DIAMETER PINCHING IN ALMOST POSITIVE RICCI CURVATURE

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Abstract. In this paper we prove a diameter sphere theorem and its corresponding λ_1 sphere theorem under L^p control of the curvature. They are generalizations of some results due to S. Ilias [8].

1. Introduction

Let (M^n, g) be a complete manifold with Ricci curvature $\mathrm{Ric} \geq n-1$. Then (M^n, g) satisfies the following classical results (the proofs can be found in [13] for instance):

- $\mathrm{Diam}(M^n, g) \leq \pi$ (S. Myers) with equality iff $(M^n, g) = (\mathbb{S}^n, can)$ (S. Cheng),
- $\lambda_1(M^n, g) \geq n$ (A. Lichnerowicz) with equality iff $(M^n, g) = (\mathbb{S}^n, can)$ (M. Obata),

where Diam is the diameter and λ_1 is the first positive eigenvalue.

Studying the properties of the sphere kept by manifold with $\mathrm{Ric} \geq n-1$ and almost extremal diameter or λ_1, S. Ilias proved in [8] the following results:

Theorem 1.1 (S. Ilias)**.** *For any $A>0$, there exists $\epsilon(A, n) > 0$ such that any n-manifolds with $\mathrm{Ric} \geq n-1$, sectional curvature $\sigma \leq A$ and $\lambda_1 \leq n+\epsilon$ is homeomorphic to $\mathbb{S}^n$.*

Theorem 1.2 (S. Ilias)**.** *For any $A>0$, there exists $\epsilon(A,n)>0$ such that any n-manifolds with $\mathrm{Ric} \geq n-1$, $\sigma \leq A$ and $\mathrm{Diam}(M) \geq \pi-\epsilon$ is homeomorphic to $\mathbb{S}^n$.*

Remark 1.3. *C. Croke proves in* [7] *that for n-manifolds with $\mathrm{Ric} \geq n-1$, $\lambda_1(M)$ close to n implies $\mathrm{Diam}(M)$ close to π. The converse is proved in* [8] *(using a spectral inequality due to S. Cheng* [6]*).*

Remark 1.4. *For $n \geq 4$, M. Anderson* [1] *and Y. Otsu* [10] *construct sequences of complete metrics g_i with $\mathrm{Ric}(g_i) \geq n-1$, $\lambda_1(g_i) \to n$ and $\mathrm{Diam}(g_i) \to \pi$ on manifolds that are not homotope to $\mathbb{S}^n$ (more precisely, Otsu shows that if $n \geq 5$, these manifolds can have infinitely many different fundamental groups).*

Remark 1.5. *The two results of S. Ilias have been improved by G. Perelman in* [11]*, where the assumption $\sigma \leq A$ is replaced by $\sigma \geq -A$ (note that under the Ilias's assumptions $\sigma \leq A$ and $\mathrm{Ric} \geq n-1$ we have $|\sigma| \leq (n-2)A$).*

Subsequently, we *denote $\underline{\mathrm{Ric}}(x)$ the lowest eigenvalue of the Ricci tensor and $\overline{\sigma}(x)$ the maximal sectional curvature at x.* In [4], we prove the following generalization of the Myers and Lichnerowicz theorems:

Theorem 1.6. *For any $p>n/2$, there exists $C(p, n)$ such that if (M^n, g) is a complete manifold with $\int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_-^p < \frac{\mathrm{Vol}\, M}{C(p,n)}$, then M is compact, has finite fundamental group and satisfies*

$$\mathrm{Diam}(M) \leq \pi\Big[1 + C(p,n)\big(\frac{\rho_p}{\mathrm{Vol}\, M}\big)^{\frac{1}{10}}\Big]$$

$$\lambda_1(M) \geq n\Big[1 - C(p,n)\big(\frac{\rho_p}{\mathrm{Vol}\, M}\big)^{\frac{1}{p}}\Big],$$

where $\rho_p = \int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_-^p$ and $x_- = \max(0, -x)$.

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Remark 1.7. *It follows from* [4] *that the constant* $C(p,n)$ *is computable, that if* $\int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_-^p$ *is finite (for* $p > n/2$*) then* $\operatorname{Vol} M$ *is finite, and that we can not bound the diameter or the first non zero eigenvalue under the assumption* $\rho_p \leq \frac{1}{C(p,n)}$ *or* $\rho_{\frac{n}{2}}$ *small (see* [4]*).*

In this paper we prove the following extensions of the Ilias's stability results.

Theorem 1.8. *Let* $n \geq 2$ *be an integer,* $A > 0$ *and* $p > n$ *be some reals. There exists a positive constant* $C(p,n,A)$ *such that any complete n-manifold which satisfies*

$$\int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_-^p < C(p,n,A) \operatorname{Vol} M, \qquad \int_M \overline{\sigma}_+^p < A \operatorname{Vol} M$$
$$\text{and} \quad \operatorname{Diam}(M) \geq \pi\big(1 - C(p,n,A)\big)$$

is homeomorphic to $\mathbb{S}^n$ *(where* $x_+ = \max(0,x)$*).*

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$$\text{and} \quad \lambda_1(M) \leq n\big(1 + C(p,n,A)\big)$$

is homeomorphic to $\mathbb{S}^n$.

Remark 1.10. *By the Hölder inequality, the two curvature assumptions of Theorem 1.9 can be replaced by*

$$\int_M \big(\underline{\mathrm{Ric}} - (n-1)\big)_- < C(p,n,A) \operatorname{Vol} M, \qquad \int_M \sigma^p < A \operatorname{Vol} M,$$

where $\sigma(x)$ *is an upper bound for the absolute value of the sectional curvatures at* x.

2. Comparison results in almost positive Ricci curvature

Subsequently we denote $B(x,r)$ (resp. $S(x,r)$) the geodesic ball (resp. sphere) of center x and radius r and $L_k(r)$ (resp. $A_k(r)$) the volume of a geodesic sphere (resp. ball) of radius r in $(\mathbb{S}^n, \frac{1}{k}g)$. Besides the theorem 1.6, we will need the following comparison results for manifolds of almost positive Ricci curvature (see [4] for a proof).

Proposition 2.1. *For any* $n \geq 2$ *and* $p > n/2$ *(*$p \geq 1$ *if* $n = 2$*) there exists a constant* $C(p,n)$ *such that for any complete Riemannian n-manifold* (M^n, g) *with* $\eta^{10} = \frac{\rho_p}{\operatorname{Vol} M} \leq \frac{1}{C(p,n)}$*, we have*

$$\Big(\frac{\operatorname{Vol}_{n-1} S(x,R)}{L_{1-\eta}(R)}\Big)^{\frac{1}{2p-1}} - \Big(\frac{\operatorname{Vol}_{n-1} S(x,r)}{L_{1-\eta}(r)}\Big)^{\frac{1}{2p-1}} \leq C(p,n)\eta^2 (R-r)^{\frac{2p-n}{2p-1}},$$
$$\frac{\operatorname{Vol} B(x,r)}{\operatorname{Vol} B(x,R)} \geq (1 - C(p,n)\eta) \frac{A_1(r)}{A_1(R)},$$
$$\operatorname{Vol}_{n-1} S(x,R) \leq \big(1+\eta^2\big) L_{1-\eta}(R),$$
$$\operatorname{Vol} B(x,R) \leq \big(1+\eta\big) A_1(R).$$

for all $x \in M$ *and all radii* $0 \leq r \leq R$.

For any $n \geq 2$ *and* $p > n/2$ *there exists a constant* $C(p,n)$ *such that if* (M^n, g) *is a complete n-manifold with* $\overline{\rho}_p \leq \frac{1}{C(p,n)}$*, then* $\|u\|_{\frac{2n}{n-2}} \leq \operatorname{Diam}(M) C(p,n) \|du\|_2 + \|u\|_2$*, for any* $u \in H^{1,2}(M)$*. In the case* $n = 2$*, we have* $\|u\|_4 \leq \operatorname{Diam}(M) C \|du\|_2 + \|u\|_2$ *if* $\overline{\rho_1} \leq \frac{1}{C}$.

Similar estimates are proved in [12] under the assumption $\operatorname{Diam}^{2p} \frac{\rho_p}{\operatorname{Vol} M} \leq \frac{1}{C(p,n)}$.

3. Theorem 1.9 implies Theorem 1.8

Proposition 3.1. *Let $n \geq 2$ and $p > n/2$. There exists $C(p,n) > 0$ such that if (M^n, g) is a complete n-manifold with $\eta^{10} = \overline{\rho}_p \leq \frac{1}{C(p,n)}$ and $\mathrm{Diam}(M) \geq \pi - \frac{1}{C(p,n)}$ then we have*

$$\lambda_1(M) \leq n + C(p,n)\big[\eta + (\mathrm{Diam}(M) - \pi)_-\big].$$

The main tool to prove this proposition is the following lemma:

Lemma 3.2. *Let $n \geq 2$ and $p > n/2$ ($p \geq 1$ if $n = 2$) and $\bar{x}_0 \in \mathbb{S}^n$. There exists a constant $C(p,n)$ such that if (M^n, g) is a complete n-manifold with $\eta^{10} = \overline{\rho}_p \leq \frac{1}{C(p,n)}$ then there exists $x_0 \in M$ such that for any C^1-function $u : [0, 2\pi] \to \mathbb{R}$ we have*

$$\left| \frac{1}{\mathrm{Vol}\, M} \int_M u \circ d_M(x_0, .)\, dv_g - \frac{1}{\mathrm{Vol}\, \mathbb{S}^n} \int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x_0}, .)\, dv_{\mathbb{S}^n} \right| \leq \|u'\|_\infty C(p,n)\big[\eta + \big(\mathrm{Diam}(M) - \pi\big)_-\big].$$

Proof. Let $(x_0, y_0) \in M^2$ such that $d = \mathrm{Diam}(M) = d(x_0, y_0)$. The functions A, L, A_1 and L_1 are defined in Proposition 2.1 and prolonged by 0 to $\mathbb{R}$ (note that the diameter of M can be greater than π). The function $r \to u(r)A(r)$ is continuous and has right differential on $\mathbb{R}$ equal to $u'A + uL$. We infer the equalities

$$u(d)\,\mathrm{Vol}\, M = \int_0^d u(r)L(r)\,dr + \int_0^d u'(r)A(r)\,dr$$
$$u(\pi)\,\mathrm{Vol}\, \mathbb{S}^n = \int_0^\pi u(r)L_1(r)\,dr + \int_0^\pi u'(r)A_1(r)\,dr$$

which imply

$$\begin{aligned}
&\left| \frac{1}{\mathrm{Vol}\, M} \int_M u \circ d_M(x_0, x)\, dv_g - \frac{1}{\mathrm{Vol}\, \mathbb{S}^n} \int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x_0}, x)\, dv_{\mathbb{S}^n} \right| \\
&= \left| \int_0^d \frac{u(r)L(r)}{\mathrm{Vol}\, M} dr - \int_0^\pi \frac{u(r)L_1(r)}{\mathrm{Vol}\, \mathbb{S}^n} \right| = \left| u(d) - u(\pi) + \int_0^\pi \frac{u' A_1}{\mathrm{Vol}\, \mathbb{S}^n} - \int_0^d \frac{u' A}{\mathrm{Vol}\, M} \right| \\
&= \left| \int_0^d u' \left(\frac{A_1}{\mathrm{Vol}\, \mathbb{S}^n} - \frac{A}{\mathrm{Vol}\, M} \right) + \int_d^\pi u' \left(\frac{A_1}{\mathrm{Vol}\, \mathbb{S}^n} - 1 \right) \right| \\
&\leq \|u'\|_\infty \left(\int_0^d \left| \frac{A_1}{\mathrm{Vol}\, \mathbb{S}^n} - \frac{A}{\mathrm{Vol}\, M} \right| dr + |\pi - d| \right)
\end{aligned}$$

By Proposition 2.1 we have, for all $r \leq d$:

$$\begin{aligned}
\big(1 - C(p,n)\eta\big)\frac{A_1(r)}{\mathrm{Vol}\, \mathbb{S}^n} \leq \frac{A(r)}{\mathrm{Vol}\, M} &\leq 1 - \frac{\mathrm{Vol}\, B(y_0, d - r)}{\mathrm{Vol}\, M} \leq 1 - \big(1 - C(p,n)\eta\big)\frac{A_1(d - r)}{\mathrm{Vol}\, \mathbb{S}^n} \\
&\leq \frac{A_1(r + \pi - d)}{\mathrm{Vol}\, \mathbb{S}^n} + C(p,n)\eta
\end{aligned}$$

Hence $\left|\frac{A(r)}{\mathrm{Vol}\, M} - \frac{A_1(r)}{\mathrm{Vol}\, \mathbb{S}^n}\right| \leq C(p,n)\eta + \frac{\big(A_1(r) - A_1(r + \pi - d)\big)_-}{\mathrm{Vol}\, \mathbb{S}^n}$. An easy computation gives $\|\frac{\left(A_1(\cdot) - A_1(\cdot + h)\right)_-}{\mathrm{Vol}\, \mathbb{S}^n}\|_\infty \leq C(n)(-h)_-$, and by Proposition 2.1 we get:

$$\left| \frac{1}{\mathrm{Vol}\, M} \int_M u \circ d_M(x_0, x)\, dv_g - \frac{1}{\mathrm{Vol}\, \mathbb{S}^n} \int_{\mathbb{S}^n} u \circ d_{\mathbb{S}^n}(\bar{x_0}, x)\, dv_{\mathbb{S}^n} \right| \leq \|u'\|_\infty C(p,n)\big[\eta + (d - \pi)_-\big].$$

□

We now finish the proof of Proposition 3.1.

Proof. Lemma 3.2 applied to $u = \sin^2$, $u = \cos^2$ and $u = \cos$ gives:

$$\Big|\int_M \frac{\sin^2 d_M(x_0,.)}{\operatorname{Vol} M} - \int_{\mathbb{S}^n} \frac{\sin^2 d_{\mathbb{S}^n}(\bar{x}_0,.)}{\operatorname{Vol}\mathbb{S}^n}\Big| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1,$$
$$\Big|\int_M \frac{\cos^2 d_M(x_0,.)}{\operatorname{Vol} M} - \int_{\mathbb{S}^n} \frac{\cos^2 d_{\mathbb{S}^n}(\bar{x}_0,.)}{\operatorname{Vol}\mathbb{S}^n}\Big| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1,$$
$$\Big|\int_M \frac{\cos d_M(x_0,.)}{\operatorname{Vol} M} - \int_{\mathbb{S}^n} \frac{\cos d_{\mathbb{S}^n}(\bar{x}_0,.)}{\operatorname{Vol}\mathbb{S}^n}\Big| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1.$$

Hence, if we set $f = \cos d_M(x_0,.)$, we get

$$\Big|\|\nabla f\|_2^2 - \frac{n}{n+1}\Big| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1,$$
$$\Big|\|f\|_2^2 - \frac{1}{n+1}\Big| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1,$$
$$\Big|\frac{1}{\operatorname{Vol} M}\int_M f\Big| \leq C(p,n)\big(\eta + (d-\pi)_-\big) \leq 1.$$

Which readily implies that

$$\lambda_1(M) \leq \frac{\|\nabla(f-\bar{f})\|_2^2}{\|f-\bar{f}\|_2} \leq n\big(1 + C(p,n)(\eta + (d-\pi)_-)\big),$$

where we have set $\bar{f} = \frac{1}{\operatorname{Vol} M}\int_M f$. □

Remark 3.3. *The same technic as in* [5] *can be used to prove that manifolds with almost positive Ricci curvature and λ_1 is close to n have a diameter close to π (see* [12]*).*

4. Proof of Theorem 1.9

4.1. **Fiber Bundle E.** Let E be the fiber bundle $TM \oplus \mathbb{R}e \to M$ endowed with the following scalar product and linear connection:

$$< X + fe, Y + he >_E = g(X,Y) + fh$$
$$D^E_Z(X + fe) = D^M_Z X + fZ + \big(df(Z) - g(Z,X)\big).e$$

Where D^M is the Levi-Civita connection of the metric g on M. *We set p the orthogonal projection of E on TM, $\operatorname{Ric}'(S) = \operatorname{Ric}_M\big(p(S)\big) - (n-1)p(S)$ and $\triangle_{sph} = \overline{\triangle}^E + \operatorname{Ric}'$.*

The following Lemma is proved in [3]:

Lemma 4.1. *If $f : M \to \mathbb{R}$ satisfies $\triangle f = \lambda f$ then $S_f = \nabla f + f.e$ satisfies $\triangle_{sph}(S_f) = (\lambda - n)(\nabla f - fe)$ and $\langle D^E_X S_f, X\rangle = Ddf(X,X) + fg(X,X)$.*

Note also that we have

$$\mathrm{R}^{\mathrm{E}}_{(\mathrm{Z},\mathrm{Y})}(\mathrm{X} + \mathrm{fe}) = \mathrm{R}^{\mathrm{M}}(\mathrm{Z},\mathrm{Y})\mathrm{X} - \big(\mathrm{g}(\mathrm{Y},\mathrm{X})\mathrm{Z} - \mathrm{g}(\mathrm{Z},\mathrm{X})\mathrm{Y}\big).$$

4.2. **Bound on the Hessian of the first eigenfunction.** To prove Theorem 1.9 we need a L^∞ bound on the Hessian of the first eigenfunction. In that purpose, we will modify the proof of Theorem 2.4 in [2] (whose proof would give us only a bound on $\|DS_f\|_{n+\epsilon}/\|S_f\|_\infty$ for a given $\epsilon = \epsilon(p,n)$). In our case we really need to perform a Möser iteration.

Proposition 4.2. *Let $n \geq 2$ and $\infty \geq p > n/2$. There exists a constant $C(p,n)$ such that if (M^n, g) is any manifold with $\overline{\rho}_p \leq \frac{1}{C(p,n)}$ and $\lambda_1 \leq n + \frac{1}{C(p,n)}$ then for $f : M \to \mathrm{R}$ such that $\triangle f = \lambda_1 f$ we have:*

$$\frac{\|D^E S_f\|_\infty}{\|S_f\|_\infty} \leq C(p,n)\big(\lambda_1 + \|R\|_{2p}\big)^\gamma \big(|\lambda_1 - n| + \overline{\rho}_p\big)^{\frac{1}{2(1+\gamma)}},$$

where $S_f = \nabla f + f \cdot e$ and $\gamma = \frac{pn}{2p-n}$.

To prove Proposition 4.2 we need a commutation Lemma (see [2]):

Lemma 4.3. *For any section* $S \in \Gamma(E)$ *we have*

$$\frac{1}{2}\triangle\left(|DS|^2\right) + |D^2S|^2 \leq \langle D^*R^ES, DS\rangle + \underline{\mathrm{Ric}}^-|DS|^2 + \langle D\overline{\triangle}S, DS\rangle + \|R^E\|.|DS|^2,$$

where $\|R^E\|$ *is the norm of the linear map* $\mathrm{R^E} : \bigwedge^2 \mathrm{T_mM} \to \bigwedge^2 \mathrm{E^*_m}$ *defined by* $R^E(u\wedge v)(T,S) = < R^E(u,v)T, S >$.

Remark 4.4. *This Lemma is valid for any Riemannian fiber bundle* $(E, D, \langle\cdot,\cdot\rangle)$.

We now give the proof of Proposition 4.2.

Proof. We set $u = \sqrt{|DS|^2 + \epsilon^2}$. we have

$$u\triangle u = \frac{1}{2}\triangle(u^2) + |du|^2 = \frac{1}{2}\triangle\big(u^2\big) + \frac{|\langle D^2S, DS\rangle|^2}{|DS|^2+\epsilon^2}$$
$$\leq \frac{1}{2}\triangle\big(|DS|^2\big) + |D^2S|^2$$

Hence, by Lemma 4.3

$$\int_M |d(u^k)|^2 \leq \frac{k^2}{2k-1}\int_M (\frac{1}{2}\triangle|DS|^2 + |D^2S|^2)u^{2(k-1)}$$
$$\leq \frac{k^2}{2k-1}\left(\int_M \underline{\mathrm{Ric}}^- u^{2k} + \int_M < D\overline{\triangle}S, DS > u^{2(k-1)}\right.$$
$$\left. + \int_M < D^*\mathrm{R^ES}, \mathrm{DS} > \mathrm{u^{2(k-1)}} + \int_{\mathrm{M}} \|\mathrm{R^E}\|\mathrm{u^{2k}}\right)$$

We now apply the divergence theorem to the form $u^{2(k-1)}\langle\overline{\triangle}S, D_\bullet S\rangle$, and get for any $k \geq 1$:

$$\int_M \langle D\overline{\triangle}S, DS\rangle u^{2(k-1)}$$
$$= \int_M |\overline{\triangle}S|^2u^{2(k-1)} - 2(k-1)\sum_i \int_M \langle\overline{\triangle}S, DS(i)\rangle du(i).u^{2k-3}$$
$$\leq \int_M |\overline{\triangle}S|^2u^{2(k-1)} + 2(k-1)\int_M |\overline{\triangle}S||du|u^{2(k-1)}$$
$$\leq \frac{k-1}{2}\int_M |du|^2u^{2(k-1)} + (2k-1)\int_M |\overline{\triangle}S|^2u^{2(k-1)}$$

We do the same with the form $u^{2(k-1)}(tr_{1,3}(\langle R^E_{(\bullet,\bullet)}S, D_\bullet S\rangle))$ and get

$$\int_M \langle D^*R^ES, DS\rangle u^{2(k-1)}$$
$$= \int_M \frac{1}{2}|\mathrm{R^ES}|^2\mathrm{u^{2(k-1)}} + 2(\mathrm{k}-1)\sum_{\mathrm{i,j}}\int_{\mathrm{M}} \langle\mathrm{R^E(i,j)S}, \mathrm{D_jS}\rangle\mathrm{du(i)u^{2k-3}}$$
$$\leq \frac{k-1}{2}\int_M |du|^2u^{2(k-1)} + (2k-1)\int_M |\mathrm{R^ES}|^2\mathrm{u^{2(k-1)}},$$

Where we have used $\sum_{i,j}\langle\mathrm{R^ES(i,j)}, \mathrm{D^2S(i,j)}\rangle = \frac{1}{2}|\mathrm{R^ES}|^2$.
Since $\int_M |du|^2u^{2(k-1)} = \frac{1}{k^2}\int_M |d(u^k)|^2$, the three last inequalities give, for any $k \geq 1$:

$$\|d(u^k)\|_2^2$$
$$\leq k\big(\int_M \underline{\mathrm{Ric}}^- u^{2k} + \int_M \|\mathrm{R^E}\|\mathrm{u^{2k}}\big) + \mathrm{k(2k-1)}\big(\int_{\mathrm{M}} \|\mathrm{R^ES}\|^2\mathrm{u^{2k-2}} + \int_{\mathrm{M}} \|\overline{\triangle}\mathrm{S}\|^2\mathrm{u^{2k-2}}\big)$$

$$\leq 4k^2\left(B_1\|u\|^{2k}_{\frac{2kp}{p-1}}+B_2\|S\|^2_\infty\|u\|^{2(k-1)}_{\frac{2(k-1)p}{p-1}}\right),$$

where we have set

$$B_1=\|\underline{\mathrm{Ric}}^-\|_p+\|R^E\|_p\leq C(n)\big(\|R^M\|^2_{2p}+\lambda_1^2\big)=B^2,$$

$$B_2=\frac{\|\overline{\triangle}S\|^2_{2p}}{\|S\|^2_\infty}+\frac{\|R^ES\|^2_{2p}}{\|S\|^2_\infty}\leq\frac{\|\triangle_{Sph}S\|^2_{2p}}{\|S\|^2_\infty}+\|\,\mathrm{Ric}'\,\|^2_{2p}+\|R^E\|^2_{2p}$$
$$\leq C(n)\big(\lambda_1^2+\|R^M\|^2_{2p}\big)=B^2.$$

By the Sobolev inequality given by Proposition 2.1, we get

$$\|DS\|^k_{\frac{2kn}{n-2}}\leq\|DS\|^k_{2k}+C(p,n)Bk\sqrt{\|DS\|^{2k}_{\frac{2kp}{p-1}}+\|S\|^2_\infty\|DS\|^{2(k-1)}_{\frac{2(k-1)p}{p-1}}},$$

and by $\|DS\|_{2k}\leq\|DS\|_{\frac{2kp}{p-1}}\leq\|DS\|^{1/k}_\infty\|DS\|^{(1-1/k)}_{\frac{2(k-1)p}{p-1}}$, we have

$$\left(\frac{\|DS\|_{\frac{2kn}{n-2}}}{\|DS\|_\infty}\right)^{\frac{2kn}{n-2}}\leq\left[1+BkC(p,n)\Big(1+\frac{\|S\|^2_\infty}{\|DS\|^2_\infty}\Big)\right]^{\frac{2n}{n-2}}\left(\frac{\|DS\|_{\frac{2(k-1)p}{p-1}}}{\|DS\|_\infty}\right)^{\nu^{\frac{2(k-1)p}{p-1}}},$$

where $\nu=\frac{n(p-1)}{2p(n-2)}>1$. We set $k=\frac{a_n(p-1)}{2p}+1$ where $(a_n)_n$ is the sequence defined by $a_0=\frac{2p}{p-1}$ and $a_{n+1}=\nu a_n+\frac{2n}{n-2}$. Then we get

$$\left(\frac{\|DS\|_{a_{n+1}}}{\|DS\|_\infty}\right)^{\frac{a_{n+1}}{\upsilon^{n+1}}}\leq\left[1+a_nC(p,n)B\Big(1+\frac{\|S\|^2_\infty}{\|DS\|^2_\infty}\Big)\right]^{\frac{2n}{(n-2)\nu^{n+1}}}\left(\frac{\|DS\|_{a_n}}{\|DS\|_\infty}\right)^{\frac{a_n}{\nu^n}},$$

Hence

$$1=\lim_{n\to+\infty}\left(\frac{\|DS\|_{a_n}}{\|DS\|_\infty}\right)^{\frac{a_n}{\nu^n}}\leq\prod_{i=1}^\infty\left(1+C(p,n)a_iB\Big(1+\frac{\|S\|^2_\infty}{\|DS\|^2_\infty}\Big)\right)^{\frac{2n}{(n-2)\nu^i}}\left(\frac{\|DS\|_{a_0}}{\|DS\|_\infty}\right)^{a_0}$$

The Hölder inequality $\|DS\|_{a_0}\leq\|DS\|_2^{1-\frac{1}{p}}\|DS\|^{\frac{1}{p}}_\infty$, gives

$$\|DS\|_\infty\leq\prod_{i=1}^\infty\left(1+C(p,n)a_iB\Big(1+\frac{\|S\|^2_\infty}{\|DS\|^2_\infty}\Big)\right)^{\frac{n}{(n-2)\nu^i}}\|DS\|_2,\quad(*)$$

If $\|DS\|_\infty\geq\|S\|_\infty$ then inequality $(*)$ gives

$$\|DS\|_\infty\leq\prod_{i=1}^\infty(1+C(p,n)a_iB)^{\frac{n}{(n-2)\nu^i}}\|DS\|_2\leq C(p,n)\big(\lambda_1+\|R\|_{2p}\big)^{\frac{pn}{2p-n}}\|DS\|_2$$

If $\|DS\|_\infty\leq\|S\|_\infty$ then inequality $(*)$ gives

$$\frac{\|DS\|_\infty}{\|DS\|_2}\leq\left(\frac{\|S\|_\infty}{\|DS\|_\infty}\right)^{\frac{2pn}{2p-n}}\prod_{i=1}^\infty(1+C(p,n)a_iB)^{\frac{n}{(n-2)\nu^i}}$$

hence

$$\frac{\|DS\|_\infty}{\|S\|_\infty}\leq C(p,n)\big(\lambda_1+\|R\|_{2p}\big)^{\frac{pn}{2p-n)}}\big(\frac{\|DS\|_2}{\|S\|_\infty}\big)^{\frac{2p-n}{2p-n+2pn}}.$$

At this stage note that, by Lemma 4.1 we have

$$\|DS\|^2_2=<\overline{\triangle}_{sph}(S),S>_{L^2}-<\mathrm{Ric}'(S),S>_{L^2}$$
$$\leq|\lambda_1-n|\,\|S\|^2_2+\int_M\frac{\big(\underline{\mathrm{Ric}}-(n-1)\big)^-}{\mathrm{Vol}\,M}|S|^2\leq\big(|\lambda_1-n|+\overline{\rho}_p\big)\|S\|^2_\infty.$$

Since we have $\frac{2p-n}{2p-n+2pn}\leq1$, we get the result. □

4.3. **Critical points of the first eigenfunction.** By Proposition 4.2 the section $S_f = \nabla f + fe$ of E satisfies $\|D^E S_f\|_\infty \leq C(p,n,A)\big(|\lambda_1 - n| + \overline{\rho}_p\big)^{\frac{1}{1+\gamma}}\|S_f\|_\infty$. Since we can suppose the pinching on $|\lambda_1 - n|$ and $\overline{\rho}_p$ small enough to have $C(p,n,A)\big(|\lambda_1 - n| + \overline{\rho}_p\big)^{\frac{1}{1+\gamma}} \leq 1/4$, the previous inequality and Theorem 1.6 give

$$\inf |S_f| \geq \big[1 - C(p,n,A)\big(|\lambda_1 - n| + \overline{\rho}_p\big)^{\frac{1}{1+\gamma}}\big]\|S_f\|_\infty$$
$$> C(p,n,A)\big(|\lambda_1 - n| + \overline{\rho}_p\big)^{\frac{1}{1+\gamma}}\|S_f\|_\infty \geq \|D^E S_f\|_\infty$$

We infer that if x_0 is a critical point of f then by Lemma 4.1 we have

$$|Ddf_{x_0}(X,X) + f(x_0)| = |\langle D^E_X S_f, X\rangle_E| \leq \|D^E S_f\|_\infty < |S_f(x_0)| = |f(x_0)|,$$

for any unit vector X of $T_{x_0}M$. Hence we have $-|f(x_0)| - f(x_0) < Ddf_{x_0}(X,X) < |f(x_0)| - f(x_0)$ for any critical point x_0 of f. So the only critical points of f are non degenerate global extrema, which implies that M is homeomorphic to $\mathbb{S}^n$ by the Reeb's theorem.

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