Hyperbolicity of mapping-torus groups and spaces *

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Abstract: This paper deals with the geometry of metric "two dimensional" spaces, equipped with semi-flows admitting transverse foliations by forests. Our main theorem relates the Gromov-hyperbolicity of such spaces, for instance mapping-telescopes of $\mathbb{R}$-trees, with the dynamical behaviour of the semi-flow. As a corollary, we give a new proof of the following theorem [3]: *Let α be an hyperbolic injective endomorphism of the rank n free group F_n. If the image of α is a malnormal subgroup of F_n, then $G_\alpha = F_n \rtimes_\alpha \mathbb{Z}$ is a hyperbolic group.*

Introduction

The subject of 3-dimensional topology has completely changed in the seventies with Thurston's geometric methods. His geometrization conjecture involves eight classes of manifolds, among which the hyperbolic manifolds play the most important rôle. In this context, a hyperbolic manifold is a compact manifold which admits (or whose interior admits in the case of non-empty boundary) a metric a constant curvature -1. According to another conjecture of Thurston, any closed hyperbolic 3-manifold should have a finite cover which is a mapping-torus. This gives a particular interest to these mapping-tori manifolds. Recall that a mapping-torus is a manifold which fibers over the circle. Namely this is a 3-manifold constructed from a homeomorphism h of a compact surface Σ as

$$M = (\Sigma \times [0,1])/((x,1) \sim (h(x),0)).$$

For these manifolds, the hyperbolization conjecture has been proved, see for instance [24]: the manifold M constructed from Σ and h as above is hyperbolic if and only if Σ has negative Euler characteristic and h is a pseudo-Anosov homeomorphism (see [12]).

In parallel to these developments in 3-dimensional topology, there has been a revival in combinatorial group theory. First introduced by Dehn at the beginning of the twentieth century, geometric methods were reintroduced in this field by Gromov in the 80's. The notion of hyperbolicity carries over in some sense from manifolds to metric spaces and groups. We speak then of Gromov hyperbolicity. Such metric spaces and groups are also termed weakly hyperbolic, or negatively curved, or word-hyperbolic, see [18] as well as

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[14], [1], [8] or [5] among others. Mapping-tori manifolds have the following analog in this setting: given a finitely presented group $F =< S\,;R >$, $S = \{x_1, \cdots, x_n\}$, and an endomorphism α of F, the mapping-torus group of (α, F) is the group with presentation $< x_1, \cdots, x_n, t\,; R, t^{-1}x_it = \alpha(x_i), i = 1, \cdots, n >$. For instance, if the 3-manifold M is the mapping-torus of (h, Σ) and if $h_\#$ is the automorphism induced by h on the fundamental group of Σ, then the fundamental group of M is the mapping-torus group of $(h_\#, \pi_1(\Sigma))$. In fact, in this case, since $h_\#$ is an automorphism of $\pi_1(\Sigma)$, the mapping-torus group is easily described as the semi-direct product $\pi_1(\Sigma) \rtimes_{h_\#} \mathbb{Z}$.

The main and central result in group theory concerning the preservation of hyperbolicity under extension is the Combination Theorem of [3] (see also a clear exposition of this theorem in [19]). Alternative proofs have been presented since the original paper of Bestvina - Feighn [17, 21], but concerning essentially the so-called 'acylindrical case', where the 'Annuli Flare Condition' of [3] is vacuously satisfied. Gersten [16] proves a converse of the Combination Theorem. At the periphery of this theorem, let us also cite [11, 23] about the hyperbolicity of other kinds of extensions or [22] who shows the existence of Cannon-Thurston maps in this context.

As a corollary of the Combination Theorem, and to illustrate it, the authors of [3] emphasize the following result: Let F be a hyperbolic group and let α be an automorphism of F. Assume that α is hyperbolic, namely there exist $m \in \mathbb{Z}$ and $\lambda \in \mathbb{R}, \lambda > 1$, such that for any element f of word-length $l(f)$ in the generators of F, we have $\max(l(\alpha^m(f)), l(\alpha^{-m}(f)))$ $\geq \lambda l(f)$. Then $F \rtimes_\alpha \mathbb{Z}$ is a hyperbolic group. This corollary lives in a different world than the above cited alternative proofs of the Combination Theorem, namely it is 'non-acylindrical'. No paper, at the exception of the Bestvina - Feighn original one, covers it. Swarup used it to give a weak hyperbolization theorem for 3-manifolds [26]. Hyperbolic automorphisms were defined by Gromov [18], see also [3]. From [25], if an hyperbolic automorphism is defined on a hyperbolic group then this hyperbolic group is the free product of two kinds of groups: free groups and fundamental groups of closed surfaces with negative Euler characteristic. Hyperbolic automorphisms of fundamental groups of closed surfaces are exactly the automorphisms induced by pseudo-Anosov homeomorphisms. Brinkmann characterized the hyperbolic automorphisms of free groups as the automorphisms without any finite invariant set of conjugacy-classes [6]. Below we consider hyperbolic injective free group endomorphisms. The notion of hyperbolic automorphism is generalized in a straightforward way to injective endomorphisms. We give a new proof of the Bestvina - Feighn's theorem in this setting:

Theorem 0.1 *Let $F_n =< x_1, \cdots, x_n >$ be the free group of rank n. Let α be a hyperbolic injective endomorphism of F_n. Assume that the image of α is malnormal, that is $w^{-1}Im(\alpha)w \cap Im(\alpha) = \{1\}$ for any $w \notin Im(\alpha)$ of F_n. Then the mapping-torus group $G_\alpha =< x_1, \cdots, x_n, t\,; t^{-1}x_it = \alpha(x_i), i = 1, \cdots, n >$ is a hyperbolic group.*

I. Kapovich [20] worked on mapping-tori of injective free group endomorphisms, trying to avoid the assumption of malnormality of the image of the endomorphism.

We consider the group given by its standard presentation of mapping-torus group. Our proof relies on an approximation of the geodesics in the Cayley complex of the group for this presentation. Let α be an automorphism of F_n. Let G_α be the mapping-torus group of (α, F_n). The above Cayley complex for G_α has a very particular structure. It carries a non-singular semi-flow and this semi-flow is transverse to a foliation of the complex by trees. A non-singular semi-flow is a one parameter family $(\sigma_t)_{t\in\mathbb{R}^+}$ of continuous maps of the 2-complex, depending continuously on the parameter and satisfying the usual properties of a flow: $\sigma_0 = \mathrm{Id}$, $\sigma_{t+t'} = \sigma_t \circ \sigma_{t'}$. Let Γ be a graph with fundamental group F_n. Let $\psi\colon \Gamma \to \Gamma$ be a simplicial map on Γ which induces α on the fundamental group of Γ. Let $K = (\Gamma \times [0,1])/((x,1) \sim (\psi(x),0))$ be the mapping-torus of (ψ,Γ). Then K is a simple example of a 2-complex equipped with a non-singular semi-flow. The orbits of the semi-flow are the concatenation of intervals $\{x\} \times [0,1]$, $x \in \Gamma$, glued together by identifying $(x,1)$ with $(\psi(x),0)$. Moreover the 2-complex is foliated with compact graphs $\Gamma \times \{t\}$ transverse to the semi-flow. The universal covering of this 2-complex is the Cayley complex of G_α for the standard presentation as a mapping-torus group. Let us describe this universal covering. The universal covering of Γ is a tree T. Let $\tilde{\psi}\colon T \to T$ be a simplicial lift of ψ. That is, if $\pi\colon \Gamma \to T$ is the covering-map, $\psi \circ \pi = \pi \circ \tilde{\psi}$. Since ψ induces an automorphism on $\pi_1(\Gamma)$, the universal covering of K is homeomorphic to the quotient of $\displaystyle\bigsqcup_{n\in\mathbb{Z}} T \times [n,n+1]$ by the identification of $(x,n+1) \in T \times [n,n+1]$ with $(\tilde{\psi}(x), n+1) \in T \times [n+1,n+2]$. Such a topological space is called the *mapping-telescope of* $(\tilde{\psi},T)$. As a corollary of our main theorem we obtain an analog for mapping-telescopes of Thurston's theorem for mapping-tori of surface homeomorphisms. The structure of graph or of 2-complex which exists when dealing, as above, with Cayley complexes of mapping-torus groups is irrelevant. We only need that T be a 0-hyperbolic metric space, that is a geodesic metric space whose geodesic triangles are tripods. Equivalently, such a T is an $\mathbb{R}$-*tree*. We refer the reader to [2] or [8] for the equivalence of these two notions and to [2] for a survey about $\mathbb{R}$-trees. Let us observe that Bowditch [4] refers, without further proof, to [3] for stating a theorem about the Gromov-hyperbolicity of mapping-telescopes of $\mathbb{R}$-graphs. A weak version of our result gives a complete proof of such a result in the case of $\mathbb{R}$-trees:

Theorem 0.2 *Let (T,d_T) be an $\mathbb{R}$-tree. Let $\tilde{\psi}\colon T \to T$ be a continuous map on T which satisfies the following properties:*

1. *There exist $\mu \geq 1$, $K \geq 0$ such that $\mu d_T(x,y) \geq d_T(\tilde{\psi}(x),\tilde{\psi}(y)) \geq \frac{1}{\mu} d_T(x,y) - K$ holds.*
2. *There exist $\lambda > 1$, $N \geq 1$, $M \geq 0$ such that for any pair of points x, y in T with $d_T(x,y) \geq M$, either $d_T(\tilde{\psi}^N(x),\tilde{\psi}^N(y)) \geq \lambda d_T(x,y)$ or for some x_N, y_N with $\tilde{\psi}^N(x_N) = x$, $\tilde{\psi}^N(y_N) = y$, $d_T(x_N,y_N) \geq \lambda d_T(x,y)$.*

Then the mapping-telescope of $(\tilde{\psi},T)$ is a Gromov-hyperbolic metric space for some mapping-telescope metric.

Let us briefly explain what is a mapping-telescope metric. Roughly speaking, at each point in the mapping-telescope we can move in two directions. Either along a leaf $T \times \{t\}$, or along a path which is a concatenation of intervals $\{x\} \times [n,n+1]$, $x \in T$. The lengths in the vertical direction are measured using the obvious parametrization. We provide the trees $T \times \{t\}$ with a metric. Then the mapping-telescope metric is defined as follows: the distance between two points x, y is the shortest way from x to y among all the paths obtained as sequences of horizontal and vertical moves.

We deal with more general spaces than mapping-telescopes. The reader will find in Section 4 the precise statement of our result. The spaces under study are called forest-stacks. We only need on the one hand the existence of a non-singular semi-flow and, on the other hand, the existence of a transverse foliation by forests. We allow the homeomorphism-types of the forests to vary along $\mathbb{R}$. We refer the reader to Remark 13.8 for a brief discussion about direct applications of our main theorem, which we chosed not to develop here for the sake of a clearer and shorter presentation.

In Section 1, we give an illustration, and a proof, of our theorem in a very particular case. However very simple, basic ideas of the work to come appear here. Sections 2 to 11 form the heart of the paper. In Sections 2 and 3 we define the objects under study. In Section 4 we state our theorem about forest-stacks. The statements of the other results, concerning mapping-telescopes and mapping-torus groups, take place in Sections 12 and 13. After some preliminary work, Section 5, we study the so-called straight quasi geodesics in forest-stacks equipped with strongly hyperbolic semi-flows, Sections 6 and 7. We rely upon these last two sections to give an approximation of straight quasi geodesics in fine position with respect to horizontal one, Section 8, and then in Section 9 to show how to put a straight quasi geodesic in fine position with respect to a horizontal one. In Section 10 we gather all these results to prove that straight quasi geodesic bigons are thin. We conclude in Section 11.

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Since they play the central rôle in this paper, we briefly precise what we mean by *Gromov hyperbolic metric spaces.* Gromov introduced the notion of (r, s)-*quasi geodesic space* in [18]: A metric space (X, d) is a (r, s)-quasi geodesic space if, for any two points x, y in X there is a (r, s)-*chain*, that is a finite set of points $x = x_0, x_1, \cdots, x_k = y$ such that $d(x_{i-1}, x_i) \leq r$ for $i = 1, \cdots, k$ and $\sum_{i=1}^{k} d(x_{i-1}, x_i) \leq sd(x, y)$. A quasi geodesic metric space is a metric space which is (r, s)-quasi geodesic for some non negative real constants r, s. A (r, s)-chain triangle in a quasi geodesic metric space is a triangle whose sides are (r, s)-chains. A chain triangle is δ-thin, $\delta \geq 0$, if any side is in the δ-neighborhood of the union of the two other sides. We say that chain triangles in a (r, s)-quasi geodesic metric space X are thin if there exists $\delta \geq 0$ such that any (r, s)-chain triangle in X is δ-thin. In this case, X is a Gromov-hyperbolic metric space, more precisely X is a δ-hyperbolic metric space.

In the whole paper, unless otherwise specified, "(quasi) geodesic(s)" means "finite length (quasi) geodesic(s)".

1 An illustration

We start considering a very particular case of our theorem. We feel this simple example might serve as an illustration of the work to come. We hope this will help the reader to figure out the contents and ideas of the paper. Our aim is to prove the Affirmation stated below.

We choose a real number $\lambda > 1$. We set d_0 the usual distance on $\mathbb{R}$. For any real r, we set $d_r = \lambda^{|r|} d_0$. The length $|I|_r$ of a real interval I is the distance, with respect to d_r, between the endpoints of I. We consider the plane $\mathbb{R}^2$. We denote by $p_x \colon \mathbb{R}^2 \to \mathbb{R}$ the projection on the x-axis and by $p_y \colon \mathbb{R}^2 \to \mathbb{R}$ the projection on the y-axis. We denote by $V_a = p_x^{-1}(a)$ the vertical line through a point a. Vertical lines (resp. horizontal line $p_y^{-1}(r)$)

are equipped with the distance d_0 (resp. with the distance d_r). Lengths of horizontal and vertical intervals are measured with respect to the distance defined on the corresponding line. A *telescopic path* is a concatenation of vertical and horizontal non degenerate intervals, where non degenerate means not reduced to a point. The horizontal (resp. vertical) length of a telescopic path is the sum of the horizontal (resp.vertical) lengths of its maximal horizontal (resp. vertical) intervals. The telescopic length of a telescopic path is the sum of its horizontal and vertical lengths. The telescopic distance between two points in $\mathbb{R}^2$ is the infimum of the telescopic lengths of the telescopic paths between these two points. We want to prove the following result:

Affirmation: The plane $\mathbb{R}^2$ equipped with the telescopic distance is a Gromov hyperbolic geodesic metric space.

Step 1: Computation of the geodesics. Let a, b be any two points in $\mathbb{R}^2$. Let I_{ab} be the compact interval of the x-axis, bounded by the projections $p_x(a)$ and $p_x(b)$ of a and b. Let g be any telescopic geodesic from a to b. On the one hand, the length of a telescopic path is never shorter than the length of its projection on a vertical line, so that g lies between V_a and V_b. On the other hand, if $c \in I_{ab}$, the vertical line V_c separates a from b, so that g intersects V_c. *Therefore the telescopic geodesic g intersects all the vertical lines separating a from b, and no other vertical line.* Given a telescopic path containing one vertical interval and two horizontal intervals I, I' at different heights, there exists a stricly shorter telescopic path with the same endpoints. It is obtained by substituting one of the horizontal intervals, say I, by another horizontal interval which intersects the same vertical lines than I, and which lies at the same height than I'. *Thus the telescopic geodesic g is the concatenation of at most one non degenerate horizontal interval with at most two non degenerate vertical intervals.* Furthermore, any horizontal interval in the x-axis minimizes the horizontal distance between the vertical lines passing through its endpoints. *Thus, if $p_y(a)p_y(b) \leq 0$, then g is the concatenation of the horizontal interval I in the x-axis which connects V_a and V_b, with the vertical intervals in V_a and V_b which connect a and b to the endpoints of I.*

In order to compute the geodesics in the case where $p_y(a)p_y(b) \geq 0$, we distinguish two cases:

Case A: $0 \leq p_y(a) = p_y(b)$ Then g is the concatenation of two vertical intervals of vertical lengths $t \geq 0$ with one horizontal interval I. The horizontal length of I is equal to $\lambda^t d_{p_y(a)}(a,b)$ if $p_y(I) \geq p_y(a)$ and to $\lambda^{-t} d_{p_y(a)}(a,b)$ if $p_y(I) \leq p_y(a)$ and $p_y(I) \geq 0$. Indeed let us recall that horizontal intervals in the x-axis are dilated both in the future and in the past. We set $f(t) = 2t + \lambda^{-t} d_{p_y(a)}(a,b)$. Let $t_\star$ be any real number such that $0 \leq t_\star \leq p_y(b)$ and $f(t_\star) = \min_{0 \leq t \leq p_y(b)} f(t)$. From which precedes g is the concatenation of two vertical intervals of length $t_\star$ with a horizontal interval in the horizontal line $p_y^{-1}(p_y(b) - t_\star)$. The function $f(t)$ attains its minimum at $t_\circ = \frac{ln(\lambda d_{p_y(a)}(a,b)/2)}{ln \lambda}$. Therefore $t_\star = \min(\max(t_\circ, 0), p_y(b))$ is unique. *We so proved that there exists a unique telescopic geodesic between a and b.* Its telescopic length is equal to $f(t_\star)$.

We distinguish below three cases.

Case (0): $t_\star > t_\circ$ *The horizontal distance between a and b is so short that the horizontal interval between a and b realizes the telescopic distance.* Indeed $t_\star >$

$t_\circ \Rightarrow t_\star = 0$. The horizontal distance between a and b, which is the horizontal length of the horizontal interval I with the notations above, is smaller than $\frac{2}{ln\lambda}$.

Case (1): $t_\star = t_\circ$ *The optimal case* The horizontal interval I of g lies in the horizontal line $p_y(a) - t_\circ$. The horizontal length of I is $\frac{2}{ln\lambda}$. The vertical intervals in g have vertical lengths $t_\circ = \frac{ln(\lambda d_{p_y(a)}(a,b)/2)}{ln\lambda}$.

Case (2): $t_\star < t_\circ$ *The horizontal distance between a and b is too large with respect to the height of the horizontal line through a and b.* Then the horizontal interval I of g lies in the x-axis. The horizontal length of I is equal to $\lambda^{-p_y(a)} d_{p_y(a)}(a,b) > \frac{2}{ln\lambda}$. It depends on $d_{p_y(a)}(a,b)$ and might be arbitrarily large.

Case B: $0 \leq p_y(a) \neq p_y(b)$ Without loss of generality we assume $p_y(a) < p_y(b)$. We consider the point $c = V_a \cap p_y^{-1}(p_y(b))$. The non negative real number $t_\star$ is greater than or equal to $p_y(b) - p_y(a)$. Therefore the telescopic geodesic from c to b computed in Case A admits a subpath from a to b. This subpath is the unique telescopic geodesic between a and b.

The same arguments apply to the case where both a and b lie in the negative half-plane. This concludes the computations of the geodesics.

Step 2: Geodesic triangles are thin. Let Δ be any geodesic triangle in the upper half-plane. Let g_1, g_2, g_3 be the sides of Δ. Let $t_\star(g_i)$ and $t_\circ(g_i)$ be the non negative real numbers for g_i defined above. Let I_1, I_2, I_3, $p_y(I_3) \geq p_y(I_2) \geq p_y(I_1)$ be the horizontal geodesics respectively in g_1, g_2 and g_3.

Case (1): $t_\star(g_1) \geq t_\circ(g_1)$ Then $t_\star(g_2) \geq t_\circ(g_2)$ and $t_\star(g_3) \geq t_\circ(g_3)$. Therefore $|I_i|_{p_y(I_i)} \leq \frac{2}{ln\lambda}$, $i = 1, 2, 3$. The vertical segment of g_2 between I_3 and I_2 is at horizontal distance smaller than $\frac{2}{ln\lambda}$ from a vertical segment in g_1. Because of the uniform contraction in λ^{-t}, this implies that I_2 is at vertical distance smaller than $\frac{ln2}{ln\lambda}$ from I_1. Therefore the union of I_1 with the two orbit-segments between its endpoints and the horizontal line $p_y^{-1}(p_y(I_2))$ is at telescopic distance smaller than $\frac{ln2}{ln\lambda} + \frac{2}{ln\lambda}$ from I_2. All the points of Δ not considered up to now belong to at least two distinct sides.

Case (2): $t_\star(g_1) < t_\circ(g_1)$ Then $p_y(I_1) = 0$, i.e. I_1 lies in the x-axis.

1. If $t_\star(g_2) = t_\circ(g_2)$ and $t_\star(g_3) = t_\circ(g_3)$, then $|I_i|_{p_y(I_i)} = \frac{2}{ln\lambda}$ for $i = 2, 3$. Thus $|I_1|_0 \leq \frac{4}{\lambda}$. We conclude as in Case (1).
2. If both $t_\star(g_2) > t_\circ(g_2)$ and $t_\star(g_3) > t_\circ(g_3)$ then both I_2 and I_3 lie in the x-axis so that $I_1 = I_2 \cup I_3$. Then any point in Δ belongs to at least two distinct sides.
3. If only $t_\star(g_3) > t_\circ(g_3)$ then $I_2 \subset I_1$. Let $I_1' \subset I_1$ be the complement of I_2 in I_1. Then $|I_1'|_0 \leq \frac{2}{ln\lambda}$. The same inequality is satisfied for the horizontal distance between the vertical segments connecting the endpoints of I_1' to I_3. This concludes Case (2).

The case where Δ lies in the negative half-plane is treated in the same way. The other cases are dealt with using similar, but simpler, arguments than above. We leave these cases as an exercise for the reader.

Remark 1.1 The above computations fail, and the space is no more Gromov-hyperbolic, if one substitutes $d_y = P(|y|)d_0$ to $d_y = \lambda^{|y|}d_0$ with $P(.)$ a polynomial function of y. Indeed, in this case, the length of the horizontal interval between the two considered orbits, evaluated at the height where the minimum of the length-function $f(t)$ is attained, depends, even in the optimal case, on the horizontal length of the interval connecting one point to the orbit of the other. Whereas in the exponential case it equals $\frac{2}{ln\lambda}$ unless it belongs to the horizontal axis.

2 Mapping-telescopes and Forest-stacks

Let X be a topological space. Call X a *topological tree* if there exists a unique arc between any two points in X. A *topological forest* is a union of disjoint topological trees. By arc, we mean the image of an injective path. A path in X is a continuous map from a bounded interval of the real line into X. A *forest-map* is a continuous map of a topological forest into itself.

Definition 2.1 Let $\psi\colon X \to X$ be a forest-map. The *mapping-telescope* K_ψ *of* (ψ, X) is the topological space resulting from $K_X = \bigsqcup_{n\in\mathbb{Z}} X \times [n, n+1]$ by the identification of each point $(x, n+1) \in X \times [n, n+1]$ with the point $(\psi(x), n+1) \in X \times [n+1, n+2]$.

Let us examine a little bit more closely the topology of these mapping-telescopes.

For any integer $n \in \mathbb{Z}$, for any $(x, r) \in X \times [n, n+1]$, for any non negative real number $t \geq 0$, we define $\tilde{\sigma}_t((x, r))$ as the point $(\psi^{E[t-(n+1-r)]+1}(x), r+t)$ in $X \times [E[r+t], E[r+t]+1]$, where $E[r]$ denotes the greatest integer smaller than r. The map $\tilde{\sigma}_t$ is defined on K_X (the disjoint union of the $X \times [n, n+1]$) for every $t \geq 0$. Moreover $\tilde{\sigma}_{t+t'} = \tilde{\sigma}_t \circ \tilde{\sigma}_{t'}$. If $a = (x, n+1) \in X \times [n+1, n+2]$, then $\tilde{\sigma}_t(a) = (\psi^{E[t]}(x), n+1+t) \in [n+1+E[t], E[t]+n+2]$. Whereas if $a = (x, n+1) \in X \times [n, n+1]$ then $\tilde{\sigma}_t(a) = (\psi^{E[t]+1}(x), n+1+t) \in X \times [n+1+E[t], E[t]+n+2]$, which is equal to $\tilde{\sigma}_t(b)$ with $b = (\psi(x), n+1) \in X \times [n+1, n+2]$. Therefore $(\tilde{\sigma}_t)_{t\in\mathbb{R}^+}$ descends to the mapping-telescope K_ψ, where it defines a family with one parameter $(\sigma_t)_{t\in\mathbb{R}^+}$ of continuous maps of K_ψ. This family depends continuously on the parameter $t \in \mathbb{R}^+$. It satisfies furthermore $\sigma_0 = \mathrm{Id}_{K_\psi}$ and $\sigma_{t+t'} = \sigma_t \circ \sigma_{t'}$. Such a family is called a *semi-flow* on K_ψ.

Let $f\colon K_\psi \to \mathbb{R}$ defined by $f(a) = r$ if $a \in X \times \{r\}$. Then f is a continuous surjective map. The pre-image of any real number r is $X \times \{r\}$, a topological forest. Furthermore, for any $t \geq 0$, $f \circ \sigma_t = \tau_t \circ f$, where $\tau_t\colon \mathbb{R} \to \mathbb{R}$ is defined by $\tau_t(r) = r + t$.

We extracted above the two properties shared by mapping-telescopes which are really important for our work. We now define a class of spaces which satisfy these two properties, and in particular generalize the mapping-telescopes.

Definition 2.2 Let X be a topological space. Let $(\sigma_t)_{t\in\mathbb{R}^+}$ be a semi-flow on X. Let $f\colon X \to \mathbb{R}$ be a surjective continuous map such that:

1. For any real number r, the *stratum* $f^{-1}(r)$ is a topological forest.

2. For any $t \geq 0$, $f \circ \sigma_t = \tau_t \circ f$, where $\tau_t(r) = r + t$ for any real number r.

Then X is a *forest-stack*, denoted by (X, f, σ_t).

Remark 2.3 All the strata of a mapping-telescope are homeomorphic. This is not required in the definition of a forest-stack.

As we just saw, a mapping-telescope is an example of a forest-stack. In Section 13, we show that a Cayley complex for the mapping-torus group of an injective free group endomorphism is a mapping-telescope of a forest-map, and so a forest-stack. The reader can also find there, and in Section 12, an illustration of the horizontal and vertical metrics on forest-stacks, that we are now going to define.

3 Metrics

The aim of this section is to introduce a particular metric on forest-stacks, called *telescopic metric*. We deal sometimes with metric spaces which are not necessarily connected, for instance forests. In this case, when considering the distance between two points, it will always be tacitly assumed that the two points lie in a same connected component of the space.

3.1 Horizontal and Vertical metrics

Let us consider a forest-stack $(\tilde{X}, f, \sigma_t)$, see Definition 2.2. We want to define a natural metric on the orbits of the semi-flow.

Definition 3.1 The *future orbit* $O^+(x)$ of a point x under the semi-flow is the set of points y such that $\sigma_t(x) = y$ for some $t \geq 0$.

The *past orbit* $O^-(x)$ of a point x under the semi-flow is the set of points y such that x is in the future orbit of y.

The *orbit* $O(x)$ of a point x under the semi-flow is the set of points y such that there exists z which lies in the future orbit of both x and y.

Let us observe that in general the orbit of a point x strictly contains the union of the future and past orbits of x.

The orbits of the semi-flow are topological trees. This is a straightforward consequence of the semi-conjugacy of the semi-flow with the translations in $\mathbb{R}$ via the map f. Let x, y be any two points in a same orbit of the semi-flow. Assume that x and y lie in a same future orbit of the semi-flow. We consider the orbit-segment between x and y, where an *orbit-segment* is a compact interval contained in the future orbit of some point. The function f is a homeomorphism from this orbit-segment onto an interval of the real line. We define the distance between x and y as the real length of this interval. Assume now that x and y do not lie in a same future orbit. The future orbits of x and y meet at some point z such that the concatenation of the orbit-segment between x and z with the orbit-segment between z and y is an injective path. We then define the distance between x and y as the sum of the distances between x and z and z and y. We so have defined a distance on the orbits of the semi-flow. This distance is termed *vertical distance*.

Definition 3.2 A *vertical path* in a forest-stack is a path contained in an orbit of the semi-flow. A *vertical geodesic* is an injective vertical path.

A *horizontal path* in a forest-stack is a path contained in a stratum. A *horizontal geodesic* is an injective horizontal path.

Definition 3.3 Let $(\tilde{X}, f, \sigma_t)$ be a forest-stack. Let $\mathcal{H} = (m_r)_{r\in\mathbb{R}}$ be a collection of metrics on the strata of $\tilde{X}$. Then $\mathcal{H}$ is a *horizontal metric* if for any $r \in \mathbb{R}$, for any $\epsilon > 0$, for any x, y in a same connected component of the stratum $f^{-1}(r)$, there exists $\mu > 0$ such that $0 \leq t \leq \mu$ implies $||\sigma_t(g_{xy})|_{r+t} - |g_{xy}|_r| \leq \epsilon$, where g_{xy} is the unique horizontal geodesic between x and y, and $|.|_r$ denotes the horizontal length with respect to m_r in the stratum $f^{-1}(r)$.

A forest-stack $\tilde{X}$ equipped with a horizontal metric $\mathcal{H}$ will be denoted by $(\tilde{X}, f, \sigma_t, \mathcal{H})$.

In other words, a horizontal metric on a forest-stack is a collection of metrics on the strata such that the length of the horizontal paths varies continuously when homotoping them along the orbits of the semi-flow. The definition of "horizontal metric" does not imply that the horizontal distance varies continuously along the orbits. See Figure 1. This figure is an illustration of what might happen because of the possible non-injectivity of the maps $\sigma_{t|f^{-1}(r)}$: if $\sigma_t(x) = \sigma_t(y)$ for two distinct points x, y in a horizontal geodesic $g \in f^{-1}(r)$ then $\sigma_t(g)$ is an horizontal path but this is not necessarily the image of an injective path. Thus the distance between the endpoints of $\sigma_t(g)$ is not realized by $\sigma_t(g)$ but by a path of smaller length, smaller of at least the length of $\sigma_t(g_{xy})$, where $g_{xy} \subset g$ is the subpath of g between x and y.

Definition 3.4 Any horizontal geodesic g_{xy} between two distinct points x, y such that $\sigma_t(x) = \sigma_t(y)$ for some $t > 0$ is a *cancellation*.

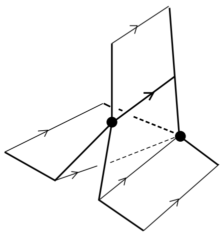

Figure 1: A cancellation

Definition 3.5 Let p be a horizontal path in the stratum $f^{-1}(r)$ of a forest-stack $(\tilde{X}, f, \sigma_t)$.

- The *pulled-tight projection* (or *image*) $[p]_{r+t}$ *of* p *on the stratum* $f^{-1}(r+t)$ is the unique horizontal geodesic between the endpoints of $\sigma_t(p)$ in the stratum $f^{-1}(r+t)$.
- A *geodesic pre-image of* p *under* σ_t is any geodesic p_{-t} with $[p_{-t}]_{f(p_{-t})+t} = p$.

If S is a path in $\tilde{X}$, the *pulled-tight projection of* S *on* $f^{-1}(r)$, $r \geq \max_{x\in S} f(x)$, is the unique horizontal geodesic which connects the images of the endpoints of S under the semi-flow in the stratum $f^{-1}(r)$.

3.2 Telescopic metric

Definition 3.6 A *telescopic path* in a forest-stack is a path which is the concatenation of non-degenerate horizontal and vertical subpaths.

The *vertical length of a telescopic path* p is equal to the sum of the vertical lengths of the maximal vertical subpaths of p.

If the considered forest-stack comes with a horizontal metric $\mathcal{H}$, the *horizontal length of a telescopic path* p is the sum of the horizontal lengths of the maximal horizontal subpaths of p.

The *telescopic length* $|p|_{(\tilde{X},\mathcal{H})}$ of a telescopic path p in $\tilde{X}$ is equal to the sum of the horizontal and vertical lengths of p.

We will always assume that our paths are equipped with an orientation, whatever it is, and we will denote by $i(p)$ (resp. $t(p)$) the initial (resp. terminal) point of a path p with respect to its orientation.

Lemma - Definition *Let $(\tilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$. For any two points x, y in $\tilde{X}$, we denote by $d_{(\tilde{X},\mathcal{H})}(x, y)$ the infimum, over all the telescopic paths p in $\tilde{X}$ between x and y, of their telescopic lengths $|p|_{(\tilde{X},\mathcal{H})}$. Then $(\tilde{X}, d_{(\tilde{X},\mathcal{H})})$ is a $(1, 2)$-quasi geodesic metric space. The map $d_{(\tilde{X},\mathcal{H})}\colon \tilde{X} \times \tilde{X} \to \mathbb{R}^+$ is a* telescopic distance *associated to $\mathcal{H}$.*

Proof of Lemma - Definition: If $d_{(\tilde{X},\mathcal{H})}(x, y) = 0$ then $f(x) = f(y)$. The distance is realized as the infimum of the telescopic lengths of an infinite sequence $(T_n)_{n\in\mathbb{N}}$ of telescopic paths. There exists a unique horizontal geodesic between x and y. Otherwise any telescopic path between x and y has vertical length, and thus telescopic length uniformly bounded away from zero. Let $\epsilon > 0$ be fixed. For some integer i all the telescopic paths $T_i, T_{i+1}, \cdots$ in the above sequence are contained in a box of height 2ϵ with horizontal boundaries the pulled-tight projection $[g]_{f(g)+\epsilon}$ and all the geodesic pre-images of g under σ_ϵ. The vertical boundaries are the orbit-segments connecting the endpoints of the above geodesic pre-images to the endpoints of $[g]_{f(g)+\epsilon}$. From the bounded-dilatation property, the horizontal length of each T_n for $n \geq i$ is greater than or equal to $\lambda_+^{-2\epsilon}|[g]_{f(g)+\epsilon}|_{f(g)+\epsilon}$. Thus for any $n \geq i$, $|T_n|_{(\tilde{X},\mathcal{H})} \geq \lambda_+^{-2\epsilon}|[g]_{f(g)+\epsilon}|_{f(g)+\epsilon}$. Since $\mathrm{Inf}_{n\in\mathbb{N}}|T_n|_{(\tilde{X},\mathcal{H})} = d_{(\tilde{X},\mathcal{H})}(x, y) = 0$, $|[g]_{f(g)+\epsilon}|_{f(g)+\epsilon} = 0$. That is $\sigma_\epsilon(x) = \sigma_\epsilon(y)$. This is satisfied for any $\epsilon > 0$. Since $(\sigma_t)_{t\in\mathbb{R}^+}$ depends continuously on t, $\sigma_0(x) = \sigma_0(y)$ so that $x = y$. We so proved that $d_{(\tilde{X},\mathcal{H})}$ does not vanish outside the diagonal of $\tilde{X} \times \tilde{X}$. The conclusion that this is a distance is now straightforward.

By definition of the telescopic distance, for any x, y in $\tilde{X}$, for any $\epsilon > 0$, there exists a telescopic path p between x and y such that $|p|_{(\tilde{X},\mathcal{H})} \leq d_{(\tilde{X},\mathcal{H})}(x, y) + \epsilon$. We choose $\epsilon < \min(d_{(\tilde{X},\mathcal{H})}(x, y), 1)$. We consider the maximal collection of points $x_0, \cdots, x_k$ in p with $x_0 = i(p)$, $x_k = t(p)$, and the telescopic length of the subpath p_i of p between x_{i-1} and x_i is equal to ϵ for $i = 1, \cdots, k-1$. The maximality of the collection $\{x_0, x_1, \cdots, x_k\}$ implies that the telescopic length of the subpath p_k of p between x_{k-1} and x_k is smaller than or equal to ϵ. By definition $d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq |p_i|_{(\tilde{X},\mathcal{H})}$ for $i = 1, \cdots, k$. Thus $d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq 1$ for any $i = 1, \cdots, k$ and $\sum_{i=1}^{k} d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq |p|_{(\tilde{X},\mathcal{H})}$. The choice of $\epsilon < d_{(\tilde{X},\mathcal{H})}(x, y)$ then implies $\sum_{i=1}^{k} d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq 2d_{(\tilde{X},\mathcal{H})}(x, y)$. Therefore $x_0, x_1, \cdots, x_k$ is a $(1, 2)$-quasi geodesic chain between x and y. $\square$

Remark 3.7 In nice cases, for instance in the case where the forest-stack is a proper metric space, the forest-stack is a true geodesic space.

4 Main Theorem

Definition 4.1 Let $(\tilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$.

1. The semi-flow is a *bounded-cancellation semi-flow* (with respect to $\mathcal{H}$) if there exist $\lambda_- \geq 1$ and $K \geq 0$ such that for any real $r \in \mathbb{R}$, for any horizontal geodesic $g \in f^{-1}(r)$, for any $t \geq 0$, $|[g]_{r+t}|_{r+t} \geq \lambda_-^{-t}|g|_r - K$.

2. The semi-flow is a *bounded-dilatation semi-flow* (with respect to $\mathcal{H}$) if there exists $\lambda_+ \geq 1$ such that for any real $r \in \mathbb{R}$, for any horizontal geodesic $g \in f^{-1}(r)$, for any $t \geq 0$, $|[g]_{r+t}|_{r+t} \leq \lambda_+^t |g|_r$.

Remark 4.2 The reader can observe a dissymetry between the bounded-cancellation and bounded-dilatation properties, in the sense that this last one does not allow any additive constant. This is really necessary, several proofs (like the proofs of Propositions 8.1 or 9.1) fail if allowing an additive constant here.

Definition 4.3 Let $(\tilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$.

1. The semi-flow is *hyperbolic* (with respect to $\mathcal{H}$) if it is a bounded-dilatation and bounded-cancellation semi-flow with respect to $\mathcal{H}$ and there exist $\lambda > 1$, $t_0, M \geq 0$ such that for any horizontal geodesic $g \in f^{-1}(r)$ with $|g|_r \geq M$:

   - Either $|[g]_{r+nt_0}|_{r+nt_0} \geq \lambda^{nt_0}|g|_r$ for any integer $n \geq 1$,
   - Or for any integer $n \geq 1$, some geodesic pre-image g_{-nt_0} of g satisfies $|g_{-nt_0}|_{r-nt_0} \geq \lambda^{nt_0}|g|_r$.

2. The semi-flow is *strongly hyperbolic* (with respect to $\mathcal{H}$) if it is hyperbolic and satisfies furthermore the following condition:

   Any horizontal geodesic $g \in f^{-1}(r)$ with $|g|_r \geq M$, which admits geodesic pre-images in distinct connected components of the stratum $f^{-1}(r-\epsilon)$ for $\epsilon > 0$ arbitrarily small, admits a pre-image g_{-nt_0} in each connected component of the stratum $f^{-1}(r - nt_0)$ such that $|g_{-nt_0}|_{r-nt_0} \geq \lambda^{nt_0}|g|_r$.

Let us observe that, if the strata are connected, then a hyperbolic semi-flow is strongly hyperbolic.

It is now possible to state the main theorem of this paper.

Theorem 4.4 *Let $(\tilde{X}, f, \sigma_t, \mathcal{H})$ be a connected forest-stack. If $(\sigma_t)_{t\in\mathbb{R}^+}$ is strongly hyperbolic with respect to $\mathcal{H}$ then $\tilde{X}$ is a Gromov-hyperbolic metric space for any telescopic metric associated to $\mathcal{H}$.*

At this point, the reader might prefer to look at Sections 12 and 13 where he will find applications, and so illustrations, of this theorem to the cases of mapping-telescopes spaces and of mapping-torus groups.

Remark 4.5 (about the necessity of the bounded-cancellation property)

Let us observe that the Cayley complex of a Baumslag - Solitar group $BS(1, m) =$

$< a, b \,;\, b^{-1}ab = a^m >$ is a forest-stack with a hyperbolic semi-flow. But this is not a Gromov hyperbolic 2-complex with respect to the telescopic metric. What happens here is that the semi-flow is hyperbolic but not strongly hyperbolic.

An example of a non Gromov-hyperbolic locally finite forest-stack with connected strata and a semi-flow satisfying all the desired properties, at the exception of the bounded-cancellation property (first item of Definition 4.1) is constructed as follows. We start with the forest-stack $\mathcal{R} = (\mathbb{R}^2, f, \sigma_t, \mathcal{H})$ defined in Section 1 and equipped with the associated telescopic metric. We consider copies $\mathcal{R}_i$, $i = 0, 1, 2, \cdots$, of $\mathcal{R}$. We glue them to $\mathcal{R}$ in the way illustrated in Figure 2, that is by creating an infinite sequence of pockets with increasing sizes.

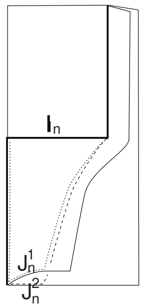

Figure 2: A pocket

We now attach copies of the negative half-plane of $\mathcal{R}$, along the horizontal lines with integer y-coordinate of the copies $\mathcal{R}_i$ of $\mathcal{R}$ considered above. In order to get a forest-stack whose strata are trees, we now identify a vertical half-line in each of the copies of the negative half-plane, ending at the horizontal line along which this copy was glued, to the corresponding vertical half-line in $\mathcal{R}$. In this way, we get a forest-stack whose strata are trees and whose semi-flow is as anounced. This forest-stack is not Gromov-hyperbolic because in each pocket (see Figure 2) the horizontal interval I_n admits two pre-images J_n^1, J_n^2 so that there are two telescopic geodesics joining the endpoints of I_n. These are the concatenation of J_n^1 and J_n^2 with the two vertical segments joining their endpoints to the endpoints of I_n. Since, by construction, there are pockets of arbitrarily large size, these two telescopic geodesics might be arbitrarily far away one from the other, so that the forest-stack is not Gromov-hyperbolic.

5 Preliminary work

We consider a forest-stack $(\tilde{X}, f, \sigma_t, \mathcal{H})$ equipped with a horizontal metric $\mathcal{H}$ such that the semi-flow $(\sigma_t)_{t\in\mathbb{R}^+}$ is strongly hyperbolic. Definition 4.3 introduces three *constants of hyperbolicity*, denoted by λ, t_0, M in all which follows. The other constants of hyperbolicity, which appear in the bounded-dilatation and bounded-cancellation properties, are denoted by λ_+, λ_-, K. Any horizontal geodesic g with horizontal length greater than M satisfies at least one of the following two properties:

- The pulled-tight image $[g]_{nt_0}$ of g after nt_0, $n \geq 1$, is λ^n times longer than g. In this case the horizontal geodesic g is *dilated in the future*, or more briefly *dilated, after* t_0.
- g admits a geodesic pre-image g_{-nt_0} under σ_{nt_0} which is λ^n times longer than g. In this case, the horizontal geodesic g is *dilated in the past after* t_0.

More generally, we will say that g is *dilated in the future after* kt_0 (resp. *dilated in the past after* kt_0), $k \geq 1$, if the same inequalities hold only for any $n \geq k$, after substituting λ^n by $\lambda^{(n+1-k)}$, and g by $[g]_{r+(k-1)t_0}$ for the dilatation in the future and by $g_{-(k-1)t_0}$ for the dilatation in the past.

When the dilatation occurs in the past, only one geodesic pre-image is required to have horizontal length λ times the horizontal length of the horizontal geodesic g considered. So it might happen, a priori, that the other geodesic pre-images of g remain short when going in the past. Lemma 5.1 below shows that the constants of hyperbolicity can be chosen so that such a situation does not occur. This is a consequence of the bounded-cancellation property.

Lemma 5.1 *Let $(\tilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack. Assume that $(\sigma_t)_{t \in \mathbb{R}^+}$ is (strongly) hyperbolic, with constants of hyperbolicity λ, t_0, M. Then:*

1. *There exist $t'_0 = jt_0$, j positive integer, and $M' \geq M$ such that any horizontal geodesic $g \in f^{-1}(r)$ dilated in the past after t'_0, with $|g|_r \geq M'$, satisfies $|g_{-nt'_0}|_{r-nt'_0} \geq 2^n |g|_r$ for any geodesic pre-image $g_{-nt'_0}$, $n \geq 1$.*
2. *The semi-flow $(\sigma_t)_{t \in \mathbb{R}^+}$ is (strongly) hyperbolic with constants of hyperbolicity λ, t'_0, M', $\lambda'_+, \lambda'_-, K'$ for any $t'_0 = jt_0$, $j \geq 1$ any positive integer, and any real numbers $M' \geq M$, $\lambda'_+ \geq \lambda_+$, $\lambda'_- \geq \lambda_-$, $K' \geq K$. Furthermore, if the semi-flow satisfies item (1) for some constants t'_0, M', then it satisfies item (1) for any $t''_0 = jt'_0$, j any positive integer, and any real number $M'' \geq M'$.*

Proof of Lemma 5.1: Item (2) is obvious. Let us check item (1). We choose $t'_0 \geq t_0, t'_0 = jt_0$ with j integer, such that $\lambda^{t'_0} > 2$. We consider any horizontal geodesic $g \in f^{-1}(r)$ with $|g|_r \geq M$. We assume that g is dilated in the past after t'_0. Since the semi-flow is strongly hyperbolic, for each $n \geq 1$, in each connected component of $f^{-1}(r - nt'_0)$, there is at least one geodesic pre-image $g_{-nt'_0}$ of g with $|g_{-nt'_0}|_{r-nt'_0} \geq \lambda^{nt'_0}|g|_r$. We need an estimate of the horizontal length of the other geodesic pre-images of g in this stratum. Lemma 5.2 below is easily deduced from the bounded-cancellation property:

Lemma 5.2 *With the assumptions and notations of Lemma 5.1, let $g \in f^{-1}(r)$ be some horizontal geodesic. If g^1_{-t} and g^2_{-t}, $t > 0$, are two geodesic pre-images of g under σ_t which belong to a same connected component of their stratum, then $||g^1_{-t}|_{r-t} - |g^2_{-t}|_{r-t}| \leq C_{5.2}(t)$ for some constant $C_{5.2}(t)$.*

Thus, from Lemma 5.2, for any $n \geq 1$, any geodesic pre-image $g_{-nt'_0}$ satisfies $|g_{-nt'_0}|_{r-nt'_0} \geq \lambda^{nt'_0}|g|_r - C_{5.2}(nt'_0)$. For $n = 1$, if $|g|_r > \frac{C_{5.2}(t'_0)}{\lambda^{t'_0}-2}$, then $|g_{-t'_0}|_{r-t'_0} > 2|g|_r$. Thus, if $|g|_r > max(M, \frac{C_{5.2}(t'_0)}{\lambda^{t'_0}-2})$ then any geodesic pre-image $g_{-t'_0}$ has horizontal length greater than $2|g|_r$. In particular $|g_{-t'_0}|_{r-t'_0} \geq M$ because $|g|_r > M$. By definition of a hyperbolic semi-flow, $g_{-t'_0}$ is dilated either in the future or in the past. This cannot be in the future since $|g_{-t'_0}|_{r-t'_0} > |g|_r$. An easy induction on n completes the proof of Lemma 5.1. It suffices to set $t'_0 = (E[\max(1, \frac{ln2}{ln\lambda})] + 1)t_0$ and $M' = \max(M, \frac{C_{5.2}(t'_0)}{\lambda^{t'_0}-2}) + 1$. □

We will assume the constants of hyperbolicity t_0 and M chosen to satisfy the conclusion of Lemma 5.1 above. Moreover the constants of hyperbolicity $t_0, M, \lambda_+, \lambda_-, K$ are chosen sufficiently large enough so that computations make sense. In what follows, we say that a path g is *C-close* to a path g' if g and g' are C-close with respect to the Hausdorff distance relative to the metric specified (the telescopic metric if no metric is specified). The indices of the constants refer to the lemmas or propositions where these constants appear.

5.1 About dilatation in cancellations

Let us recall that a *cancellation* is a horizontal geodesic whose endpoints are identified under some σ_t, $t > 0$.

Lemma 5.3 *Let $g \in f^{-1}(r)$ be any horizontal geodesic which is dilated in the future after nt_0 for some integer $n \geq 1$. There exists a constant $C_{5.3}(n) \geq M$, increasing with n, such that, if g is contained in a cancellation, then $|g|_r \leq C_{5.3}(n)$.*

Proof of Lemma 5.3: Let c be the cancellation containing g. Let $c = c_1 \cup c_2$, with $[c_1]_{r+t} = [c_2]_{r+t}$ for some $t > 0$. We assume for a while that $c_1 \cap c_2$ is an endpoint of g. The bounded-cancellation property implies that the horizontal length of a cancellation "killed" in time t_0, that is a cancellation whose pulled-tight projection after t_0 is a point, is a constant $C(t_0)$. This constant does not depend on the horizontal length of g. Let us consider the pulled-tight image $[g]_{r+t_0}$. Let $p \subset [g]_{r+t_0}$ be the maximal subpath outside the pulled-tight image of c. This subpath p is the image of a cancellation killed in time t_0. From the observation above and the bounded-dilatation property, $|p|_{r+t_0} \leq \lambda_+^{t_0} C(t_0)$. The same arguments lead to the upper bound $(\lambda_+^{nt_0} + \lambda_+^{(n-1)t_0} + \cdots + \lambda_+^{t_0})C(t_0)$ for the horizontal length of the subpath of $[g]_{r+nt_0}$ outside $[c]_{r+nt_0}$. Since g is dilated in the future after nt_0, $|[g]_{r+nt_0}|_{r+nt_0} \geq \lambda^{t_0}|g|_r$. From the last two inequalities, if $|g|_r > \frac{(\lambda_+^{nt_0} + \lambda_+^{(n-1)t_0} + \cdots + \lambda_+^{t_0})C(t_0)}{\lambda^{t_0} - 1}$, then the horizontal length of the subpath q of $[g]_{r+nt_0}$ in $[c]_{r+nt_0}$ is greater than $|g|_r$. If $|g|_r \geq M$, $|q|_{r+nt_0} \geq M$ is dilated in the future after t_0 by the convention that M satisfies the conclusion of Lemma 5.1. We so obtain, for any $j \geq n$, the existence of a geodesic with horizontal length greater than $|g|_r$ in $[c]_{r+jt_0}$. This is impossible.

Let us now consider the case where $c_1 \cap c_2$ is not an endpoint of g. After some time $t > 0$, the situation will be the one described above, that is a cancellation $c' = c'_1 \cup c'_2$ with $c'_1 \cap c'_2$ an endpoint of $[g]_{r+t}$. The arguments above, together with the bounded-cancellation and bounded-dilatation properties, lead to the conclusion. □

We will often encounter situations where the pulled-tight projection of a horizontal geodesic p_1 is identified with the pulled-tight projection of another horizontal geodesic p_2 in the same stratum. In this case, p_1, p_2 are not necessarily contained in cancellations. But, if they lie in the same connected component of their stratum, both are contained in the union of two cancellations. Lemma 5.4 below will allow us to deal with similar situations.

Lemma 5.4 *Let p be a horizontal geodesic which admits a decomposition in r subpaths p_i such that for some constant $L \geq 0$, for any $i = 1, \cdots, r$, either $|[p_i]_{r+nt_0}|_{r+nt_0} \leq |p_i|_r$ or $L \geq |[p_i]_{r+nt_0}|_{r+nt_0} > |p_i|_r$. There exists a constant $C_{5.4}(n, r, L)$, increasing in each variable, such that, if p is dilated in the future after nt_0, then $|p|_r \leq C_{5.4}(n, r, L)$.*

Proof of Lemma 5.4: We set $n = 1$ in order to simplify the notations, the general case is done in the same way. Up to permuting the indices, $|[p_i]_{r+t_0}|_{r+t_0} > |p_i|_r$ for $i = 1, \cdots, j$. Since p is dilated in the future after t_0, $jL + \sum_{i=j+1}^{r} |p_i|_r \geq \lambda^{t_0} \sum_{i=1}^{r} |p_i|_r$. Therefore $|p|_r \leq \frac{jL}{\lambda^{t_0} - 1}$. □

5.2 Straight telescopic paths

Definition 5.5 A *straight* telescopic path is a telescopic path S such that if x, y are any two points in S with $x \in O^+(y) \cup O^-(y)$ then the subpath of S between x and y is equal to the orbit-segment of the semi-flow between x and y.

If S is a path containing a point x, let $S_{x,t} \subset S$ be the maximal subpath of S containing x, whose pulled-tight projection $[S_{x,t}]_{f(x)+t}$ on $f^{-1}(f(x)+t)$ is well defined. The point $\sigma_t(x)$ does not necessarily belong to $[S_{x,t}]_{f(x)+t}$. However there exists a unique point in $[S_{x,t}]_{f(x)+t}$ which minimizes the horizontal distance between $\sigma_t(x)$ and $[S_{x,t}]_{f(x)+t}$. This point is denoted by $\overline{x}_t$. Lemma 5.6 below gives an upper-bound, depending on t, for the telescopic distance between x and $\overline{x}_t$.

Lemma 5.6 *Let S be any straight telescopic path. If t is any non negative real number, there exists a constant $C_{5.6}(t) \geq t$, increasing with t, such that any point $x \in S$ is at telescopic distance smaller than $C_{5.6}(t)$ from the point $\overline{x}_t$ (see above).*

Proof of Lemma 5.6: If $\sigma_t(x) \in [S_{x,t}]_{f(x)+t}$, we set $C_{5.6}(t) = t$. Since S is straight, if $\sigma_t(x) \notin [S_{x,t}]_{f(x)+t}$, x belongs to a cancellation c whose endpoints lie in the past orbits of $\overline{x}_t$. The bounded-cancellation property gives an upper-bound on the horizontal length of c. This leads to the conclusion. □

6 About straight quasi geodesics

Definition 6.1 Let $(\tilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack. A (J, J')-*quasi geodesic*, $J \geq 1, J' \geq 0$, in $(\tilde{X}, d_{(\tilde{X},\mathcal{H})})$ is a telescopic path S whose every subpath S' satisfies:

$$|S'|_{(\tilde{X},\mathcal{H})} \leq J d_{(\tilde{X},\mathcal{H})}(i(S'), t(S')) + J'$$

.

Lemma 6.2 *Let p be a straight (J, J')-quasi geodesic with $|r_{max} - f(i(p))| \leq t_0$, where $r_{max} = max_{x \in p} f(x)$. There exists a constant $C_{6.2}(J, J') \geq M$, increasing with J and J', such that if $|[p]_{r_{max}}|_{r_{max}} \geq C_{6.2}(J, J')$ then $[p]_{r_{max}}$ is dilated both in the future and in the past after $C_{6.2}(J, J')t_0$.*

Proof of Lemma 6.2: By the bounded-dilatation property, $|p|_{(\tilde{X},\mathcal{H})} \geq \lambda_+^{-t_0}|[p]_{r_{max}}|_{r_{max}} + t_0$. We choose n_* so that $\lambda_+^{-t_0} - J\lambda^{-n_* t_0} > 0$. For any n greater than n_*, the inequality

$$J(2t_0 + 2nt_0 + \lambda^{-nt_0}|[p]_{r_{max}}|_{r_{max}}) + J' < \lambda_+^{-t_0}|[p]_{r_{max}}|_{r_{max}} + t_0$$

is satisfied for $|[p]_{r_{max}}|_{r_{max}} > \frac{(2J-1)t_0 + 2nJt_0 + J'}{\lambda_+^{-t_0} - J\lambda^{-nt_0}}$. This is a contradiction with p being a (J, J')-quasi geodesic. If $|[p]_{r_{max}}|_{r_{max}} > \lambda_+^{n_* t_0} M$, then, by the bounded-dilatation property, the geodesic pre-images of $[p]_{r_{max}}$ under $\sigma_{n_* t_0}$ have horizontal length greater or equal to M. From which precedes, if moreover $|[p]_{r_{max}}|_{r_{max}} > \frac{(2J-1)t_0 + 2n_* J t_0 + J'}{\lambda_+^{-t_0} - J\lambda^{-n_* t_0}}$ the hyperbolicity of the semi-flow then implies that they are dilated in the past after t_0. The bounded-dilatation property implies that these geodesic pre-images have horizontal length greater or equal to $\lambda_+^{-n_* t_0}|[p]_{r_{max}}|_{r_{max}}$. Choosing N_* such that $\lambda^{N_* t_0} \geq \lambda_+^{n_* t_0}$, we get that $[p]_{r_{max}}$ is dilated in the past after $(N_* + 1)t_0$. The same arguments allow us to find a lower bound on $|[p]_{r_{max}}|_{r_{max}}$ for $[p]_{r_{max}}$ being dilated in the future after some fixed finite time. □

Definition 6.3 Let $(\tilde{X}, f, \sigma_t)$ be a forest-stack. A *stair* in $\tilde{X}$ is a telescopic path along which the function f is monotone.

Lemma 6.4 *Let p be a straight (J, J')-quasi geodesic stair between two points a and b, $f(a) \leq f(b)$. There exists a constant $C_{6.4}(J, J') \geq M$, increasing with J and J', such that, if the horizontal length of a horizontal geodesic I between a and $O^-(b)$ (resp. b and $O^+(a)$) is greater or equal to $C_{6.4}(J, J')$ then I is dilated in the past (resp. in the future) after t_0.*

Proof of Lemma 6.4: Let X be such that $\lambda^{t_0} X > X + \lambda_+^{t_0} C_{6.2}(J, J')$. Assume that the horizontal length of some horizontal geodesic I between a and $O^-(b)$ is greater or equal to X. By Lemma 6.2, the choice of X implies that, if I is dilated in the future after t_0, then the first point a_1 along p satisfying $f(a_1) = f(a) + t_0$ is at horizontal distance greater than X from $O^-(b)$. By induction, we so obtain an infinite sequence of points $a_1, a_2, \cdots, a_n, \cdots$ in p such that $f(a_i) = f(a_{i-1}) + t_0$ and each a_i is at horizontal distance greater or equal to X from $O^-(b)$. This is absurd. The other case of Lemma 6.4 is treated in the same way. □

Definition 6.5 Let S_0, S_1 be two telescopic paths whose pulled-tight projections agree after some finite time. We say that S_0 and S_1 *are in fine position* if, for any two points $x, y, x \neq y$, satisfying $x \in S_i \cap O(y)$, $y \in S_{i+1}$, $i = 0, 1$ mod 2, then $x \in O^+(y) \cup O^-(y)$.

Let us observe that a path is always is fine position with respect to any of its pulled-tight projections.

Definition 6.6 A $+$-*hole* (resp. $-$-*hole*) is a telescopic path with both endpoints in a same stratum, which is in fine position with respect to the horizontal geodesic I between its endpoints, and which satisfies furthermore $\min_{x \in p} f(x) \geq f(I)$ (resp. $\max_{x \in p} f(x) \leq f(I)$).

Lemma 6.7 *Let p be a straight (J, J')-quasi geodesic $+$-hole (resp. $-$-hole). There exists a constant $C_{6.7}(J, J') \geq M$, increasing with J and J', such that, if I is the horizontal geodesic between the endpoints of p and $|I|_{f(I)} \geq C_{6.7}(J, J')$, then I is dilated in the past (resp. future) after $C_{6.7}(J, J') t_0$.*

Proof of Lemma 6.7: We consider a decomposition $p_1 p_2 \cdots p_l$ of p such that $\max_{x \in p_i} |f(x) - f(i(p_i))| \leq t_0$. We consider a decomposition of $I = I_1 \cdots I_l$ where I_k joins the past orbits of the endpoints of p_k. We denote by I_D the union of the I_k's which are dilated in the past after $C_{6.2}(J, J') t_0$. We denote by I_C the union of the other intervals in I. From Lemma 6.2, the horizontal length of any interval in I_C is less or equal to $C_{6.2}(J, J')$.

Let n be some positive integer. We consider a horizontal geodesic h with $I = [h]_{f(h) + n C_{6.2}(J, J') t_0}$ and we assume that h is dilated in the future after t_0. Therefore:

$$\lambda^n |I_D|_{f(I)} + \lambda_+^{-n} |I_C|_{f(I)} \leq |h|_{f(h)} \leq \lambda^{-n} (|I_D|_{f(I)} + |I_C|_{f(I)}).$$

Hence $|I_C|_{f(I)} \geq \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda^{-n}} |I_D|_{f(I)}$, so that $|I_C|_{f(I)} \geq \frac{X(n)}{1 + X(n)} |I|_{f(I)}$ with $X(n) = \frac{\lambda^n - \lambda_+^{-n}}{\lambda^n - \lambda^{-n}}$. Observe $\lim_{n \to +\infty} \frac{X(n)}{1 + X(n)} = 1$, so that for some $n_* \geq 1$, for any $n \geq n_*$, $\frac{X(n)}{1 + X(n)} \geq \frac{1}{2}$. Since the horizontal length of any interval I_k in I_C is less or equal to $C_{6.2}(J, J')$, and the telescopic length of the associated $p_k \subset p$ is at least t_0, we obtain

$$|p|_{(\tilde{X}, \mathcal{H})} \geq \frac{t_0}{2 C_{6.2}(J, J')} |I|_{f(I)}.$$

On the other hand $|p|_{(\tilde{X},\mathcal{H})} \leq 2Jnt_0 + \lambda^{-n}J|I|_{f(I)} + J'$ for any $n \geq n_*$. The last two inequalities give, for $n \geq n_*$, $2Jnt_0 + \lambda^{-n}J|I|_{f(I)} + J' \geq \frac{t_0}{2C_{6.2}(J,J')}|I|_{f(I)}$, equivalently $2Jnt_0+J' \geq (\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n}J)|I|_{f(I)}$. We choose $n_\circ \geq n_*$ so that $\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ}J > 0$. We get:

$$\frac{2Jn_\circ t_0 + J'}{\frac{t_0}{2C_{6.2}(J,J')} - \lambda^{-n_\circ}J} \geq |I|_{f(I)}.$$

Thus, for $|I|_{f(I)} > \frac{2Jn_\circ t_0+J'}{\frac{t_0}{2C_{6.2}(J,J')}-\lambda^{-n_\circ}J}$, h is not dilated in the future after t_0. If $|I|_{f(I)} > \lambda_+^{n_\circ}M$, $|h|_{f(h)} \geq M$. Therefore h is dilated in the past after t_0. We choose N so that $\lambda^N\lambda_+^{-n_\circ} > \lambda$. Thus, if $|I|_{f(I)} \geq \max(\lambda_+^{n_\circ}M, \frac{2Jn_\circ t_0+J'}{\frac{t_0}{2C_{6.2}(J,J')}-\lambda^{-n_\circ}J})$ then I is dilated in the past after $(n_\circ C_{6.2}(J,J') + N)t_0$. The arguments and computations in the case where $\max_{x\in p} f(x) \leq f(I)$ are the same. $\square$

7 Substitution of quasi geodesics

Lemma 7.1 *Let p be a (J,J')-quasi geodesic. Let q be obtained from p by substituting subpaths $p_i \subset p$ by (L,L')-quasi geodesics q_i satisfying the following properties:*

- *q_i has the same endpoints than p_i,*
- *q_i is L-close to p_i,*
- *$|q_i|_{(\tilde{X},\mathcal{H})} \leq L|p_i|_{(\tilde{X},\mathcal{H})}$.*

There exists a constant $C_{7.1}(L,L',J,J')$, increasing in each variable, such that q is a $(C_{7.1}(L,L',J,J'),C_{7.1}(L,L',J,J'))$-quasi geodesic which is L-close to p.

Proof of Lemma 7.1: Since each q_i is L-close to a p_i, and with the same endpoints, q is L-close to p. Let us consider any two points x,y in q and let $q_{xy} \subset q$ be the subpath of q between x and y. If both x and y lie in a q_i, or in a same subpath in the closed complement of the union of the q_i's, $|q_{xy}|_{(\tilde{X},\mathcal{H})} \leq \max(L,J)d_{(\tilde{X},\mathcal{H})}(x,y) + \max(L',J')$. Otherwise $q_{xy} = w_1w_2w_3$, where w_1, w_3 are contained either in some q_i or in p, and w_2 begins and ends with the initial or terminal point of some q_i. The third property given about the q_i's leads to: $|w_2|_{(\tilde{X},\mathcal{H})} \leq L|p_2|_{(\tilde{X},\mathcal{H})}$ where $p_2 \subset p$ is the subpath of p with the same endpoints than w_2. Thus $|q_{xy}|_{(\tilde{X},\mathcal{H})} \leq LJd_{(\tilde{X},\mathcal{H})}(x,y) + 2\max(L',LJ')$. This completes the proof of Lemma 7.1. $\square$

Lemma 7.2 *Let p be a straight (J,J')-quasi geodesic $-$-hole such that $\max_{x\in p} f(I) - f(x) \leq L$, where I is the horizontal geodesic joining the endpoints of p. Then there exists a constant $C_{7.2}(L,J,J') \geq M$, increasing in each variable, such that:*

1. *$|I|_{f(I)} \leq C_{7.2}(L,J,J')|p|_{(\tilde{X},\mathcal{H})}$.*
2. *I is a straight $(C_{7.2}(L,J,J'),C_{7.2}(L,J,J'))$-quasi geodesic which is $C_{7.2}(L,J,J')$-close to p.*

Proof of Lemma 7.2: A horizontal geodesic is always straight. The horizontal geodesic I is the pulled-tight projection of p. Thus, by the bounded-dilatation property, $|I|_{f(I)} \leq \lambda_+^L |p|_{(\tilde{X},\mathcal{H})}$. From Lemma 5.6, I is $C_{5.6}(L)$-close to p. Let us consider any subpath I' of I, this is the pulled-tight projection of some subpath p' of p. By the bounded-dilatation property, $|I'|_{f(I)} \leq \lambda_+^L |p'|_{(\tilde{X},\mathcal{H})}$. Since p is a (J, J')-quasi geodesic, $|I'|_{f(I)} \leq \lambda_+^L (J d_{(\tilde{X},\mathcal{H})}(i(p'), t(p')) + J')$. Since I' is $C_{5.6}(L)$-close to p', $|I'|_{f(I)} \leq \lambda_+^L J d_{(\tilde{X},\mathcal{H})}(i(I'), t(I'))$ $+\lambda_+^L (2JC_{5.6}(L) + J')$. The proof of Lemma 7.2 is complete. □

Lemma 7.3 *Let p be a straight (J, J')-quasi geodesic $-$-hole such that the horizontal length of the horizontal geodesic I between its endpoints is less or equal to L. Then there exists a constant $C_{7.3}(L, J, J') \geq M$, increasing in each variable, such that:*

1. $|I|_{f(I)} \leq C_{7.3}(L, J, J') |p|_{(\tilde{X},\mathcal{H})}$.
2. *I is a straight $(C_{7.3}(L, J, J'), C_{7.3}(L, J, J'))$-quasi geodesic which is $C_{7.3}(L, J, J')$-close to p.*

Proof of Lemma 7.3: Since p is a (J, J')-quasi geodesic, $\max_{x \in p} |f(x) - f(I)| \leq J|I|_{f(I)} + J'$. Lemma 7.2 then gives Lemma 7.3. □

Lemma 7.4 *Let p be a straight (J, J')-quasi geodesic stair. For any $L \geq 0$, there exists a constant $C_{7.4}(L, J, J')$, increasing in each variable, such that, if q is a straight stair whose points are at horizontal distance less or equal to L from p, and with the same endpoints than p, then:*

1. *q is a straight $(C_{7.4}(L, J, J'), C_{7.4}(L, J, J'))$-quasi geodesic stair which is L-close to p.*
2. $|q|_{(\tilde{X},\mathcal{H})} \leq C_{7.4}(L, J, J') |p|_{(\tilde{X},\mathcal{H})}$.

Proof of Lemma 7.4: We consider a stair S, in the disc bounded by $p \cup q$, with endpoints the endpoints of p and q, and whose vertical geodesics end at q, all the stairs being oriented so that f is increasing along them. Let us consider a subpath S' of S which is the concatenation of a vertical segment followed by an horizontal one. By assumption, the horizontal length X of S' is bounded above by L. Let t be its vertical length. The bounded-dilatation property implies that the quotient of $|S'|_{(\tilde{X},\mathcal{H})}$ by the telescopic length of the subpath of p between the endpoints of S' is bounded above by $Q = \frac{t+X}{t+\lambda_+^{-t} X}$. Since $X \leq L$, Q tends toward 1 with $t \to +\infty$. One so obtains a constant T such that for $t \geq T$, Q is bounded above by some constant, depending on L. When both t and X are close to 0 then Q is also close to 1. Thus, since Q is continuous, Q admits an upper bound, denoted by $A(L)$, for all the t and X considered. This upper bound will be the same for all the subpaths S' as above. The stair S is a concatenation of such subpaths S', possibly with one or two subpaths of p at the extremities. Thus the additivity of the telescopic length gives $|S|_{(\tilde{X},\mathcal{H})} \leq A(L)|p|_{(\tilde{X},\mathcal{H})}$. Let S'' be a subpath of S which is the concatenation of a horizontal subpath followed by a vertical one. The path S is the concatenation of such subpaths S'' possibly with one or two subpaths of q at the extremities. Exactly the same arguments than above give $|q|_{(\tilde{X},\mathcal{H})} \leq A(L)|S|_{(\tilde{X},\mathcal{H})}$. We so get $|q|_{(\tilde{X},\mathcal{H})} \leq A(L)^2 |p|_{(\tilde{X},\mathcal{H})}$. It only remains to prove that q is a quasi geodesic with constants of quasi geodesicity depending only on L, J, J'. Let x, y be any two points in q. As usually q_{xy} is the subpath of q between x and y and we denote by $p_{x'y'}$ the subpath of p between the two points x', y' in p which lie at horizontal distance less or equal to L from x and y. We consider a stair S between q_{xy} and $p_{x'y'}$, with the same endpoints than q_{xy}. The same arguments than above apply and give $|q_{xy}|_{(\tilde{X},\mathcal{H})} \leq A(L)^2 |p_{x'y'}|_{(\tilde{X},\mathcal{H})}$. Since p is a (J,J')-quasi geodesic, we conclude $|q_{xy}|_{(\tilde{X},\mathcal{H})} \leq JA(L)^2 d_{(\tilde{X},\mathcal{H})}(x',y') + J'A(L)^2$. Since $d_{(\tilde{X},\mathcal{H})}(x',y') \leq d_{(\tilde{X},\mathcal{H})}(x,y) + 2L$, the proof of Lemma 7.4 is complete. $\square$

8 Approximation of straight quasi geodesics in fine position

Proposition 8.1 *Let h be a horizontal geodesic. Let g be a straight (J,J')-quasi geodesic, between the orbits of the endpoints of h. There exists a constant $C_{8.1}(|h|_r, J, J')$ such that, if g is in fine position with respect to h, then g is $C_{8.1}(|h|_r, J, J')$-close to the orbit-segments between its endpoints and those of h. Moreover $C_{8.1}(L,J,J') \leq C_{8.1}(M,J,J')$ if $0 \leq L \leq M$, and $C_{8.1}(L,J,J') > C_{8.1}(L',J,J')$ if $L > L' \geq M$.*

Proof of Proposition 8.1: We consider any maximal (in the sense of the inclusion) $+$-hole b in g, with $\min_{x \in b} f(x) \geq f(h) + C_{6.7}(J,J')t_0$. From Lemma 6.7, the horizontal geodesic I between its endpoints is dilated in the past after $C_{6.7}(J,J')t_0$ if $|I|_{f(I)} \geq C_{6.7}(J,J')$. Since g and h are in fine position, this implies $|I|_{f(I)} \leq \max(|h|_r, C_{6.7}(J,J'))$. If $f(h) \leq f(I) \leq f(h) + C_{6.7}(J,J')t_0$, the bounded-dilatation property gives $|I|_{f(I)} \leq \lambda_+^{C_{6.7}(J,J')t_0} |h|_r$.

With the same notations, assume now that b is a maximal $-$-hole with $f(I) \leq f(h) - C_{6.7}(J,J')t_0$. The pulled-tight image of I in the stratum of h is not necessarily contained in h. However, if it is not, then $I = I_1 I_2 I_3$ such that: I_1 and I_3 are contained in cancellations, the pulled-tight image of I_2 in the stratum of h is contained in h. This is a consequence of the fact that h and g are in fine position. If $|I|_{f(I)} \geq C_{6.7}(J,J')$ then, from Lemma 6.7, I is dilated in the future after $C_{6.7}(J,J')t_0$. On the other hand, $|[I_2]_{f(h)}|_{f(h)} \leq |h|_r$, and either $|I_i|_{f(I)} \leq C_{5.3}((C_{6.7}(J,J')+1)t_0)$ or $|[I_i]_{f(I)+C_{6.7}(J,J')t_0}|_{f(I)+C_{6.7}(J,J')t_0} \leq |I_i|_{f(I)}$ for $i=1$ or $i=3$. Indeed $|[I_i]_{f(I)+C_{6.7}(J,J')t_0}|_{f(I)+C_{6.7}(J,J')t_0} > |I_i|_{f(I)} > C_{5.3}((C_{6.7}(J,J')+1)t_0)$ gives a contradiction to Lemma 5.3 since the left inequality implies that $[I_i]_{f(I)+C_{6.7}(J,J')t_0}$ is dilated in the future after t_0, thus I_i would be dilated in the future after $(C_{6.7}(J,J')+1)t_0$. From Lemma 5.4, we get:

If $|I|_{f(I)} \geq C_{6.7}(J,J')$, then $|I|_{f(I)} \leq C_{5.4}(C_{6.7}(J,J'), 3, \max(|h|_r, C_{5.3}((C_{6.7}(J,J')+1)t_0)))$. It remains to consider the case where $f(h) \geq f(I) \geq f(h) - C_{6.7}(J,J')t_0$. The bounded-cancellation property gives an upper-bound for $|I|_{f(I)}$.

We so proved that, whatever maximal $+$-hole b in g which lies above h, or whatever maximal $-$-hole b in g which lies below h, the horizontal distance between the endpoints of b is bounded above by some constant $A(|h|_r, J, J')$. Lemmas 7.3 and 7.1 give then a constant $B(|h|_r, J, J') = C_{7.1}(C_{7.3}((A(|h|_r,J,J'),J,J'), C_{7.3}((A(|h|_r,J,J'),J,J'),J,J')$ such that after substituting maximal $-$-holes in g by the horizontal geodesics between their endpoints, we get a straight $(B(|h|_r,J,J'), B(|h|_r,J,J'))$-quasi geodesic, with the same endpoints, in fine position with respect to h, which is $C_{7.3}(A(|h|_r,J,J'),J,J')$-close to g and which is a stair or the concatenation of two stairs. Lemma 6.4, together with Lemma 5.4 used as above, give then $C_{6.4}(B(|h|_r,J,J'), B(|h|_r,J,J'))$ and

$$D(|h|_r,J,J') = C_{5.4}(1,3,C_{6.4}(B(|h|_r,J,J'), B(|h|_r,J,J'))$$

such that this, or these, stair(s) are $D(|h|_r,J,J')$-close to the orbit-segments between h and their endpoints. We conclude that g is $C_{7.3}(A(|h|_r,J,J'),J,J') + D(|h|_r,J,J')$-close to these orbit-segments. The last point of the proposition is clear. $\square$

9 Putting paths in fine position

Proposition 9.1 *Let h be a horizontal geodesic. Let g be a straight (J, J')-quasi geodesic, which joins the future or past orbits of the endpoints of h. There exist a constant $C_{9.1}(J, J')$ and a $(C_{9.1}(J, J'), C_{9.1}(J, J'))$-quasi geodesic $\mathcal{G}$ which is $C_{9.1}(J, J')$-close to g, which has the same endpoints than g, and which is in fine position with respect to h.*

Proof of Proposition 9.1: We consider a maximal subpath g' of g whose endpoints lie in the future or past orbits of some points in h, and such that no other point of g' satisfies this property. Let us consider any maximal $-$-hole b in g'. Let us denote by I the horizontal geodesic between the endpoints of b.

Case 1*:* either I is contained in a cancellation or I is the concatenation of two horizontal geodesics, each one contained in a cancellation.

Lemma 6.7 gives $C_{6.7}(J, J')$ such that, if $|I|_{f(I)} \geq C_{6.7}(J, J')$ then I is dilated in the future after $C_{6.7}(J, J')t_0$. Lemma 5.3 gives $C_{5.3}(C_{6.7}(J, J'))$ such that the horizontal length of any horizontal geodesic contained in a cancellation and dilated in the future after $C_{6.7}(J, J')t_0$ is less or equal to $C_{5.3}(C_{6.7}(J, J'))$. From Lemma 5.4, we get an upper-bound $C_{5.4}(C_{6.7}(J, J'), 2, C_{5.3}(C_{6.7}(J, J')))$ on the horizontal length of I.

Case 2*:* There exists another horizontal geodesic in another connected component of the same stratum whose pulled-tight projection agrees with the pulled-tight projection of I after some finite time.

We consider the maximal geodesic pre-image I' of I under $\sigma_{C_{6.7}(J,J')t_0}$ which connects two points of b. It admits a decomposition in subpaths I'_α connecting points in b such that the subpath of b between the endpoints of each I'_α is a $-$-hole. The strong hyperbolicity of the semi-flow implies, by Lemma 6.7, that the horizontal length of each I'_α is bounded above by $C_{6.7}(J, J')$. Since g is a (J, J')-quasi geodesic, we get $\max_{x \in b} f(I) - f(x) \leq JC_{6.7}(J, J') + J' + C_{6.7}(J, J')$.

Case 3*:* Some subpath of I connects the future or past orbits of points in h.

The only possibility is that I be a pulled-tight image of h, i.e. $g' = b$. Consider a geodesic pre-image I' of I under $\sigma_{C_{6.7}(J,J')t_0}$ between two points in b. Then proceed as in Case 2, the only difference being that for each subpath I_α, *either* there exists a horizontal geodesic in another connected component of the same stratum, whose pulled-tight projection agrees with the pulled-tight projection of I_α after some finite time, this is exactly Case 2, *or* I_α is contained in a cancellation or in the union of two cancellations, and the arguments are exactly those of Case 1. The bounded-dilatation property then gives an upper-bound on the horizontal length of I.

We denote by $A(J, J')$ the maximum of the constants found in Cases 1, 2 and 3. We denote by $A'(J, J')$ the maximum of the constants $A(J, J')$, $C_{7.3}(A(J, J'), J, J')$ and $C_{7.2}(A(J, J'), J, J')$. Lemmas 7.2, 7.3 and 7.1 give then $B(J, J') = C_{7.1}(A'(J, J'), A'(J, J'), J, J')$ such that, substituting the maximal $-$-holes in g' by the horizontal geodesic between their endpoints yields a straight $(B(J, J'), B(J, J'))$-quasi geodesic stair S, with the same endpoints, which is $A'(J, J')$-close to g'. Let I' be a horizontal geodesic between S and a future or past orbit of some point in h, which is minimal in the sense of the inclusion, i.e. does not contain any subpath connecting S to a future or past orbit of a point in h. This horizontal geodesic I' is a pulled-tight image of a subpath of S in the stratum considered. It is either contained in a cancellation, or is the union of two horizontal geodesics contained in a cancellation. Lemma 6.4 gives $C_{6.4}(B(J,J'),B(J,J'))$ such that, if $|I'|_{f(I')} \geq C_{6.4}(B(J,J'),B(J,J'))$ then I' is dilated in the futur after t_0. From Lemmas 5.3 and 5.4 we get $|I'|_{f(I')} \leq C_{5.4}(1,2,C_{5.3}(1))$. Therefore S is at horizontal distance less or equal to $D(J,J') = \max(C_{6.4}(B(J,J'),B(J,J')),C_{5.4}(1,2,C_{5.3}(1)))$ from a straight stair $\mathcal{S}(g')$, with the same endpoints and in fine position with respect to h. Lemmas 7.4 and 7.1 then give $E(J,J') = C_{7.1}(C_{7.4}(D(J,J'),B(J,J'),B(J,J')),C_{7.4}(D(J,J'),B(J,J'),B(J,J')),J,J')$ such that substituting the maximal subpaths g' as above by the given stair $\mathcal{S}(g')$ gives a straigth $(E(J,J'),E(J,J'))$-quasi geodesic, with the same endpoints than g, in fine position with respect to h, and which is $D(J,J')$-close to g. $\square$

10 Straight quasi geodesic bigons are thin

Proposition 10.1 *There exists a constant $Bi(J,J')$ such that any straight (J,J')-quasi geodesic bigon is $Bi(J,J')$-thin.*

Proof of Proposition 10.1: We denote by g,g' the two sides of a (J,J')-quasi geodesic bigon. We assume for a while that some horizontal geodesic connects the past orbits of the endpoints of the bigon. We choose such a horizontal geodesic h satisfying $f(h) \leq \min_{x \in g \cup g'} f(x) - C_{9.1}(J,J')$. Proposition 9.1 gives a $(C_{9.1}(J,J'),C_{9.1}(J,J'))$-quasi geodesic bigon, with the same vertices, which is $C_{9.1}(J,J')$-close to $g \cup g'$. We denote by $\mathcal{G}$ and $\mathcal{G}'$ the sides of this bigon.

Let us call a *diagonal* a horizontal geodesic which minimizes the horizontal distance between the future and past orbits of its endpoints. From the hyperbolicity of the semi-flow, any diagonal with horizontal length greater or equal to M is dilated both in the future and in the past after $2t_0$.

We choose a real number $L_0 \geq C_{5.4}(2,3,\lambda_+^{2t_0}C_{5.3}(2)) \geq M$ (the signification of the constant $C_{5.4}(2,3,\lambda_+^{2t_0}C_{5.3}(2))$ will become clear later). Let $P \in \mathcal{G}$. We assume that there exist two points $P_1,P_2 \in h$, whose future orbits intersect $\mathcal{G}$, such that P is at telescopic distance $L_1 > C_{8.1}(L_0,C_{9.1}(J,J'),C_{9.1}(J,J'))$ from $O^+(P_i) \cup O^-(P_i)$, $i=1,2$.

We consider a diagonal D between $O^+(P_1) \cup O^-(P_1)$ and $O^+(P_2) \cup O^-(P_2)$. This diagonal is in fine position with respect to h. Since $\mathcal{G}$ is in fine position with respect to h, and D connects the future or past orbits of points in h, and the future or past orbits of points in $\mathcal{G}$, then $\mathcal{G}$ is in fine position with respect to D. Since the point P is at telescopic distance $L_1 > C_{8.1}(L_0,C_{9.1}(J,J'),C_{9.1}(J,J'))$ from $O^+(P_1) \cup O^-(P_1)$ and from $O^+(P_2) \cup O^-(P_2)$, Proposition 8.1 implies $|D|_{f(D)} > L_0$.

Since $\mathcal{G}$ is in fine position with respect to D, and connects the union of the future and past orbits of the endpoints of D, some horizontal geodesics connect $P \in \mathcal{G}$ to $O^+(P_1)$ and to $O^+(P_2)$. These horizontal geodesics either are contained in the pulled-tight image of D, or some pulled-tight image of their concatenation contains D. Because of the bounded-cancellation and bounded-dilatation properties, the telescopic distance between a point and an orbit tends toward infinity with the horizontal distance between this point and that orbit. Since the telescopic distance between P and $O^+(P_1) \cup O^-(P_1)$, and between P and $O^+(P_2) \cup O^-(P_2)$ is L_1, this easy observation gives an upper-bound X, depending on L_1, for the horizontal length of each one of these horizontal geodesics. Therefore some horizontal geodesic connecting $O^+(P_1) \cup O^-(P_1)$ to $O^+(P_2) \cup O^-(P_2)$ has horizontal length smaller or equal to some constant $2X$ (depending on L_1). In particular, $|D|_{f(D)} \leq 2X$.

We noted that a diagonal D with $|D|_{f(D)} \geq M$ is dilated both in the future and in the past after $2t_0$. Here $|D|_{f(D)} > L_0 \geq M$. Since the concatenation of the above two horizontal geodesics, which lie in the future or in the past of D, has horizontal length less or equal to $2X$, a straightforward computation gives $Y > 0$, still depending on L_1 such that $|f(P) - f(D)| \leq Y$. Lemma 5.6 then implies that P is at telescopic distance smaller than $C_{5.6}(Y)$ from some point in D.

Since $\mathcal{G}'$ and D are in fine position, if no point of $\mathcal{G}'$ lies in the future or past orbit of an endpoint of D, this endpoint belongs to a cancellation. Thus we can write $D = D_1 D_2 D_3$ with:

- D_1 (resp. D_3) is non trivial if and only if no point of $\mathcal{G}'$ lies in the future or past orbit of the initial (resp. terminal) point of D.
- D_1 and D_3, if non trivial, are contained in cancellations.
- $\mathcal{G}'$ connects the future or past orbits of the endpoints of D_2.

Let us assume that D_1 and D_3 both are trivial. Then, since $2X \geq |D|_{f(D)} \geq L_0$, Proposition 8.1 tells us that some subpath of $\mathcal{G}'$ is $C_{8.1}(2X, C_{9.1}(J, J'), C_{9.1}(J, J'))$-close to the orbit-segments which connect its endpoints to the endpoints of D. We observed that D is dilated both in the future and in the past after $2t_0$. We proved $2X \geq |D|_{f(D)} \geq L_0$. An easy computation gives a time t_* such that the pulled-tight images and the geodesic pre-images of D after t_* have horizontal length greater or equal to $3C_{8.1}(2X, C_{9.1}(J, J'), C_{9.1}(J, J'))$. Thus some point Q of the above subpath of $\mathcal{G}'$ satisfies $|f(Q) - f(D)| \leq t_*$. Lemma 5.6 gives $C_{5.6}(t_*)$ such that Q is $C_{5.6}(t_*)$-close to D. Therefore $P \in \mathcal{G}$ and $Q \in \mathcal{G}'$ are $C_{5.6}(t_*) + C_{5.6}(Y) + X$-close.

Consider now $D = D_1 D_2 D_3$ with at least D_1 or D_3 non trivial. Since $|D|_{f(D)} \geq C_{5.4}(2, 3, \lambda_+^{2t_0} C_{5.3}(2))$, and D is dilated in the future after $2t_0$, Lemmas 5.3 and 5.4, together with the bounded-dilatation property, give $|D_2|_{f(D)} \geq \lambda_+^{-2t_0} \lambda_+^{2t_0} C_{5.3}(2) \geq M$. Also obviously $|D_2|_{f(D)} \leq 2X$. As in the case where D_1 and D_3 are trivial, by substituting D_2 to D in the above arguments, Proposition 8.1 and Lemma 5.6 eventually give a constant $C_{5.6}(t_\circ)$ such that some point $Q \in \mathcal{G}'$ is $C_{5.6}(t_\circ)$-close to D_2. Thus $P \in \mathcal{G}$ and $Q \in \mathcal{G}'$ are $C_{5.6}(t_\circ) + C_{5.6}(Y) + X$-close.

Let us now consider the case where the points P_1, P_2 do not exist. Then P is L_1-close from some point P' in the orbit of an endpoint, say a, of the bigon. With similar arguments than above (putting paths in fine position and applying Proposition 8.1), we find a horizontal geodesic h', with one endpoint in the future or past orbit of a, such that both paths $\mathcal{G}$ and $\mathcal{G}'$ have one point A-close from h', for some constant A. Since $\mathcal{G}$ and $\mathcal{G}'$ both end or begin at the point a, this implies that $\mathcal{G}'$ admits a point B-close to each point of the orbit-segment between a and h'. In particular there exists $Q \in \mathcal{G}'$ which is $B + L_1$-close to $P \in \mathcal{G}$.

It remains to consider the case where no horizontal geodesic connects the past orbits of the endpoints of the considered (J, J')-quasi geodesic bigon. Then, in the future orbit of the initial endpoint there exists a point z whose past orbit can be connected to the past orbit of the terminal endpoint, and this property is not satisfied by the point w with $f(z) - f(w) = t_0$, which is either in the future or past orbit of the initial endpoint. The strong hyperbolicity of the semi-flow and Proposition 8.1 give then a constant $C_{8.1}(M, J, J')$ such that initial subpaths of both sides of the bigon are $C_{8.1}(M, J, J') + t_0$-close to the orbit-segment connecting the initial endpoint of the bigon to z. From which precedes, any (R, R')-quasi geodesic bigon between z and the terminal endpoint of the considered bigon is $X(R, R')$-thin, for some constant $X(R, R')$. This easily implies that the given bigon is $2(C_{8.1}(M, J, J') + t_0) + X(R, R' + C_{8.1}(M, J, J') + t_0)$-thin, and completes the proof of Proposition 10.1. $\square$

11 Geodesic triangles are thin

The following lemma was suggested to the author by I. Kapovich, and allows us to simplify the conclusion. Let us recall that, in the context of quasi geodesic metric spaces, a (r', s')-chain bigon is a bigon whose sides are (r', s')-chains. Still with this terminology, a (r, s)-chain triangle is a triangle whose sides are (r, s)-chains.

Lemma 11.1 *Let X be a (r, s)-quasi geodesic metric space. If (r', s')-chain bigons are $\delta(r', s')$-thin, $r' \geq r$, $s' \geq s$, then X is $2\delta(r, 3s)$-hyperbolic.*

Proof of Lemma 11.1: We consider a (r, s)-chain triangle with vertices a, b, c and sides $[ab]$, $[ac]$ and $[bc]$. We consider a point x in the (r, s)-chain $[ab]$ which is closest to c. We claim that $[cx] \cup [xb]$ is a $(r, 3s)$-chain, where $[cx]$ and $[xb]$ denote (r, s)-chains from c to x and from x to b. Indeed, for any points u, v in $[xb]$ or $[cx]$, one obviously has: $r d_X(u, v) \geq |[uv]|_X$. Let us thus assume $u \in [cx]$ and $v \in [xb]$. Since x is a point in $[ab]$ closest to c, x is a point in $[ab]$ closest to u. Thus $|[ux]|_X \leq |[uv]|_X$. Moreover $|[xv]|_X \leq |[xu]|_X + |[uv]|_X$. Therefore $|[ux]|_X + |[xv]|_X \leq 3|[uv]|_X$. Whence the claim. The given (r, s)-chain triangle decomposes in two $(r, 3s)$-chain bigons. Therefore this triangle is $2\delta(r, 3s)$-thin. $\square$

Lemma 11.2 *Let $(\tilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack. There exists a constant $C_{11.2}(r, s)$ such that any (r, s)-chain in $(\tilde{X}, d_{(\tilde{X},\mathcal{H})})$ is contained in a $(C_{11.2}(r, s), C_{11.2}(r, s))$-quasi geodesic.*

Proof of Lemma 11.2: Any consecutive pair of points x_{i-1}, x_i, $i = 1, \cdots, k$, in a (r, s)-chain $c = x_0, \cdots, x_k$ can be connected by a telescopic path p_i which is the concatenation of exactly one vertical geodesic and one horizontal geodesic. The vertical length of the vertical geodesic is bounded above by $d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i)$. By the bounded-dilatation property, the horizontal length of the horizontal geodesic is bounded above by $\lambda_+^{d_{(\tilde{X},\mathcal{H})}(x_{i-1},x_i)} d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i)$. If p is the concatenation of the p_i's then p is a telescopic path containing the chain c whose telescopic length satisfies: $|p|_{(\tilde{X},\mathcal{H})} \leq \sum_{i=1}^{k}(1 + \lambda_+^{d_{(\tilde{X},\mathcal{H})}(x_{i-1},x_i)}) d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i)$. Since we consider (r, s)-chains $d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq r$. Thus $|p|_{(\tilde{X},\mathcal{H})} \leq (1+\lambda_+^r) \sum_{i=1}^{k} d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i)$. By definition of a (r, s)-chain $\sum_{i=1}^{k} d_{(\tilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq s d_{(\tilde{X},\mathcal{H})}(x_0, x_k)$. Thus $|p|_{(\tilde{X},\mathcal{H})} \leq s(1+\lambda_+^r) d_{(\tilde{X},\mathcal{H})}(x_0, x_k)$. Any subpath p' of p decomposes as a concatenation $q p_i p_{i+1} \cdots p_m q'$ where q, q' are proper subpaths respectively of p_{i-1} and p_{m+1}. The same arguments than above prove $|p_i p_{i+1} \cdots p_m|_{(\tilde{X},\mathcal{H})} \leq s(1 + \lambda_+^r) d_{(\tilde{X},\mathcal{H})}(i(p_i), t(p_m))$. Furthermore $|q|_{(\tilde{X},\mathcal{H})} \leq (1+\lambda_+^r) r$ and $|q'|_{(\tilde{X},\mathcal{H})} \leq (1+\lambda_+^r) r$. This implies $|p'|_{(\tilde{X},\mathcal{H})} \leq |p_i p_{i+1} \cdots p_m|_{(\tilde{X},\mathcal{H})} + 2r(1+\lambda_+^r)$ and $d_{(\tilde{X},\mathcal{H})}(i(p_i), t(p_m)) \leq d_{(\tilde{X},\mathcal{H})}(i(p'), t(p')) + 2r$. We conclude $|p'|_{(\tilde{X},\mathcal{H})} \leq s(1 + \lambda_+^r) d_{(\tilde{X},\mathcal{H})}(i(p'), t(p')) + 2r(1 + s)(1 + \lambda_+^r)$. Setting $C_{11.2}(r, s) = \max(s, 2r(1 + s))(1 + \lambda_+^r)$, we get Lemma 11.2. $\square$

Lemma 11.3 *There exists a constant $C_{11.3}(J,J')$ such that any (J,J')-quasi geodesic $\mathcal{G}$ is $C_{11.3}(J,J')$-close to a straight $(C_{11.3}(J,J'), C_{11.3}(J,J'))$-quasi geodesic.*

Proof of Lemma 11.3: Let us call *bad subpath* of $\mathcal{G}$ any "maximal" subpath p of $\mathcal{G}$ whose endpoints lie in a same orbit-segment of the semi-flow, where maximal means that, if p_0 (resp. p_1) are arbitrarily small, non trivial, subpaths preceding (resp. following) p in $\mathcal{G}$, then the endpoints of p_0 and p_1 do not lie in a same orbit-segment. We consider a bad subpath p. It might happen that p contains other bad subpaths p_α. If this happens, we choose one of them, denoted by q, and we substitute all the other bad subpaths in p by the orbit-segment between their endpoints. Since orbit-segments are telescopic geodesics, the resulting path, denoted by p', is a (J,J')-quasi geodesic. Since p' does not contain any bad subpath other than q, there exists a point $a \in q \subset p'$ such that p' is the concatenation of two straight (J,J')-quasi geodesics g_0, g_1, where g_0 goes from its initial point $i(p')$ to a, and g_1 goes from a to its terminal point $t(p')$. We now consider the (J,J')-quasi geodesic triangle of vertices $i(p'), t(p'), a$, and with sides g_0, g_1 and the orbit-segment O between $i(p')$ and $t(p')$. We consider any point $z \in g_1$ which minimizes the telescopic distance between $i(p')$ and g_1. We choose a telescopic geodesic g_2 between $i(p')$ and g_1.

We denote by u (resp. v) the path from $i(p')$ to a (resp. $t(p')$) which is the concatenation of g_2 with the subpath of g_1 between z and a (resp. $t(p')$). As in the proof of Lemma 11.1, we prove that the bigon of vertices $i(p')$ and a, with sides g_0 and u, and the bigon of vertices $i(p')$ and $t(p')$ with sides v and O are straight $(3J, 3J')$-quasi geodesic bigons. From Proposition 10.1, these bigons are $Bi(3J,3J')$-thin. Thus there exist two points $x \in g_0$ and $y \in g_1$ which are $2Bi(3J,3J')$-close, and such that the subpaths of g_0 (resp. of g_1) between $i(p')$ and x (resp. between $t(p')$ and y) are $2Bi(3J,3J')$-close to O. Since p' is a (J,J')-quasi geodesic, we conclude that p' is $(2J+2)Bi(3J,3J')+J'$-close to O. The same conclusion holds if one considers any bad subpath other than q in p. Thus any point in p is $(2J+2)Bi(3J,3J')+J'$-close to O. Since the choice of the bad subpath p is arbitrary, the proof of Lemma 11.3 is complete. $\square$

Proof of Theorem 4.4: Let $(\tilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$ such that $(\sigma_t)_{t\in\mathbb{R}^+}$ is strongly hyperbolic with respect to $\mathcal{H}$. From the Lemma - Definition of Section 3.2, this forest-stack is a $(1,2)$-quasi geodesic metric space. Let us consider any (r,s)-chain bigon, $r \geq 1, s \geq 2$. From Lemma 11.2, it is contained in a $(C_{11.2}(r,s), C_{11.2}(r,s))$-quasi geodesic bigon. From Lemma 11.3, this bigon is $A(r,s)$-close, with $A(r,s) = C_{11.3}(C_{11.2}(r,s), C_{11.2}(r,s))$, to a straight $(A(r,s), A(r,s))$-quasi geodesic bigon. Proposition 10.1 gives $\kappa(r,s) = Bi(A(r,s), A(r,s))$ such that this bigon is $\kappa(r,s)$-thin. Thus the given (r,s)-chain bigon is $\delta(r,s)$-thin, with $\delta(r,s) = \kappa(r,s) + 2A(r,s))$. From Lemma 11.1, the given forest-stack, which is a $(1,2)$-quasi geodesic metric space, is $2\delta(1,6)$-hyperbolic. $\square$

12 Back to mapping-telescopes

We elucidate in this section the relationships between forest-stacks and mapping-telescopes.

12.1 Statement of Theorem

An $\mathbb{R}$-*tree*, see [9, 2] among many others, is a metric space such that any two points are joined by a unique arc and this arc is a geodesic for the metric. In particular an $\mathbb{R}$-tree is a topological tree. A $\mathbb{R}$-*forest* is a union of disjoint $\mathbb{R}$-trees.

Lemma 12.1 *Let (Γ, d_Γ) be a $\mathbb{R}$-forest and let $\psi\colon \Gamma \to \Gamma$ be a forest-map of Γ. Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ) equipped with a structure of forest-stack as given in Section 2. Then there is a horizontal metric $\mathcal{H} = (m_r)_{r\in\mathbb{R}}$ on K_ψ such that:*

1. *The $\mathbb{R}$-forests $(f^{-1}(r), m_r)$ and $(f^{-1}(r+1), m_{r+1})$ are isometric. Each stratum $(f^{-1}(n), m_n)$, n integer, is isometric to (Γ, d_Γ).*
2. *For any real r, for any horizontal geodesic $g \in f^{-1}(r)$, the map*
$$l_{r,g}\colon \left\{ \begin{array}{ccc} [0, E[r]+1-r] & \to & \mathbb{R}^+ \\ t & \mapsto & |\sigma_t(g)|_{r+t} \end{array} \right. \text{ is monotone.}$$

Such a horizontal metric is called a horizontal d_Γ-metric. *The telescopic metric associated to a horizontal d_Γ-metric is called a* mapping-telescope d_Γ-metric.

Proof of Lemma 12.1: We make each $\Gamma \times \{n\}$, n integer, a $\mathbb{R}$-forest isometric to Γ. We consider a cover of Γ by geodesics of length 1 which only intersect at their endpoints. Each $\Gamma \times \{n\}$ inherits the same cover. There is a disc $D_{e,n}$ in K_ψ for each such horizontal geodesic e in $\Gamma \times \{n\}$. This disc is bounded by e, $\psi(e)$ and the orbit-segments between the endpoints of e and the endpoints of $\psi(e)$. We foliate this disc by segments with endpoints in, and transverse to, the orbit-segments in its boundary. Then we assign a length to each such segment so that the collection of lengths varies continuously, in a monotonic way, from the length of e to the length of $\psi(e)$. We so obtain a horizontal metric on the mapping-telescope. Furthermore each stratum $f^{-1}(n)$, n integer, is isometric to (Γ, d_Γ). And the maps denoted by $l_{r,g}$ in Lemma 12.1 are monotone by construction. By definition of a mapping-telescope, the discs $D_{e,n}$ between $\Gamma \times \{n\}$ and $\Gamma \times \{n+1\}$ are copies of the discs $D_{e,n'}$ between $\Gamma \times \{n'\}$ and $\Gamma \times \{n'+1\}$, for any n, n' in $\mathbb{Z}$. This allows us to choose the horizontal metric to satisfy moreover that $(f^{-1}(r), m_r)$ is isometric with $(f^{-1}(r+1), m_{r+1})$ for any real number r. $\square$

We now define dynamical properties for $\mathbb{R}$-forests maps.

Definition 12.2 Let (Γ, d_Γ) be a $\mathbb{R}$-forest. A forest-map ψ of (Γ, d_Γ) is *weakly bi-Lipschitz* if there exist $\mu \geq 1$, $K \geq 0$ such that $\mu d_\Gamma(x, y) \geq d_\Gamma(\psi(x), \psi(y)) \geq \frac{1}{\mu} d_\Gamma(x, y) - K$ holds.

Definition 12.3 Let (Γ, d_Γ) be a $\mathbb{R}$-forest.

A forest-map ψ of (Γ, d_Γ) is *hyperbolic* if it is weakly bi-Lipschitz and there exist $\lambda > 1$, $N \geq 1$, $M \geq 0$ such that for any pair of points x, y in Γ with $d_\Gamma(x, y) \geq M$, either $d_\Gamma(\psi^N(x), \psi^N(y)) \geq \lambda d_\Gamma(x, y)$ or for some x_N, y_N with $\psi^N(x_N) = x$, $\psi^N(y_N) = y$, $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$.

A forest-map ψ of (Γ, d_Γ) is *strongly hyperbolic* if it is a hyperbolic forest-map such that, if x, y are any pair of points with $d_\Gamma(x, y) \geq M$, then in each connected component containing both a pre-image of x and a pre-image of y under ψ^N, there is at least one pair of such pre-images x_N, y_N such that $d_\Gamma(x_N, y_N) \geq \lambda d_\Gamma(x, y)$.

If the forest Γ is a tree then a hyperbolic forest-map is strongly hyperbolic (similarly we saw that a hyperbolic semi-flow on a forest-stack whose strata are connected is strongly hyperbolic).

Our theorem about mapping-telescopes is stated as follows:

Theorem 12.4 *Let (Γ, d_Γ) be a $\mathbb{R}$-forest. Let ψ be a strongly hyperbolic forest-map of (Γ, d_Γ) whose mapping-telescope K_ψ is connected. Then K_ψ is a Gromov-hyperbolic metric space for any mapping-telescope d_Γ-metric.*

12.2 Proof of Theorem 12.4

Lemma 12.5 *Let (Γ, d_Γ) be a $\mathbb{R}$-forest. Let ψ be a weakly bi-Lipschitz forest-map of (Γ, d_Γ). Let (K_ψ, f, σ_t) be the mapping-telescope of (ψ, Γ), equipped with a structure of forest-stack as given in Section 2. Then the semi-flow $(\sigma_t)_{t\in\mathbb{R}^+}$ is a bounded-cancellation and bounded-dilatation semi-flow with respect to any horizontal d_Γ-metric (see Lemma 12.1).*

Proof of Lemma 12.5: The horizontal metric $\mathcal{H}$ agrees with the metric d_Γ on all the strata $f^{-1}(n)$, n integer (see Lemma 12.1). Let us consider any horizontal geodesic g in the stratum $f^{-1}(0)$. If ψ is weakly bi-Lipschitz with constants μ_0 and K_0, for any integer $n \geq 0$, $|[g]_n|_n \geq \frac{1}{\mu_0^n}|g|_0 - K_0(\frac{1}{\mu_0^{n-1}} + \frac{1}{\mu_0^{n-2}} + \cdots + 1)$. Since $\frac{1}{\mu_0} < 1$ the sum converges toward $\frac{\mu_0}{\mu_0 - 1}$ with $n \to +\infty$. Setting $\lambda_- = \frac{1}{\mu_0}$ and $K = K_0\frac{\mu_0}{\mu_0-1}$, this proves the inequality of item (1) for horizontal geodesics in $f^{-1}(n)$, n integer, and integer times t. For the case t any positive real number and $g \in f^{-1}(r)$, r any real number, just decompose $\sigma_t = \sigma_{t-E[t]} \circ \sigma_{E[t-(E[r]+1-r)]} \circ \sigma_{E[r]+1-r}$. The map σ_t is a homeomorphism from $f^{-1}(r)$ onto $f^{-1}(r+t)$ for any $t \in [0, E[r]+1-r)$. That is, for any real r, $|[g]_{r+t}|_{r+t} = |\sigma_t(g)|_{r+t}$ for $t \in [0, E[r]+1-r)$. The monoticity of the maps $l_{r,g}$ (see Lemma 12.1, item (2)) implies, for any r and $t \in [0, E[r]+1-r)$, $|\sigma_t(g)|_{r+t} \geq \frac{1}{\mu_0}|g|_r$. The conclusion follows. □

Lemma 12.6 *With the assumptions and notations of Lemma 12.5, if the map ψ is a (strongly) hyperbolic forest-map of (Γ, d_Γ) then the semi-flow $(\sigma_t)_{t\in\mathbb{R}^+}$ is (strongly) hyperbolic with respect to any horizontal d_Γ-metric.*

The proof of this lemma goes in the same way than the proof of Lemma 12.5. □

Proof of Theorem 12.4: From Lemmas 12.5 and 12.6, a mapping-telescope admits a structure of forest-stack $(\tilde{X}, f, \sigma_t, \mathcal{H})$ with horizontal metric $\mathcal{H}$ such that the semi-flow $(\sigma_t)_{t\in\mathbb{R}^+}$ is a strongly hyperbolic semi-flow with respect to $\mathcal{H}$. Theorem 4.4 gives Theorem 12.4. □

13 About mapping-torus groups

We first recall the definition of a *hyperbolic endomorphism* of a group introduced by Gromov [18].

Definition 13.1 ([18, 3]) An injective endomorphism α of the rank n free group F_n is hyperbolic if there exist $\lambda_\alpha > 1$ and $j_\alpha > 0$ such that for any $w \in F_n$, either $\lambda_\alpha|w| \leq |\alpha^{j_\alpha}(w)|$ or w admits a pre-image $\alpha^{-j_\alpha}(w)$ which satisfies $\lambda_\alpha|w| \leq |\alpha^{-j_\alpha}(w)|$, where $|.|$ denotes the usual word-metric.

We recall that a subgroup H in a group G is *malnormal* if for any element $w \notin H$ of G, $w^{-1}Hw \cap H = \{1\}$. Our theorem about mapping-torus groups is stated as follows:

Theorem 13.2 *Let α be an injective hyperbolic endomorphism of the rank n free group F_n. If the image of α is a malnormal subgroup of F_n then the mapping-torus group $G_\alpha = < x_1, \cdots, x_n, t \,;\, t^{-1}x_it = \alpha(x_i), i = 1, \cdots, n >$ is a hyperbolic group.*

13.1 Relationships with mapping-telescopes

We consider the rank n free group $F_n =< x_1, \cdots, x_n >$. Let α be an injective endomorphism of F_n. Let $G_\alpha =< x_1, \cdots, x_n, t\,; t^{-1}x_i t = \alpha(x_i), i = 1, \cdots, n >$ be the mapping-torus group of (α, F_n). We consider the Cayley graph Γ associated to the given system of generators. Let l be a loop in Γ whose associated word in the edges of Γ reads a relation $t^{-1}x_i t\alpha(x_i)^{-1}$. We attach a 2-cell by its boundary circle along any such loop l. The resulting topological space is a 2-complex. This is the Cayley complex of the mapping-torus group G_α for the given presentation.

Let us check that the above Cayley complex is a mapping-telescope of a forest-map. We consider the rose with n petals $\mathcal{R}_n$. We label each edge by a generator x_i of F_n. We denote by ψ the simplicial map on $\mathcal{R}_n$ such that $\psi(x_i)$ is a locally injective path whose associated word in the edges of $\mathcal{R}_n$ reads $\alpha(x_i)$. Let us denote by T the universal covering of $\mathcal{R}_n$ (T is a tree) and $\pi\colon T \to \mathcal{R}_n$ the associated covering-map. We denote by $\hat{\psi}\colon T \to T$ a simplicial lift of ψ to T, that is $\pi \circ \hat{\psi} = \psi \circ \pi$. We consider the mapping-torus of $(\psi, \mathcal{R}_n)$, this is the 2-complex $\mathcal{R}_n \times [0,1]/(x,1) \sim (\psi(x), 0)$. Then the universal covering of this mapping-torus is the mapping-telescope of $\tilde{\psi}\colon F \to F$ where F and $\tilde{\psi}$ are defined as follows:

- We denote by I the set of integers from 1 to Card$(F_n/\text{Im}(\alpha))$. The different classes are written $w_i\text{Im}(\alpha)$, $i = 0, 1, \cdots$. We denote by $\gamma\colon I \to \{w_0, w_1, \cdots\}$ the bijection. Then the connected components of F are in bijection with $\mathbb{N}^{\text{Card}(I)}$. Each connected component is the image, by a bijection μ, of a sequence of Card(I) integers. Each connected component $\mu(x_0, x_1, \cdots)$ of F is homeomorphic to T via $\beta_{(x_0,x_1,\cdots)}\colon \mu(x_0, x_1, \cdots) \to T$.

- We define the restriction of $\tilde{\psi}$ to any connected component $\mu((x_0, x_1, \cdots))$ as follows:

  - If Card$(I) < +\infty$ then

$$\tilde{\psi}_{|\mu((x_0,x_1,\cdots))}\colon \left\{ \begin{array}{ccc} \mu((x_0, x_1, \cdots)) & \to & \mu((E[\frac{x_0}{\text{Card}(I)}], x_1, \cdots)) \\ x & \to & (\gamma(j)\beta^{-1}_{(x_0,x_1,\cdots)}\hat{\psi}\beta_{(x_0,x_1,\cdots)})(x) \end{array} \right.$$

    where $j < \text{Card}(I)$ satisfies $E[\frac{x_0}{\text{Card}(I)}] = k\text{Card}(I) + j$.

  - If Card$(I) = +\infty$ then

$$\tilde{\psi}_{|\mu((x_0,x_1,\cdots))}\colon \left\{ \begin{array}{ccc} \mu((x_0, x_1, \cdots)) & \to & \mu((x_1, x_2, \cdots)) \\ x & \to & (\gamma(x_0)\beta^{-1}_{(x_0,x_1,\cdots)}\hat{\psi}\beta_{(x_0,x_1,\cdots)})(x) \end{array} \right.$$

The mapping-torus of $(\psi, \mathcal{R}_n)$ is a 2-complex whose 1-skeleton is the rose with $n+1$ petals in bijection with $\{x_1, \cdots, x_n, t\}$. There is one 2-cell for each relation $t^{-1}x_i t\alpha(x_i)^{-1}$. Thus the universal covering described above is the Cayley complex for G_α with the presentation $G_\alpha =< x_1, \cdots, x_n, t\,; t^{-1}x_i t = \alpha(x_i), i = 1, \cdots, n >$. We so proved the following:

Lemma 13.3 *Let α be an injective endomorphism of $F_n =< x_1, \cdots, x_n >$. Let $G_\alpha = < x_1, \cdots, x_n, t\,; t^{-1}x_i t = \alpha(x_i), i = 1, \cdots, n >$ be the mapping-torus group of α. Let $\mathcal{C}(G_\alpha)$ be the Cayley complex of G_α for the given presentation. Then $\mathcal{C}(G_\alpha)$ is the mapping-telescope of a forest-map.*

Remark 13.4 If the endomorphism α is an automorphism then the above Cayley complex is the mapping-telescope of a tree-map. The tree is the universal covering of the rose with n petals. If the endomorphism α is not injective then some element $w \in F_n$ satisfies $w = 1$ in G_α; the construction above fails because of the corresponding loops in the Cayley graph.

Let α be an injective free group endomorphism. Let G_α be the mapping-torus group of α. Let $\mathcal{C}(G_\alpha)$ be the Cayley complex of G_α for the usual presentation $G_\alpha =< x_1, \cdots, x_n, t\,; t^{-1}x_it = \alpha(x_i), i = 1, \cdots, n >$. From Lemma 13.3 $\mathcal{C}(G_\alpha)$ is a mapping-telescope of a forest-map. We now want to see what happens with respect to metrics and dynamics. The Cayley graph of a group is equipped with a metric which makes each edge isometric to the interval $(0, 1)$. More generally, given a graph Γ, we call *standard metric*, and denote by d^s_Γ, such a metric on Γ. We will call *mapping-telescope standard metric* any mapping-telescope d^s_Γ-metric on $\mathcal{C}(G_\alpha)$.

Lemma 13.5 *The mapping-torus group G_α of an injective free group endomorphism acts cocompactly, properly discontinuously and isometrically on the Cayley complex $\mathcal{C}(G_\alpha)$ equipped with any mapping-telescope standard metric.*

Proof of Lemma 13.5: We consider the usual action by left translations of the group on its Cayley graph. This action is extended in a natural way to a free action on the Cayley complex $\mathcal{C}(G_\alpha)$. Let us denote by f the map giving the strata for the structure of forest-stack of $\mathcal{C}(G_\alpha)$, see Lemma 13.3. For a mapping-telescope metric, all the strata $f^{-1}(r)$ and $f^{-1}(r + 1)$ are isometric. And for a mapping-telescope standard metric all the strata $f^{-1}(n)$, $n \in \mathbb{Z}$, are equipped with the standard metric. This readily implies that the above action is isometric. □

13.2 Free group endomorphisms and forest-maps

The key-point of Lemma 13.6 below is the so-called "bounded-cancellation lemma" of [7] for free group automorphisms, and of [10] for the injective free group endomorphisms.

Lemma 13.6 *Let α be an injective free group endomorphism. Let F and $\tilde{\psi}$ be the forest and the forest-map on F given by Lemma 13.3. Then $\tilde{\psi}$ is a weakly bi-Lipschitz forest-map of F equipped with the standard metric d^s_F.*

Proof of Lemma 13.6: If w is any element in $F_n =< x_1, \cdots, x_n >$, denoting $|.|_{F_n}$ the word-metric on F_n, then $|\alpha(w)|_{F_n} \leq (\max_{i=1,\cdots,n}|\alpha(x_i)|_{F_n})|w|_{F_n}$. By definition of the standard metric, and setting $\mu_0 = \max_{i=1,\cdots,n}|\alpha(x_i)|_{F_n}$, the map $\tilde{\psi}$ satisfies $d^s_F(\tilde{\psi}(x), \tilde{\psi}(y)) \leq \mu_0\, d^s_F(x, y)$ for any pair of *vertices* x, y. If x, y are not vertices, then they are joined in their stratum by a horizontal geodesic which is the concatenation of a path between two vertices, with two proper subset of edges. By construction and simpliciality of $\tilde{\psi}$, proper subset of edges are dilated by a bounded factor when applying $\tilde{\psi}$, so that the conclusion follows for the upper-bound.

If w is any element in F_n then $|\alpha^{-1}(w)|_{F_n} \leq (\max_{i=1,\cdots,n}|\alpha^{-1}(x_i)|_{F_n})|w|_{F_n}$. Setting $\mu_1 = \max_{i=1,\cdots,n}|\alpha^{-1}(x_i)|_{F_n}$ we get $|\alpha(w)|_{F_n} \geq \frac{1}{\mu_1}|w|_{F_n}$. Therefore $d^s_F(\tilde{\psi}(x), \tilde{\psi}(y)) \geq \frac{1}{\mu_1}d^s_F(x, y)$ for any pair of *vertices* x, y. Unlike the upper-bound, the inequality for all points x, y does not follow so easily. Because the map $\tilde{\psi}$ might identify points, which could make the distance brutally decreasing. However assume the existence of a constant K_0 such that $\tilde{\psi}(x) = \tilde{\psi}(y) \Rightarrow d^s_F(x, y) \leq K_0$. Any geodesic in F is the concatenation of a geodesic between two vertices with two proper subsets of edges of F. Thus the inequality $d^s_F(\tilde{\psi}(x), \tilde{\psi}(y)) \geq \frac{1}{\mu_1} d^s_F(x, y) - 2K_0$ follows in a straightforward way from the preceding assertions. Injective free group endomorphisms satisfy the so-called "bounded-cancellation lemma" (see [10], and [7] for the particular case of automorphisms), i.e. there exists $A_\alpha > 0$ such that for any w_1, w_2 in F_n with $|w_1 w_2|_{F_n} = |w_1|_{F_n} + |w_2|_{F_n}$ then $|\alpha(w_1 w_2)|_{F_n} \geq |\alpha(w_1)|_{F_n} + |\alpha(w_2)|_{F_n} - A_\alpha$. This inequality gives a constant $K_0 = A_\alpha + 2$ as required above, that is such that, if $\tilde{\psi}(x) = \tilde{\psi}(y)$ then $d^s_F(x, y) \leq K_0$. Setting $\mu = \max(\mu_0, \mu_1)$ and $K = 2K_0$, we get Lemma 13.6. □

Lemma 13.7 *With the assumptions and notations of Lemma 13.6:*

1. *If α is hyperbolic then the forest-map is hyperbolic.*
2. *If α is hyperbolic and its image Im(α) is malnormal, then the forest-map is strongly hyperbolic.*

Proof of Lemma 13.7: Item (1) is plain to check. Let us prove item (2). The notations used are the notations introduced in Section 13 when defining the forest F and the map $\tilde{\psi}$. If the map is not strongly hyperbolic, there exists an infinite sequence of pairs of connected components (T_i, T'_i) such that T_i and T'_i get identified under $\tilde{\psi}$ along a geodesic g_i and the length of g_i tends toward $+\infty$ with $i \to +\infty$. Thus there exists an infinite number of elements $(u_i, u'_i) \in F_n - \text{Im}(\alpha) \times F_n - \text{Im}(\alpha)$ such that some geodesic word $a_i w_i b_i$ (resp. $a'_i w_i b'_i$) connects two vertices associated to elements in $u_i \text{Im}(\alpha)$ (resp. in $u'_i \text{Im}(\alpha)$) where the length of the w_i's tends toward $+\infty$ with $i \to +\infty$. Observe that in particular $a_i w_i b_i \in \text{Im}(\alpha)$, $a'_i w_i b'_i \in \text{Im}(\alpha)$, whereas $a_i w_i b'_i \notin \text{Im}(\alpha)$ and $a'_i w_i b_i \notin \text{Im}(\alpha)$ because they carry an element of $u_i \text{Im}(\alpha)$ (resp. $u'_i \text{Im}(\alpha)$) to an element of $u'_i \text{Im}(\alpha)$ (resp. of $u_i \text{Im}(\alpha)$). The lengths of the a_i, b_i, a'_i, b'_i can be assumed less or equal to the maximum of the length of the images under α of the generators of F_n, which is finite. Since there are only a finite number of pairs of elements of bounded lengths, a same pair a_I, b_I (resp. a'_I, b'_I) appears an infinite number of times when listing the sequence of words $a_i w_i b_i$ (resp. $a'_i w_i b'_i$). The same argument of finiteness then gives two words $\omega_1 \subsetneq \omega_2$ with $\omega_2 = \omega \omega_1$ such that $a_I \omega_j b_I \in \text{Im}(\alpha)$, $a'_I \omega_j b'_I \in \text{Im}(\alpha)$, $a_I \omega_j b'_I \notin \text{Im}(\alpha)$ and $a'_I \omega_j b_I \notin \text{Im}(\alpha)$, $j = 1, 2$. Thus $a_I \omega_1 b_I b_I^{-1} \omega_1^{-1} \omega^{-1} a_I^{-1} \in \text{Im}(\alpha)$, $a'_I \omega_1 b'_I b'^{-1}_I \omega_1^{-1} \omega^{-1} a'^{-1}_I \in \text{Im}(\alpha)$, $a_I \omega_1 b'_I b'^{-1}_I \omega_1^{-1} \omega^{-1} a'^{-1}_I \notin \text{Im}(\alpha)$. Now $(a_I \omega^{-1} a'^{-1}_I)^{-1} a_I \omega^{-1} a_I^{-1} (a_I \omega^{-1} a'^{-1}_I) = a'_I \omega^{-1} a'^{-1}_I \in \text{Im}(\alpha)$ whereas $a_I \omega^{-1} a'^{-1}_I \notin \text{Im}(\alpha)$, $a_I \omega^{-1} a_I^{-1} \in \text{Im}(\alpha)$ and $a'_I \omega^{-1} a'^{-1}_I \in \text{Im}(\alpha)$. We so get a contradiction with the malnormality of $\text{Im}(\alpha)$ in F_n. This completes the proof of Lemma 13.7. □

13.3 Proof of Theorem 13.2

From Lemmas 13.6 and 13.7, the Cayley complex $\mathcal{C}(G_\alpha)$ is the mapping-telescope of a strongly hyperbolic forest-map, equipped with the standard metric. A Cayley complex is connected. Thus, from Theorem 12.4, $\mathcal{C}(G_\alpha)$ is a Gromov-hyperbolic metric space for any mapping-telescope standard metric. From Lemma 13.5 the group G_α acts cocompactly, properly discontinuously and isometrically on $\mathcal{C}(G_\alpha)$ equipped with a mapping-telescope standard metric. A classical lemma of geometric group theory (usually attributed to Effremovich, Svàrc, Milnor - see [18] or [15] for instance), applied to quasi geodesic metric spaces, tells us that G_α and $\mathcal{C}(G_\alpha)$ are quasi-isometric so that G_α is a hyperbolic group. □

Remark 13.8 Another way to state our main theorem about "forest-stacks", using the language of trees of spaces, goes roughly as follows: "An oriented $\mathbb{R}$-tree of $\mathbb{R}$-trees with the gluing-maps satisfying the conditions of hyperbolicity and strong hyperbolicity with uniform constants is Gromov-hyperbolic." Where by "oriented $\mathbb{R}$-tree" we mean an $\mathbb{R}$-tree T equipped with an orientation going from the domain to the image of each attaching-map, and a surjective continuous map $f\colon T \to \mathbb{R}$ respecting this orientation. As a corollary of our theorem, and in order to illustrate it, we chosed to concentrate on mapping-telescopes. We could as well consider spaces similar to mapping-telescopes but where we allow the attaching-maps not to be the same at each step. That we only need is to have uniform constants of quasi-isometry, hyperbolicity and so on. Also, with respect to groups, a corollary could have been stated dealing with HNN-extensions rather than just semi-direct products.

Another result which easily follows from our work could be more or less stated as follows. "Let T be a tree of spaces X_i, $i = 0, 1, \cdots$. Let $\psi\colon T \to T$ be a map of T such that the mapping-telescope of each X_i under ψ is Gromov-hyperbolic. If ψ induces a hyperbolic map on the tree resulting of the collapsing of each X_i to a point, then the mapping-telescope of the tree of spaces T under ψ is Gromov-hyperbolic." We leave the precise statement of such corollaries to the reader. Together with [13] where a new proof of the Bestvina-Feighn's theorem is given for mapping-tori of surface groups, the last one gives, thanks to [25], a new proof of the full version of the Combination Theorem for mapping-tori of hyperbolic groups, namely: "If G is a hyperbolic group and α a hyperbolic automorphism of G, then $G \rtimes_\alpha \mathbb{Z}$ is a hyperbolic group".

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