Dynamical n-complexes

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Abstract. Using the special n-polyhedra of Matveev ([2]), we give a generalization to any dimension $n \geq 2$ of the dynamical 2-complexes introduced in the author's thesis ([1]). We are particularly interested in codim 1-foliations with compact leaves of these complexes. One is lead to define a particular class of homotopy equivalences between special polyhedra, which are a generalization, in high dimension, of the Whitehead moves on graphs and of the Matveev moves on special 2-polyhedra ([3]).

n-complexes dynamiques

Résumé. On donne une généralisation à toute dimension $n \geq 2$ de la notion de 2-complexe dynamique introduite dans [1]. On utilise pour cela les n-polyhèdres spéciaux dus à Matveev ([2]). On s'intéresse plus particulièrement aux feuilletages en $(n-1)$-complexes compacts de ces polyhèdres. Cette généralisation amène à définir une classe particulière d'équivalences d'homotopie entre ces complexes. Ce sont une généralisation en toute dimension des mouvements de Whitehead sur les graphes et des mouvements de Matveev ([3]) sur les 2-polyhèdres spéciaux.

Version française abrégée

La notion de 2-*complexe dynamique*, introduite et étudiée dans [1], et généralisée à toute dimension dans cette Note, permet en particulier d'obtenir une représentation topologique de tout automorphisme de groupe libre. De même qu'en dimension 2, la définition donnée ici d'un n-complexe dynamique est combinatoire, et permet, pour une large classe (complexes dynamiques *standards* - voir ci-dessous), l'introduction d'un semi-flot non-singulier sur le complexe. D'autre part, les n-complexes dynamiques fournissent une représentation topologique et dynamique de produits semi-direct de groupes non nécessairement libres avec $\mathbf{Z}$, et de tels produits ne sont pas forcément groupes fondamentaux de 2-complexes dynamiques.

Les n-*polyhèdres spéciaux* de [2] sont des CW-complexes de dimension n, linéaires par morceaux, satisfaisant la propriété suivante : Tout point admet un voisinage homéomorphe au voisinage d'un point intérieur au cone sur le $(n-1)$-squelette du bord du $(n+1)$-simplexe (noté $Con((\partial\Delta^{n+1})^{(n-1)})$). L'*ensemble singulier* $K^{(i)}_{sing}$ d'un tel complexe K est l'ensemble des points dont l'image par l'homéo--morphisme ci-dessus appartient à une i-cellule de l'intérieur de $Con((\partial\Delta^{n+1})^{(n-1)})$. Une $(i+1)$-*composante* est une composante connexe de $K^{(i+1)}_{sing} - K^{(i)}_{sing}$. Un n-*complexe plat* est un n-

polyhèdre spécial tel que toute j-composante est soit une j-cellule D^j, soit la suspension $(D^{j-1} \times [0,1])/((x,1) \sim (f(x),0))$ d'une $(j-1)$-cellule D^{j-1} par f, où f est soit l'application identité soit une réflexion relative au diamètre $D^{j-2} \subset D^{j-1}$ (D^0 est un point). Si toutes les composantes sont des cellules, on dit que le n-complexe est *standard*. Le germe d'une j-composante en un sommet de $K^{(0)}_{sing}$ (ou *croisement*) contient exactement j germes de 1-composantes. Si on munit les 1-composantes de K d'une orientation, alors on dira qu'une j-composante C *admet un croisement* v *comme attracteur* (resp. *répulseur*) dans son bord si les j germes de 1-composantes contenues dans l'un des germes de C en v admettent toutes v comme sommet terminal (resp. initial).

Définition 1 *Un n-*complexe dynamique *est un n-complexe plat muni d'une orientation sur les 1-composantes de K telle que tout croisement est le sommet terminal et initial d'au plus n 1-composantes, et toute j-composante ($2 \leq j \leq n$) qui est une cellule a exactement un attracteur et un répulseur dans son bord, les autres composantes n'en ont pas.*

De même que dans [1], on s'intéresse aux feuilletages de codimension 1, transversalement orientables et en feuilles compactes de ces complexes. Une fois définis les *feuilletages réguliers*, généralisation naturelle des feuilletages non singuliers de variétés, on a les résultats suivants :

Théorème 1 *Un n-complexe plat K admet un feuilletage régulier $\mathcal{F}$, de codimension 1, transversalement orientable et en feuilles compactes ayant toutes la même caractéristique d'Euler, si et seulement si il existe une orientation des 1-composantes du complexe telle que K, muni de cette orientation, est un n-complexe dynamique admettant un* cocycle positif.
Toutes les feuilles d'un tel feuilletage sont des $(n-1)$-complexes standard homotopiquement équivalents. Pour toute feuille $\mathcal{L}$, il existe une équivalence d'homotopie $\psi : \mathcal{L} \to \mathcal{L}$ telle que K soit la suspension de $\mathcal{L}$ par ψ.

Proposition 1 *Pour toute équivalence d'homotopie ψ d'un n-complexe standard K, donnée comme une composition de* WM-mouvements généralisés, *il existe un $(n+1)$-complexe dynamique K_S qui est la suspension de K par ψ, et admettant un cocycle positif.*

Ces résultats font intervenir les *cocycles positifs* et les *WM-mouvements généralisés*. Un cocycle positif est un cocycle dans $C^1(K; \mathbf{Z})$, qui est positif ou nul sur les 1-cellules de $K^{(1)}_{sing}$ (munies de l'orientation qui fait de K un complexe dynamique) et strictement positif sur ses boucles positives. Un WM-mouvement généralisé sur un n-complexe standard K consiste à écraser un j-simplexe "singulier" de K sur un sommet v, puis exploser ce sommet en un $(n+1-j)$-simplexe singulier (par "simplexe singulier" on entend un j-simplexe de K dont toutes les i-faces, $i \leq j$, sont dans $K^{(i)}_{sing}$). Si $n = 1$, on retrouve les mouvements de Whitehead sur les graphes. La relation entre cocycle positif et WM-mouvements généralisés est à la base du théorème ci-dessus. Une feuille d'un feuilletage régulier définit un cocycle. S'il est positif, on peut la "pousser" le long du graphe singulier, dans le sens donné par l'orientation des branches, de telle façon que le type d'homéomorphisme de la feuille ne change que lorsque l'on atteint un croisement. La topologie d'un polyhèdre spécial au voisinage d'un croisement, et le fait que tout croisement d'un complexe dynamique K est le sommet initial et terminal d'au plus n 1-composantes, impliquent alors que le passage par le croisement est un WM-mouvement généralisé. Pour la réciproque, la préservation de la caractéristique d'Euler assure que toute orientation des 1-cellules donnée par une quelconque orientation transverse au feuilletage fait de K un complexe dynamique. La conclusion est alors triviale. La preuve de la proposition 1 est combinatoire. On définit une déformation continue d'un complexe standard, qui réalise un WM-mouvement généralisé. La concaténation des déformations correspondant à chaque WM-mouvement dans la décomposition de ψ permet de construire le $(n+1)$-complexe voulu.

The notion of *dynamical* 2*-complex*, introduced and studied in [1], and generalized to any dimension in this Note, allows in particular to obtain a topological representation of any free group automorphism. As in the 2-dimensional case, the definition of a n-dynamical complex is combinatorial, and allows, for a large class of complexes (the *standard* dynamical complexes - see below), to introduce a non-singular semi-flow on the complex. An other aspect is to obtain a topological and dynamical representation of certain semi-direct products of groups, which are not necessarily free, with **Z**. Such products are not necessarily fundamental groups of dynamical 2-complexes. In all the note, we only give sketches of the proofs, the missing ingredients being similar to those used for the 2-dimensional case.

1 Basic definitions

We denote by $Con(X)$ the cone over a space X, by Δ^n the closed n-simplex and, if X is a CW-complex, by $X^{(n)}$ the n-skeleton of X. A *special n-polyhedron K* ([2]) is a n-polyhedron satisfying the following property: For any point $x \in K$, there is a neighborhood $N(x)$ of x in K, a neighborhood $N(y)$ of a point y in the interior of $Con((\partial\Delta^{n+1})^{(n-1)})$, and a homeomorphism $h_x : N(x) \to N(y)$ such that $h_x(x) = y$. If one allows the point y to belong to the boundary of $Con((\partial\Delta^{n+1})^{(n-1)})$, one obtains a notion of *special n-polyhedron with boundary.* The *boundary points of* K are the points $x \in K$ such that $h_x(x)$ belongs to the boundary of $Con((\partial\Delta^{n+1})^{(n-1)})$. The *singular set* $K^{(i)}_{sing}$ $(0 \leq i \leq n)$ is the closure of the set of points x in K whose image under h_x belongs to a i-cell of the interior of $Con((\partial\Delta^{n+1})^{(n-1)})$. The set $K^{(1)}_{sing}$ is called the *singular graph.* The vertices in $K^{(0)}_{sing}$ are called *crossings.* The connected components of $K^{(m+1)}_{sing} - K^{(m)}_{sing}$, $0 \leq m \leq n-1$, are called the *$(m+1)$-components* of the complex (a 0-component is a crossing). Unless otherwise specified, in what follows the complexes considered have an empty boundary.
Let us recall that, if $f : X \to X$ is a continuous map of a topological space, then the *suspension* of $f : X \to X$, denoted by $X \times_f I$, is the space $X \times [0, 1]$, quotiented by the equivalence relation $(x, 1) \sim (f(x), 0)$.
A *flat n-complex* $(n \geq 1)$ K is a special n-polyhedron such that for any integer $1 \leq j \leq n-1$, the $(j+1)$-components are either $(j+1)$-cells or are homeomorphic to $D^j \times_f I$, where f is either the identity or the reflexion relative to a diameter $D^{j-1} \subset D^j$ (D^0 is a single point). If all the components are cells, the n-polyhedron is called *standard.*
One equips any flat complex K with a structure of CW-complex which makes its 1-components to be the edges contained in $K^{(1)}_{sing}$. Let K be a flat n-complex, together with an orientation on the edges of the singular graph. Let C be any i-component of K $(2 \leq i \leq n)$. We will say that C contains an *attractor* (resp. a *repellor*) in its boundary if there is a crossing v of K and a germ $g_v(C)$ of C at v such that all the germs of edges of $K^{(1)}_{sing}$ at v contained in $g_v(C)$ are incoming (resp. outgoing) at v.

Definition 1 *A* dynamical n-complex *is a flat n-complex K, $n \geq 2$, together with an orientation on the edges of $K^{(1)}_{sing}$ satisfying the following two properties:*

1. *Each crossing of K is the initial and terminal crossing of at most n edges of $K^{(1)}_{sing}$.*
2. *Any j-component, $2 \leq j \leq n$, which is a j-cell has exactly one attractor and one repellor for this orientation in its boundary. The other components have no attractor and no repellor in their boundary.*

A *type j-crossing* is a crossing which is the terminal vertex of exactly j 1-components.

Lemma 1 *The Euler characteristic of a dynamical n-complex is zero.*

PROOF: The Euler characteristic of K is equal to $\sum_{i=0}^{n}(-1)^i N_i$, where N_i is the number of i-disc components. One proves by a descending induction on the type of the crossings that there is the same number of type j- and of type $(n+2-j)$-crossings. For $2 \leq i \leq n$, for each pair type j-, type $(n+2-j)$-crossing, there are $C_j^i + C_{n+2-j}^i$ i-disc components in K ($C_p^q = 0$ if $q > p$). Furthermore, $N_1 = \frac{n+2}{2}N_0$. A simple calculation completes the proof. ∎

2 Cocycles and Foliations

Definition 2 *Let K be a flat n-complex ($n \geq 2$), together with an orientation on the edges of its singular graph. A* non-negative cocycle *is a cocycle $u \in C^1(K; \mathbf{Z})$ such that $u(e) \geq 0$ holds for any positive edge e in $K_{sing}^{(1)}$ and there is at least one such edge e with $u(e) > 0$. A* positive cocycle *is a non-negative cocycle which is positive on all the positive embedded loops in $K_{sing}^{(1)}$.*

A *δ_v-move* on a cocycle u of a flat n-complex K consists of removing 1 from the value of u on all the incoming edges at some crossing v of K, and adding 1 to its value on the outgoing one. A *non-negative δ_v-move* on a non-negative cocycle u is a δ_v-move on u such that $\delta_v(u)$ is non-negative.

For any j-component C of K, there is a continuous map h_C from a compact connected j-manifold with boundary $\overline{S}_C$ to the closure $\overline{C}$ of C in K, which is a homeomorphism from the interior of $\overline{S}_C$ onto C and which sends the boundary of $\overline{S}_C$ onto the boundary of $\overline{C}$ in K.

Definition 3 *A* regular codim 1-foliation *$\mathcal{F}$ of a flat n-complex is a decomposition of K into disjointly embedded $(n-1)$-complexes satisfying the following properties:*

1. *For any j-component C which is not a cell ($j \geq 2$), $h_C^{-1}(\mathcal{F} \cap \overline{C})$ is a non-singular foliation of $\overline{S}_C$ by $(j-1)$ cells transverse to the boundary of $\overline{S}_C$.*

2. *For any j-component C which is a cell ($j \geq 2$), $h_C^{-1}(\mathcal{F} \cap \overline{C})$ is a foliation of $\overline{S}_C$ with exactly two leaves reduced to two distinct points, which are in the boundary of $\overline{S}_C$, and whose images under h_C are one or two crossings of K. All the other leaves of $h_C^{-1}(\mathcal{F} \cap \overline{C})$ are disjointly embedded $(n-1)$-cells transverse to the boundary of $\overline{S}_C$ which define a non-singular foliation of the interior of $\overline{S}_C$.*

If K is a flat n-complex with boundary, one requires the leaves of a regular foliation of K to be either disjoint, or contained in a connected component of the boundary of K. The leaves of a codim 1-regular foliation are standard $(n-1)$-complexes embedded in K in a particular way. Such embeddings are called *r-embeddings.* A r-embedding is *degenerate* if it contains a crossing of K. The following lemma establishes the relationship between r-embeddings and integer cocycles.

Lemma 2 *Let K be a flat n-complex ($n \geq 2$). Any cocycle $u \in C^1(K; \mathbf{Z})$ defines a transversely oriented flat $(n-1)$-complex K_u which is r-embedded in K. Conversely, any transversely oriented flat $(n-1)$-complex K_u r-embedded in K in a non-degenerate way defines a unique cocycle $u \in C^1(K; \mathbf{Z})$.*

3 Generalized WM-moves

We call below *singular i-simplex* of a standard n-complex K a closed i-cell of K which is a i-simplex, and whose all j-faces are in $K_{sing}^{(j)}$ for any $1 \leq i \leq n$ and any $0 \leq j \leq i$.

Definition 4 *A* generalized WM-move $W_{\Delta_0\Delta_1}$ *from a standard n-complex K_0 ($n \geq 1$) to a standard n-complex K_1 consists of a collapsing C_{Δ_0} of a singular j-simplex Δ_0 ($n \geq j \geq 1$) of K_0 followed by a splitting S_{Δ_1} at $C_{\Delta_0}(\Delta_0)$ of a singular $(n+1-j)$-simplex Δ_1, such that S_{Δ_1} is not a homotopy-inverse of C_{Δ_0}.* Clearly, a generalized WM-move is a homotopy equivalence.

Lemma 3 below gives a topological realization of such a WM-move.

Lemma 3 *With the assumptions and notations of definition 4, we denote by $H^t_{\Delta_0\Delta_1}$, $t \in [-1, 1]$, a continuous deformation which realizes a generalized WM-move $W_{\Delta_0\Delta_1}$, with $H^0_{\Delta_0\Delta_1}(K_0) = C_{\Delta_0}(K_0)$, and with support in a neighborhood of Δ_0. If $\mathcal{C}_{K_0K_1} = \bigcup_{t=-1}^{1} H^t_{\Delta_0\Delta_1}(K_0)$, then $\mathcal{C}_{K_0K_1}$ is a flat $(n+1)$-complex with two boundary components $K_0 \times \{-1\}$ and $K_1 \times \{1\}$, called an* elementary foliated complex. *It admits a transversely orientable codim 1-regular foliation with compact leaves $\mathcal{F}$. All the non-degenerate leaves of $\mathcal{F}$ are homeomorphic either to K_0 or to K_1. There is exactly one degenerate leaf, containing the only crossing of $\mathcal{C}_{K_0K_1}$, which is homeomorphic to the complex $H^0_{\Delta_0\Delta_1}(K_0)$.*

The restriction on the collapsing C_{Δ_0} and the splitting S_{Δ_1} not to be homotopy-inverses one from the other (see definition 4), is crucial for a neighborhood of $H^0_{\Delta_0\Delta_1}(\Delta_0)$ in $\mathcal{C}_{K_0K_1}$ to be as desired. Lemma 4 outlines the connection between non-negative δ_v-moves and generalized WM-moves.

Lemma 4 *Let K be a dynamical n-complex ($n \geq 2$) which admits a non-negative cocycle $u \in C^1(K; \mathbf{Z})$. If v is a crossing of K such that $\delta_v(u)$ is a non-negative cocycle, then there is a subcomplex of K containing v which is homeomorphic to an elementary foliated complex $\mathcal{C}_{K_uK_{\delta_v(u)}}$, where K_u and $K_{\delta_v(u)}$ are as given by lemma 2. In particular, $K_{\delta_v(u)}$ is obtained from K_u by a generalized WM-move. If v is a type j-crossing, this WM-move consists of the collapsing of a $(j-1)$-simplex followed by the splitting of a $(n-j+1)$-simplex.*

Proof: Lemma 2 associates a transversely oriented flat $(n-1)$-complex K_u (resp. $K_{\delta_v(u)}$) to the cocycle u (resp. $\delta_v(u)$) whose transverse orientation agrees with the orientation of the edges of the singular graph. The cocycles u and $\delta_v(u)$ only differ on the edges of $K^{(1)}_{sing}$ which are incident to v. This allows to define a continuous deformation in K from K_u to $K_{\delta_v(u)}$, with support in a neighborhood in K of v. By definition of the orientation of the edges of the singular graph, and thanks to the topology of a dynamical n-complex in a neighborhood of the crossings, this deformation realizes a generalized WM-move as indicated. ■

4 Main results

Theorem 1 *A flat n-complex K, $n \geq 2$, admits a transversely orientable codim 1-regular foliation $\mathcal{F}$, whose leaves are compact and all have the same Euler characteristic, if and only if there exists an orientation of the edges of the singular graph such that the complex K together with this orientation is a dynamical n-complex admitting a positive cocycle.*
In addition, if such a foliation $\mathcal{F}$ exists, then all its leaves are homotopically equivalent standard $(n-1)$-complexes. If $\mathcal{L}$ is any leaf of $\mathcal{F}$, there is a homotopy equivalence $\psi : \mathcal{L} \to \mathcal{L}$ such that the complex K is homotopically equivalent to the suspension of $\mathcal{L}$ under ψ.

PROOF: One orients the edges of $K^{(1)}_{sing}$ by some transverse orientation to $\mathcal{F}$. The Euler characteristic condition on the leaves implies that condition (1) of definition 1 is satisfied by this orientation. This choice implies also that, if there is an attractor or a repellor y in the boundary of a j-component C, then there is a leaf of $h_C^{-1}(\mathcal{F} \cap \overline{C})$ reduced to a single point x in the boundary of $\overline{S}_C$, and such that $h_C(x) = y$. By definition of a regular foliation, there are exactly two such leaves for each component. One concludes that K equipped with this orientation is a dynamical n-complex. The union of the leaves of $\mathcal{F}$ clearly defines a positive cocycle in $C^1(K; \mathbf{Z})$.
The definition of a regular foliation forces the leaves of $\mathcal{F}$ to be standard $(n-1)$-complexes. Because the corresponding cocycles are obtained one from an other by non-negative δ_v-moves, the leaves are obtained one from an other by sequences of WM-moves (see lemma 4). Thus they are homotopically equivalent. The last point of theorem 1 is classical.
Conversely, let u be a positive cocycle of a dynamical n-complex K. Then there is a crossing v of K such that u is positive on all the incoming 1-components at v. Otherwise a simple induction allows to find a positive loop l in $K^{(1)}_{sing}$ with $u(l) = 0$. Therefore $\delta_v(u)$ is a positive cocycle. A similar induction proves that such a non-negative δ_v-move can be applied exactly once for each crossing, and that one so returns to the original cocycle u. Lemma 4 allows then to build a foliation of K as anounced. ■

Proposition 1 *Let $\psi : K \to K$ be a continuous map of a standard n-complex K $(n \geq 1)$, such that $\psi = \alpha \circ \sigma_r \circ \cdots \circ \sigma_1$ is a composition of generalized WM-moves $\sigma_i : K_{i-1} \to K_i$ $(i = 1, \cdots, r)$, and of a homeomorphism $\alpha : K_r \to K_0$. Then there is a dynamical $(n+1)$-complex K_S which is homotopically equivalent to the suspension of K under ψ. Moreover, K_S admits a positive cocycle u such that a n-complex K_u associated to u by lemma 2 is homeomorphic to K. This cocycle defines on K_u the above sequence of generalized WM-moves.*

PROOF: By lemma 3, each WM-move σ_i defines an elementary foliated $(n+1)$-complex E_i. For each $i = 1, \cdots, r-1$, E_i is glued to E_{i+1} along K_i by the identity-map. Lemma 3 implies that the resulting complex is a flat $(n+1)$-complex with two boundary components, K_0 and K_r. Identifying K_r to K_0 by α gives a flat $(n+1)$-complex. By construction, it is dynamical and satisfies the anounced properties. ■

Proposition 2 *Any standard dynamical n-complex K carries a non-singular semi-flow.*

One defines a semi-flow in a neighborhood of $K^{(1)}_{sing}$, whose orientation agrees with the orientation of the edges. Definition 1, item (2) allows to extend it to a non-singular semi-flow on K. It is important here all the j-components to be j-cells for any $j \geq 2$. ■

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