METRIC SHAPE OF HYPERSURFACES WITH SMALL EXTRINSIC RADIUS OR LARGE λ_1

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Abstract. We determine the metric shape of Euclidean hypersurfaces with large λ_1 or small extrinsic radius. The description of the shape is improved when we assume an a priori bound on the L^p norm of the mean curvature with $p+1$ not less than the dimension of the hypersurfaces.

1. Introduction

Throughout the paper, $X\colon M^n \to \mathbb{R}^{n+1}$ is a closed, connected, immersed Euclidean hypersurface (with $n \geqslant 2$). We set v_M its volume, B its second fundamental form, $\mathrm{H} = \frac{1}{n}\mathrm{tr}\,\mathrm{B}$ its mean curvature, r_M its extrinsic radius (i.e. the least radius of the Euclidean balls containing M), $0 = \lambda_0^M < \lambda_1^M \leqslant \lambda_2^M \leqslant \cdots$ the non-decreasing sequence of its eigenvalues labelled with multiplicities, $Sp(M) = (\lambda_i^M)_{i\in\mathbb{N}}$ and $\overline{X} := \frac{1}{v_M}\int_M X$ its center of mass. For any function $f : M \to \mathbb{R}$, we set $\|f\|_\alpha = \big(\frac{1}{v_M}\int_M |f|^\alpha\big)^{\frac{1}{\alpha}}$. We denote by m_1 the 1-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$ and by $B_x(R)$ the open Euclidean ball with center x and radius R.

The Hasanis-Koutroufiotis inequality ([7]) asserts that

$$r_M\|\mathrm{H}\|_2 \geqslant 1, \tag{1.1}$$

with equality if and only if M is the Euclidean sphere S_M with center $\overline{X}$ and radius $\frac{1}{\|\mathrm{H}\|_2}$. The Reilly inequality ([12]) asserts that

$$\lambda_1^M \leqslant n\|\mathrm{H}\|_2^2, \tag{1.2}$$

once again with equality if and only if M is the sphere S_M.

In this paper, we characterize the limit-points for the Hausdorff distance of the extremizing sequences of Euclidean-hypersurfaces for the Reilly or the Hasanis-Koutroufiotis inequalities. Our study of these almost extremal hypersurfaces began in [2], where their limit-spectrum was described.

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1.1. **Weak Hausdorff convergence vs Hausdorff convergence.** The results described in this subsection arise as a technical tool to deal with our main problem, but we consider it to be of general interest for stability problems involving submanifolds.

Let us remind some basic facts about Hausdorff-Attouch-Wetts topology on closed sets of $\mathbb{R}^{n+1}$. For any subset $A \subset \mathbb{R}^{n+1}$ and any positive real number $\varepsilon > 0$, we set $A_\varepsilon = \{x \in \mathbb{R}^{n+1} / \, d(A, x) \leqslant \varepsilon\}$ the tubular neighbourhood of radius ε of A. $d_H(A, B) = \inf\{\varepsilon > 0 / \, A \subset B_\varepsilon \text{ and } B \subset A_\varepsilon\}$ defines a complete distance on the compact subsets of $\mathbb{R}^{n+1}$ called the Hausdorff distance. If $d_A : \mathbb{R}^{n+1} \to \mathbb{R}$ denotes the distance function to the subset A, we have $d_H(A, B) = \|d_A - d_B\|_\infty$ and so the Hausdorff topology on compact subset of $\mathbb{R}^{n+1}$ coincides with the topology of the uniform convergence on $\mathbb{R}^{n+1}$ of the associated distance functions. Seemingly, on the set of closed subset of $\mathbb{R}^{n+1}$ we consider the Attouch-Wetts topology , that is the topology of the uniform convergence on compact subset of the distance functions. It is a complete, metrizable topology induced by the distance $d_{AW}(A, B) = \sum_{R \in \mathbb{N}^*} 2^{-R} \inf\bigl(1, \sup_{x \in B_0(R)} |d_A(x) - d_B(x)|\bigr)$. We have $\lim_k d_{AW}(A_k, B) = 0$ if and only if $\lim_k d'_R(A_k, B) = 0$ for any $R \in \mathbb{N}$ large enough, where $d'_R(A, B) = \inf\{\varepsilon > 0 / \, A \cap B_0(R) \subset B_\varepsilon \text{ and } B \cap B_0(R) \subset A_\varepsilon\}$ (see the proof of Proposition 3.1.6 in [5]). If (A_n) is a sequence of closed, connected subsets of $\mathbb{R}^{n+1}$ that converges to a closed, bounded limit Z, then Z is connected, the (A_n) are uniformly bounded for n large enough and we have $d_H(A_n, Z) \to 0$ (see Lemma 3.2.2 in [5]). Note also that when $d_{AW}(A_n, B) \to 0$ then we have the relation

$$B = \bigcup_{(a_l) \in \prod_{l \in \mathbb{N}} A_l} \text{limit set of } (a_l)_{l \in \mathbb{N}}$$

In this paper, a sequence $(M_k^m)_{k \in \mathbb{N}}$ of immersed submanifolds of dimension m in $\mathbb{R}^{n+1}$ is said to weakly converge in Hausdorff topology to a non empty closed subset $Z \subset \mathbb{R}^{n+1}$ if there exists a sequence of closed subsets $A_k \subset M_k$ such that $d_{AW}(A_k, Z) \to 0$ and $\mathrm{Vol}\,(M_k \setminus A_k)/\mathrm{Vol}\, M_k \to 0$. Note that the weak limit of a sequence is not unique a priori. Any sequence (M_k) that weakly converges to Z has a non-empty limit-set for the Attouch-Wetts distance that is made of closed, connected subset of $\mathbb{R}^{n+1}$ that contain Z (it is an easy consequence of the Ascoli theorem and Lemma 3.1.1 of [5]). Of course, this limit-set is not always equal to $\{Z\}$. Our aim in this part is to describe this limit-set when an a priori L^p bound on the mean curvature is assumed.

Theorem 1.1. *Let $A > 0$ and $p > m - 1$ and $(M_k)_{k \in \mathbb{N}}$ be any sequence of immersed, compact submanifolds of dimension m which weakly converges to $Z \subset \mathbb{R}^{n+1}$.*

If $\mathrm{Vol}\,(M_k)\|\mathrm{H}\|_p^{m-1} \leqslant A$ for any k, then Z is compact, $d_H(M_k, Z) \to 0$ and so the limit-set of $(M_k)_{k \in \mathbb{N}}$ for the Hausdorff distance is reduced to $\{Z\}$.

If $\mathrm{Vol}\,(M_k)\|\mathrm{H}\|_{m-1}^{m-1} \leqslant A$ for any k, then any limit point of $(M_k)_{k \in \mathbb{N}}$ for the Hausdorff distance is a compact, connected subset of the form $Z \cup T \subset \mathbb{R}^{n+1}$ with $m_1(T) \leqslant C(m)A$ where $C(m)$ is a (computable) constant that depends only on the dimension m.

Note that it derives from the proof that in the case $p = m - 1$, we actually have $m_1(T) \leqslant C(m) \sup_{\varepsilon > 0} \liminf_k \int_{M_k \setminus Z_\varepsilon} |\mathrm{H}|^{m-1}$.

This theorem is a consequence of the following decomposition result (see section 4.1), which asserts that a submanifold M can be approximated in Hausdorff distance by the union of any subset $A \subset M^m$ of large relative volume with a finite number of geodesic subtrees, whose total length is bounded by the L^{m-1} norm of the mean curvature. The proof is a refinement of an argument developed by P.Topping in [15] to get an upper bound of $\mathrm{Diam}(M)$ by $\int_M |\mathrm{H}|^{m-1}$.

Lemma 1.2. *There exists a (computable) constant $C = C(m)$ such that, for any compact submanifold M^m of $\mathbb{R}^{n+1}$ and any closed subset $A \subset M$, there exists a finite family $(T_i)_{\in \in I}$ of geodesic trees in M with $A \cap T_i \neq \emptyset$ for any $i \in I$, $d_H\big(A \cup (\cup_{i\in I} T_i), M\big) \leqslant C\big(\mathrm{Vol}\,(M \setminus A)\big)^{\frac{1}{m}}$ and $\sum_{i\in I} m_1(T_i) \leqslant C^{m(m-1)} \int_{M\setminus A} |\mathrm{H}|^{m-1}$.*

The description of the Hausdorff limit-point of weakly convergent sequence given by Theorem 1.1 is rather optimal since we have the following result.

Proposition 1.3. *Let $Z \subset Z'$ be two closed sets of $\mathbb{R}^{n+1}$ with Z' connected and M be a compact, immersible hypersurface of $\mathbb{R}^{n+1}$.*

For any $\alpha \in]0, n-1[$ and any $A > 0$, there exists a sequence of immersion $i_k : M \to \mathbb{R}^{n+1}$ such that $\mathrm{Vol}\, i_k(M) \|B\|^{n-1}_{\alpha, i_k(M)} \leqslant A$ and $i_k(M)$ weakly converges to Z and strongly to Z'.

For any $A > 0$, there exists a sequence of immersion $i_k : M \to \mathbb{R}^{n+1}$ such that $\mathrm{Vol}\, i_k(M) \|B\|^{n-1}_{n-1, i_k(M)} \leqslant A + \mathrm{Vol}\,\mathbb{S}^{n-1} m_1(Z' \setminus Z)$, $\mathrm{Vol}\, i_k(M) \|H\|^{n-1}_{n-1, i_k(M)} \leqslant A + (\frac{n-1}{n})^{n-1} \mathrm{Vol}\,\mathbb{S}^{n-1} m_1(Z' \setminus Z)$ and $i_k(M)$ weakly converges to Z and strongly to Z'.

It shows that if a sequence (M_k) weakly converges to Z with $\mathrm{Vol}\, M_k \|\mathrm{H}\|^{m-1}_p$ bounded for some $p < m-1$ then nothing can be said a priori about the strong limit points for the Attouch-Wetts topology except that they have to be closed, connected subsets of $\mathbb{R}^{n+1}$ that contains Z. In the case $p = m-1$, it proves that the limit points can be essentially any closed, connected Euclidean subset obtained by attaching hair to Z with total length bounded by the L^{m-1} norm of the mean curvature. Note however that the constant $C(m)$ obtained in Theorem 1.1 was larger than $\big((\frac{m-1}{m})^{m-1} \mathrm{Vol}\,\mathbb{S}^{m-1}\big)^{-1}$. The previous proposition is a corollary of the following, more general result.

Theorem 1.4. *Let $M_1^m, M_2^m \hookrightarrow \mathbb{R}^{n+1}$ be two immersed compact submanifolds, $M_1 \# M_2$ be their connected sum and T be any closed subset of $\mathbb{R}^{n+1}$ such that $M_1 \cup T$ is connected. Then there exists a sequence of immersions $i_k : M_1 \# M_2 \hookrightarrow \mathbb{R}^{n+1}$ such that*

1. *$i_k(M_1 \# M_2)$ weakly converges to M_1 and strongly converges to $M_1 \cup T$,*
2. *the curvatures of $i_k(M_1 \# M_2)$ satisfy*

$$\int_{i_k(M_1\#M_2)} |\mathrm{H}|^{m-1} \to \int_{M_1} |\mathrm{H}|^{m-1} + (\frac{m-1}{m})^{m-1} \mathrm{Vol}\,\mathbb{S}^{m-1} m_1(T'),$$
$$\int_{i_k(M_1\#M_2)} |\mathrm{B}|^{m-1} \to \int_{M_1} |\mathrm{B}|^{m-1} + \mathrm{Vol}\,\mathbb{S}^{m-1} m_1(T'),$$
$$\int_{i_k(M_1\#M_2)} |\mathrm{H}|^{\alpha} \to \int_{M_1} |\mathrm{H}|^{\alpha} \qquad \textit{for any } \alpha \in [1, m-1),$$
$$\int_{i_k(M_1\#M_2)} |\mathrm{B}|^{\alpha} \to \int_{M_1} |\mathrm{B}|^{\alpha} \qquad \textit{for any } \alpha \in [1, m-1),$$

   where $T' = \overline{T \setminus M_1}$,
3. *$\lambda_p(i_k(M_1 \# M_2)) \to \lambda_p(M_1)$ for any $p \in \mathbb{N}$,*
4. *$\mathrm{Vol}\,(i_k(M_1 \# M_2)) \to \mathrm{Vol}\, M_1$.*

Note that $T' \subset T$, $M_1 \cup T' = M_1 \cup T$ and that $m_1(T') \leqslant m_1(T)$. Conditions (3) and (4) in Theorem 1.4 are designed on purpose for our study of almost extremal Euclidean hypersurfaces for the Reilly or Hasanis-Koutroufiotis Inequalities. Of course, the main difficulty in the proof of Theorem 1.4 is to get condition (3).

All the results of this section can be easily extended to the case where $\mathbb{R}^{n+1}$ is replaced by any fixed Riemannian manifold (N, g).

1.2. **Hypersurfaces with large λ_1 or small Extrinsic radius.** Our aim in this section is to study the metric shape of the Euclidean hypersurfaces with almost extremal extrinsic radius or λ_1.

1.2.1. *Almost extremal hypersurfaces weakly converge to S_M.* Our first result describes some volume and curvature concentration properties of almost extremal hypersurfaces that imply weak convergence to S_M. Note that in this result we do not assume any bound on the mean curvature. It easily implies that convex, almost extremal hypersurfaces are Lipschitz close to a Euclidean sphere.

We set $B_x(r)$ the closed ball with center x and radius r in $\mathbb{R}^{n+1}$ and A_η the annulus $\left\{X \in \mathbb{R}^{n+1} / \big| \|X - \bar{X}\| - \frac{1}{\|\mathrm{H}\|_2} \big| \leqslant \frac{\eta}{\|\mathrm{H}\|_2}\right\}$. Throughout the paper we shall adopt the notation that $\tau(\varepsilon|n, p, h, \cdots)$ is a positive function which depends on $n, p, h, \cdots$ and which converges to zero as $\varepsilon \to 0$. Note that these functions τ will always be explicitly computable.

Theorem 1.5. *Any immersed hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $r_M \|\mathrm{H}\|_2 \leqslant 1 + \varepsilon$ (or with $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1^M} \leqslant 1 + \varepsilon$) satisfies*

$$\big\| |\mathrm{H}| - \|\mathrm{H}\|_2 \big\|_2 \leqslant 100 \sqrt[8]{\varepsilon} \|\mathrm{H}\|_2, \tag{1.3}$$

$$\mathrm{Vol}\,(M \setminus A_{\sqrt[8]{\varepsilon}}) \leqslant 100 \sqrt[8]{\varepsilon} v_M. \tag{1.4}$$

Moreover, for any $r > 0$ and any $x \in S_M = \overline{X} + \frac{1}{\|\mathrm{H}\|_2} \cdot \mathbb{S}^n$, we have

$$\Big| \frac{\mathrm{Vol}\,\big(B_x(\frac{r}{\|\mathrm{H}\|_2}) \cap M\big)}{v_M} - \frac{\mathrm{Vol}\,\big(B_x(\frac{r}{\|\mathrm{H}\|_2}) \cap S_M\big)}{\mathrm{Vol}\, S_M} \Big| \leqslant \tau(\varepsilon|n, r) \frac{\mathrm{Vol}\,\big(B_x(\frac{r}{\|\mathrm{H}\|_2}) \cap S_M\big)}{\mathrm{Vol}\, S_M}. \tag{1.5}$$

Note that (1.5) implies not only that M goes near any point of the sphere S_M, but also that the density of M near each point of S_M converges to $v_M / \mathrm{Vol}\, S_M$ at any scale. However, the convergence is not uniform with respect to the radius r. We infer that $A_{\tau(\varepsilon|n)} \cap M$ is Hausdorff close to S_M, which implies weak convergence to S_M of almost extremal hypersurfaces.

Corollary 1.6. *For any immersed hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $r_M \|\mathrm{H}\|_2 \leqslant 1 + \varepsilon$ (or with $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1^M} \leqslant 1 + \varepsilon$) there exists a subset $A \subset M$ such that* $\mathrm{Vol}\,(M \setminus A) \leqslant \tau(\varepsilon|n) v_M$ *and* $d_H(A, S_M) \leqslant \frac{\tau(\varepsilon|n)}{\|\mathrm{H}\|_2}$.

In the case where M is the boundary of a convex body in $\mathbb{R}^{n+1}$ with $r_M \|\mathrm{H}\|_2 \leqslant 1 + \varepsilon$ (or with $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1^M} \leqslant 1 + \varepsilon$), the previous result implies easily the following.

Theorem 1.7. *Any convex, compact hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $r_M \|\mathrm{H}\|_2 \leqslant 1 + \varepsilon$ (or with $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1^M} \leqslant 1 + \varepsilon$) satisfies $d_L(M, S_M) \leqslant \frac{\tau(\varepsilon|n)}{\|\mathrm{H}\|_2}$, where d_L is the Lipschitz distance.*

1.2.2. *Hausdorff limit-set of extremizing sequences.* Constructions similar to that made in the proof of Theorem 1.4 shows that we can not expect any control on the topology of almost extremal hypersurfaces nor on the metric shape (even on the diameter) of the part $M \setminus A$ of Corollary 1.6 if we do not assume a strong enough upper bound on the curvature of almost extremal hypersurfaces.

Theorem 1.8. *Let M be any hypersurface immersible in $\mathbb{R}^{n+1}$ and T be a closed subset of $\mathbb{R}^{n+1}$, such that $\mathbb{S}^n \cup T$ is connected (resp. and $T \cup \mathbb{S}^n \subset B_0(1)$). There exists a sequence of immersions $j_i : M \hookrightarrow \mathbb{R}^{n+1}$ of M which satisfies*
1) $\lambda_1^{j_i(M)} \to \lambda_1(\mathbb{S}^n)$ (resp. $r_{j_i(M)} \to 1$),
2) $\|\mathrm{B}_i - \mathrm{Id}\|_p \to 1$ for any $p < n-1$,
3) $\mathrm{Vol}\, j_i(M) \to \mathrm{Vol}\,\mathbb{S}^n$,
4) $j_i(M)$ converges to $\mathbb{S}^n \cup T$ in Hausdorff distance,
5) $\mathrm{Vol}\, j_i(M)\|\mathrm{H}_i\|_{n-1}^{n-1} \to C(n)m_1(T) + \mathrm{Vol}\,\mathbb{S}^n$.

Note that in the constructions of almost extremal hypersurfaces made in Theorem 1.8, the only way to keep $\|\mathrm{H}_i\|_{n-1}^{n-1}$ bounded is to take a limit $\mathbb{S}^n \cup T$ with T a set of Hausdorff dimension 1 and length bounded (due to point 5)). On the other hand, Corollary 1.6 and Theorem 1.1 imply the following metric shape stability result.

Theorem 1.9. *For any $n \geqslant 3$ and any immersed hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $v_M\|\mathrm{H}\|_{n-1}^n \leqslant A$ and $r_M\|\mathrm{H}\|_2 \leq 1+\varepsilon$ (or with $v_M\|\mathrm{H}\|_{n-1}^n \leqslant A$ and $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1} \leqslant 1+\varepsilon$) there exists a subset T of 1-dimensional Haussdorff measure less than $C(n)\int_M |\mathrm{H}|^{n-1} \leqslant C(n)A\|\mathrm{H}\|_2^{-1}$ such that $T \cup S_M$ is connected and $d_H(M, S_M \cup T) \leqslant \tau(\varepsilon|n, A)\|\mathrm{H}\|_2^{-1}$.*

More precisely, for any sequence $(M_k)_{k\in\mathbb{N}}$ of immersed hypersurfaces normalized by $\|\mathrm{H}_k\|_2 = 1$ and $\overline{X}_k = 0$, which satisfies $v_{M_k}\|\mathrm{H}_k\|_{n-1}^n \leqslant A$ and $r_{M_k} \to 1$ (or $v_{M_k}\|\mathrm{H}_k\|_{n-1}^n \leqslant A$ and $\frac{n}{\lambda_1(M_k)} \to 1$) there exist a closed subset $T \subset \mathbb{R}^{n+1}$ and a subsequence $M_{k'}$ such that $m_1(T) \leqslant C(n)A$, $T\cup\mathbb{S}^n$ is connected and $d_H(M_{k'}, \mathbb{S}^n \cup T) \to 0$.

Here also the constant $C(n)$ of this theorem is not the same as in Theorem 1.8. So we do not have an exact computation of the Hausdorff limit point in the case $p = n-1$ but we conjecture that it is just a mater of non optimality of the constant $C(m)$ in the bound on $m_1(T)$ in Theorem 1.1.

Finally, as a direct consequence of Theorem 1.1, we get the following result.

Theorem 1.10. *Let $2 \leqslant n-1 < p \leqslant +\infty$. Any immersed hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $v_M\|\mathrm{H}\|_p^n \leqslant A$ and $r_M\|\mathrm{H}\|_2 \leq 1+\varepsilon$ (or with $v_M\|\mathrm{H}\|_p^n \leqslant A$ and $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1} \leqslant 1+\varepsilon$) satisfies $d_H(M, S_M) \leqslant \tau(\varepsilon|n, p, A)\|\mathrm{H}\|_2^{-1}$.*

Theorem 1.10 was already proved in the case $p = +\infty$ and under the stronger assumption $(1+\varepsilon)\lambda_1 \geqslant n\|\mathrm{H}\|_4^2$ in [6], and in the case $p = +\infty$ and under the stronger assumption $r_M\|\mathrm{H}\|_4 \leq 1+\varepsilon$ in [13]. It is also proved in an unpublished previous version of this paper [3] in the case $p > n$. In all these papers, the Hausdorff convergence is obtained by first proving that $\|X\|$ is almost constant in L^2 norm and then by applying a Moser iteration technique to infer that $\|X\|$ is almost constant is L^∞-norm. This scheme of proof cannot be applied in the case $n \geqslant p > n-1$ since the critical exponent for the iteration is $p = n$.

Note that by Theorem 1.9, in the case $v_M\|\mathrm{H}\|_p^n \leqslant A$ with $p > n-1$, almost extremal hypersurfaces for the Reilly inequality are almost extremal hypersurfaces for the Hasanis-Koutroufiotis inequality. Actually, in that case, an hypersurface is Hausdorff close to a sphere if and only if it is almost extremal for the Hasanis-Koutroufiotis inequality. In [2], we prove that an hypersurface Hausdorff close to a sphere or almost extremal for the Hasanis-Koutroufiotis inequality is not necessarily almost extremal for the Reilly inequality, even under the assumption $v_M\|\mathrm{B}\|_p^n \leqslant A$, for any $p \leqslant n$.

The structure of the paper is as follows: in Section 2, we recall some concentration properties for the volume and the mean curvature of almost extremal hypersurfaces (in particular Inequalities (1.4) and (1.3)) and some estimates on the restrictions to hypersurfaces of the homogeneous, harmonic polynomials of $\mathbb{R}^{n+1}$, proved in [2]. They are used in Section 3 to prove Inequality (1.5). Theorem 1.1 is proved in Section 4. We end the paper in section 5 by the proof of Theorem 1.4.

Throughout the paper we adopt the notation that $C(n, k, p, \cdots)$ is function greater than 1 which depends on p, q, n, $\cdots$. It eases the exposition to disregard the explicit nature of these functions. The convenience of this notation is that even though C might change from line to line in a calculation it still maintains these basic features.

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2. Some estimates on almost extremal hypersurfaces

We recall some estimate on almost extremal hypersurfaces proved in [2]. From now on, we assume, without loss of generality, that $\bar{X} = 0$. Let $X^T(x)$ denote the orthogonal projection of $X(x)$ on the tangent space T_xM.

Lemma 2.1 ([2]). *If $n\|\mathrm{H}\|_2^2/\lambda_1^M \leqslant 1+\varepsilon$ or $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ holds, then we have $\|X^T\|_2 \leqslant \sqrt{3\varepsilon}\|X\|_2$ and $\|X - \frac{\mathrm{H}}{\|\mathrm{H}\|_2^2}\nu\|_2 \leqslant \sqrt{3\varepsilon}\|X\|_2$.*

We set $A_\eta = B_0(\frac{1+\eta}{\|\mathrm{H}\|_2}) \setminus B_0(\frac{1-\eta}{\|\mathrm{H}\|_2})$.

Lemma 2.2 ([2]). *If $n\|\mathrm{H}\|_2^2/\lambda_1^M \leqslant 1+\varepsilon$ or $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ holds (with $\varepsilon \leqslant \frac{1}{100}$), then we have $\big\|\|X\| - \frac{1}{\|\mathrm{H}\|_2}\big\|_2 \leqslant \frac{C}{\|\mathrm{H}\|_2}\sqrt[8]{\varepsilon}$, $\||\mathrm{H}| - \|\mathrm{H}\|_2\|_2 \le C\sqrt[8]{\varepsilon}\|\mathrm{H}\|_2$ and $\mathrm{Vol}\,(M \setminus A_{\sqrt[8]{\varepsilon}}) \le C\sqrt[8]{\varepsilon}v_M$, where $C = 6 \times 2^{\frac{2p}{p-2}}$ in the case $(P_{p,\varepsilon})$ and $C = 100$ in the other cases.*

We set $\mathcal{H}^k(M)$ the set of functions $\{P_{|M}\}$, where P is any harmonic, homogeneous polynomials of degree k of $\mathbb{R}^{n+1}$. We also set ψ:$[0,\infty) \to [0,1]$ a smooth function, which is 0 outside $[\frac{(1-2\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}, \frac{(1+2\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}]$ and 1 on $[\frac{(1-\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}, \frac{(1+\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}]$, and φ the function on M defined by $\varphi(x) = \psi(|X_x|^2)$.

Lemma 2.3 ([2]). *For any hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ isometrically immersed with $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ (or $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1} \leqslant 1+\varepsilon$) and for any $P \in \mathcal{H}^k(M)$, we have*

$$\big|\|\mathrm{H}\|_2^{2k}\|\varphi P\|_2^2 - \|P\|_{\mathbb{S}^n}^2\big| \leqslant C\sqrt[32]{\varepsilon}\,\|P\|_{\mathbb{S}^n}^2\,,$$

where $C = C(n,k)$.

If moreover $\varepsilon \leqslant \frac{1}{(2C)^{32}}$*, then we have* $\left\|\Delta(\varphi P) - \mu_k^{S_M}\varphi P\right\|_2 \leqslant C\sqrt[16]{\varepsilon}\mu_k^{S_M}\|\varphi P\|_2$.

3. Proof of Inequality 1.5

By a homogeneity, we can assume $\|\mathrm{H}\|_2 = 1$. Let $\theta \in (0,1)$, $x \in \mathbb{S}^n$ and set $V^n(s) = \mathrm{Vol}\left(B(x,s)\cap\mathbb{S}^n\right)$. Let $\beta(\theta,r) > 0$ small enough so that $(1+\theta/2)V^n\big((1+2\beta)r\big) \leqslant (1+\theta)V^n(r)$ and $(1-\theta/2)V^n\big((1-2\beta)r\big) \geqslant (1-\theta)V^n(r)$. Let $f_1 : \mathbb{S}^n \to [0,1]$ (resp. $f_2 : \mathbb{S}^n \to [0,1]$) be a smooth function such that $f_1 = 1$ on $B_x\big((1+\beta)r\big)\cap\mathbb{S}^n$ (resp. $f_2 = 1$ on $B_x\big((1-2\beta)r\big)\cap\mathbb{S}^n$) and $f_1 = 0$ outside $B_x\big((1+2\beta)r\big)\cap\mathbb{S}^n$ (resp. $f_2 = 0$ outside $B_x\big((1-\beta)r\big)\cap\mathbb{S}^n$). There exist an integer $N(\theta,r)$ and a family $(P_k^i)_{k\leqslant N}$ such that $P_k^i \in \mathcal{H}^k(\mathbb{R}^{n+1})$ and $A = \sup_{\mathbb{S}^n}\big|f_i - \sum_{k\leqslant N}P_k^i\big| \leqslant \|f_i\|_{\mathbb{S}^n}\theta/18$. We extend f_i to $\mathbb{R}^{n+1}\setminus\{0\}$ by $f_i(X) = f_i\big(\frac{X}{|X|}\big)$. Then we have

$$\Big|\|\varphi f_i\|_2^2 - \frac{1}{\mathrm{Vol}\,\mathbb{S}^n}\int_{\mathbb{S}^n}|f_i|^2\Big| \leqslant I_1 + I_2 + I_3$$

where

$$I_1 := \Big|\frac{1}{v_M}\int_M\Big(|\varphi f_i|^2 - \varphi^2\big(\sum_{k\leqslant N}|X|^{-k}P_k^i\big)^2\Big)dv\Big|$$

$$I_2 := \Big|\frac{1}{v_M}\int_M\varphi^2\big(\sum_{k\leqslant N}|X|^{-k}P_k^i\big)^2dv - \sum_{k\leqslant N}\|P_k^i\|_{\mathbb{S}^n}^2\Big|$$

and

$$I_3 := \Big|\frac{1}{\mathrm{Vol}\,\mathbb{S}^n}\int_{\mathbb{S}^n}\Big(\big(\sum_{k\leqslant N}P_k^i\big)^2 - f_i^2\Big)\Big|.$$

On $\mathbb{S}^n$ we have $\big|f_i^2 - (\sum_{k\leqslant N}P_k^i)^2\big| \leqslant A\big(2\sup_{\mathbb{S}^n}|f_i| + A\big) \leqslant \|f_i\|_{\mathbb{S}^n}^2\theta/6$ and on M we have

$$\varphi^2\Big|f_i^2(X) - \big(\sum_{k\leqslant N}|X|^{-k}P_k^i(X)\big)^2\Big| \leqslant \Big|f_i^2\big(\frac{X}{|X|}\big) - \Big(\sum_{k\leqslant N}P_k^i\big(\frac{X}{|X|}\big)\Big)^2\Big| \leqslant \|f_i\|_{\mathbb{S}^n}^2\theta/6$$

.

Hence $I_1 + I_3 \leqslant \|f_i\|_{\mathbb{S}^n}^2\theta/3$. Now

$$\begin{aligned}
I_2 \leqslant&\Big|\frac{1}{v_M}\int_M\varphi^2\sum_{k\leqslant N}\frac{(P_k^i)^2}{|X|^{2k}}dv - \sum_{k\leqslant N}\|P_k^i\|_{\mathbb{S}^n}^2\Big| + \frac{1}{v_M}\Big|\int_M\varphi^2\sum_{1\leqslant k\neq k'\leqslant N}\frac{P_k^iP_{k'}^i}{|X|^{k+k'}}dv\Big|\\
\leqslant&\frac{1}{v_M}\int_M\varphi^2\sum_{k\leqslant N}\Big|\frac{1}{|X|^{2k}} - \|\mathrm{H}\|_2^{2k}\Big|(P_k^i)^2dv\\
&+\frac{1}{v_M}\int_M\sum_{1\leqslant k\neq k'\leqslant N}\varphi^2\Big|\frac{1}{|X|^{k+k'}} - \|\mathrm{H}\|_2^{k+k'}\Big||P_k^iP_{k'}^i|dv\\
&+\sum_{k\leqslant N}\big|\|\mathrm{H}\|_2^{2k}\|\varphi P_k^i\|_2^2 - \big\|P_k^i\big\|_{\mathbb{S}^n}^2\big| + \sum_{1\leqslant k\neq k'\leqslant N}\frac{\|\mathrm{H}\|_2^{k+k'}}{v_M}\Big|\int_M\varphi^2P_k^iP_{k'}^idv\Big|
\end{aligned}$$

We have $\varphi^2\big|\frac{1}{|X|^{k+k'}}-\|\mathrm{H}\|_2^{k+k'}\big|\leqslant\varphi^2(k+k')2^{k+k'+2}\sqrt[16]{\varepsilon}\|\mathrm{H}\|_2^{k+k'}$ by assumption on φ. From this and Lemma 2.3, we have

$$\begin{aligned}I_2\leqslant&N^2 4^{N+1}\sqrt[16]{\varepsilon}\sum_{k\leqslant N}\|\mathrm{H}\|_2^{2k}\|\varphi P_k^i\|_2^2+\sqrt[32]{\varepsilon}\sum_{k\leqslant N}C(n,k)\left\|P_k^i\right\|_{\mathbb{S}^n}^2\\&+\sum_{1\leqslant k\neq k'\leqslant N}\frac{\|\mathrm{H}\|_2^{k+k'}}{v_M}\Big|\int_M\varphi^2P_k^iP_{k'}^i dv\Big|\\\leqslant&C(n,N)\sqrt[32]{\varepsilon}+\sum_{1\leqslant k\neq k'\leqslant N}\frac{\|\mathrm{H}\|_2^{k+k'}}{v_M}\Big|\int_M\varphi^2P_k^iP_{k'}^i dv\Big|\end{aligned}$$

and, by Lemma 2.3, we have

$$\begin{aligned}&\Big|\frac{\|\mathrm{H}\|_2^2(\mu_k-\mu_{k'})}{v_M}\int_M\varphi^2P_k^iP_{k'}^i dv\Big|\\&\leqslant\int_M\frac{|\varphi P_k^i\big(\Delta(\varphi P_{k'}^i)-\|\mathrm{H}\|_2^2\mu_{k'}\varphi P_{k'}^i\big)|}{v_M}dv+\int_M\frac{|\varphi P_{k'}^i\big(\Delta(\varphi P_k^i)-\|\mathrm{H}\|_2^2\mu_k\varphi P_k^i\big)|}{v_M}dv\\&\leqslant\|\varphi P_k^i\|_2\big\|\Delta(\varphi P_{k'}^i)-\|\mathrm{H}\|_2^2\mu_{k'}\varphi P_{k'}^i\big\|_2+\|\varphi P_{k'}^i\|_2\big\|\Delta(\varphi P_k^i)-\|\mathrm{H}\|_2^2\mu_k\varphi P_k^i\big\|_2\\&\leqslant C(n,N)\sqrt[16]{\varepsilon}\|\mathrm{H}\|_2^2\|\varphi P_{k'}^i\|_2\|\varphi P_k^i\|_2\end{aligned}$$

under the condition $\varepsilon\leqslant(\frac{1}{2C(n,N)})^{32}$. Since $\mu_k-\mu_{k'}\geqslant n$ when $k\neq k'$, we get

$$\sum_{1\leqslant k\neq k'\leqslant N}\Big|\frac{1}{v_M}\int_M\varphi^2P_k^iP_{k'}^i dv\Big|\leqslant\sum_{1\leqslant k\neq k'\leqslant N}C(n,N)\sqrt[16]{\varepsilon}\|\varphi P_{k'}^i\|_2\|\varphi P_k^i\|_2\leqslant\frac{C(n,N)\sqrt[16]{\varepsilon}}{\|\mathrm{H}\|_2^{k+k'}}$$

hence $I_2\leqslant C(n,N)\sqrt[32]{\varepsilon}$ and

$$\Big|\|\varphi f_i\|_2^2-\frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n}f_i^2\Big|\leqslant C(n,N)\sqrt[32]{\varepsilon}+\frac{\theta}{3}\|f_i\|_{\mathbb{S}^n}^2.$$

We infer that if $\sqrt[32]{\varepsilon}\leqslant\frac{V^n((1-2\beta)r)\theta}{6C(n,N)\operatorname{Vol}\mathbb{S}^n}\leqslant\frac{\|f_i\|_{\mathbb{S}^n}^2\theta}{6C(n,N)}$, then we have

$$\Big|\|\varphi f_i\|_2^2-\frac{1}{\operatorname{Vol}\mathbb{S}^n}\int_{\mathbb{S}^n}|f_i|^2\Big|\leqslant\theta\|f_i\|_{\mathbb{S}^n}^2/2$$

Note that N depends on r and θ but not on x since $O(n+1)$ acts transitively on $\mathbb{S}^n$. By assumption on f_1 and f_2, we have

$$\begin{aligned}\frac{\operatorname{Vol}\big(B_x((1+\beta)r-\sqrt[16]{\varepsilon})\cap M\cap A_{\sqrt[16]{\varepsilon}}\big)}{v_M}&\leqslant\|\varphi f_1\|_2^2\leqslant(1+\frac{\theta}{2})\|f_1\|_{\mathbb{S}^n}^2\\&\leqslant(1+\frac{\theta}{2})\frac{V^n((1+2\beta)r)}{\operatorname{Vol}\mathbb{S}^n}\leqslant(1+\theta)\frac{V^n(r)}{\operatorname{Vol}\mathbb{S}^n}\\\frac{\operatorname{Vol}\big(B_x((1-\beta)r+2\sqrt[16]{\varepsilon})\cap M\cap A_{2\sqrt[16]{\varepsilon}}\big)}{v_M}&\geqslant\|\varphi f_2\|_2^2\geqslant(1-\frac{\theta}{2})\|f_2\|_{\mathbb{S}^n}^2\\&\geqslant(1-\frac{\theta}{2})\frac{V^n((1-2\beta)r)}{\operatorname{Vol}\mathbb{S}^n}\geqslant(1-\theta)\frac{V^n(r)}{\operatorname{Vol}\mathbb{S}^n}\end{aligned}$$

In the second estimates, we can replace ε by $\varepsilon/2^{16}$ as soon as we assume that $\varepsilon\leqslant\big(\min(\frac{1}{4^{16}},\frac{1}{(2C(n,N))^{32}},(\beta r)^{16},(\frac{\|f_i\|_{\mathbb{S}^n}^2\theta}{6(C(n,N)})^{32}\big)=K(\theta,r,n)$. Then we have $(1-\beta)r+\sqrt[16]{\varepsilon}\leqslant$

$r \leqslant (1+\beta)r - \sqrt[16]{\varepsilon}$ and get

$$\Big|\frac{\mathrm{Vol}\,(B_x(r)\cap M\cap A_{\sqrt[16]{\varepsilon}})}{v_M} - \frac{V^n(r)}{\mathrm{Vol}\,\mathbb{S}^n}\Big| \leqslant \theta\frac{V^n(r)}{\mathrm{Vol}\,\mathbb{S}^n}$$

Combined with Lemma 2.2, we get the result with $\tau(\varepsilon|r,n) = \min\{\theta/\,2^{16}\varepsilon \leqslant K(\theta,r,n)\}$.

4. Proof of Theorem 1.1

4.1. **Proof of Lemma 1.2.**

Proof. In [15], using the Michael-Simon Sobolev inequality as a differential inequation on the volume of intrinsic spheres, P.Topping prove the following lemma.

Lemma 4.1 ([15])**.** *Suppose that M^m is a submanifold smoothly immersed in $\mathbb{R}^{n+1}$, which is complete with respect to the induced metric. Then there exists a constant $\delta(m)>0$ such that for any $x\in M$ and $R>0$, at least one of the following is true:*

(i) $M(x,R) := \sup_{r\in(0,R]}\int_{B_x(r)}|\mathrm{H}|^{m-1}/r > \delta^{m-1}$;

(ii) $\kappa(x,R) := \inf_{r\in(0,R]}\frac{\mathrm{Vol}\,B_x(r)}{r^m} > \delta$.

Where $B_x(r)$ is the geodesic ball in M for the intrinsic distance.

In this section, d stands for the intrinsic distance on M. If $d_H(A,M)\leqslant 10(\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}$, then we just set $T=\emptyset$. Otherwise, there exists $x_0\in M$ such that $d(A,x_0) = d_H(A,M)\geqslant 10(\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}$. Let $\gamma_0:[0,l_0]\to M\setminus A$ be a normal minimizing geodesic from x_0 to A. For any $t\in I_0=[0,l_0-(\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}]$, we have $B_{\gamma_0(t)}\big((\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}\big)\subset M\setminus A$ and by the previous lemma, there exists $r_{0,t}\leqslant(\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}$ such that $r_{0,t}\leqslant\frac{1}{\delta^{m-1}}\int_{B_{\gamma_0(t)}(r_{0,t})}|\mathrm{H}|^{m-1}$. By compactness of $\gamma_0(I_0)$ and by Wiener's selection principle, there exists a finite family $(t_j)_{j\in J_0}$ of elements of I_0 such that the balls of the family $\mathcal{F}_0=\big(B_{\gamma_0(t_j)}(r_{0,t_j})\big)_{j\in J_0}$ are disjoint and $\gamma(I_0)\subset\cup_{j\in J_0}B_{\gamma_0(t_j)}(3r_{0,t_j})$. Hence we have

$$\frac{\delta^{m-1}(l_0-(\frac{\mathrm{Vol}\,M\backslash A}{\delta})^{\frac{1}{m}})}{6}\leqslant\delta^{m-1}\sum_{j\in J_0}r_{0,t_j}\leqslant\sum_{j\in J_0}\int_{B_{\gamma_0(t_j)}(r_{0,t_j})}|\mathrm{H}|^{m-1}$$

And by assumption on l_0, we get $10(\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}\leqslant l_0\leqslant\frac{10}{\delta^{m-1}}\sum_{j\in J_0}\int_{B_{\gamma_0(t_j)}(r_{0,t_j})}|\mathrm{H}|^{m-1}$.

If $d_H\big(A\cup\gamma_0([0,l_0]),M\big)\leqslant 10(\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}$, we set $T=\gamma_0([0,l_0])$. Otherwise, we set x_1 a point of $M\backslash A$ at maximal distance l_1 from $A\cup\gamma_0([0,l_0])$ and γ_1 the corresponding minimal geodesic. We set $I_1=[2(\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}, l_1-2(\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}]$. Once again, by the Wiener Lemma applied to $\gamma_1(I_1)$ we get a family of disjoint balls $\mathcal{F}_1=\big(B_{\gamma_1(t_j)}(r_{1,t_j})\big)_{j\in J_1}$ such that

$$\frac{\delta^{m-1}(l_1-4(\frac{\mathrm{Vol}\,M\backslash A}{\delta})^{\frac{1}{\delta}})}{6}\leqslant\delta^{m-1}\sum_{j\in J_1}r_{1,t_j}\leqslant\sum_{j\in J_1}\int_{B_{\gamma_1(t_j)}(r_{1,t_j})}|\mathrm{H}|^{m-1}$$

which gives $10(\frac{\mathrm{Vol}\,M\backslash A}{\delta(m)})^{\frac{1}{m}}\leqslant l_1\leqslant\frac{10}{\delta^{m-1}}\sum_{j\in J_1}\int_{B_{\gamma_1(t_j)}(r_{1,t_j})}|\mathrm{H}|^{m-1}$. Note also that the balls of the family $\mathcal{F}_1\cup\mathcal{F}_2$ are disjoint.

If $d_H\big(A\cup\gamma_0([0,l_0])\cup\gamma_1([0,l_1]),M\big)\leqslant 10(\frac{\mathrm{Vol}\,M\setminus A}{\delta(m)})^{\frac{1}{m}}$, we set $T=\gamma_0([0,l_0])\cup\gamma_1([0,l_1])$. Note that T is a geodesic tree (if $\gamma_1(l_1)\in\gamma_0([0,l_1])$) or the disjoint union of 2 geodesic trees.

If $d_H\big(A\cup\gamma_0([0,l_0])\cup\gamma_1([0,l_1]),M\big)\geqslant 10(\frac{\mathrm{Vol}\,M\setminus A}{\delta(m)})^{\frac{1}{m}}$, then by iteration of what was made for x_1, γ_1 and $\mathcal{F}_1$, we construct a family $(x_j)_j$ of points, a family $(\gamma_j)_j$ of geodesics and a family $(\mathcal{F}_j)_j$ of sets of disjoint balls. Since the $(x_j)_j$ are $10(\frac{\mathrm{Vol}\,M\setminus A}{\delta(m)})^{\frac{1}{m}}$-separated in M and since M is compact, the families are finite and only a finite step of iterations can be made. The set $T=\cup_j\gamma_j([0,l_j])$ is the disjoint union of a finite set of finite geodesic trees and we have

$$m_1(T)\leqslant\frac{10}{\delta^{m-1}}\sum_j\sum_{k\in J_j}\int_{B_{\gamma_j(t_j)}(r_{j,t_k})}|\mathrm{H}|^{m-1}\leqslant\frac{10}{\delta^{m-1}}\int_{M\setminus A}|\mathrm{H}|^{m-1}. \tag{4.1}$$

□

4.2. **Proof of Theorem 1.1.** We begin the proof by the case where $\int_{M_k}|\mathrm{H}|^{m-1}\leqslant A$. By Topping's upper bound on the diameter [15] the sequence (M_k) is contained in a fixed ball. By Blaschke selection theorem, we can assume that the sequence M_k converges in Hausdorff topology to a compact, connected limit set M_∞, which contains Z. Note also that the classical Michael-Simon Sobolev inequality applied to $f=1$ gives us $(\mathrm{Vol}\,M_k)^{1-\frac{1}{n}}\leqslant C(n)\int_{M_k}|\mathrm{H}|$ and so by Hölder, we get $\mathrm{Vol}\,M_k\leqslant C(n)(\int_{M_k}|\mathrm{H}|^{n-1})^n\leqslant C(n,A)$.

It just remain to prove that $m_1(M_\infty\setminus Z)\leqslant C(m)A$. Let $\ell\in\mathbb{N}^*$ fixed. We set $Z_r=\{x\in\mathbb{R}^{n+1}/\,d(x,Z)\leqslant r\}$. By weak convergence of $(M_k)_k$ to Z and the above upper bound on the volume, we have $\lim_k\mathrm{Vol}\,(M_k\setminus Z_{\frac{1}{3\ell}})=0$ and by Lemma 1.2, there exists a finite union of geodesic trees T_k^ℓ such that $\lim_k d_H\big((M_k\cap Z_{\frac{1}{3\ell}})\cup T_k^\ell,M_\infty\big)=0$ and $m_1(T_k^\ell)\leqslant C(m)\int_{M_k\setminus Z_{\frac{1}{3\ell}}}|\mathrm{H}|^{m-1}$ for any k. Moreover, by construction of the part T in the proof of Lemma 1.2, each connected part of T_k^ℓ is a geodesic tree intersecting $Z_{\frac{1}{3\ell}}\cap M_k$, and by Inequality (4.1), the number of such component intersecting $\mathbb{R}^{n+1}\setminus Z_{\frac{2}{3\ell}}$ is bounded above by $3\ell C(m)\int_{M_k\setminus Z_{\frac{1}{3\ell}}}|\mathrm{H}|^{m-1}$. We can assume that this number is constant up to a subsequence. Their union forms a sequence of compact sets $(\tilde{T}_k^\ell)$ which, up to a subsequence, converges to a set Y that contains $M_\infty\setminus Z_{\frac{1}{\ell}}$. By lower semi-continuity of the m_1-measure for sequence of trees (see Theorem 3.18 in [8]), we get that $m_1(M_\infty\setminus Z_{\frac{1}{\ell}})\leqslant m_1(Y)\leqslant\liminf_k m_1(\tilde{T}_k^\ell)\leqslant C(m)\liminf_k\int_{M_k\setminus Z_{\frac{1}{3\ell}}}|\mathrm{H}|^{m-1}$. Since $M_\infty\setminus Z$ is the monotone union of the $M_\infty\setminus Z_{\frac{1}{\ell}}$, we get that $M_\infty=Z\cup T$ with T a 1-dimensional subset of $\mathbb{R}^{n+1}$ of measure less than $C(m)\sup_l\liminf_k\int_{M_k\setminus Z_{\frac{1}{3l}}}|\mathrm{H}|^{m-1}\leqslant C(m)A$.

In the case $\int_{M_k}|\mathrm{H}|^p\leqslant A$ with $p>m-1$, we have

$$\int_{M_k\setminus Z_{\frac{1}{3\ell}}}|\mathrm{H}|^{m-1}\leqslant\Big(\frac{\mathrm{Vol}\,M_k\setminus Z_{\frac{1}{3\ell}}}{\mathrm{Vol}\,M_k}\Big)^{\frac{p-m+1}{p}}\mathrm{Vol}\,M_k\|\mathrm{H}\|_p^{m-1}$$

So the weak convergence to Z implies that $m_1(M_\infty \setminus Z_{\frac{1}{3\ell}}) = 0$ for any ℓ. Since $M_\infty \setminus Z_{\frac{1}{3\ell}} \neq \emptyset$ implies $m_1(M_\infty \setminus Z) \geqslant \frac{1}{3\ell}$ by what precedes, we get $M_\infty \subset Z_{\frac{1}{3l}}$ for any l, hence $M_\infty = Z$.

4.3. **Proof of Theorems 1.9 and 1.10.** We can assume that $\overline{X}(M_k) = 0$ and $\|\mathrm{H}\|_2 = 1 \leqslant \|\mathrm{H}\|_p$ by scaling. Hence we have $v_{M_k}\|\mathrm{H}\|_p^{n-1} \leqslant v_{M_k}\|\mathrm{H}\|_p^n \leqslant A$ and $S_{M_k} = \mathbb{S}^n$ for any k. We now conclude by Corollary 1.6 and Theorem 1.1.

5. Proof of Theorem 1.4

We first deal the case where T is a segment $[x_0, x_0 + l\nu]$ with $x_0 \in M_1$ and ν a normal vector to M_1 at x_0. The general case will be obtained by iterating this simple case.

5.1. case where T is a segment.

5.1.1. *basic construction.* We take off a small ball of M_2 and glue smoothly instead a curved cylinder that is isometric to the product $[0,1] \times \frac{1}{10}\mathbb{S}^{m-1}$ at the neighbourhood of its left boundary component.

We note H_1 the resulting submanifold and $H_\varepsilon = \varepsilon H_1$. Let $c : [0,l] \to \mathbb{R}^+$ be a $\mathcal{C}^1$ positive function, constant equal to $\frac{1}{10}$ at the neighbourhoods of 0 and l, $T_{c,\varepsilon}$ be a cylinder of revolution isometric to $\{(t,u) \in [0,l] \times \mathbb{R}^m / |u| = \varepsilon c(t)\}$ and J_1 be a cylinder of revolution isometric to $[0,1/4] \times \frac{1}{10}\mathbb{S}^{m-1}$ at the neighbourhood of one of its boundary component and isometric to the flat annulus $B_0(\frac{3}{10}) \setminus B_0(\frac{2}{10}) \subset \mathbb{R}^m)$ at the neighbourhood of its other boundary component. We also set $J_\varepsilon = \varepsilon J_1$ and $N_{c,\varepsilon}$ the submanifold obtained by gluing H_ε, $T_{c,\varepsilon}$ and J_ε.

Since the second fundamental form of $T_{c,\varepsilon}$ is given by $|B|^2 = \frac{(\varepsilon c'')^2}{(1+(\varepsilon c')^2)^3} + \frac{m-1}{\varepsilon^2 c^2 (1+(\varepsilon c')^2)}$, we get

$$\int_{N_{c,\varepsilon}} |\mathrm{B}|^\alpha dv = a(H_1, J_1)\varepsilon^{m-\alpha} + \mathrm{Vol}\,\mathbb{S}^{m-1}\varepsilon^{m-1-\alpha}(m-1)^{\frac{\alpha}{2}} \int_0^l c^{m-1-\alpha} + O_{c,\alpha}(\varepsilon^{m+1-\alpha}),$$

with $a(H_1, J_1)$ a constant that depends only on H_1 and J_1 (not on c, l and ε).

We set M_1^ε the submanifold of $\mathbb{R}^{n+1}$ obtained by flattening M_1 at the neighbourhood of a point $x_0 \in M_1$ and taking out a ball centred at x_0 and of radius $\frac{3\varepsilon}{10}$. More precisely, M_1 is locally equal to $\{x_0 + w + f(w),\ w \in B_0(\varepsilon_0) \subset T_{x_0}M_1\}$ where $f : B_0(\varepsilon_0) \subset T_{x_0}M_1 \to N_{x_0}M_1$ is a smooth function and $N_{x_0}M_1$ is the normal bundle M_1 at x_0. Let $\varphi : \mathbb{R}_+ \to [0,1]$ be a smooth function such that $\varphi = 0$ on $[0, \frac{\varepsilon_0}{3}]$ and $\varphi = 1$ on $[\frac{2\varepsilon_0}{3}, +\infty)$. We set M_1^ε the submanifold obtained by replacing the subset $\{x_0 + w + f(w),\ w \in B_0(\varepsilon_0) \subset T_{x_0}M_1\}$ by $\{x_0 + w + f_\varepsilon(w),\ w \in B_0(\varepsilon_0) \setminus B_0(3\varepsilon/10) \subset T_{x_0}M_1\}$, with $f_\varepsilon(w) = f\big(\varphi(\frac{\varepsilon_0\|w\|}{\varepsilon})w\big)$ for any $\varepsilon \leqslant 3\varepsilon_0/2$. Note that M_1^ε is a smooth deformation of M_1 in a neighbourhood of x_0 and its boundary has a neighbourhood isometric to a flat annulus $B_0(\varepsilon/3) \setminus B_0(3\varepsilon/10)$ in $\mathbb{R}^m$. Note that for ε small enough, $M_1^\varepsilon \setminus \{x \in M_1^\varepsilon / d(x, \partial M_1^\varepsilon) \leqslant 8\varepsilon\}$ is a subset of M_1. This fact will be used below. As a graph of a function, the curvatures of M_1^ε at the neighbourhood of x_0 are given by the formulae

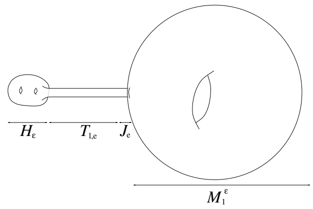
$$|\mathrm{B}_\varepsilon|^2 = \sum_{i,j,k,l=1}^{m} \sum_{p,q=m+1}^{n+1} Ddf_p(e_i, e_k) Ddf_q(e_j, e_l) H^{i,j} H^{k,l} G^{p,q}$$

$$\mathrm{H}_\varepsilon = \frac{1}{m} \sum_{k,l=m+1}^{n+1} \sum_{i,j=1}^{m} Ddf_k(e_i, e_j) H^{i,j} G^{k,l} (\nabla f_l - e_l)$$

where $(e_1, \cdots, e_m)$ is an ONB of $T_{x_0}M_1$, $(e_{m+1}, \cdots, e_{n+1})$ an ONB of $N_{x_0}M_1$, $f_\varepsilon(w) = \sum_{i=m+1}^{n+1} f_i(w) e_i$, $G_{kl} = \delta_{kl} + \langle \nabla f_k, \nabla f_l \rangle$ and $H_{kl} = \delta_{kl} + \langle df_\varepsilon(e_k), df_\varepsilon(e_l) \rangle$. Now f_ε converges in $\mathcal{C}^\infty$ norm to f on any compact subset of $B_0(\varepsilon_0) \setminus \{0\}$, while $|df_\varepsilon|$ and $|Ddf_\varepsilon|$ remain uniformly bounded on $B_0(\varepsilon_0)$ when ε tends to 0. By the Lebesgue convergence theorem, we get

$$\int_{M_1^\varepsilon} |\mathrm{H}_\varepsilon|^\alpha dv \to \int_{M_1} |\mathrm{H}|^\alpha dv \qquad \int_{M_1^\varepsilon} |\mathrm{B}_\varepsilon|^\alpha dv \to \int_{M_1} |\mathrm{B}|^\alpha dv$$

We set M_ε the m-submanifold of $\mathbb{R}^{n+1}$ obtained by gluing M_1^ε and $N_{c,\varepsilon}$ along their boundaries in a fixed direction $\nu \in N_{x_0}M_1$. Note that M_ε is a smooth immersion of $M_1 \# M_2$.



By the computations above, the sequence of immersion $i_k(M_1 \# M_2) = M_{\frac{1}{k}}$ satisfies the properties 1), 2) an 4) announced in Theorem 1.4 when k tends to ∞ (in the case where T is a segment).

5.1.2. *Computation of the limit spectrum of the basic construction.* Let $(\lambda_k)_{k\in\mathbb{N}}$ be the spectrum with multiplicities obtained by union the spectrum of M_1 and of the spectrum $Sp(P_c)$ of the operator $P(f) = -f'' - (m-1)\frac{c'}{c} f'$ on $[0, l]$ with Dirichlet condition at 0 and Neumann condition at l. We will adapt the method developed by C.Anné in [4] to prove that the spectrum of the immersions constructed in the previous toy case converges to $(\lambda_k)_{k\in\mathbb{N}}$. We denote by $(\mu_k)_{k\in\mathbb{N}}$ the eigenvalues of M_1 counted with multiplicities and by $(P_k)_{k\in\mathbb{N}}$ a L^2-ONB of eigenfunctions of M_1. We set $(\nu_k, h_k)_{k\in\mathbb{N}}$ and $(\lambda_k^\varepsilon, f_k^\varepsilon)_{k\in\mathbb{N}}$ the corresponding data on $([0, l], c^{n-1}(t)\,dt)$ and M_ε. We set $\tilde{h}_k^\varepsilon$ the function on M_ε obtained by considering h_k as a function on the cylinder $T_{c,\varepsilon}$, extending it continuously by 0 on J_ε and M_1^ε, and by $h_k(l)$ on H_ε. We also set $\tilde{P}_k^\varepsilon$ the function on M_ε which is equal to $\psi_\varepsilon\big(d(\partial M_1^\varepsilon, \cdot)\big) P_k$ on M_1^ε (with $\psi_\varepsilon(t) = 0$ when $t \leqslant 8\varepsilon$, $\psi_\varepsilon(t) = \frac{\ln t - \ln(8\varepsilon)}{-\ln(8\sqrt{\varepsilon})}$ when $t \in [8\varepsilon, \sqrt{\varepsilon}]$ and $\psi_\varepsilon(t) = 1$ otherwise) and is extended by 0 outside M_1^ε. Using the family $(\tilde{h}_k^\varepsilon, \tilde{P}_k^\varepsilon)$ as test functions, the min-max principle easily gives us

$$\lambda_k^\varepsilon \leqslant \lambda_k \big(1 + \tau(\varepsilon | k, n, c, M_1)\big) \tag{5.1}$$

For any $k \in \mathbb{N}$, we set $\alpha_k = \liminf_{\varepsilon\to 0} \lambda_k^\varepsilon$, $\varphi^{(1)}_{k,\varepsilon}(x) = \varepsilon^{\frac{m}{2}}(f_k^\varepsilon)_{|H_\varepsilon\cup J_\varepsilon}(\varepsilon x)$, seen as a function on $H_1\cup J_1$, $\varphi^{(2)}_{k,\varepsilon}(t,x) = \varepsilon^{\frac{m-1}{2}}(f_k^\varepsilon)_{|T_{c,\varepsilon}}(t,\varepsilon c(t)x)$ seen as a function on $[0,l]\times\mathbb{S}^{m-1}$ and $\varphi^{(3)}_{k,\varepsilon}$ the function on M_1 equal to f_k^ε on $\{x\in M_1^\varepsilon / \, d(x,\partial M_1^\varepsilon)\geqslant 8\varepsilon\}$ and extended harmonically to M_1.

Easy computations give us

(5.2)
$$\int_{H_1\cup J_1} |\varphi^{(1)}_{k,\varepsilon}|^2 = \int_{H_\varepsilon\cup J_\varepsilon} |f_k^\varepsilon|^2, \qquad \int_{H_1\cup J_1} |d\varphi^{(1)}_{k,\varepsilon}|^2 = \varepsilon^2 \int_{H_\varepsilon\cup J_\varepsilon} |df_k^\varepsilon|^2$$
(5.3)
$$\int_{T_{c,\varepsilon}} |f_k^\varepsilon|^2 = \int_0^l \Big(\int_{\mathbb{S}^{m-1}} |\varphi^{(2)}_{k,\varepsilon}(t,u)|^2 du\Big)\sqrt{1+\varepsilon^2(c'(t))^2}\, c^{m-1}(t)dt,$$
(5.4)
$$\int_{T_{c,\varepsilon}} |df_k^\varepsilon|^2 = \int_0^l \Big[\frac{c^{m-1}}{\sqrt{1+\varepsilon^2(c')^2}}\int_{\mathbb{S}^{m-1}} |\frac{\partial\varphi^{(2)}_{k,\varepsilon}}{\partial t}|^2 + \frac{\sqrt{1+\varepsilon^2(c')^2}c^{m-1}}{\varepsilon^2c^2}\int_{\mathbb{S}^{m-1}} |d_{\mathbb{S}^{m-1}}\varphi^{(2)}_{k,\varepsilon}|^2\Big].$$

The argument of C. Anne in [4] (or of Rauch and Taylor in [11]) can be adapted to get that there exists a constant $C(M_1)$ such that $\|\varphi^{(3)}_{k,\varepsilon}\|_{H^1(M_1)} \leqslant C\|f_k^\varepsilon\|_{H^1(M_\varepsilon)}$. Since we have $\|f_k^\varepsilon\|_{H^1(M_\varepsilon)} = 1+\lambda_k^\varepsilon$, (5.1) gives us $\|\varphi^{(3)}_{k,\varepsilon}\|_{H^1(M_1)} \leqslant C(k,M_1,l)$ for $\varepsilon\leqslant\varepsilon_0(k,M_1,l)$. We infer that for any $k\in\mathbb{N}$ there is a subsequence $\varphi^{(3)}_{k,\varepsilon_i}$ which weakly converges to $\tilde f_k^{(3)} \in H^1(M_1)$ and strongly in $L^2(M_1)$ and such that $\lim_i \lambda_k^{\varepsilon_i} = \alpha_k$. By definitions of M_1^ε and $\varphi^{(3)}_{k,\varepsilon}$, and since $\mathcal{C}_0^\infty(M_1\setminus\{x_0\})$ is dense in $\mathcal{C}^\infty(M_1)$, it is easy to see that $\tilde f_k^{(3)}$ is a distributional (hence a strong) solution to $\Delta\tilde f_k^{(3)} = \alpha_k\tilde f_k^{(3)}$ on M_1 (see [14], p.206). In particular, either $\tilde f_k^{(3)}$ is 0 or α_k is an eigenvalue of M_1.

By the same compactness argument, there exists a subsequence $\varphi^{(1)}_{k,\varepsilon_i}$ which weakly converges to $\tilde f_k^{(1)}$ in $H^1(H_1\cup J_1)$ and strongly in $L^2(H_1\cup J_1)$. By Equalities (5.2), we get that $\|d\tilde f_k^{(1)}\|_{L^2(H_1)} = 0$ and so $\tilde f_k^{(1)}$ is constant on H_1 and on J_1 and $\varphi^{(1)}_{k,\varepsilon_i}$ strongly converges to $\tilde f_k^{(1)}$ in $H^1(H_1\cup J_1)$. Let $\eta:[0,10]\to[0,1]$ be a smooth function such that $\eta(x)=1$ for any $x\leqslant 1/2$, $\eta(x)=0$ for any $x\geqslant 1$ and $|\eta'|\leqslant 4$. We set s_ε the distance function to $\partial S_\varepsilon = \{0\}\times\frac{\varepsilon}{10}\mathbb{S}^{m-1}$ in $S_\varepsilon = M_1^\varepsilon\cup J_\varepsilon$ and θ_ε the volume density of S_ε in normal coordinate to ∂S_ε. We set L the distance between the two boundary components of J_1. By construction of S_ε, we have $\frac{3}{10}\geqslant\theta_\varepsilon(s_\varepsilon,u) = \theta_1(s_\varepsilon/\varepsilon)\geqslant 1$ for any $s_\varepsilon\in[0,L\varepsilon]$ and any u normal to ∂S_ε, and $c(M_1)(\frac{s_\varepsilon}{\varepsilon})^{m-1}\geqslant\theta_\varepsilon(s_\varepsilon,u)\geqslant\frac{1}{c(M_1)}(\frac{s_\varepsilon}{\varepsilon})^{m-1}$ for $s_\varepsilon\in[\varepsilon L, 8\varepsilon]$. Hence, if we denote by $S_{\partial S_\varepsilon}(r)$ the set of points in S_ε at distance r

from ∂S_ε, we get for any $r \leqslant 8+L$ that

$$\int_{S_{\partial S_\varepsilon}(\varepsilon r)} (f_k^\varepsilon)^2 = \int_{\frac{\varepsilon}{10}\mathbb{S}^{m-1}} \Big(\int_{\varepsilon r}^1 \frac{\partial}{\partial s_\varepsilon}[\eta(\cdot) f_k^\varepsilon(\cdot,u)]ds_\varepsilon\Big)^2 \theta_\varepsilon(r\varepsilon,u)du$$
$$= \frac{\varepsilon^{m-1}}{10^{m-1}} \int_{\mathbb{S}^{m-1}} \Big(\int_{\varepsilon r}^1 \frac{\partial}{\partial s_\varepsilon}[\eta(\cdot) f_k^\varepsilon(\cdot,\frac{\varepsilon}{10}u)]ds_\varepsilon\Big)^2 \theta_\varepsilon(r\varepsilon,\frac{\varepsilon}{10}u)du$$
$$\leqslant \frac{c(M_1)\varepsilon^{m-1}}{10^{m-1}} \int_{\mathbb{S}^{m-1}} \Big(\int_0^1 \big(\frac{\partial}{\partial s_\varepsilon}[\eta(\cdot) f_k^\varepsilon(\cdot,\frac{\varepsilon}{10}u)]\big)^2 \theta_\varepsilon(s_\varepsilon,\frac{\varepsilon}{10}u)ds_\varepsilon\Big)\Big(\int_0^1 \frac{1}{\theta_\varepsilon(s_\varepsilon,\frac{\varepsilon}{10}u)}ds_\varepsilon\Big)du$$
$$\int_{S_{\partial S_\varepsilon}(\varepsilon r)} (f_k^\varepsilon)^2 \leqslant c(M_1)\|f_k^\varepsilon\|^2_{H^1(S_\varepsilon)}\varepsilon|\ln\varepsilon| \tag{5.5}$$

which gives us $\varepsilon_i\int_{\partial S_{\varepsilon_i}}(f_k^{\varepsilon_i})^2 = \int_{\partial S_1}(\varphi_{k,\varepsilon_i}^{(1)})^2 \to \int_{\partial S_1}(\tilde f_k^{(1)})^2 = 0$ (by the trace inequality and the compactness of the trace operator) and so $\tilde f_k^{(1)}$ is null on J_1.

By (5.4), and since c is positive and $\mathcal{C}^1$ on $[0,l]$, there exists a subsequence $\varphi_{k,\varepsilon_i}^{(2)}$ which converges weakly to $\tilde f_k^{(2)}$ in $H^1([0,l]\times\mathbb{S}^{m-1})$ and strongly in $L^2([0,l]\times\mathbb{S}^{m-1})$. By the trace inequality applied on $[0,l]\times\mathbb{S}^{m-1}$, we also have that $\|\varphi_{k,\varepsilon_i}^{(2)}\|_{L^2(\{l\}\times\mathbb{S}^{m-1})}$ is bounded. Now, since

$$10^{1-m}\varepsilon_i\int_{\{l\}\times\mathbb{S}^{m-1}}|\varphi_{k,\varepsilon_i}^{(2)}|^2 = \varepsilon_i\int_{\{l\}\times\frac{\varepsilon_i}{10}\mathbb{S}^{m-1}}|f_k^{\varepsilon_i}|^2 = \varepsilon_i\int_{\partial H_{\varepsilon_i}}|f_k^{\varepsilon_i}|^2 = \int_{\partial H_1}|\varphi_{k,\varepsilon_i}^{(1)}|^2$$

we get that $\tilde f_k^{(1)} = 0$ on H_1.

We set $h_i(t) = \int_{\mathbb{S}^{m-1}}\varphi_{k,\varepsilon_i}^{(2)}(t,x)dx$ and $h(t) = \int_{\mathbb{S}^{m-1}}\tilde f_k^{(2)}(t,x)dx$, we have $h, h_i \in H^1([0,l])$ (with $h_i'(t) = \int_{\mathbb{S}^{m-1}}\frac{\partial\varphi_{k,\varepsilon_i}^{(2)}}{\partial t}(t,x)dx$), $h_i\to h$ strongly in $L^2([0,l])$ and weakly in $H^1([0,l])$. For any $\psi\in\mathcal{C}^\infty([0,l])$ with $\psi(0)=0$ and $\psi'(l)=0$, seen as a function on $T_{c,\varepsilon}$ and extended by 0 to S_ε and by $\psi(l)$ to H_ε, we have

$$\int_0^l h'(\psi c^{m-1})'\,dt - (m-1)\int_0^l h'\frac{c'}{c}\psi c^{m-1}\,dt = \int_0^l h'\psi' c^{m-1}\,dt$$
$$= \lim_i \int_0^l h_i'(t)\psi'(t)\frac{c^{m-1}}{\sqrt{1+\varepsilon_i^2(c')^2}}\,dt = \lim_i\int_{M_{\varepsilon_i}}\varepsilon_i^{\frac{1-m}{2}}\langle df_k^{\varepsilon_i}, d\psi\rangle = \lim_i\int_{M_{\varepsilon_i}}\varepsilon_i^{\frac{1-m}{2}}\lambda_k^{\varepsilon_i} f_k^{\varepsilon_i}\psi$$
$$= \alpha_k\lim_i\Big(\int_{[0,l]\times\mathbb{S}^{m-1}}\varphi_{k,\varepsilon_i}^{(2)}\psi c^{m-1}\sqrt{1+\varepsilon_i^2(c')^2} + \psi(l)\varepsilon_i^{\frac{1-m}{2}}\int_{H_{\varepsilon_i}} f_k^{\varepsilon_i}\Big)$$
$$= \alpha_k\int_0^l h\psi c^{m-1}\,dt$$

where we have used that $\varepsilon_i^{\frac{1-m}{2}}|\int_{H_{\varepsilon_i}} f_k^{\varepsilon_i}| \leqslant \sqrt{\varepsilon_i}\sqrt{\mathrm{Vol}\,(H_1)\int_{H_{\varepsilon_i}}(f_k^{\varepsilon_i})^2}$. Since c is positive, we get that h is a weak solution to $y'' + (m-1)\frac{c'}{c}y' + \alpha_k y = 0$ on $[0,l]$ and that $h'(l) = 0$. Since we have $10^{m-1}\int_{\partial S_{\varepsilon_i}}(f_k^{\varepsilon_i})^2 = \int_{\{0\}\times\mathbb{S}^{m-1}}(\varphi_{k,\varepsilon_i}^{(2)})^2 \to \int_{\{0\}\times\mathbb{S}^{m-1}}(\tilde f_k^{(2)})^2$ (by compactness of the trace operator) and $\int_{\partial S_{\varepsilon_i}}(f_k^{\varepsilon_i})^2\to 0$ by (5.5), we get $|h(0)|^2 \leqslant \mathrm{Vol}\,\mathbb{S}^{m-1}\int_{\{0\}\times\mathbb{S}^{m-1}}(\tilde f_k^{(2)})^2 = 0$, and so $h(0)=0$. Since $d_{\mathbb{S}^{m-1}}\varphi_{k,\varepsilon_i}^{(2)}$ converges weakly to $d_{\mathbb{S}^{m-1}}\tilde f_k^{(2)}$ in $L^2([0,l]\times\mathbb{S}^{m-1})$, Inequality (5.4) gives $\|d_{\mathbb{S}^{m-1}}\tilde f_k^{(2)}\|_{L^2([0,l]\times\mathbb{S}^{m-1})} = 0$, i.e.

$\tilde{f}_k^{(2)}$ is constant on almost every sphere $\{t\} \times \mathbb{S}^{m-1}$ of $[0,l] \times \mathbb{S}^{m-1}$. We infer that $\tilde{f}_k^{(2)}$ is equal to $\frac{1}{\operatorname{Vol}\mathbb{S}^{m-1}} h$ seen as a function on $[0,l] \times \mathbb{S}^{m-1}$ and so, either $\tilde{f}_k^{(2)} = 0$ or α_k is an eigenvalue of P_c for the Dirichlet condition at 0 and the Neumann condition at l.

To conclude, we have

$$\begin{aligned}
&\int_{M_1} \tilde{f}_k^{(3)} \tilde{f}_l^{(3)} + \int_{[0,l]\times\mathbb{S}^{m-1}} \tilde{f}_k^{(2)} \tilde{f}_l^{(2)} c^{m-1} \\
&= \lim_i \int_{M_1} \varphi_{k,\varepsilon_i}^{(3)} \varphi_{l,\varepsilon_i}^{(3)} + \int_{J_1 \cup H_1} \varphi_{k,\varepsilon_i}^{(1)} \varphi_{l,\varepsilon_i}^{(1)} + \int_{[0,l]\times\mathbb{S}^{m-1}} \varphi_{k,\varepsilon_i}^{(2)} \varphi_{l,\varepsilon_i}^{(2)} c^{m-1} \sqrt{1+\varepsilon_i^2 (c')^2} \\
&= \lim_i \int_{M_{\varepsilon_i}} f_k^{\varepsilon_i} f_l^{\varepsilon_i} - \lim_i \int_{M_1^{\varepsilon_i} \cap B(\partial M_1^{\varepsilon_i}, 8\varepsilon_i)} f_k^{\varepsilon_i} f_l^{\varepsilon_i} + \lim_i \int_{M_1 \setminus \left(M_1^{\varepsilon_i} \setminus B(\partial M_1^{\varepsilon_i}, 8\varepsilon_i)\right)} \varphi_{k,\varepsilon_i}^{(3)} \varphi_{l,\varepsilon_i}^{(3)} \\
&= \delta_{kl},
\end{aligned}$$

where, in the last equality, we have used that $\varphi_{k,\varepsilon_i}^{(3)}$ and $\varphi_{l,\varepsilon_i}^{(3)}$ converge strongly to $\tilde{f}_k^{(3)}$ and $\tilde{f}_l^{(3)}$ in $L^2(M_1)$, that $\operatorname{Vol}\left(M_1 \setminus (M_1^{\varepsilon_i} \setminus B(\partial M_1^{\varepsilon_i}, 8\varepsilon_i)\right)$ tends to 0 with ε_i, and the inequality

$$\int_{M_1^{\varepsilon_i} \cap B(\partial M_1^{\varepsilon_i}, 8\varepsilon_i)} (f_k^{\varepsilon_i})^2 \leqslant c(M_1) \| f_k^{\varepsilon_i} \|_{H^1(M_{\varepsilon_i})} \varepsilon_i^2 |\ln \varepsilon_i|$$

which is obtained by integration of Inequality (5.5) with respect on $r \in [L, L+8]$. Note that we need the inclusion. Hence, by the min-max principle, we have $\alpha_k \geqslant \lambda_k$ for any $k \in \mathbb{N}$. We conclude that $\lim_{\varepsilon \to 0} \lambda_k(M_\varepsilon) = \lambda_k$ for any $k \in \mathbb{N}$. Note that in the case $c \equiv \frac{1}{10}$, the spectrum of P_c with Dirichlet condition at 0 and Neumann condition at l is $\{\frac{\pi^2}{l^2}(k+\frac{1}{2})^2,\ k \in \mathbb{N}\}$ with all the multiplicities equal to 1.

5.1.3. *End of the proof of Theorem 1.4 in the case where T is a segment.* The sequence of basic immersions (M_ε) gives Theorem 1.4 for $T = [x_0, x_0 + l\nu]$, except for the point 3) since all the eigenvalues of $[0,l]$ appear in the spectrum of the limit. To get also point 3) of Theorem 1.4, we will iterate the basic construction. We fix $k \in \mathbb{N}$ and l_k small enough such that $\lambda_1([0,l_k]) > 2k$ and with $l/l_k \in \mathbb{N}$. Applying the basic construction to $M_1' = M_1$, $M_2' = \mathbb{S}^n$ and $T' = [x_0, x_0 + l_k\nu]$, we get an immersion of $N_1 = M_1 \# \mathbb{S}^m$ such that $d_H(M_1 \cup [x_0, x_0 + l_k\nu], N_1) \leqslant 2^{-\frac{l}{l_k}}$, $|\lambda_p(N_1) - \lambda_p(M_1)| \leqslant 2^{-\frac{l}{l_k}}$ for any p such that $\lambda_p(M_1) \leqslant k$, $|\operatorname{Vol} N_1 \setminus M_1^{\varepsilon_0}| \leqslant 2^{-\frac{l}{l_k}} \operatorname{Vol} M_1$, $|\int_{N_1 \setminus M_1^{\varepsilon_0}} |\mathrm{B}|^{m-1} - \operatorname{Vol}\mathbb{S}^{m-1} l_k| \leqslant 2^{-\frac{l}{l_k}} \int_{M_1} |\mathrm{B}|^{m-1}$, $|\int_{N_1 \setminus M_1^{\varepsilon_0}} |\mathrm{H}|^{m-1} - \operatorname{Vol}\mathbb{S}^{m-1} (\frac{m-1}{m})^{m-1} l_k| \leqslant 2^{-\frac{l}{l_k}} \int_{M_1} |\mathrm{H}|^{m-1}$, where $\varepsilon_0 = \varepsilon_0(k)$ and $\lim_k \varepsilon_0 = 0$ and $\int_{N_1 \setminus M_1^{\varepsilon_0}} |\mathrm{B}|^{(m-1)\frac{k-1}{k}} \leqslant 2^{-\frac{l}{l_k}} \int_{M_1} |\mathrm{H}|^{(m-1)\frac{k-1}{k}}$. We now iterate the basic construction (with $M_1' = N_i$, $M_2' = \mathbb{S}^n$ and $T' = [x_i, x_i + l_k\nu]$, where $\{x_i\} = N_i \cap (x_0 + \mathbb{R}^+\nu))$ to get a sequence of $\frac{l}{l_k}$ immersions $N_2 = N_1 \# \mathbb{S}^m, \cdots,\ N_{\frac{l}{l_k}-1} =$

$N_{\frac{l}{l_k}-2}\#\mathbb{S}^m$, $N_{\frac{l}{l_k}}=N_{\frac{l}{l_k}-1}\#M_2$ such that

$$\begin{aligned}
&d_H(N_i, M_1\cup[x_0,x_0+il_k\nu])\leqslant i2^{-\frac{l}{l_k}},\quad |\mathrm{Vol}\,N_{i+1}\setminus N_i^{\varepsilon_i}|\leqslant 2^{-\frac{l}{l_k}}\mathrm{Vol}\,M_1,\\
&\Big|\int_{N_{i+1}\setminus N_i^{\varepsilon_i}}|\mathrm{B}|^{m-1}-\mathrm{Vol}\,\mathbb{S}^{m-1}l_k\Big|\leqslant 2^{-\frac{l}{l_k}}\int_{M_1}|\mathrm{B}|^{m-1},\\
&\Big|\int_{N_{i+1}\setminus N_i^{\varepsilon_i}}|\mathrm{H}|^{m-1}-\mathrm{Vol}\,\mathbb{S}^{m-1}\big(\frac{m-1}{m}\big)^{m-1}l_k\Big|\leqslant 2^{-\frac{l}{l_k}}\int_{M_1}|\mathrm{H}|^{m-1},\\
&\int_{N_{i+1}\setminus N_i^{\varepsilon_i}}|\mathrm{B}|^{(m-1)\frac{k-1}{k}}\leqslant 2^{-\frac{l}{l_k}}\int_{M_1}|\mathrm{H}|^{(m-1)\frac{k-1}{k}},\\
&|\lambda_p(N_i)-\lambda_p(N_{i+1})|\leqslant 2^{-\frac{l}{l_k}}\text{ for any } i\leqslant\frac{l}{l_k}-1\text{ and any } p \text{ such that } \lambda_p(M_1)\leqslant k.
\end{aligned}$$

By gathering these estimates, we derive that the sequence of immersion $i_k(M_1\#M_2):=N_{\frac{l}{l_k}}$ satisfies

$$\begin{aligned}
&d_H\big(i_k(M_1\#M_2), M_1\cup[x_0,x_0+l\nu]\big)\leqslant\frac{l}{l_k}2^{-\frac{l}{l_k}},\\
&\big|\mathrm{Vol}\,\big(i_k(M_1\#M_2)\big)-\mathrm{Vol}\,M_1^{\varepsilon_0}\big|\leqslant\frac{l}{l_k}2^{-\frac{l}{l_k}}\mathrm{Vol}\,M_1,\\
&\Big|\int_{i_k(M_1\#M_2)}|\mathrm{B}|^{m-1}-\int_{M_1^{\varepsilon_0}}|\mathrm{B}|^{m-1}-\mathrm{Vol}\,\mathbb{S}^{m-1}l\Big|\leqslant\frac{l}{l_k}2^{-\frac{l}{l_k}}\int_{M_1}|\mathrm{B}|^{m-1},\\
&\Big|\int_{i_k(M_1\#M_2)}|\mathrm{H}|^{m-1}-\int_{M_1^{\varepsilon_0}}|\mathrm{H}|^{m-1}-\mathrm{Vol}\,\mathbb{S}^{m-1}\big(\frac{m-1}{m}\big)^{m-1}l\Big|\leqslant\frac{l}{l_k}2^{-\frac{l}{l_k}}\int_{M_1}|\mathrm{H}|^{m-1},\\
&\int_{N_{i+1}\setminus N_i^{\varepsilon}}|\mathrm{B}|^{(m-1)\frac{k-1}{k}}\leqslant 2^{-\frac{l}{l_k}}\int_{M_1}|\mathrm{H}|^{(m-1)\frac{k-1}{k}},\\
&|\lambda_p(N_i)-\lambda_p(N_{i+1})|\leqslant 2^{-\frac{l}{l_k}}\text{ for any } i\leqslant\frac{l}{l_k}-1\text{ and any } p \text{ such that } \lambda_p(M_1)\leqslant k.
\end{aligned}$$

By Hölder inequality, we get for any $\alpha\leqslant\frac{(m-1)(k-1)}{k}$

$$\Big|\int_{i_k(M_1\#M_2)}|\mathrm{B}|^\alpha-\int_{M_1^{\varepsilon_0}}|\mathrm{B}|^\alpha\Big|\leqslant\sum_i\int_{N_{i+1}\setminus N_i^{\varepsilon}}|\mathrm{B}|^\alpha\leqslant\frac{l}{l_k}2^{-\frac{l}{l_k}}\mathrm{Vol}\,(M_1)\|\mathrm{H}\|^\alpha_{M_1,m-1}.$$

Since we have $\lim_k\varepsilon_0=0$, we get that $\lim_k\int_{i_k(M_1\#M_2)}|\mathrm{B}|^\alpha=\int_{M_1}|\mathrm{B}|^\alpha$ for any $\alpha<m-1$. This gives Theorem 1.4 in the case $T=[x_0,x_0+l\nu]$.

5.2. **Case where T is a finite Euclidean tree.** The iteration of the basic construction used to finish the proof of Theorem 1.4 in the case of a segment can easily be generalized to get Theorem 1.4 for any finite union of finite Euclidean trees $T=\cup_iT_i$ each intersecting M_1 only once, and such that $\sum_i m_1(T_i)\leqslant l$ (note that since $n+1\geqslant 3$, we can assume up to small perturbations still converging to $M_1\cup T$, that the trees are disjoint and by adding some vertices, that the edges intersecting M_1 are orthogonal to M_1).

5.3. **Case $m_1(T)$ finite.** When T is a closed subset with $m_1(T) < \infty$ and $M_1 \cup T$ connected, then each connected component of $T' = \overline{T \setminus M_1}$ intersects M_1. As in the proof of Theorem 1.1, the family $(F_i)_{i \in I_k}$ of the connected components of T' that intersect $\mathbb{R}^{n+1} \setminus (M_1)_{\frac{1}{k}}$ is finite. Moreover, we have $T' \supset \cup_{i\in I_k} F_i \supset T \setminus M_{\frac{1}{k}}$, hence $T' = \cup_k \cup_{i \in I_k} F_i$. Since $m_1(F_i)$ is finite, for any $i \in I_k$, there exists a finite Euclidean tree $T_{i,k}$ such that $d_H(T_{i,k}, F_i) \leqslant \frac{1}{k}$ and $|m_1(F_i) - m_1(T_{i,k})| \leqslant \frac{1}{k \# I_k}$ (see [8]). Since F_i intersects M_1, we can assume that each $T_{i,k}$ intersects M_1 orthogonally (by adding a segment and vertices if necessary, and small perturbations) only once (by suppressing unnecessary open segments of $T_{i,k}$). Then the sequence $\big(M_1 \cup (\cup_{i \in I_k} T_{i,k})\big)_{k \in \mathbb{N}}$ converges to $M_1 \cup T' = M_1 \cup T$ in Hausdorff distance. Since Theorem 1.4 is valid for $M_1 \cup (\cup_{i\in I_k} T_{i,k})$, there exists for any $k \in \mathbb{N}^*$ an immersion $i_k(M_1 \# M_2)$ such that

$$
\begin{aligned}
& d_H\big(i_k(M_1\#M_2), M_1 \cup (\cup_{i \in I_k} T_{i,k})\big) \leqslant \frac{1}{k} \\
& \Big|\int_{i_k(M_1\#M_2)} |\mathrm{H}|^{m-1} - \int_{M_1} |\mathrm{H}|^{m-1} - (\frac{m-1}{m})^{m-1} \mathrm{Vol}\, \mathbb{S}^{m-1} \sum_i m_1(T_{i,k})\Big| \leqslant \frac{1}{k}, \\
& \Big|\int_{i_k(M_1\#M_2)} |\mathrm{B}|^{m-1} - \int_{M_1} |\mathrm{B}|^{m-1} - \mathrm{Vol}\, \mathbb{S}^{m-1} \sum_i m_1(T_{i,k})\Big| \leqslant \frac{1}{k}, \\
& \Big|\int_{i_k(M_1\#M_2)} |\mathrm{H}|^{\alpha} - \int_{M_1} |\mathrm{H}|^{\alpha}\Big| \leqslant \frac{1}{k} \qquad \text{for any } \alpha \in [1, m-1), \\
& \Big|\int_{i_k(M_1\#M_2)} |\mathrm{B}|^{\alpha} - \int_{M_1} |\mathrm{B}|^{\alpha}\Big| \leqslant \frac{1}{k} \qquad \text{for any } \alpha \in [1, m-1), \\
& \big|\lambda_p(i_k(M_1\#M_2)) - \lambda_p(M_1)\big| \leqslant \frac{1}{k} \text{ for any } p \leqslant k, \\
& \big|\mathrm{Vol}\,(i_k(M_1\#M_2)) - \mathrm{Vol}\, M_1\big| \leqslant \frac{1}{k}.
\end{aligned}
$$

Hence the sequence $i_k(M_1\#M_2)$ converges to $M_1 \cup T$ and since $\lim_k \sum_{i \in I_k} m_1(T_{i,k}) = \lim_k m_1(\cup_{i\in I_k} F_i) = m_1(\cup_k \cup_{i \in I_k} F_i) = m_1(T')$ (by the monotone convergence theorem).

5.4. **Case $m_1(T) = \infty$.** The L^{m-1} control of the curvature in condition 2) are automatically fulfilled. To deal with the remaining conditions, we approximate $M_1 \cup T$ in Attouch-Wetts distance by some unions of M_1 with finite number of finite Euclidean trees. Firstly, $M_1 \cup T$ is the d_{AW}-limit of the sequence of compact, connected sets $M_1 \cup T'_k := \big((M_1 \cup T) \cap B_0(k)\big) \cup k\mathbb{S}^n$. Let N_k be a maximal set of points of T'_k such that any two different points of N_k are at distance larger than $\frac{1}{k}$ (note that N_k is finite since $M_1 \cup T'_k$ is bounded), N'_k the family of points of N_k that are at distance from M_1 less than $\frac{6}{k}$ and for any $x \in N'_k$, let $y_x \in M_1$ be a point such that $\|x - y_x\| = d(x, M_1)$. Let G_k be a graph whose vertices are the points of $N_k \cup \{y_x,\, x \in N'_k\}$ and whose edges are the Euclidean segments between any couple of points of N_k at distance less than $6/k$ and the euclidean segments $\{[x, y_x],\, x \in N'_k\}$. Then $M_1 \cup G_k$ is closed and connected. We finally consider $M_1 \cup T_k$ obtained from $M_1 \cup G_k$ by suppressing some open-edges from T_k as long as $M_1 \cup T_k$ remains connected. Note that the set of vertices of T_k is the same vertices as for G_k, hence contains N_k, and that T_k has no cycle, hence is a finite union of Euclidean trees. So $M_1 \cup T_k$ is closed, connected, with T_k a finite union of finite trees each intersecting M_1 and the sequence converge to $M_1 \cup T$ in d_{AW}-distance.

Applying Theorem 1.4 to $M_1 \cup T_k$ and arguing as in the previous case, we get Theorem 1.4 for $M \cup T$.

6. Proof of Proposition 1.3

In the case where Z' is reduced to a point $\{z\}$, we just take $M_k = z + \frac{1}{k}M$.

We now suppose that Z' is not reduced to a point. Z is the limit in Attouch-Wetts topology of the sequence $\big(Z \cap B(R)\big)_{R\in\mathbb{N}}$, which itself is the limit of the sequence $M_R = \cup_{x\in N_R}\partial B_x(\frac{1}{R^{3n}})$, where N_R is a maximal set of points of $Z \cap B(R)$ such that any two different points of N_R are at distance larger than $\frac{1}{R}$. Then M_R is a hypersurface of $\mathbb{R}^{n+1}$. Note that by connectedness of Z', any sphere of M_R intersect Z' except if N_R is reduced to a point z and if $Z' \subset B_z(R)$. Since Z' contains at least two points, we infer that $M_R \cup Z'$ is connected for any R large enough. Applying the same procedure as in the proof on Theorem 1.4, we get a disjoint, finite family of Euclidean finite trees $(T_{i,R})_{i\in I_R}$ such that the sequence $(M_R \cup (\cup_{i\in I_R} T_{i,R}))_R$ of simply connected set (since we have suppressed all the cycles by sutting unnecessary edges) converges to Z' in d_{AW} distance (and each tree intersects the connected component of M_R at most once). We can then iterate the basic construction to approximate the set $M_R \cup (\cup_{i\in I_R} T_{i,R})$ by a submanifold $M'_R = M_R \cup (\cup_{i\in I_R} N_{i,R})$ with all vertices of the trees replace by some small sphere and each edges replaced by a pinched cylinder. Then M'_R is diffeomorphic to $\mathbb{S}^n$, and so can be appoximated in distance d_{AW} by an immersion of M by connected sum of M'_R with a scaled copy of M. So we get a sequence of immersions of M that converge strongly to Z' and weakly to Z.

By construction we have $\#N_R = O(R^{2n})$ and so $\mathrm{Vol}\, M_R \|B\|^{n-1}_{\alpha, M_R} = O(\frac{1}{R^n})$, we get the bounds on curvature as in the proof of Theorem 1.4.

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