Calculation of eddy currents in moving structures by a sliding mesh-finite element method

A. Buffa*, Y.Maday*, F. Rapetti*°

*ASCI - UPR 9029 CNRS, Paris Sud University, Building 506, 91403 Orsay cedex, France

°LGEP - UMR 8507 CNRS, Supelec University Plateau de Moulon, 91192 Gif-Sur-Yvette cedex, France

Abstract – **In this paper we present some theoretical and numerical results concerning the simulation of eddy currents in non-stationary structures. Both $2D$ and $3D$ models are considered. The approximation is based on the sliding-mesh mortar method combined with node elements in $2D$ and edge elements in $3D$. An implicit Euler scheme is used to discretize in time.**

Key words – **eddy currents, finite element approximation on non-matching grids, magnetic field as primary variable, moving structures.**

I. Introduction

The subject of our research activity is the analysis and development of simulation tools to effectively predict the induced current distribution in non-stationary geometries with sliding interfaces. We then study the flux of eddy currents in a domain composed of two rigid conductive parts, a rotating one (rotor) in sliding contact with a fixed one (stator). Our system is a sample of induction motor.

We work in space-time variables and we consider both the $2D$ *transverse electric (TE) formulation* and the $3D$ complete system of the eddy current problem. These formulations are obtained from Maxwell's equations where the displacement currents are neglected with respect to the conducting ones. The magnetic field $\mathbf{H}$ is chosen as primary variable.

The problem equations are solved in their own frames (i.e. we work in Lagrangian variables). This choice allows us to avoid the presence of convection terms which give rise to instabilities in the numerical solution when the speed of the moving part is high. In this context, we propose a new sliding mesh method based on the *mortar element method* technique [1], [4-3] which handles in an optimal way (in a sense that will be clear soon) non-matching grids at the interface between the static and moving part. No constraint needs to be imposed between the mesh size h and the time step δt at the stator-rotor interface. In space, the $2D$ problem formulation is discretized by means of Lagrangian finite elements and the $3D$ one by applying the second family of first order edge elements (see [6] for the definition). If the approximation does not allow for non-matching grids either a "lock-step" technique (see [5]), or an overlapping one (see [8]) or the local distortion and partial re-meshing has to be adopted. All these remedies are rather expensive and with limitations (see [7] for a review on the subject).

We propose an implicit Euler discretization in time: the final linear system is characterized by having a sparse, symmetric and positive definite matrix.

The outline of the paper is the following: in Section II we present the continuous problem in transmission form, in Section III we concentrate on the $2D$ formulation and both convergence results and numerical tests are presented. In section IV, we present the 3D discrete formulation. A convergence result is stated in Section V and finally, Section VI is devoted to some considerations and conclusions.

II. The continuous problem

Let $\Omega \subset \mathbf{R}^3$ be a Lipschitz bounded (at least in two directions) open set, $\Omega_1 \subset \Omega$ be its moving part and Ω_2 its complement in Ω, $\Omega_2 = \Omega \backslash \bar{\Omega}_1$. The set Ω_1 is a axisymmetric subdomain, strictly included in Ω_2. We call Γ the interface, namely $\Gamma = \partial\Omega_1$, $\mathbf{n}_\Gamma$ the inward normal to Ω_1 at Γ. Let $\theta(t)$ be the rotation angle at time t, $r_t : \Omega_1 \to \Omega_1$ the rotation operator which turns the domain Ω_1 of the angle $\theta(t)$ and r_{-t} its inverse (see Fig.1).

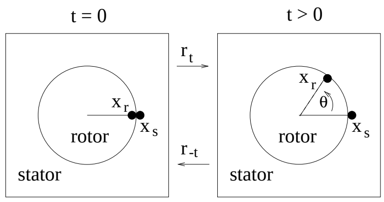

Fig.1 This is a transversal section of the domain. In the moving geometries formulation, the system configuration at time t = 0 is taken as reference configuration for the problem solution: at time $t > 0$, the rotor domain has rotated with an angle $\theta = \omega\, t$ and we go back to the reference configuration thanks to the operator r_{-t}. In particular, $\mathbf{x}_r(t) = r_t(\mathbf{x}_r(0))$ and $\mathbf{x}_r(0) = r_{-t}(\mathbf{x}_r(t))$.

The eddy current problem equations with the magnetic field $\mathbf{H}$ as primary variable read:

$$\mu \partial_t \mathbf{H}_i + \mathbf{curl}\, \sigma^{-1} \mathbf{curl}\, \mathbf{H} = \mathbf{curl}\, (\sigma^{-1} \mathbf{J}_i) \ , \ \Omega_i \times I \quad (1)$$

$$\mathbf{H}(\mathbf{x}, t) \wedge \mathbf{n}_{\partial\Omega} = 0 \ , \ \partial\Omega \times I \ , \quad (2)$$

$$\mathbf{H}(\mathbf{x}, 0) = \mathbf{H}_0 \ , \ \Omega \times \{0\} \quad (3)$$

with two transmission conditions imposed on $\Gamma \times I$

$$r_t \mathbf{H}_1(r_{-t}\mathbf{x}, t) \wedge \mathbf{n}_\Gamma = \mathbf{H}_2(\mathbf{x}, t) \wedge \mathbf{n}_\Gamma \ , \quad (4)$$

$$r_t \mathbf{curl}\, \mathbf{H}_1(r_{-t}\mathbf{x}, t) \wedge \mathbf{n}_\Gamma = \mathbf{curl}\, \mathbf{H}_2(\mathbf{x}, t) \wedge \mathbf{n}_\Gamma \ . \quad (5)$$

In the system (1)-(5), $I=]0,T[$ is the considered time interval, $\sigma>0$ the electric conductivity and $\mu>0$ the magnetic permeability. Moreover, $\mathbf{H}_i=\mathbf{H}_{|\Omega_i}$ $(i=1,2)$, $\mathbf{J}$ is the current density and the magnetic field $\mathbf{H}_0$ involved in the initial condition is supposed to be divergence free. For simplicity of the presentation, we consider only homogeneous boundary conditions. The equations (4)-(5) are the transmission conditions, the essential and the natural one, and take into account the movement of the system. They translate the physical property of continuity of the tangential component of the magnetic and electric field across the interface Γ.

We set $H(\mathbf{curl},\Omega_i)=\{\mathbf{u}\in\mathbf{L}^2(\Omega_i)\ s.t.\ \mathbf{curl}\,\mathbf{u}\in\mathbf{L}^2(\Omega_i)\}$, $\mathcal{H}=H(\mathbf{curl},\Omega_1)\times H_{0,\partial\Omega}(\mathbf{curl},\Omega_2)$ where the sub-index $_{0,\partial\Omega}$ means that we are considering homogeneous boundary condition over $\partial\Omega$. We set:

$$\begin{aligned}\mathcal{H}^t &= \{\mathbf{H}=(\mathbf{H}_1,\mathbf{H}_2)\in\mathcal{H}\ s.t. \\ &\quad r_t\mathbf{H}_1(r_{-t}\mathbf{x},t)\wedge\mathbf{n}_\Gamma=\mathbf{H}_2(\mathbf{x},t)\wedge\mathbf{n}_\Gamma\quad\forall\mathbf{x}\in\Gamma\}.\end{aligned}\tag{6}$$

Problem (1-5) admits then a unique solution in the following variational sense: *Find $\boldsymbol{H}\in\mathcal{H}^t$ such that $\forall\mathbf{v}\in\mathcal{H}^t$*

$$\sum_{i=1}^{2}\int_{\Omega_i}(\mu\,\partial_t\mathbf{H}_i\cdot\mathbf{v}_i+\sigma^{-1}\mathbf{curl}\,\mathbf{H}_i\cdot\mathbf{curl}\,\mathbf{v}_i)d\Omega=\int_\Omega\mathbf{curl}\,(\sigma^{-1}\mathbf{J})\cdot\mathbf{v}d\Omega\,.\tag{7}$$

Note that here the essential transmission condition (4) is strongly imposed in the definition of the functional space, while the natural one (5) is weakly imposed through the variational formulation (this is a consequence of the integration by parts.) We are now in the position of making a discretization of this problem and the key point will be the discrete counterpart of the time dependent constraint characterizing the definition of the space $\mathcal{H}^t$.

III. Setting and discretization of the 2D problem

We suppose here that Ω is an infinitely long cylinder and z the axis direction; the data $\mathbf{H}_0$ and $\mathbf{J}$ respect this symmetry, namely $\mathbf{H}_0(x,y,z)=(0,0,\mathrm{H}_{0z}(x,y))$ and $\mathbf{J}(x,y,z)=(J_x(x,y),J_y(x,y),0)$.

The vector problem for the magnetic field (1)-(5) becomes a scalar one with variational formulation: *Find $t\mapsto u(t)\in\mathcal{U}^t$ such that $\forall v\in\mathcal{U}^t$*

$$\sum_{i=1}^{2}\int_{\Omega_i}(\mu\partial_t u_i\,v_i+\sigma^{-1}\nabla u_i\cdot\nabla v_i)d\Omega=\int_\Omega f\,v\,d\Omega\tag{8}$$

where $f=(\partial_y(\sigma^{-1}J_x)-\partial_x(\sigma^{-1}J_y))$ and the functional space $\mathcal{U}^t$ is defined as

$$\begin{aligned}\mathcal{U}^t &= \{u=(u_1,u_2)\in H^1(\Omega_1)\times H^1_{0,\partial\Omega}(\Omega_2)\ s.t. \\ &\quad u_1(r_{-t}\mathbf{x},t)=u_2(\mathbf{x},t)\quad\forall\mathbf{x}\in\Gamma\}.\end{aligned}$$

Remark that we have changed the notation both for the solution and the functional space now, i.e., u stands for the "scalar" magnetic field and the solution $\mathbf{H}$ of the problem (7) reads $\mathbf{H}=(0,0,u)$. A different notation might have been misleading.

Regarding to the space discretization of the variational problem (8), we propose a Lagrangian finite element discretization in both domains Ω_1 and Ω_2. Two decompositions $\mathcal{T}_{i,h_i}$ are chosen independently on the domains Ω_i, $i=1,2$ (being h_i the mesh size). Let X_{i,h_i} be the space of continuous piecewise linear functions on the mesh $\mathcal{T}_{i,h_i}$, $i=1,2$. The space $X_h=X_{1,h_1}\times X_{2,h_2}$ gives a good finite element approximation of the continuous one $H^1(\Omega_1)\times H^1(\Omega_2)$.

The time-dependent constraint characterizing the space $\mathcal{U}^t$ can not be directly imposed at the discrete level due to the fact that, given $(u_{1,h},u_{2,h})\in X_h$, the two traces $u_{1,h}(r_{-t}\mathbf{x})_{|\Gamma}$ and $u_{2,h}(\mathbf{x})_{|\Gamma}$ live on different nonmatching grids. In what follows, this condition is weakly imposed by means of a suitable space of Lagrange multipliers, defined as:

$$M_h=\{\varphi_h\in C^0(\Gamma)\mid\varphi_{h\,|e}\in\mathcal{P}_1(e)\,,\forall e\in\mathcal{T}_{2,h_2|\Gamma}\},\tag{9}$$

where e represents one mesh edge. According to [3], we define the following non-conforming approximation space:

$$\begin{aligned}\mathcal{U}_h^t=\{u_h\in X_{1,h}\times X_{2,h}\mid u_{2,h\,\partial\Omega}=0\,,\forall\varphi_h\in M_h \\ \int_\Gamma(v_{1,h}(r_{-t}\mathbf{x})-v_{2,h}(\mathbf{x}))\,\varphi_h(\mathbf{x})\,d\Gamma=0\ \}.\end{aligned}\tag{10}$$

The constraint (10) is time dependent, and it is the discrete weak version of the equality constraint characterizing $\mathcal{U}^t$. With an abuse of notation we wrote the integral (10) although the functions $v_{i,h}$ are not defined on Γ (the sense of this integral is made precise in [2]). Concerning the time discretization we use an implicit Euler scheme and we denote by δt the time step. We set, at $t=0$, $u_h^0=\mathcal{I}_hH_0$ where $\mathcal{I}_h$ is the interpolation operator associated to $\mathcal{T}_{1,h_1}$ on Ω_1 and to $\mathcal{T}_{2,h_2}$ on Ω_2. The fully discrete variational problem reads: for $n=0,..,N-1$, *find $u_h^{n+1}\in\mathcal{U}_h^0(n+1)$ such that $\forall v_h^{n+1}\in\mathcal{U}_h^0(n+1)$*

$$\int_\Omega\mu\frac{u_h^{n+1}-u_h^n}{\delta t}v_h^{n+1}\,d\Omega+a(u_h^{n+1},v_h^{n+1})=\int_\Omega j_s v_h^{n+1}\,d\Omega\tag{11}$$

where $a(u_h,v_h)=\sum_{i=1}^2\int_{\Omega_i}\sigma^{-1}\nabla u_{i,h}\cdot\nabla v_{i,h}\,d\Omega$. Note that, due to the time-dependence of the test functions, the use of an explicit Euler scheme, for example, would have destroyed the symmetry of the problem.

Theorem: *Let u and $\{u_h^n\}_n$ be the solutions of problem (8) and (11) respectively. With suitable regularity conditions on u, the following error estimate holds $\forall n$:*

$$||u^n-u_h^n||_{0,\Omega}^2+\alpha\sum_{i=0}^{n}||u^i-u_h^i||_{\mathcal{H}}^2\le C(T)(h^2+\delta t^2).$$

Of course the same error estimate is true when the homogeneous boundary conditions are replaced by nonhomegeneous ones and/or of different type (Dirichlet, Dirichlet-Neumann.) Here following, the results of our numerical simulation are presented.

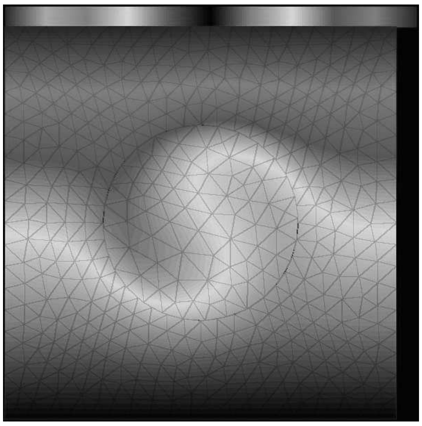

Fig.1 Magnetic isolines distribution at the transient state

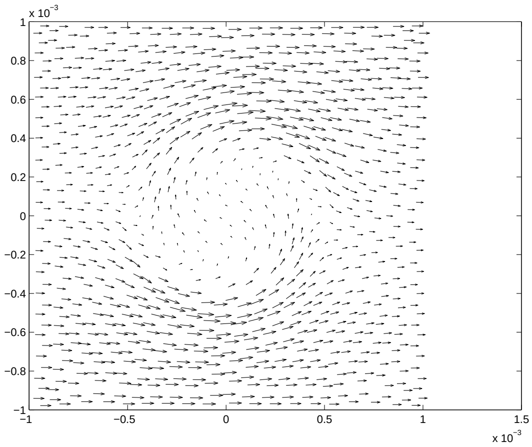

Fig.2 Electric field distribution at the steady state

In Fig.1 and Fig.2 we present the results of a simulation with mixed boundary conditions: $u_2 = 1$ at the top, $u_2 = 0$ at the bottom and homogeneous Neumann conditions on the vertical sides. These figures show the effect of the rotor movement on the distribution of the magnetic field isolines at the transient state (Fig.1) and of the electric field vector at steady state (Fig.2). The gray scale refers to different magnetic field intensity values and the arrows represent instead the electric field vector, both computed at the mesh triangles barycentres.

IV. Discretization of the 3D problem

In this section we go back to the $3D$ vector problem (7). The description of the algorithm in the case of $3D$ simulation is more complicated and requires some preliminary definitions. Basically, we would like to repeat the same reasoning of the previous section but some difficulties occur:
– we need to use edge elements in both domains Ω_i
– the interface Γ between the moving and static part is now a two dimensional surface (that may have edges)
– a clever choice of the space of discrete Lagrange multipliers is necessary.

We are given two triangulations $\mathcal{T}_{1,h_1}$ and $\mathcal{T}_{2,h_2}$ nonmatching at the interface Γ and we concentrate on the first order edge element approximation. Let us first recall the set of degrees of freedom for the edge elements proposed in [6]. Let $\hat{K}$ be the reference tetrahedron and $\hat{e}$ its edges, we have:

$$\hat{\mathbf{v}} \in \mathbb{P}_1(\hat{K})^3 \,, \quad \int_{\hat{e}} \hat{\mathbf{v}} \cdot \hat{\tau}_{\hat{e}}\, \varphi ds \,, \quad \forall \varphi \in P_1(\hat{e}) \,. \tag{12}$$

We define

$$\mathbf{X}_{i,h_i} = \{\mathbf{v}_{i,h_i} \in H(\mathbf{curl}\,,\Omega_i) \mid \mathbf{v}_{2,h_2} \wedge \mathbf{n} = 0 \text{ at } \partial\Omega\,; \quad \mathbf{v}_{i,h_i}{}_{|K} \in \mathcal{N}_1(K)\ ,\ \forall K \in \mathcal{T}_{i,h_i}\}$$

where we denote by $\mathcal{N}_1$ the second family of edge elements of first degree defined in (12). The space $X_h = X_{1,h_1} \times X_{2,h_2}$ is a good approximation of $\mathcal{H}$ and now we have to deal with the time dependent constraint condition characterizing $\mathcal{H}^t$ defined in (6).

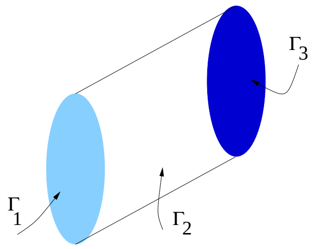

Fig.4 The stator-rotor interface in the $3D$ case.

As in Fig.4, we split the interface Γ in three parts, Γ_j, $j = 1, 2, 3$. Our purpose now is to define the Lagrange multiplier space and, in order, to obtain optimal approximation properties of the finite element space we need to do it locally at each part Γ_j. Namely, we want to define three spaces $M_{j,h}$ whose elements have support on Γ_j respectively.

Let $T_{j,h} := \{\mathbf{v}_{2,h} \wedge \mathbf{n}_{\Gamma_j}, \mathbf{v}_{2,h} \in X_{2,h}\}$ be the space of traces on Γ_j of the edge element spaces $X_{2,h}$. We choose $M_{j,h} \subset T_{j,h}$ and verifying some additional constraint. Let BT^j be the set of triangles of $\mathcal{T}_{i,h_{|\Gamma_j}}$ which share at least one edge with $\partial\Gamma_j$. For all $K \in BT^j$, we assume that the mapping $F_K : K \to \hat{K}$ $(F_K(\mathbf{x}) = B_K\mathbf{x} + \mathbf{c}_K)$ associates to (one of) the boundary edge(s) $(\bar{K} \cap \partial\Gamma_j)$ an edge of $\hat{T}$ that is horizontal (this is exhaustive up to a rotation). The space $M_{j,h}$ is then defined as:

$$M_{j,h} = \{\varphi_h \in T_{j,h} \ :\ B_K^{-1}(\varphi_h \circ F_K) \in \mathbb{P}_1(\hat{K}) \times \mathbb{P}_0(\hat{K})\}. \tag{13}$$

This definition seems to be more complicated then it is in reality; in Fig.5 we give a graphical representation of this space. Finally, we are now in the position to define the discrete space we use as approximation of $\mathcal{H}^t$:

$$\mathcal{H}_h^t = \{\mathbf{H}_h \in \mathbf{X}_{1,h_1} \times \mathbf{X}_{2,h_2} \mid \forall j\,, \forall \boldsymbol{\varphi}_{j,h} \in M_{j,h} \quad \int_\Gamma (r_t \mathbf{H}_{1,h_1}(r_{-t}\mathbf{x},t) - \mathbf{H}_{2,h_2}(\mathbf{x},t)) \wedge \mathbf{n}_\Gamma \cdot \boldsymbol{\varphi}_{j,h} d\Gamma = 0\}.$$

The semi-discrete formulation (discrete in space and continuous in time) of the variational equation (7) reads

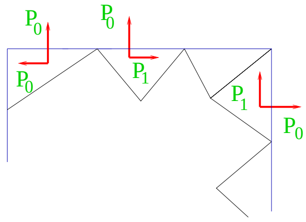

Fig. 5 Graphical representation of the Lagrange multipliers on a portion of Γ.

then: *find* $\boldsymbol{H}_h \in \mathcal{H}_h^t$ *such that:* $\forall \mathbf{v}_h \in \mathcal{H}_h^t$

$$\sum_{i=1}^{2}\int_{\Omega_i}(\mu\,\partial_t\mathbf{H}_{i,h}\cdot\mathbf{v}_{i,h}+\sigma^{-1}\mathbf{curl}\,\mathbf{H}_{i,h}\cdot\mathbf{curl}\,\mathbf{v}_{i,h})d\Omega=\int_\Omega\mathbf{f}\cdot\mathbf{v}d\Omega$$

Finally, by using again the implicit Euler scheme in time, we obtain the fully-discrete formulation. We set $\mathbf{H}_h^0 = \mathcal{I}_h\mathbf{H}_0$ where $\mathcal{I}_h$ is the interpolation operator for edge elements associated to $\mathcal{T}_{i,h_i}$ on Ω_i respectively and for every n=0,...,N-1, we solve: *find* $\boldsymbol{H}_h^{n+1} \in \mathcal{H}_t^{n+1}$ *such that* $\forall \mathbf{v}_h^{n+1} \in \mathcal{H}_t^{n+1}$*:*

$$\int_\Omega \mu\frac{\mathbf{H}_h^{n+1}-\mathbf{H}_h^n}{\delta t}\cdot\mathbf{v}_h^{n+1}\,d\Omega+a(\mathbf{H}_h^{n+1},\mathbf{v}_h^{n+1})=\int_\Omega\mathbf{f}\cdot\mathbf{v}_h^{n+1}\,d\Omega \tag{14}$$

where $a(\mathbf{u},\mathbf{v})=\sum_{i=1}^2\int_{\Omega_i}\sigma^{-1}\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v}\,d\Omega$.
Basically, we have introduced a non-conforming domain decomposition method which allows for non-matching grids at the interface and we have coupled this method with a time scheme. In the next Section, we prove that this method has good approximation properties by giving an error estimate for a static model problem.

V. Error for the 3D scheme on a model problem

We consider the problem that has to be solved at each iteration after the time discretization: *find* $\boldsymbol{H} \in H_0(\boldsymbol{curl},\Omega)$ *such that* $\forall \mathbf{v} \in H_0(\boldsymbol{curl},\Omega)$

$$\sum_{i=1}^{2}\int_{\Omega_i}(\mathbf{H}_i\cdot\mathbf{v}_i+\mathbf{curl}\,\mathbf{H}_i\cdot\mathbf{curl}\,\mathbf{v}_i)d\Omega=\int_\Omega\mathbf{f}\cdot\mathbf{v}\,d\Omega\,. \tag{15}$$

And, at the discrete level we use the non-conforming edge element method proposed above. We are given two non-matching grids $\mathcal{T}_{i,h_i}$ and we set

$$\begin{aligned} X_h &= \{\mathbf{H}=(\mathbf{H}_{1,h},\mathbf{H}_{2,h})\in X_{1,h_1}\times X_{2,h_2}\ s.t.\\ &\int_{\Gamma_i}(\mathbf{H}_{1,h_1}-\mathbf{H}_{2,h_2})\wedge\mathbf{n}_\Gamma\cdot\boldsymbol{\varphi}_h d\Gamma=0\ ,\ \forall\boldsymbol{\varphi}_h\in M_h\}. \end{aligned}$$

We solve the problem: *find* $\boldsymbol{H}_h \in X_h$ *s.t.* $\forall \mathbf{v} \in X_h$*:*

$$\sum_{i=1}^{2}\int_{\Omega_i}(\mathbf{H}_{i,h}\cdot\mathbf{v}_{i,h}+\mathbf{curl}\,\mathbf{H}_{i,h}\cdot\mathbf{curl}\,\mathbf{v}_{i,h})d\Omega=\int_\Omega\mathbf{f}\cdot\mathbf{v}_h\,d\Omega. \tag{16}$$

The following theorem holds:

Theorem 2: *Let* $\boldsymbol{H}$ *and* $\boldsymbol{H}_h$ *be the solutions of problem (15) and (16). Under suitable regularity conditions on* $\boldsymbol{H}$*, the following error estimate holds for every n:*

$$\sum_{i=1}^{2}||\mathbf{H}_i-\mathbf{H}_{i,h}||_{H(\mathbf{curl}\,,\Omega_i)}\le\mathcal{C}h\sqrt{|\ln h|}$$

where the constant $\mathcal{C}$ *does not depend on the mesh sizes.* This result has been proven in [4] and it is only slightly sub-optimal due to the presence of the factor $\sqrt{|\ln h|}$.

Remark Another and easier approach would have been to consider only one interface, the whole Γ and, as in the 2D case, to choose as Lagrange multipliers, the space of tangential traces of $X_{2,h}$ over Γ. Unfortunately the resulting scheme have very poor convergence properties.

VI. Conclusions

The sliding mesh mortar element method has been extended here both in the $2D$ and $3D$ cases for the simulation of rotating engines. It is theoretically justified and illustrated by numerical results in $2D$. The implementation of the $3D$ edge element mortar method is under progress [7].

VII. References

[1] C. Bernardi, Y. Maday and A. Patera, “A new nonconforming approach to domain decomposition: The mortar elements method”, in *Nonlinear partial differential equations and their applications*, H. Brezis and J. Lions, eds., Pitman, pp. 13–51,1990.
[2] A. Bossavit, *Computational Electromagnetism: Variational Formulations, Complementarity, Edge Elements*, 1st ed., Academic Press, 1998.
[3] A. Buffa, Y. Maday and F. Rapetti, “ A sliding mesh-mortar method for a two-dimensional eddy currents model of electric engines”, submitted to M^2AN, 1999.
[4] F. Ben Belgacem, A. Buffa and Y. Maday, “The mortar element method for 3d Maxwell's equations: first results”, submitted to *SIAM J. Numer. Anal.*, 1999.
[5] C.R.I.Emson, C.P.Riley, D.A.Walsh, K.Ueda and T.Kumano, “ Modelling eddy currents induced in rotating systems “, *IEEE Trans. on Magn.*, vol.34, pp.2593–2596, 1998.
[6] J. Nédélec, “A new family of mixed finite element in $\mathbf{R}^3$”, *Numer. Math.*, vol.35, pp. 57–81, 1986.
[7] F. Rapetti, “ Approximation des équations de la magnéto dynamique á l' aide des méthodes numériques sur maillages non conformes”, Phd Thesis, Université de Pierre et Marie Curie, Laboratoire d' Analyse Numérique. In preparation, 1999.
[8] I. J. Tsukerman, “ Overlapping finite elements for problems with movement ”, *IEEE Trans. Mag.*, Vol. 28, pp. 2247-2249, 1992.