}{p}-1} \leqslant \frac{1}{\bar{\rho}^{1-\frac{n}{p}} C_3^{\frac{p-n}{p}} K^{1/n}} \leqslant \frac{C(n,p)}{\bar{\rho}^{1-\frac{n}{p}}}$$

which gives with 6.2 the inequality 6.4.

(6) We want to apply Theorem 7 to $\partial\Omega$ and $B_x(\bar\rho)$ and so need $\bar\rho\in[C(n)\delta(\Omega)^{1/8}R_\Omega,R_\Omega]$. Note that $\bar\rho\leqslant R_\Omega$ was already obtained in (2). On the other hand, we have that

$$\frac{\bar\rho}{R_\Omega}=\frac{C(n)\delta(\Omega)^{\frac{1}{8n}}}{C(n)\delta(\Omega)^{\frac{1}{8n}}C_3(n,p)K^{\frac{p}{n(p-n)}}}=\frac{C(n)\delta(\Omega)^{\frac{1}{8n}}C_1(n,p)^{\frac{1}{8n}}}{C(n)C_3(n,p)[\delta(\Omega)C_1(n,p)K^{\alpha(n,p)}]^{\frac{1}{8n}}}$$
$$\geqslant C(n)\delta(\Omega)^{\frac18}\frac{C_1(n,p)^{\frac18}}{C(n)C_3(n,p)}.$$

Now it is clear that for $C_1(n,p)$ large enough, we have $\frac{\bar\rho}{R_\Omega}\geqslant C(n)\delta(\Omega)^{\frac{1}{8n}}\geqslant C(n)\delta(\Omega)^{\frac18}$.

(7) Now we prove that for $C_1(n,p)$ large enough, $\partial\Omega$ satisfies (1) for $\bar\rho=\frac{R_\Omega}{C_3(n,p)K^{\frac{p}{n(p-n)}}}$ with $C_3(n,p)$ fixed in (2). Let $x\in S_{x_\Omega}(R_\Omega)$. Then Theorem 7 gives us

$$\frac{\mathcal{H}^n\big(B_x(\bar\rho)\cap S_{x_\Omega}(R_\Omega)\big)}{P(\Omega)}\leqslant C(n)\delta(\Omega)^{1/4}+\frac{\mathcal{H}^n(B_x(\bar\rho)\cap S_{x_\Omega}(R_\Omega)}{R_\Omega^n\mathcal{H}^n(\mathbb{S}^n)}$$
$$\leqslant C(n)\delta(\Omega)^{1/4}+\frac{\mathcal{H}^n\Big(B_{x'}(\frac{\bar\rho}{R_\Omega})\cap S_{x_\Omega}(1)\Big)}{\mathcal{H}^n(\mathbb{S}^n)}$$

where $x'=x_\Omega+\frac{1}{R_\Omega}(x-x_\Omega)$. Now by the condition (2) above, we have

$$\mathcal{H}^n\Big(B_{x'}(\frac{\bar\rho}{R_\Omega})\cap S_{x_\Omega}(1)\Big)\leqslant(1+\eta/4)|\mathbb{B}^n|\frac{\bar\rho^n}{R_\Omega^n}$$

and so

$$\mathcal{H}^n\big(B_x(\bar\rho)\cap S_{x_\Omega}(R_\Omega)\big)\leqslant C(n)P(\Omega)\delta(\Omega)^{1/4}+(1+\eta/4)\frac{P(\Omega)|\mathbb{B}^n|\bar\rho^n}{\mathcal{H}^n(\mathbb{S}^n)R_\Omega^n}$$
$$\leqslant\Big(\frac{2C(n)\mathcal{H}^n(\mathbb{S}^n)R_\Omega^n}{|\mathbb{B}^n|\bar\rho^n}\delta(\Omega)^{1/4}+(1+\eta/4)\big(1+\delta(\Omega)\big)\Big)|\mathbb{B}^n|\bar\rho^n$$
$$\leqslant\Big(\frac{2\mathcal{H}^n(\mathbb{S}^n)}{(C(n))^{n-1}|\mathbb{B}^n|}\delta(\Omega)^{1/8}+(1+\eta/4)\big(1+\delta(\Omega)\big)\Big)|\mathbb{B}^n|\bar\rho^n$$

where we have used the fact that $\frac{P(\Omega)}{R_\Omega^n\mathcal{H}^n(\mathbb{S}^n)}=1+\delta(\Omega)\leqslant 2$, and $\dfrac{R_\Omega^n}{\bar\rho^n}\leqslant\dfrac{1}{C(n)^n\delta(\Omega)^{1/8}}$ proved in (5). Now from the condition $\delta(\Omega)\leqslant\min\big(\frac{\eta}{\eta+4},(\frac{|\mathbb{B}^n|(C(n))^{n-1}\eta}{8\mathcal{H}^n(\mathbb{S}^n)})^8\big)$ we deduce that

$$\mathcal{H}^n\big(B_x(\bar\rho)\cap S_{x_\Omega}(R_\Omega)\big)\leqslant\left(\eta/4+(1+\eta/4)(1+\frac{\eta}{\eta+4}\right)|\mathbb{B}^n|\bar\rho^n\leqslant(1+\eta)|\mathbb{B}^n|\bar\rho^n$$

□

Now, using Duggan's regularity theorem, we can show a Calderon-Zygmund property of almost isoperimetric manifolds with L^p bounded mean curvature:

Lemma 5. *Let $p>n$. There exists $C(n,p)>0$ such that for any domain Ω with smooth boundary $\partial\Omega$ satisfying $P(\Omega)\,\|\mathrm{H}\|_p^n\leqslant K$ and $\delta(\Omega)\leqslant\frac{1}{C(n,p)K^{\alpha(n,p)}}$ we have*

$$P(\Omega)\|\mathrm{B}\|_p^n\leqslant C(n,p)K^{\frac{p+1}{p-n}}$$

Remark 10. *We can improve the proof below to get $P(\Omega)\|\mathrm{B}\|_p^n\leqslant C(n,p)K^{\frac{p}{p-n}}$.*

Proof. Let $(x_i)_i$ be a maximal family of points of $\partial\Omega$ such that the balls $B_{x_i}(\gamma\bar\rho/2)$ are disjoints in $\mathbb{R}^{n+1}$. Then the family $\big(\partial\Omega\cap B_{x_i}(\gamma\bar\rho)\big)_i$ covers $\partial\Omega$. By (6.3), all the balls $B_{x_i}(\gamma\bar\rho/2)$ are included in $B_{x_i}\left(\left(\frac{\gamma}{2C_3k^{\frac{p}{p-n}}}+\frac{C_2}{C_1^{\beta}k^{\alpha\beta}}+1\right)R_\Omega\right)$ and for C_1 and C_3 large enough, they are included in $B_{x_\Omega}(3R_\Omega)$. And so the family has at most $(\frac{6R_\Omega}{\gamma\bar\rho})^{n+1}\leqslant C(n,p)K^{\frac{(n+1)p}{n(p-n)}}$ elements (note that using the fact that $\partial\Omega$ is Hausdorff close to $S_{x_\Omega}(R_\Omega)$ we could replace $K^{\frac{(n+1)p}{n(p-n)}}$ by the better $K^{\frac{p}{p-n}}$).

By Theorem 12, denoting by u_i each corresponding function we then have $|B|\leqslant\sqrt{n}\frac{|d^2u_i|}{\sqrt{1+|du_i|^2}}$ on $\partial\Omega\cap B_{x_i}(\gamma\bar\rho)$ and

$$\begin{aligned}\int_{\partial\Omega\cap B_{x_i}(\gamma\bar\rho)}|\mathrm{B}|^pd\mathcal{H}^n&\leqslant\int_{B_0^{\mathbb{R}^n}(\gamma\bar\rho)}n^{p/2}\frac{|d^2u_i|^p}{(1+|du_i|^2)^{\frac{p-1}{2}}}d\mathcal{H}^n\\&\leqslant\int_{B_0^{\mathbb{R}^n}(\gamma\bar\rho)}n^{p/2}|d^2u_i|^pd\mathcal{H}^n\leqslant\frac{C(n,p)}{\bar\rho^{p-n}}\end{aligned}$$

from which we get

$$\begin{aligned}P(\Omega)\|\mathrm{B}\|_p^n=P(\Omega)^{1-\frac{n}{p}}\left(\int_{\partial\Omega}|\mathrm{B}|^pd\mathcal{H}^n\right)^{n/p}&\leqslant C(n,p)\left(\frac{P(\Omega)}{\bar\rho^n}\right)^{\frac{p-n}{p}}K^{\frac{n+1}{p-n}}\\&=C(n,p)\left(\frac{P(\Omega)}{R_\Omega^n}C_3^nK^{\frac{p}{p-n}}\right)^{\frac{p-n}{p}}K^{\frac{n+1}{p-n}}\leqslant C(n,p)K^{\frac{p+1}{p-n}}\end{aligned}$$

□

Using Duggan's Theorem we now improve the L^2 smallness of Z given by Lemma 3 in an L^∞ one.

Lemma 6. *Let $p>n$. There exists $C(n,p)>0$ such that for any domain Ω with smooth boundary $\partial\Omega$ satisfying $P(\Omega)\,\|\mathrm{H}\|_p^n\leqslant K$ and $\delta(\Omega)\leqslant\frac{1}{C(n,p)K^{\alpha(n,p)}}$, we have*

$$\sup_{x\in\partial\Omega}|Z_x|\leqslant C(n,p)K^{\frac{1}{n}}\delta(\Omega)^{\frac{1}{n\alpha(n,p)}}\tag{6.6}$$

Here $\alpha(n,p)$ is the same as in Lemma 4.

Proof. Let

$$C_4(n,p)=\max\Big(\frac{C(n)}{\gamma},\frac{C_2(n,p)}{\gamma},\frac{1}{\gamma}\left(\frac{4C(n)}{|\mathbb{B}^n|}\right)^{\frac{1}{n}}\Big)\tag{6.7}$$

where $C(n)$ is the constant of Theorem 7 and $C_2(n,p)$ is the constant of Lemma 4. We set $\rho'=2\gamma C_4(n,p)\delta(\Omega)^{\frac{1}{8n}}R_\Omega$.

Assume now that $\delta(\Omega)\leqslant\frac{1}{C_1'(n,p)K^{\alpha(n,p)}}$ where $C_1'\geqslant C_1$. For $C_1'(n,p)$ large enough we have $\delta(\Omega)\leqslant\frac{1}{C_1'(n,p)K^{\alpha(n,p)}}\leqslant\frac{1}{(2C_4C_3)^{8n}K^{\alpha(n,p)}}$ and $\rho'\leqslant\gamma\bar\rho$. As explained in the point (2) of the proof of Lemma 4, we can assume $C_1'(n,p)$ large enough to get that

$$\delta(\Omega)^{\frac{1}{8n}}\leqslant\min\Big(\frac{\gamma C_4}{C_2},\frac{1}{\gamma C_4}\Big)\tag{6.8}$$

where $C_3(n,p)$ is the constant used in the proof of Lemma 4. For any $x\in\partial\Omega$ and for any $y,z\in\partial\Omega\cap B_x(\rho')$, Inequality (6.4) and the value of $\bar\rho$ give us

$$|Z_y-Z_z|\leqslant\frac{C(n,p)\eta(n,p)^{1/4n}}{\bar\rho^{\frac{p-n}{p}}}|y-z|^{1-\frac{n}{p}}\leqslant C(n,p)K^{1/n}\left(\frac{\rho'}{R_\Omega}\right)^{1-\frac{n}{p}}$$

Since, $K\geqslant k(n)$, we can assume $C_1'(n,p)$ large enough so that Lemma 3 applies and then for any $x\in\Omega$

$$\begin{aligned}|Z_x|&\leqslant\frac{1}{\mathcal{H}^n(B_x(\rho')\cap\partial\Omega)}\Big(\int_{B_x(\rho')\cap\partial\Omega}\big|Z_x-Z_y\big|d\mathcal{H}^n(y)+\int_{B_x(\rho')\cap\partial\Omega}\big|Z_y\big|d\mathcal{H}^n(y)\Big)\\&\leqslant C(n,p)K^{\frac1n}\big(\frac{\rho'}{R_\Omega}\big)^{1-\frac{n}{p}}+\Big(\frac{1}{\mathcal{H}^n(B_x(\rho')\cap\partial\Omega)}\int_{B_x(\rho')\cap\partial\Omega}|Z_y|^2d\mathcal{H}^n(y)\Big)^{1/2}\\&\leqslant C(n,p)K^{\frac1n}\left(\frac{\rho'}{R_\Omega}\right)^{1-\frac{n}{p}}+C(n)\left(\frac{P(\Omega)}{\mathcal{H}^n(B_x(\rho')\cap\partial\Omega)}\right)^{\frac12}\delta(\Omega)^{\frac14}\end{aligned}$$

Now let $x'=x_\Omega+R_\Omega\frac{x-x_\Omega}{|x-x_\Omega|}\in S_{x_\Omega}(R_\Omega)$. From (6.3), an easy computation shows that $B_{x'}\big(\frac{\rho'}{2}\big)\subset B_x(\rho')$. Indeed if $y\in B_{x'}\big(\frac{\rho'}{2}\big)$, then

$$|x-y|\leqslant\big||x-x_\Omega|-R_\Omega\big|+\frac{\rho'}{2}\leqslant C_2R_\Omega\delta(\Omega)^\beta+\frac{\rho'}{2}$$

From the choices made in (6.7) and (6.8) we have $\delta(\Omega)^\beta\leqslant\frac{\gamma C_4}{C_2}\delta(\Omega)^{1/8n}$ and $|x-y|\leqslant\rho'$. We then get

$$|Z_x|\leqslant C(n,p)K^{1/n}\delta(\Omega)^{\frac{1}{n\alpha}}+C(n)\left(\frac{P(\Omega)}{\mathcal{H}^n(B_{x'}(\rho'/2)\cap\partial\Omega)}\right)^{\frac12}\delta(\Omega)^{\frac14}\tag{6.9}$$

Now (6.7) and (6.8) imply that $\delta(\Omega)^{1/8n}\leqslant 1/\gamma C_4$ and $C_4\geqslant C(n)/\gamma$ which gives $\frac{\rho'}{2}\in[C(n)\delta(\Omega)^{1/8}R_\Omega,R_\Omega]$. So we can apply Theorem 7, and since we have $\frac{\rho'}{R_\Omega}\leqslant r(n,p)$ (see (2) in the proof of the previous lemma), we get $\mathcal{H}^n\left(B_x\left(\frac{\rho'}{2R_\Omega}\right)\cap\mathbb{S}^n\right)\geqslant\frac{|\mathbb{B}^n|}{2}\big(\frac{\rho'}{2R_\Omega}\big)^n$ and

$$\begin{aligned}\frac{\mathcal{H}^n(B_{x'}\left(\frac{\rho'}{2}\right)\cap\partial\Omega)}{P(\Omega)}&\geqslant\frac{\mathcal{H}^n(B_{x'}\left(\frac{\rho'}{2}\right)\cap S_{x_\Omega}(R_\Omega))}{R_\Omega^n|\mathbb{S}^n|}-C(n)\delta(\Omega)^{1/4}\\&\geqslant\frac{\mathcal{H}^n(B_{x''}\left(\frac{\rho'}{2R_\Omega}\right)\cap S_{x_\Omega}(1))}{|\mathbb{S}^n|}-C(n)\delta(\Omega)^{1/4}\\&\geqslant\frac{|\mathbb{B}^n|}{2}\left(\frac{\rho'}{2R_\Omega}\right)^n-C(n)\delta(\Omega)^{1/4}\\&=\frac{|\mathbb{B}^n|(\gamma C_4(n,p))^n}{2}\delta(\Omega)^{\frac18}-C(n)\delta(\Omega)^{1/4}\\&\geqslant C(n)\delta(\Omega)^{1/8}\end{aligned}$$

where $x'' = x_\Omega + \frac{1}{R_\Omega}(x' - x_\Omega)$ and in the last inequality we used again (6.7). Reporting this in (6.9) we obtain

$$\begin{aligned}|Z_x| &\leqslant C(n,p)K^{1/n}\delta(\Omega)^{\frac{1}{n\alpha(n,p)}} + C(n,p)\delta(\Omega)^{3/16}\\ &\leqslant C(n,p)K^{1/n}\delta(\Omega)^{\frac{1}{n\alpha(n,p)}} + \frac{C(n,p)}{k(n)^{1/n}}K^{1/n}\delta(\Omega)^{\frac{1}{n\alpha(n,p)}}\\ &\leqslant C_5(n,p)K^{1/n}\delta(\Omega)^{\frac{1}{n\alpha(n,p)}}\end{aligned}$$

which gives the desired inequality by putting $C(n,p) = \max(C_1'(n,p), C_5(n,p))$. □

Since we have an upper bound on the second fundamental form, we could also perform a Moser iteration as in [3] to prove the previous lemma.

Let Ω be an almost isoperimetric domain. We consider the map $F : \partial\Omega \longrightarrow S_{x_\Omega}(R_\Omega)$ defined by

$$F(x) = R_\Omega \frac{x - x_\Omega}{|x - x_\Omega|}$$

Proof of Theorem 4 : In this proof, $C(n,p)$ is the constant of the Lemma 6. For more convenience up to a translation we can assume $x_\Omega = 0$. Under the assumptions of Lemma 4, we have $|x| \geqslant \frac{1}{2}R_\Omega$. Hence F is well defined on $\partial\Omega$. Moreover, for any $x \in \partial\Omega$ and $u \in T_x\partial\Omega$, we have $dF_x(u) = \frac{R_\Omega}{|x|}\left(u - \frac{\langle x,u\rangle}{|x|^2}x\right) = \frac{R_\Omega}{|x|}\left(u - \langle Z_x, u\rangle\frac{x}{|x|}\right)$ and we have

$$|dF_x(u)|^2 = \frac{R_\Omega^2}{|x|^2}(|u|^2 - \langle Z_x, u\rangle^2)$$

Let $D(n,p) \geqslant C(n,p)$ large enough and assume $\delta(\Omega)^{1/2} \leqslant \frac{1}{DK^\alpha}$. Since by Inequality (6.6) of Lemma 6 we have

$$|Z_x| \leqslant CK^{1/n}\delta(\Omega)^{\frac{1}{n\alpha}} \leqslant \frac{CK^{1/n}}{D^{\frac{1}{n\alpha}}K^{\frac{1}{n}}} = \frac{C}{D^{\frac{1}{n\alpha}}}$$

Hence we can assume $\|Z\|_\infty < 1/2$ for $D(n,p)$ large enough, which infer that F is a local diffeomorphism form $\partial\Omega$ into $S_0(R_\Omega)$. Let $\partial\Omega_0$ be a connected component of $\partial\Omega$. Since $\partial\Omega_0$ is compact and $S_0(R_\Omega)$ is simply connected, we get that F is a diffeomorphism. Moreover since $||x| - R_\Omega| \leqslant C_2R_\Omega\delta(\Omega)^\beta$ we have $\left|\frac{R_\Omega}{|x|} - 1\right| \leqslant 2C_2\delta(\Omega)^\beta$ and

$$\begin{aligned}||dF_x(u)|^2 - |u|^2| &\leqslant \left|\frac{R_\Omega}{|x|} - 1\right|\left|\frac{R_\Omega}{|x|} + 1\right||u|^2 + \frac{R_\Omega^2}{|x|^2}|Z_x|^2|u|^2\\ &\leqslant \left(6C_2\delta(\Omega)^\beta + 2\|Z\|_\infty\right)|u|^2\\ &\leqslant \left(\frac{6C_2}{k(n)^{1/n}}\delta(\Omega)^{\beta - \frac{1}{n\alpha}} + 2C\right)K^{1/n}\delta(\Omega)^{\frac{1}{n\alpha}}|u|^2\\ &\leqslant \left(\frac{6C_2}{k(n)^{1/n}} + 2C\right)\frac{\delta(\Omega)^{\frac{1}{2n\alpha}}}{D^{1/n\alpha}}|u|^2\\ &\leqslant C_6(n,p)\delta(\Omega)^{\frac{1}{2n\alpha}}|u|^2 \qquad (6.10)\end{aligned}$$

Now if $\partial\Omega$ as at least 2 connected components $\partial\Omega_0$ and $\partial\Omega_1$ we have for any $i \in \{0,1\}$

$$\mathcal{H}^n(S_0(R_\Omega)) = \int_{\partial\Omega_i} F^\star d\mathcal{H}^n = \int_{\partial\Omega_i} \frac{|\langle x, \nu_x\rangle|}{|x|}\left(\frac{R_\Omega}{|x|}\right)^n d\mathcal{H}^n \leqslant \frac{\mathcal{H}^n(\partial\Omega_i)}{(1 - C_2\delta(\Omega)^\beta)^n}$$

and

$$P(\Omega) \geqslant \mathcal{H}^n(\partial\Omega_0) + \mathcal{H}^n(\partial\Omega_1) \geqslant 2(1 - C_2\delta(\Omega)^\beta)^n \mathcal{H}^n(S_0(R_\Omega)) \geqslant 2\frac{(1 - C_2\delta(\Omega)^\beta)^n}{1 + \delta(\Omega)} P(\Omega)$$

Where we have used the fact that $\frac{\mathcal{H}^n(\partial\Omega)}{\mathcal{H}^n(S_0(R_\Omega))} = \frac{I(\Omega)}{I(\mathbb{B}^{n+1})} = 1+\delta(\Omega)$. Now we can prove easily that $\frac{(1-C_2\delta(\Omega)^\beta)^n}{1+\delta(\Omega)} > 1/2$ for D great enough and we deduce that $\partial\Omega$ has one connected component.

Actually Inequality 6.10 gives for D great enough that $d_L(\partial\Omega, S_0(R_\Omega)) \leqslant C_6(n,p)\delta(\Omega)^{\frac{1}{2n\alpha}}$ $= C_6(n,p)\delta(\Omega)^{\frac{16pn}{p-n}}$. But we can improve this bound in order to have sharp estimates with respect to the powers of $\delta(\Omega)$ involved in the estimates on d_L and d_H.

Let $\varphi : S_0(R_\Omega) \to \mathbb{R}$ given by $\varphi(w) = \|F^{-1}(w)\|/R_\Omega$. Then we have $\partial\Omega = \{\varphi(w)w, w \in S_0(R_\Omega)\}$, $\varphi \geqslant 1/2$ and from 6.3 $\|\varphi - 1\|_\infty \leqslant C_2\delta(\Omega)^\beta$. Moreover for any $u \in T_wS_0(R_\Omega)$ we have :

$$d\varphi_w(u) = \frac{1}{R_\Omega^2}\langle dF_w^{-1}(u), w\rangle = \frac{\langle dF_w^{-1}(u), Z_{F^{-1}(w)}\rangle}{R_\Omega}$$

Consequently $R_\Omega|d\varphi_w| \leqslant |dF_w^{-1}||Z_{F^{-1}(w)}|$ and for D great enough we deduce from 6.10 that $|dF_w^{-1}|^2 \leqslant \frac{1}{1-C_6\delta(\Omega)^{1/2n\alpha}} \leqslant \frac{1}{2}$ and from 6.6 we get

$$R_\Omega\|d\varphi\|_\infty \leqslant C(n,p)K^{1/n}\delta(\Omega)^{1/n\alpha}$$

Now the second fundamental form B of the boundary can be expressed by the formulae

$$(F^{-1})^\star \mathrm{B} = \frac{\varphi\nabla d\varphi - d\varphi \otimes d\varphi - \frac{1}{R_\Omega^2}(F^{-1})^\star g}{\sqrt{|d\varphi|^2 + \frac{\varphi^2}{R_\Omega^2}}}$$

which gives

$$\begin{aligned}
|\nabla d\varphi| &\leqslant \frac{1}{\varphi}\left(\sqrt{\|d\varphi\|_\infty^2 + \frac{\|\varphi\|_\infty^2}{R_\Omega^2}}|(F^{-1})^\star\mathrm{B}| + \|d\varphi\|_\infty^2 + \frac{1}{R_\Omega^2}|(F^{-1})^\star g|\right)\\
&\leqslant \frac{1}{\varphi}\left(\sqrt{\|d\varphi\|_\infty^2 + \frac{\|\varphi\|_\infty^2}{R_\Omega^2}}|\mathrm{B}\circ F^{-1}||dF^{-1}|^2 + \|d\varphi\|_\infty^2 + \frac{1}{R_\Omega^2}|dF^{-1}|^2\right)\\
&\leqslant \frac{1}{2\varphi}\left(\sqrt{\|d\varphi\|_\infty^2 + \frac{\|\varphi\|_\infty^2}{R_\Omega^2}}|\mathrm{B}\circ F^{-1}| + 2\|d\varphi\|_\infty^2 + \frac{1}{R_\Omega^2}\right)\\
&\leqslant \frac{C(n,p,K)}{R_\Omega}\left(|\mathrm{B}\circ F^{-1}| + \frac{1}{R_\Omega}\right)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|\mathrm{B}\circ F^{-1}\|_p^p &= \frac{1}{\mathcal{H}^n(S_0(R_\Omega))}\int_{S_0(R_\Omega)}|\mathrm{B}\circ F^{-1}|^p d\mathcal{H}^n = \frac{1}{\mathcal{H}^n(S_0(R_\Omega))}\int_{\partial\Omega}|\mathrm{B}|^p F^\star d\mathcal{H}^n\\
&\leqslant \frac{1}{\mathcal{H}^n(S_0(R_\Omega))}\int_{\partial\Omega}|\mathrm{B}|^p\frac{|\langle x,\nu_x\rangle|}{|x|}\left(\frac{R_\Omega}{|x|}\right)^n d\mathcal{H}^n \leqslant \frac{2^n\mathcal{H}^n(\partial\Omega)}{\mathcal{H}^n(S_0(R_\Omega))}\|\mathrm{B}\|_p^p
\end{aligned}$$

Now $\frac{\mathcal{H}^n(\partial\Omega)}{\mathcal{H}^n(S_0(R_\Omega))} = \frac{I(\Omega)}{I(\mathbb{B}^{n+1})} = 1+\delta(\Omega) \leqslant 2$ which gives with Lemma 5 and the fact that $P(\Omega)^{1/n} = R_\Omega|\mathbb{B}^{n+1}|^{1/n+1}I(\Omega)^{1/n}$

$$\|\nabla d\varphi\|_p \leqslant C(n,p,K)\left(\frac{1}{R_\Omega^2}+\frac{1}{R_\Omega P(\Omega)^{1/n}}\right) \leqslant \frac{C(n,p,K)}{R_\Omega^2}$$

If we set $u:\mathbb{S}^n\to\mathbb{R}$ defined by $u(x)=\varphi(R_\Omega x)-1$ we have for D large enough $\|u\|_\infty = \|\varphi-1\|_\infty \leqslant C_2\delta(\Omega)^\beta \leqslant \frac{3}{20(n+1)}$ and $\|du\|_\infty = R_\Omega\|d\varphi\|_\infty \leqslant CK^{1/n}\delta(\Omega)^{1/n\alpha} \leqslant 1/2$ and so Ω is a nearly spherical domain in the sense of Fuglede.

Moreover since $\|\nabla du\|_p = R_\Omega^2\|\nabla d\varphi\|_p \leqslant C(n,p,K)$ we have $\|du\|_{W^{1,p}} \leqslant C(n,p,K)$ and by the Campanato-Morrey estimate, we then get for any $x,x_0\in\mathbb{S}^n$ that

$$\frac{||du_x|-|du_{x_0}||}{d_{\mathbb{S}^n}(x,x_0)^{1-\frac{n}{p}}} \leqslant C(n,p,K)$$

and choosing x_0 such that $|du_{x_0}| = \|du\|_\infty$ we have

$$|du_x| \geqslant \|du\|_\infty - C(n,p,K)(d_{\mathbb{S}^n}(x,x_0))^{1-\frac{n}{p}}$$

Let $r_0 := \left(\frac{\|du\|_\infty}{C(n,p,K)}\right)^{\frac{1}{1-\frac{n}{p}}}$. We can assume $r_0 < \frac{\pi}{2}$ by taking $C(n,p,K)$ large enough. Integrating the above inequality on the ball of $\mathbb{S}^n$ of center x_0 and radius r_0 and using the estimates of [11] and then Inequality (I.a) of [11] we get that

$$\begin{aligned}
10\delta(\Omega) \geqslant \|du\|_2^2 &\geqslant \frac{1}{|\mathbb{S}^n|}\int_{B_{x_0}^{\mathbb{S}^n}(r_0)} |du|^2 d\mathcal{H}^n \\
&\geqslant \frac{|\mathbb{S}^{n-1}|}{|\mathbb{S}^n|}\int_0^{r_0}(\|du\|_\infty - C(n,p,K)t^{1-\frac{n}{p}})^2\sin^{n-1}t dt \\
&\geqslant \left(\frac{2}{\pi}\right)^{n-1}\frac{|\mathbb{S}^{n-1}|}{|\mathbb{S}^n|}\int_0^{r_0}(\|du\|_\infty - C(n,p,K)t^{1-\frac{n}{p}})^2 t^{n-1}tdt \\
&\geqslant \frac{1}{C'(n,p,K)}\|du\|_\infty^{2+\frac{n}{1-\frac{n}{p}}} \qquad (6.11)
\end{aligned}$$

From which we infer that

$$(R_\Omega\|d\varphi\|_\infty)^{2+\frac{n}{1-\frac{n}{p}}} = \|du\|_\infty^{2+\frac{n}{1-\frac{n}{p}}} \leqslant C(n,p,K)\delta(\Omega) \qquad (6.12)$$

Using Inequalities (I.b) of [11] and the above inequality (6.12), we get that

$$\begin{aligned}
d_H(\partial\Omega, S_{x_\Omega}(R_\Omega)) = R_\Omega\|u\|_\infty &\leqslant R_\Omega C(n)\|du\|_\infty^{\frac{n-2}{n}}\delta(\Omega)^{1/n} \\
&\leqslant C(n,p,K)R_\Omega\delta(\Omega)^{\frac{2p-n}{2p-2n+np}}
\end{aligned}$$

for $n\geqslant 3$ and $d_H(\partial\Omega, S_{x_\Omega} \leqslant C(p,K)R_\Omega(-\delta(\Omega)\ln\delta(\Omega))^{\frac{1}{2}}$ for $n=2$.

Now since for any $x\in\partial\Omega$, $F^{-1}(x) = x\varphi(x)$, and so $|dF^{-1}_{R_\Omega x}(v)|^2 = (1+u(x))^2|v|^2 + (du_x(v))^2$, we can use the previous estimates on u to obtain

$$||dF^{-1}_{R_\Omega x}(v)|^2 - |v|^2| \leqslant (\|du\|_\infty^2 + 2\|u\|_\infty + \|u\|_\infty^2)|v|^2 \leqslant C(n,p,K)\delta(\Omega)^{\frac{2p-2n}{2p-2n+np}}|v|^2$$

From this and the definition of the Lipschitz distance we conclude that for $\delta(\Omega)$ small enough

$$\begin{aligned}d_L(\partial\Omega, S_{x_\Omega}(R_\Omega)) &\leqslant (|\ln \mathrm{d}il(F)| + |\ln \mathrm{d}il(F^{-1})|)\\ &\leqslant \max(|\ln(1+C\delta(\Omega)^{\frac{2p-2n}{2p-2n+np}}|, |\ln(1-C\delta(\Omega)^{\frac{2p-2n}{2p-2n+np}})|)\\ &\leqslant C(n,p,K)\delta(\Omega)^{\frac{2p-2n}{2p-2n+np}}\end{aligned}$$

where for any diffeomorphism f from $\partial\Omega$ into $S_0(R_\Omega)$, $\mathrm{d}il(f) = \sup_{x\in\partial\Omega} |df(x)|$ (for more details on the Lipschitz distance see [16]). □

We end this section by the construction of simple examples that prove the sharpness of Theorem 4 with respect of the power of delta involved in our estimates:

The sharpness in the case $n = 3$ is already contained in Fuglede's work [11]. In the case $n \geqslant 3$, let $\varphi : \mathbb{R}^n \to \mathbb{R}$ the function defined by

$$\varphi(x) = \begin{cases} 0 & \text{if } |x| \geqslant r := \delta^{\frac{p}{2p-2n+pn}} \\ \frac{1}{3}(r-|x|)^{2-\frac{n}{p}} & \text{if } \frac{r}{2} \leqslant |x| \leqslant r \\ \frac{1}{3}(2(\frac{r}{2})^{2-\frac{n}{p}} - |x|^{2-\frac{n}{p}}) & \text{if } |x| \leqslant r/2 \end{cases} \tag{6.13}$$

φ is a $C^{1,1-\frac{n}{p}}$ function on $\mathbb{R}^n$ with

$$\nabla\varphi(x) = \begin{cases} 0 & \text{if } |x| \geqslant r \\ \frac{1}{3}(\frac{n}{p}-2)(r-|x|)^{1-\frac{n}{p}}\frac{x}{|x|} & \text{if } \frac{r}{2} \leqslant |x| \leqslant r \\ \frac{1}{3}(\frac{n}{p}-2)|x|^{1-\frac{n}{p}}\frac{x}{|x|} & \text{if } |x| \leqslant r/2 \end{cases} \tag{6.14}$$

from which we infer that $\|\varphi\|_\infty = C(n,p)\delta^{\frac{2p-n}{2p-2n+pn}}$, $\|d\varphi\|_\infty \leqslant C(n,p)\delta^{\frac{p-n}{2p-2n+pn}}$ and $\frac{1}{C(n,p)}\delta \leqslant \int_{\mathbb{R}^n} |d\varphi|^2 d\mathcal{H}^n \leqslant C(n,p)\delta$. φ can be transposed to a function defined on $\mathbb{S}^n$ (via the exponential map at a fixed point of $\mathbb{S}^n$) for δ small enough. The previous estimates will be preserved and the surface $S_\varphi = \{(1+\varphi(x))x, x \in \mathbb{S}^n\}$ will be an almost spherical surface in the sense of Fuglede. In particular, according to the inequality (I.a) of [11], the isoperimetric deficit of the domain Ω_φ bounded by S_φ satisfies $\frac{\delta}{C(n,p)} \leqslant \delta(\Omega_\varphi) \leqslant C(p,n)\delta$. Since $d_H(S_\varphi, S_{x_\Omega}(R_\Omega)) = \|\varphi\|_\infty$, and for any $q < p$ there exists $K(n,q)$ such that $\|\nabla d\varphi\|_q \leqslant K(n,q)$ for any $\delta > 0$, we infer that $\|B\|_q \leqslant K(n,q)$ for any $\delta > 0$. These examples prove that the estimate of Theorem 4 are sharp with respect to the powers of δ involved in the estimate on d_H. An easy computation show that it is the same way for the estimate on d_L.

7. Almost extremal domains for Chavel's inequality

Proof of Theorem 5 Let Σ be an embedded compact hypersurface bounding a domain Ω in $\mathbb{R}^{n+1}$ and let X be the vector position. Up to a translation we can assume that

$\int_\Sigma X d\mathcal{H}^n = 0$ which allows us to use the variational characterization. Then

$$(n+1)^2 \frac{|\Omega|^2}{\mathcal{H}^n(\Sigma)^2} = \left(\frac{1}{\mathcal{H}^n(\Sigma)} \int_\Omega \frac{1}{2}\Delta |X|^2 d\mathcal{H}^{n+1}\right)^2 = \left(\frac{1}{\mathcal{H}^n(\Sigma)} \int_\Sigma \langle X, \nu\rangle \, d\mathcal{H}^n\right)^2$$
$$\leqslant \|X\|_1^2 \leqslant \|X\|_2^2 \leqslant \frac{\|dX\|_2^2}{\lambda_1^\Sigma} = \frac{n}{\lambda_1^\Sigma}$$
$$= \frac{(n+1)^2 \mathcal{H}^n(\Sigma)^{2/n}}{I(\Omega)^{2(\frac{n+1}{n})}}(1+\gamma(\Omega)) = (n+1)^2 \frac{|\Omega|^2}{\mathcal{H}^n(\Sigma)^2}(1+\gamma(\Omega))$$

Let us put $\rho_\Omega := (n+1)\frac{|\Omega|}{\mathcal{H}^n(\Sigma)}$. From the inequalities above we deduce that

$$|\,\|X\|_2^2 - \rho_\Omega^2| \leqslant \rho_\Omega^2 \gamma(\Omega) \quad \text{and} \quad |\,\|X\|_1^2 - \rho_\Omega^2| \leqslant \rho_\Omega^2 \gamma(\Omega)$$

which gives for $\gamma(\Omega) < 1$

$$\||X| - \rho_\Omega\|_2^2 = \|X\|_2^2 - 2\rho_\Omega \|X\|_1 + \rho_\Omega^2$$
$$\leqslant \rho_\Omega^2((1+\gamma(\Omega)) - 2(1-\gamma(\Omega))^{1/2} + 1) \leqslant 3\rho_\Omega^2 \gamma(\Omega)$$

Now by the divergence theorem to the field $Z = (|X| - \rho_\Omega)\frac{X}{|X|}$, we get

$$|\Omega \setminus B_0(\rho_\Omega)| \leqslant \int_{\Omega \setminus B_0(1)(\rho_\Omega)} \mathrm{d}ivZ d\mathcal{H}^{n+1} = \int_{\partial\Omega \setminus B_0(\rho_\Omega)} (|X| - \rho_\Omega)\langle \frac{X}{|X|}, \nu\rangle d\mathcal{H}^n$$
$$\leqslant \mathcal{H}^n(\Sigma)\big\||X| - \rho_\Omega\big\|_1 \leqslant 3^{1/2}\mathcal{H}^n(\Sigma)\rho_\Omega \gamma(\Omega)^{1/2}$$

Now since $\rho_\Omega = \frac{1}{1+\delta(\Omega)} R_\Omega \leqslant R_\Omega$ and $|B_0(R_\Omega)| = |\Omega|$ we have

$$|B_0(\rho_\Omega) \setminus \Omega| \leqslant |\Omega \setminus B_0(\rho_\Omega)| \leqslant 3^{1/2}\mathcal{H}^n(\Sigma)\rho_\Omega \gamma(\Omega)^{1/2}$$

It follows that $\big||\Omega| - |B_0(\rho_\Omega)|\big| \leqslant 2(3^{1/2})\mathcal{H}^n(\Sigma)\rho_\Omega\gamma(\Omega)^{1/2}$. From the expression of ρ_Ω, $I(\Omega)$ and the fact that $I(B_0(1)) = (n+1)|B_0(1)|^{\frac{1}{n+1}}$, the last inequality can be rewritten as

$$\left|1 - \frac{I(B_0(1))^{n+1}}{I(\Omega)^{n+1}}\right| \leqslant 2(n+1)3^{1/2}\gamma(\Omega)^{1/2}$$

which gives the desired result. □

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