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Numerical analysis of a model for an axisymmetric guide for electromagnetic waves Part I: The continuous problem and its Fourier expansion

Faker Ben Belgacem[1,‡], Christine Bernardi[2,*,†] and Francesca Rapetti[3,§]

[1]*Mathématiques pour l'Industrie et la Physique* (*U.M.R. 5640 C.N.R.S.*), *Université Paul Sabatier*, *31062 Toulouse Cedex 04*, *France*
[2]*Laboratoire Jacques-Louis Lions*, *C.N.R.S. & Université Pierre et Marie Curie*, *B.C. 187*, *4 place Jussieu*, *75252 Paris Cedex 05*, *France*
[3]*Laboratoire Jean-Alexandre Dieudonné* (*U.M.R. 6621 C.N.R.S.*), *Université de Nice Sophia-Antipolis*, *Parc Valrose*, *06108 Nice Cedex 02*, *France*

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SUMMARY

As a first model for an electromagnetic wave guide, we consider Maxwell's system in a three-dimensional axisymmetric domain provided with appropriate boundary conditions on different parts of the boundary. We check the well-posedness of the corresponding variational problem. We write the Fourier expansion of the solution as a function of the angular variable and derive the well-posedness of the two-dimensional problems satisfied by each Fourier coefficient. The first step for approximating the three-dimensional solution relies on Fourier truncation, and we prove optimal estimates for the error issued from this truncation. Copyright © 2005 John Wiley & Sons, Ltd.

RÉSUMÉ

Comme modèle de base d'un guide d'ondes électromagnétiques, nous considérons les équations de Maxwell dans un domaine tri-dimensionnel axisymétrique, munies de conditions aux limites adéquates sur différentes parties de la frontière. Nous prouvons que le problème variationnel correspondant est bien posé. Nous écrivons le développement en série de Fourier de la solution par rapport à la variable angulaire et constatons que les problèmes bi-dimensionnels vérifiés par chaque coefficient de Fourier sont également bien posés. La première idée pour construire une approximation de la solution tri-dimensionnelle consiste en une troncature en Fourier, et nous établissons des majorations optimales de l'erreur dues a cette troncature.

KEY WORDS: electromagnetic wave guide; axisymmetric geometry; Fourier expansion

*Correspondence to: Christine Bernardi, Laboratoire Jacques-Louis Lions, C.N.R.S. & Université Pierre et Marie Curie, B.C. 187, 4 place Jussieu, 75252 Paris Cedex 05, France.
† E-mail: bernardi@ann.jussieu.fr
‡ E-mail: belgacem@mip.ups-tlse.fr
§ E-mail: frapetti@math.unice.fr

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1. INTRODUCTION

Time harmonic Maxwell's system models the propagation of electromagnetic waves in three-dimensional domains. Let Ω be a connected bounded open set in $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$, and let $\boldsymbol{n}$ denote the unit outward normal to Ω on $\partial\Omega$. Without restriction, we assume that $\partial\Omega$ is divided into three disjoint parts Γ_{i}, Γ_{c} and Γ_{t} (the indices i, c and t stand for incident, conducting and transparent, respectively) and we consider the following system:

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e} - \kappa^2\,\boldsymbol{e} = \mathbf{0} & \text{in } \Omega \\ \boldsymbol{e} \times \boldsymbol{n} = \boldsymbol{E}_{\mathrm{i}} & \text{on } \Gamma_{\mathrm{i}} \\ \boldsymbol{e} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{\mathrm{c}} \\ (\mathbf{curl}\,\boldsymbol{e} \times \boldsymbol{n} - \mathrm{i}\kappa\,\boldsymbol{e}) \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{\mathrm{t}} \end{cases} \tag{1}$$

The unknown here is the electric field $\boldsymbol{e}$. The parameter κ, called wave number, is positive, equal to $2\pi f\,\sqrt{\mu\varepsilon}$, where f is the frequency of the wave, μ is the magnetic permeability of the medium and ε its electric permittivity. The data $\boldsymbol{E}_{\mathrm{i}}$ consist of a given wave, indeed Γ_{i} denotes the part of the boundary where the source is located. Γ_{c} represents a lateral boundary, and the conditions there correspond to a perfectly conducting material. Finally, Γ_{t} is an artificial boundary and the boundary conditions on it are transparent, of Silver–Müller type.

One of the early milestones in microwave engineering was the development of wave guides and other transmission lines for the low-loss transmission of microwave power. A wave guide restricts the three-dimensional 'free space' propagation of the electromagnetic wave to a leading direction. Usually the wave travels along the guide without greatly attenuating as it goes. Moreover, we can gently bend the guiding structure without losing contact with the wave, without generating reflections, and without incurring much additional loss.

There are a great many different wave guiding structures and different co-ordinate systems are associated with different wave guide cross section shapes. Cylindrical co-ordinates are used to describe circular cross-section waveguides, and coaxial cables. Rectangular Cartesian co-ordinates are preferred for rectangular waveguides. In the case of more exotic structures such as microstrip or coplanar waveguide, it is standard to use rectangular co-ordinates and solve approximately for the cross-sectional field distributions by using 'conformal mapping', a technique borrowed from complex variable theory.

Wave guides, often consisting of a single conductor, support transverse electric (TE) and/or transverse magnetic (TM) waves, characterized by the presence of, respectively, longitudinal magnetic or electric field components. The lateral surface of the wave guide is supposed to be an electric wall and this is mathematically modelled by the fact that the tangential component of the wave vanishes. Due to the need of a finite computational domain, the *a priori* infinite wave guide has to be 'truncated', thus transparent conditions are imposed on a boundary cross section. Finally, the other cross section represents the place where a source is located. Then, time harmonic Maxwell's equations model the propagation of electromagnetic waves in the computational domain (i.e. the interior of the truncated wave guide). The simulation of electromagnetic propagation in wave guides can be made more involved due to the presence of slots and apertures (to construct antennas in wave guides) as well as of different dielectric materials (doped wave guides to enhance the transmission of some specific signals).

Oliver Heaviside (1850–1925) in 1893 was the first to consider the possibility of propagation of electromagnetic waves inside a closed hollow tube [1]. In 1897, John William Strutt (1842–1919, known as Lord Rayleigh) mathematically proved that wave propagation in wave guides was possible for both circular and rectangular cross sections. Rayleigh also noted the infinite set of modes of Transverse Electric (TE) and Transverse Magnetic (TM) type that were possible and the existence of a cutoff frequency. No experimental verification was made at the time. Later in 1936, George C. Southworth (1890–1972) of the AT & T Company (Bell Telephone Labs) in New York and L.W. Barrow of the Massachusetts Institute of Technology separately presented their work on wave guides with experimental confirmation of propagation. Since then, many applications and studies have been done on the subject, see [2] and the references therein.

Problem (1) describes the propagation of TE electromagnetic waves in a conductor Ω filled with dielectric material of physical parameters ε and μ. We refer to [3, Chapter 9; 4, Chapter 5] for the derivation of this type of model. We first write its variational formulation and prove the equivalence of the system of partial differential equations with the variational problem. In a second step, we check the existence and uniqueness of the solution. Note that this is not obvious for all values of κ and not true for other types of boundary conditions, for instance if Γ_t is empty (see for instance [5, Chapter II, Section 3; 6] and the references therein). Fortunately, the boundary conditions on Γ_t allow for deriving the well-posedness of the problem. More general results on this subject are proved for instance in Reference [7]. However, we are interested in a rather specific geometry: The domain Ω that we consider is invariant by rotation around an axis, and both Γ_i and Γ_t are plane faces. By taking into account the axisymmetry of Ω, we also state some basic regularity properties of the solution, according to the ideas in References [8,9].

The main interest of working with an axisymmetric domain is that the three-dimensional solution admits a Fourier expansion with respect to the angular variable and that each Fourier coefficient is the solution of a two-dimensional problem set in the meridian domain. As explained in Reference [10], one of the difficulties for this dimension reduction is that the Cartesian measure is replaced by a weighted one, which is due to the use of cylindrical coordinates. Thus we introduce the corresponding weighted Sobolev spaces and write the variational formulation of the two-dimensional problems in these spaces. We derive their well-posedness and the regularity of the solution from the three-dimensional results.

The discretization we intend to work on relies on Fourier truncation, i.e., we only consider the approximation of a finite number of Fourier coefficients. So we conclude by establishing an estimate of the error issued from Fourier truncation. According to the ideas in Reference [10, Section II.4.b], this error only involves the regularity of the data, more precisely the maximal order of anisotropic Sobolev spaces to which these data belong. Moreover, in practical situations, the Fourier coefficients of the data cannot be computed in an exact way. So we propose to use a quadrature formula to compute an approximation of them and also prove an estimate for the resulting error.

The analysis of the finite element and spectral element discretizations of the two-dimensional problems on the meridian domain is the subject of the second and third parts of this work. By combining these results with the previous ones, we obtain an accurate approximation of the three-dimensional solution. The main interest of such an approach is that solving a few two-dimensional discrete problems is much less expensive than solving a three-dimensional one. In our opinion, the discretizations we propose by taking into account the specific geometry of the domain are very efficient.

The outline of the paper is as follows:

- In Section 2, we write the variational formulation of problem (1) and check its equivalence with the system of partial differential equations.
- Section 3 is devoted to the proof of the existence and uniqueness of a solution of the variational problem. We also state the regularity properties.
- In Section 4, we write the two-dimensional problems satisfied by each Fourier coefficient. Next we describe the weighted spaces and the corresponding variational formulation of these problems. We prove their well-posedness.
- In Section 5, we derive estimates for the error due to Fourier truncation and also for the error due to the approximation of the Fourier coefficients of the data.
- A technical proof is given in the appendix.

2. THE VARIATIONAL FORMULATION

Let $\boldsymbol{x} = (x, y, z)$ denote a set of Cartesian co-ordinates in $\mathbb{R}^3$ such that Ω is invariant by rotation around the axis $x = y = 0$, together with $\Gamma_{\rm i}$, $\Gamma_{\rm c}$ and $\Gamma_{\rm t}$. We make the further assumptions that:

- the domain Ω is contained in the strip $z_{\rm i} < z < z_{\rm t}$ for two real numbers $z_{\rm i}$ and $z_{\rm t}$, with $z_{\rm i} < z_{\rm t}$,
- $\Gamma_{\rm i}$ coincides with an open disk contained in the plane $z = z_{\rm i}$, with centre $(0, 0, z_{\rm i})$ and radius $R_{\rm i}$, and $\overline{\Gamma}_{\rm i}$ is equal to the intersection of $\overline{\Omega}$ with the plane $z = z_{\rm i}$,
- $\Gamma_{\rm t}$ coincides with an open disk contained in the plane $z = z_{\rm t}$, with centre $(0, 0, z_{\rm t})$ and radius $R_{\rm t}$, and $\overline{\Gamma}_{\rm t}$ is equal to the intersection of $\overline{\Omega}$ with the plane $z = z_{\rm t}$,
- $\Gamma_{\rm c}$ is open and generated by the rotation around the axis $x = y = 0$ of a broken line, made of a finite union of segments (only for simplicity, since all the results of this paper still hold when the segments are replaced by regular curves).

Since $\partial\Omega$ is the union of $\overline{\Gamma}_{\rm i}$, $\overline{\Gamma}_{\rm c}$ and $\overline{\Gamma}_{\rm t}$, the intersection of $\overline{\Gamma}_{\rm c}$ with $\overline{\Gamma}_{\rm i}$ and $\overline{\Gamma}_{\rm t}$ are circles contained in the planes $z = z_{\rm i}$ and $z = z_{\rm t}$, respectively. In order to take into account possible obstacles in the guide, we assume that $\Gamma_{\rm c}$ is the union of $L+1$ connected components $\Gamma_{{\rm c}\ell}$, $0 \leqslant \ell \leqslant L$, such that the intersection of each $\overline{\Gamma}_{{\rm c}\ell}$ with $\overline{\Gamma}_{\rm i}$ or $\overline{\Gamma}_{\rm t}$ is empty except for $\ell = 0$. Figure 1 illustrates this type of geometry. Even if most results in this section are still valid in more complex geometries, we are specifically interested in this one.

On the domain Ω, we introduce the space

$$H(\mathbf{curl}, \Omega) = \left\{ \boldsymbol{v} \in L^2(\Omega)^3;\, \mathbf{curl}\, \boldsymbol{v} \in L^2(\Omega)^3 \right\} \tag{2}$$

We recall from Reference [11, Chapter I, Theorems 2.10 and 2.11] that the space $\mathscr{C}^\infty(\overline{\Omega})^3$ of indefinitely differentiable functions in $\overline{\Omega}$ is dense in $H(\mathbf{curl}, \Omega)$ and also that the tangential trace operator: $\boldsymbol{v} \mapsto \boldsymbol{v} \times \boldsymbol{n}$ is continuous from $H(\mathbf{curl}, \Omega)$ into the dual space $H^{-1/2}(\partial\Omega)^3$ of $H^{1/2}(\partial\Omega)^3$. By obvious extension, for any open surface Γ contained in $\partial\Omega$, it is also continuous from $H(\mathbf{curl}, \Omega)$ into the dual space $(H^{1/2}_{00}(\Gamma)^3)'$ of $H^{1/2}_{00}(\Gamma)^3$ (we refer to Reference [12, Chapter 1, Theorem 11.7] for the definition of $H^{1/2}_{00}(\Gamma)$ and to Reference [13] for more

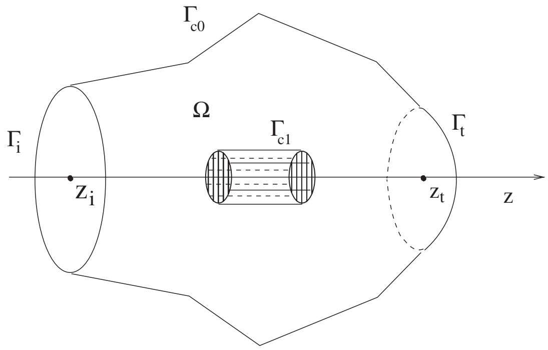

Figure 1. An example of domain Ω.

complete results). Moreover, the following Green's formula holds for any $\boldsymbol{v}$ in $H(\mathbf{curl},\Omega)$ and $\boldsymbol{w}$ in $H^1(\Omega)^3$:

$$\langle \boldsymbol{v}\times\boldsymbol{n},\boldsymbol{w}\rangle = \int_\Omega (\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{w} - \boldsymbol{w}\cdot\mathbf{curl}\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} \tag{3}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\partial\Omega)^3$ and $H^{1/2}(\partial\Omega)^3$. From now on, we denote by $H_0(\mathbf{curl},\Omega)$ the set of functions in $H(\mathbf{curl},\Omega)$ with a null tangential trace on $\partial\Omega$.

We are in a position to define the variational spaces. Let $X(\Omega)$ denote the space of functions $\boldsymbol{v}$ in $H(\mathbf{curl},\Omega)$ such that $(\boldsymbol{v}\times\boldsymbol{n})_{|\Gamma_\mathrm{t}}$ belongs to $L^2(\Gamma_\mathrm{t})^3$. We define its subspaces $X_0(\Omega)$ made of functions in $X(\Omega)$ such that

$$\boldsymbol{v}\times\boldsymbol{n}=\mathbf{0} \quad \text{on } \Gamma_\mathrm{i}\cup\Gamma_\mathrm{c} \tag{4}$$

and $X_{\boldsymbol{E}_\mathrm{i}}(\Omega)$ made of functions in $X(\Omega)$ such that

$$\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{E}_\mathrm{i} \quad \text{on } \Gamma_\mathrm{i} \quad \text{and} \quad \boldsymbol{v}\times\boldsymbol{n}=\mathbf{0} \quad \text{on } \Gamma_\mathrm{c} \tag{5}$$

These spaces are equipped with the norm

$$\|\boldsymbol{v}\|_{X(\Omega)} = (\|\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\boldsymbol{v}\times\boldsymbol{n}\|^2_{L^2(\Gamma_\mathrm{t})^3})^{1/2} \tag{6}$$

and it can be noted for instance that $X(\Omega)$ and $X_0(\Omega)$ are Hilbert spaces for the scalar product associated with this norm.

We consider the following variational problem:
Find $\boldsymbol{e}$ in $X_{\boldsymbol{E}_\mathrm{i}}(\Omega)$ such that

$$\forall \boldsymbol{v}\in X_0(\Omega), \quad \int_\Omega (\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} - \kappa^2\,\boldsymbol{e}\cdot\bar{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} + \mathrm{i}\kappa\int_{\Gamma_\mathrm{t}} (\boldsymbol{e}\times\boldsymbol{n})\cdot(\bar{\boldsymbol{v}}\times\boldsymbol{n})\,\mathrm{d}\boldsymbol{\tau} = 0 \tag{7}$$

where $\mathrm{d}\boldsymbol{\tau}$ denotes the Lebesgue measure associated with the tangential co-ordinates $\boldsymbol{\tau}=(x,y)$ on Γ_t. Indeed, in a first step, we are going to prove the equivalence of this problem with system (1). This requires a density result which is an extension of the main theorem in Reference [14]. Since its proof is rather technical, we write it in the appendix.

Lemma 2.1
The space $\mathscr{D}(\Omega \cup \Gamma_t)^3$ of infinitely differentiable vector-valued functions with a compact support in $\Omega \cup \Gamma_t$ is dense in $X_0(\Omega)$.

We are now in a position to check the equivalence property.

Proposition 2.2
A vector field $\boldsymbol{e}$ in $X(\Omega)$ is a solution of system (1) if and only if it is a solution of problem (7).

Proof
We prove successively the two parts of the assertion.

(1) Let $\boldsymbol{e}$ be a solution of problem (7). From definition (5) of $X_{\boldsymbol{E}_i}(\Omega)$, this vector field satisfies the second and third line in (1). Moreover, letting $\boldsymbol{v}$ run through $\mathscr{D}(\Omega)^3$ yields that it also satisfies the first line of (1) in the distribution sense. Finally, for $\boldsymbol{v}$ in $\mathscr{D}(\Omega \cup \Gamma_t)^3$, using formula (3) leads to

$$\int_{\Gamma_t} (\mathbf{curl}\, \boldsymbol{e} \times \boldsymbol{n}) \cdot \bar{\boldsymbol{v}} \,\mathrm{d}\tau + \mathrm{i}\kappa \int_{\Gamma_t} (\boldsymbol{e} \times \boldsymbol{n}) \cdot (\bar{\boldsymbol{v}} \times \boldsymbol{n}) \,\mathrm{d}\tau = 0$$

The antisymmetry properties of the exterior product thus leads to

$$\int_{\Gamma_t} ((\mathbf{curl}\, \boldsymbol{e} \times \boldsymbol{n} - \mathrm{i}\kappa\, \boldsymbol{e}) \times \boldsymbol{n}) \cdot \bar{\boldsymbol{v}} \,\mathrm{d}\tau = 0$$

whence the fourth line of (1).

(2) Conversely, let $\boldsymbol{e}$ be a function of $X(\Omega)$ which is a solution of system (1). The second and third lines in (1) imply that $\boldsymbol{e}$ belongs to $X_{\boldsymbol{E}_i}(\Omega)$. Moreover, the same integration by parts as previously gives, for any $\boldsymbol{v}$ in $\mathscr{D}(\Omega \cup \Gamma_t)^3$,

$$\int_{\Omega} (\mathbf{curl}\, \boldsymbol{e} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} - \kappa^2\, \boldsymbol{e} \cdot \bar{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} + \mathrm{i}\kappa \int_{\Gamma_t} (\boldsymbol{e} \times \boldsymbol{n}) \cdot (\bar{\boldsymbol{v}} \times \boldsymbol{n}) \,\mathrm{d}\tau = 0$$

So the density result of Lemma 2.1 yields Equation (7) in an obvious way. □

3. WELL-POSEDNESS OF THE VARIATIONAL PROBLEM

In order to prove the existence and uniqueness of the solution of problem (7), we first use a lifting of the boundary data. In the following lemma we recall a result due to Tartar [15, Theorem 2] in the case of a general three-dimensional domain with a Lipschitz-continuous boundary (see also References [16, Theorem 4.1, 17, Theorem 6.6] for a slightly different statement). Let $\boldsymbol{v}_i$ stand for the unit outward normal vector to Γ_i on $\partial\Gamma_i$ contained in the plane $z = z_i$.

Lemma 3.1
For any $\boldsymbol{E}_i = (E_{ix}, E_{iy}, 0)$ satisfying

$$(E_{ix}, E_{iy}) \in H^{-1/2}(\Gamma_i)^2, \quad \partial_x E_{i,x} + \partial_y E_{i,y} \in H^{-1/2}(\Gamma_i), \quad \boldsymbol{E}_i \cdot \boldsymbol{v}_i = 0 \text{ on } \partial\Gamma_i \tag{8}$$

there exists a function $\boldsymbol{e}_{\mathrm{i}}^{\dagger}$ in $X(\Omega)$ such that

$$\boldsymbol{e}_{\mathrm{i}}^{\dagger} \times \boldsymbol{n} = \boldsymbol{E}_{\mathrm{i}} \quad \text{on } \Gamma_{\mathrm{i}} \quad \text{and} \quad \boldsymbol{e}_{\mathrm{i}}^{\dagger} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{c}} \cup \Gamma_{\mathrm{t}} \tag{9}$$

Moreover the function $\boldsymbol{e}_{\mathrm{i}}^{\dagger}$ satisfies

$$\|\boldsymbol{e}_{\mathrm{i}}^{\dagger}\|_{X(\Omega)} \leqslant c\,(\|E_{\mathrm{i}x}\|_{H^{-1/2}(\Gamma_{\mathrm{i}})} + \|E_{\mathrm{i}y}\|_{H^{-1/2}(\Gamma_{\mathrm{i}})} + \|\partial_x E_{\mathrm{i}x} + \partial_y E_{\mathrm{i}y}\|_{H^{-1/2}(\Gamma_{\mathrm{i}})}) \tag{10}$$

From now on, we suppose that the function $\boldsymbol{E}_{\mathrm{i}}$ satisfies the assumptions of Lemma 3.1 and we denote by $\|\boldsymbol{E}_{\mathrm{i}}\|_{W(\Gamma_{\mathrm{i}})}$ the norm in the right-hand side of (10). We note that, when solving the problem
Find p in $H_0^1(\Omega)$ such that

$$\forall q \in H_0^1(\Omega), \quad \int_{\Omega} \mathbf{grad}\, p \cdot \mathbf{grad}\, q \,\mathrm{d}\boldsymbol{x} = \int_{\Omega} \boldsymbol{e}_{\mathrm{i}}^{\dagger} \cdot \mathbf{grad}\, q \,\mathrm{d}\boldsymbol{x} \tag{11}$$

the new function $\boldsymbol{e}_{\mathrm{i}}^{\dagger} - \mathbf{grad}\, p$ still satisfies (9) and (10) and is divergence-free on Ω. Finally, in order to take into account the possible non-connexity of Γ_{c} and following Reference [18, Proposition 3.18], we introduce the solutions q_ℓ, $1 \leqslant \ell \leqslant L$, of the problems

$$\begin{cases} -\Delta q_\ell = 0 & \text{in } \Omega \\ q_\ell = 0 & \text{on } \Gamma_{\mathrm{i}} \cup \Gamma_{\mathrm{t}} \cup \Gamma_{\mathrm{c}0} \\ q_\ell = \text{constant} & \text{on } \Gamma_{\mathrm{c}\ell'}, \quad 1 \leqslant \ell' \leqslant L \\ \displaystyle\int_{\Gamma_{\mathrm{c}\ell'}} \partial_n q_\ell(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau} = \delta_{\ell\ell'}, \quad 1 \leqslant \ell' \leqslant L & \end{cases} \tag{12}$$

where by extension the duality pairings on the $\Gamma_{\mathrm{c}\ell}$ are denoted by an integral. The function $\boldsymbol{e}_{\mathrm{i}}^{*}$ defined by

$$\boldsymbol{e}_{\mathrm{i}}^{*} = \boldsymbol{e}_{\mathrm{i}}^{\dagger} - \mathbf{grad}\, p - \sum_{\ell=1}^{L} \left(\int_{\Gamma_{\mathrm{c}\ell}} (\boldsymbol{e}_{\mathrm{i}}^{\dagger} \cdot \boldsymbol{n} - \partial_n p)(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau} \right) \mathbf{grad}\, q_\ell$$

still satisfies (9) and (10) and moreover is such that

$$\operatorname{div} \boldsymbol{e}_{\mathrm{i}}^{*} = 0 \quad \text{in } \Omega \quad \text{and} \quad \int_{\Gamma_{\mathrm{c}\ell}} (\boldsymbol{e}_{\mathrm{i}}^{*} \cdot \boldsymbol{n})(\boldsymbol{\tau})\,\mathrm{d}\boldsymbol{\tau} = 0, \quad 1 \leqslant \ell \leqslant L \tag{13}$$

We set: $\boldsymbol{e}_0 = \boldsymbol{e} - \boldsymbol{e}_{\mathrm{i}}^{*}$ and we observe that $\boldsymbol{e}$ is a solution of problem (7) if and only if $\boldsymbol{e}_0$ is a solution
Find $\boldsymbol{e}_0$ in $X_0(\Omega)$ such that

$$\forall \boldsymbol{v} \in X_0(\Omega),$$
$$\int_{\Omega} (\mathbf{curl}\, \boldsymbol{e}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} - \kappa^2\, \boldsymbol{e}_0 \cdot \overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} + \mathrm{i}\kappa \int_{\Gamma_{\mathrm{t}}} (\boldsymbol{e}_0 \times \boldsymbol{n}) \cdot (\overline{\boldsymbol{v}} \times \boldsymbol{n})\,\mathrm{d}\boldsymbol{\tau} = \langle \boldsymbol{f}_{\mathrm{i}}, \boldsymbol{v} \rangle \tag{14}$$

where $\langle\cdot,\cdot\rangle$ here denotes the duality pairing between $X_0(\Omega)'$ and $X_0(\Omega)$, while $\boldsymbol{f}_{\rm i}$ is defined by

$$\forall \boldsymbol{v}\in X_0(\Omega),\quad \langle \boldsymbol{f}_{\rm i},\boldsymbol{v}\rangle = -\int_\Omega (\mathbf{curl}\, \boldsymbol{e}_{\rm i}^*\cdot \mathbf{curl}\,\bar{\boldsymbol{v}} - \kappa^2\, \boldsymbol{e}_{\rm i}^*\cdot\bar{\boldsymbol{v}})\,{\rm d}\boldsymbol{x} \tag{15}$$

It follows from (10) that

$$\|\boldsymbol{f}_{\rm i}\|_{X_0(\Omega)'} \leqslant c\,\|\boldsymbol{E}_{\rm i}\|_{W(\Gamma_{\rm i})} \tag{16}$$

We now check the uniqueness of the solution.

Lemma 3.2
For any data $\boldsymbol{E}_{\rm i}$ in $H^{-1/2}(\Gamma_{\rm i})^3$ satisfying (8), problem (7) has at most a solution $\boldsymbol{e}$ in $X_{\boldsymbol{E}_{\rm i}}(\Omega)$.

Proof
Since problem (7) is linear, it suffices to check that its only solution with data $\boldsymbol{E}_{\rm i}$ equal to zero is zero. When $\boldsymbol{E}_{\rm i}$ is zero, the solution $\boldsymbol{e}$ of problem (7) belongs to $X_0(\Omega)$ and satisfies

$$\forall \boldsymbol{v}\in X_0(\Omega),\quad \int_\Omega (\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\bar{\boldsymbol{v}} - \kappa^2\,\boldsymbol{e}\cdot\bar{\boldsymbol{v}})\,{\rm d}\boldsymbol{x} + {\rm i}\kappa\int_{\Gamma_{\rm t}} (\boldsymbol{e}\times\boldsymbol{n})\cdot(\bar{\boldsymbol{v}}\times\boldsymbol{n})\,{\rm d}\boldsymbol{\tau} = 0$$

Choosing $\boldsymbol{v}$ equal to $\boldsymbol{e}$ in the previous line and taking the imaginary part of the corresponding equation yield that $\boldsymbol{e}\times\boldsymbol{n}$ is zero on $\Gamma_{\rm t}$. So $\boldsymbol{e}$ is now a solution of the system

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{e} - \kappa^2\,\boldsymbol{e} = \mathbf{0} & \text{in } \Omega \\ \boldsymbol{e}\times\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{\rm i}\cup\Gamma_{\rm c} \\ \mathbf{curl}\,\boldsymbol{e}\times\boldsymbol{n} = \boldsymbol{e}\times\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma_{\rm t} \end{cases}$$

Thus, the fact that $\boldsymbol{e}$ is zero follows from Holmgren's theorem, see for instance References [7, Chapter 2, Theorem 13; 19, Theorem 5.3.1; 20, Section 21]. □

In a further step, we introduce the kernel

$$V(\Omega) = \left\{\boldsymbol{v}\in X_0(\Omega);\ {\rm div}\,\boldsymbol{v} = 0\ \text{ in } \Omega \text{ and } \int_{\Gamma_{{\rm c}\ell}} (\boldsymbol{v}\cdot\boldsymbol{n})(\boldsymbol{\tau})\,{\rm d}\boldsymbol{\tau} = 0,\ 1\leqslant \ell\leqslant L\right\} \tag{17}$$

There also, it can be checked that $V(\Omega)$ provided with the inner product associated with the norm of $X(\Omega)$ is a Hilbert space. Moreover, we have the following result.

Lemma 3.3
Any solution $\boldsymbol{e}_0$ of problem (14) belongs to $V(\Omega)$.

Proof
Taking $\boldsymbol{v}$ equal to $\mathbf{grad}\, p$ in (14) for any p in $\mathscr{D}(\Omega)$ yields

$$\int_\Omega (\boldsymbol{e}_0 + \boldsymbol{e}_{\rm i}^*)\cdot\mathbf{grad}\, p\,{\rm d}\boldsymbol{x} = 0$$

so that $\mathbf{div}\,\boldsymbol{e}_0$ is zero thanks to (13). The previous line is still valid when p belongs to $H^1(\Omega)$ and vanishes on $\partial\Omega$ except on one of the $\Gamma_{{\rm c}\ell}$ where it is equal to 1. Using once more (13) yields that $\boldsymbol{e}_0$ satisfies the second part of the definition of $V(\Omega)$ for $1 \leqslant \ell \leqslant L$, hence belongs to $V(\Omega)$.

The same arguments as previously yield that any function of $X_0(\Omega)$ is the sum of a function in $V(\Omega)$, of a $\mathbf{grad}\, p$ for p in $H_0^1(\Omega)$, and of a linear combination of the gradients of the functions q_ℓ introduced in (12). So, $\boldsymbol{e}_0$ is a solution of problem (14) if and only if it is a solution of

Find $\boldsymbol{e}_0$ in $V(\Omega)$ such that

$$\forall \boldsymbol{v} \in V(\Omega), \quad \int_\Omega (\mathbf{curl}\, \boldsymbol{e}_0 \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} - \kappa^2 \boldsymbol{e}_0 \cdot \bar{\boldsymbol{v}})\, \mathrm{d}\boldsymbol{x} + \mathrm{i}\kappa \int_{\Gamma_\mathrm{t}} (\boldsymbol{e}_0 \times \boldsymbol{n}) \cdot (\bar{\boldsymbol{v}} \times \boldsymbol{n})\, \mathrm{d}\tau = \langle \boldsymbol{f}_\mathrm{i}, \boldsymbol{v} \rangle \tag{18}$$

We are now interested in the well-posedness of problem (18).

We define the sesquilinear form

$$a_\kappa(\boldsymbol{e}, \boldsymbol{v}) = \int_\Omega (\mathbf{curl}\, \boldsymbol{e} \cdot \mathbf{curl}\, \bar{\boldsymbol{v}} - \kappa^2\, \boldsymbol{e} \cdot \bar{\boldsymbol{v}})\, \mathrm{d}\boldsymbol{x} + \mathrm{i}\kappa \int_{\Gamma_\mathrm{t}} (\boldsymbol{e} \times \boldsymbol{n}) \cdot (\bar{\boldsymbol{v}} \times \boldsymbol{n})\, \mathrm{d}\tau \tag{19}$$

The following property can be derived from the Peetre–Tartar lemma.

Lemma 3.4

There exists a positive constant γ such that the following inequality holds:

$$\forall \boldsymbol{v} \in V(\Omega), \quad \|\boldsymbol{v}\|_{L^2(\Omega)^3} \leqslant \gamma\, (\|\mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\boldsymbol{v} \times \boldsymbol{n}\|^2_{L^2(\Gamma_\mathrm{t})^3})^{1/2} \tag{20}$$

Proof

The norm $\|\cdot\|_{X(\Omega)}$ is equivalent to the quantity

$$\|\boldsymbol{v}\|_{L^2(\Omega)^3} + (\|\mathbf{curl}\, \boldsymbol{v}\|^2_{L^2(\Omega)^3} + \|\boldsymbol{v} \times \boldsymbol{n}\|^2_{L^2(\Gamma_\mathrm{t})^3})^{1/2}$$

Moreover, we have the following properties:

(1) Let $\boldsymbol{v}$ be a function in $V(\Omega)$ such that $\mathbf{curl}\, \boldsymbol{v}$ is zero on Ω and $\boldsymbol{v} \times \boldsymbol{n}$ is zero on Γ_t. Since $\boldsymbol{v}$ is divergence-free and has a zero flux through each $\Gamma_{\mathrm{c}\ell}$, it follows from Reference [18, Proposition 3.18] that $\boldsymbol{v}$ is zero.

(2) It is proven in Reference [21] that $V(\Omega)$ is imbedded into $H^{1/2}(\Omega)^3$. So, the imbedding of $V(\Omega)$ into $L^2(\Omega)^3$ is compact.

Combining these two properties allows for applying the Peetre–Tartar lemma, see Reference [11, Chapter I, Theorem 2.1], which leads to the desired result.

Lemma 3.4 allows for proving the following inf–sup condition when κ is small enough.

Lemma 3.5

There exist a real number $\kappa_0 > 0$ only depending on Ω and a constant $\beta > 0$ such that the following inf–sup condition holds:

$$\forall \boldsymbol{e} \in V(\Omega), \quad \sup_{\boldsymbol{v} \in V(\Omega)} \frac{\mathrm{Re}\, a_{\kappa_0}(\boldsymbol{e}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{X(\Omega)}} \geqslant \beta\, \|\boldsymbol{e}\|_{X(\Omega)} \tag{21}$$

Proof
For each $\boldsymbol{e}$ in $V(\Omega)$, we set: $\boldsymbol{v}=\boldsymbol{e}-\mathrm{i}\lambda\,\boldsymbol{e}$, for an appropriate positive real constant λ. This gives

$$\operatorname{Re} a_{\kappa_0}(\boldsymbol{e},\boldsymbol{v}) = \|\mathbf{curl}\,\boldsymbol{e}\|^2_{L^2(\Omega)^3} - \kappa_0^2\,\|\boldsymbol{e}\|^2_{L^2(\Omega)^3} + \lambda\,\kappa_0\,\|\boldsymbol{e}\times\boldsymbol{n}\|^2_{L^2(\Gamma_\mathrm{t})^3}$$

Using Lemma 3.4 and taking κ_0 such that $\kappa_0^2 = 1/2\gamma$ gives

$$\operatorname{Re} a_{\kappa_0}(\boldsymbol{e},\boldsymbol{v}) \geqslant \tfrac{1}{2}\|\mathbf{curl}\,\boldsymbol{e}\|^2_{L^2(\Omega)^3} + (\lambda\,\kappa_0 - \tfrac{1}{2})\,\|\boldsymbol{e}\times\boldsymbol{n}\|^2_{L^2(\Gamma_\mathrm{t})^3}$$

We now choose λ equal to $1/\kappa_0$ so that using once more Lemma 3.4 gives

$$\operatorname{Re} a_{\kappa_0}(\boldsymbol{e},\boldsymbol{v}) \geqslant \frac{1}{4}\,\min\left\{1,\frac{1}{\gamma}\right\}\,\|\boldsymbol{e}\|^2_{X(\Omega)}$$

Combining this with the inequality

$$\|\boldsymbol{v}\|_{X(\Omega)} \leqslant (1+\lambda^2)^{1/2}\,\|\boldsymbol{e}\|_{X(\Omega)}$$

leads to the desired inf–sup condition. □

Note that condition (21) holds for any κ, $0<\kappa\leqslant\kappa_0$, but that even for small values of κ, its proof requires Lemma 3.4. Due to the formula

$$\forall \boldsymbol{e}\in X(\Omega),\quad \forall \boldsymbol{v}\in X(\Omega),\quad a_\kappa(\boldsymbol{e},\boldsymbol{v}) = \overline{a_{-\kappa}(\boldsymbol{v},\boldsymbol{e})}$$

the proof of the next inf–sup condition relies on exactly the same arguments as previously (with λ replaced by $-\lambda$).

Lemma 3.6
The following inf–sup condition holds for the real number $\kappa_0>0$ and the constant $\beta>0$ introduced in Lemma 3.5

$$\forall \boldsymbol{v}\in V(\Omega),\qquad \sup_{\boldsymbol{e}\in V(\Omega)} \frac{\operatorname{Re} a_{\kappa_0}(\boldsymbol{e},\boldsymbol{v})}{\|\boldsymbol{e}\|_{X(\Omega)}} \geqslant \beta\,\|\boldsymbol{v}\|_{X(\Omega)} \tag{22}$$

Corollary 3.7
For the real number κ_0 introduced in Lemma 3.5 and for any data $\boldsymbol{g}$ in the dual space of $V(\Omega)$, the problem
Find $\boldsymbol{e}$ in $V(\Omega)$ such that

$$\forall \boldsymbol{v}\in V(\Omega),\qquad \int_\Omega (\mathbf{curl}\,\boldsymbol{e}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}} - \kappa_0^2\,\boldsymbol{e}\cdot\overline{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x} + \mathrm{i}\kappa_0\int_{\Gamma_\mathrm{t}}(\boldsymbol{e}\times\boldsymbol{n})\cdot(\overline{\boldsymbol{v}}\times\boldsymbol{n})\,\mathrm{d}\boldsymbol{\tau} = \langle \boldsymbol{g},\boldsymbol{v}\rangle \tag{23}$$

has a unique solution.

Proof
It follows from the inf–sup conditions (21) and (22) that there exists a unique $\boldsymbol{e}$ in $V(\Omega)$ such that

$$\forall \boldsymbol{v}\in V(\Omega),\quad \operatorname{Re} a_{\kappa_0}(\boldsymbol{e},\boldsymbol{v}) = \operatorname{Re}\,\langle \boldsymbol{g},\boldsymbol{v}\rangle$$

We have for any $\boldsymbol{e}$ and $\boldsymbol{v}$ in $X(\Omega)$,

$$\operatorname{Im} a_\kappa(\boldsymbol{e},\boldsymbol{v}) = \operatorname{Re} a_\kappa(\boldsymbol{e},\mathrm{i}\boldsymbol{v}) \tag{24}$$

So the function $\boldsymbol{e}$ also satisfies, for all $\boldsymbol{v}$ in $V(\Omega)$,

$$a_{\kappa_0}(\boldsymbol{e},\boldsymbol{v}) = \operatorname{Re} a_{\kappa_0}(\boldsymbol{e},\boldsymbol{v}) + \mathrm{i}\operatorname{Im} a_{\kappa_0}(\boldsymbol{e},\boldsymbol{v}) = \operatorname{Re}\langle \boldsymbol{g},\boldsymbol{v}\rangle + \mathrm{i}\operatorname{Re}\langle \boldsymbol{g}, i\boldsymbol{v}\rangle = \langle \boldsymbol{g},\boldsymbol{v}\rangle$$

hence is a solution of problem (23). Its uniqueness follows from Lemma 3.2.

Let T_{κ_0} denote the operator from the dual space $V(\Omega)'$ of $V(\Omega)$ into $V(\Omega)$ which associates with the data $\boldsymbol{g}$ the unique solution of problem (23). As already recalled, it follows from Reference [21] that the space $V(\Omega)$ is included in $H^{1/2}(\Omega)^3$, hence that the embedding of $V(\Omega)$ into $L^2(\Omega)^3$ is compact. So the operator T_{κ_0} is compact from $V'(\Omega)$ into $L^2(\Omega)^3$. We are now in a position to state the main result of this section.

Theorem 3.8
For any data $\boldsymbol{E}_{\mathrm{i}}$ satisfying (8), problem (7) admits a unique solution $\boldsymbol{e}$ in $X_{\boldsymbol{E}_{\mathrm{i}}}(\Omega)$. Moreover, this solution satisfies

$$\|\boldsymbol{e}\|_{X(\Omega)} \leqslant c\,\|\boldsymbol{E}_{\mathrm{i}}\|_{W(\Gamma_{\mathrm{i}})} \tag{25}$$

Proof
Since the uniqueness of the solution follows from Lemma 3.2, it is sufficient to establish the existence of a solution, or equivalently the existence of a solution of problem (18). This problem can be written as

$$\boldsymbol{e}_0 + T_{\kappa_0} A \boldsymbol{e}_0 = T_{\kappa_0}\boldsymbol{f}_{\mathrm{i}} \tag{26}$$

where the operator A is defined from $V(\Omega)$ into $V(\Omega)'$ by

$$\forall \boldsymbol{v}\in V(\Omega),\quad \langle A\boldsymbol{e},\boldsymbol{v}\rangle = -(\kappa^2-\kappa_0^2)\int_\Omega \boldsymbol{e}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x} + \mathrm{i}(\kappa-\kappa_0)\int_{\Gamma_{\mathrm{t}}}(\boldsymbol{e}\times\boldsymbol{n})\cdot(\bar{\boldsymbol{v}}\times\boldsymbol{n})\,\mathrm{d}\boldsymbol{\tau}$$

The operator $\mathrm{Id}+T_{\kappa_0}A$ is defined from $V(\Omega)$ into $L^2(\Omega)^3$ and it follows from Lemma 3.2 that its kernel is reduced to $\{\boldsymbol{0}\}$. Moreover the operator $T_{\kappa_0}A$ is compact from $V(\Omega)$ into $L^2(\Omega)^3$. So the operator $\mathrm{Id}+T_{\kappa_0}A$ admits an inverse from $L^2(\Omega)^3$ into $V(\Omega)$. Thus problem (26) admits a unique solution $\boldsymbol{e}_0$, which satisfies

$$\|\boldsymbol{e}_0\|_{L^2(\Omega)^3} \leqslant c\,\|\boldsymbol{f}_{\mathrm{i}}\|_{V(\Omega)'}$$

where the constant c only depends on the norms of $(\mathrm{Id}+T_{\kappa_0}A)^{-1}$ and T_{κ_0}. It then follows from problem (18) that

$$\|\boldsymbol{e}_0\|_{X(\Omega)} \leqslant c\,\|\boldsymbol{f}_{\mathrm{i}}\|_{V(\Omega)'} \tag{27}$$

The function $\boldsymbol{e}=\boldsymbol{e}_0+\boldsymbol{e}_{\mathrm{i}}^*$ is a solution of problem (7), and estimate (25) is derived by combining (10) applied to $\boldsymbol{e}_{\mathrm{i}}^*$, (16) and (27).

We conclude this section with some regularity properties of the solution $\boldsymbol{e}$.

Proposition 3.9
For any data $\boldsymbol{E}_{\mathrm{i}}$ in $L^2(\Gamma_{\mathrm{i}})^2$ satisfying (8), the solution $\boldsymbol{e}$ of problem (7) belongs to $H^{1/2}(\Omega)^3$. If moreover the data $\boldsymbol{E}_{\mathrm{i}}$ are such that $\partial_x E_{\mathrm{i}x}+\partial_y E_{\mathrm{i}y}$ belongs to $L^2(\Omega)$, then $\mathbf{curl}\,\boldsymbol{e}$ also belongs to $H^{1/2}(\Omega)^3$.

Proof
It is performed in two steps.

(1) The solution $\boldsymbol{e}$ belongs to $H(\mathbf{curl},\Omega)$, is divergence-free on Ω and it results from the assumption on $\boldsymbol{E}_{\mathrm{i}}$ that $\boldsymbol{e}\times\boldsymbol{n}$ belongs to $L^2(\partial\Omega)^3$. So it follows from Reference [21] that it belongs to $H^{1/2}(\Omega)^3$.

(2) On Γ_{i}, $\mathbf{curl}\,\boldsymbol{e}\cdot\boldsymbol{n}$ is equal to $\partial_x e_y-\partial_y e_x$, hence to $-(\partial_x E_{\mathrm{i}x}+\partial_y E_{\mathrm{i}y})$. If this last quantity belongs to $L^2(\Gamma_{\mathrm{i}})$, $\mathbf{curl}\,\boldsymbol{e}$ satisfies

$$\mathbf{curl}\,\boldsymbol{e}\cdot\boldsymbol{n}\in L^2(\Gamma_{\mathrm{i}}),\quad \mathbf{curl}\,\boldsymbol{e}\cdot\boldsymbol{n}=0 \text{ on } \Gamma_{\mathrm{c}},\quad \mathbf{curl}\,\boldsymbol{e}\times\boldsymbol{n}\in L^2(\Gamma_{\mathrm{t}})$$

where the last condition follows from the fourth line in (1). The function $\mathbf{curl}\,\boldsymbol{e}$ also belongs to $H(\mathbf{curl},\Omega)$ thanks to the first line in (1) and is divergence-free on Ω. Thus it is proven in References [8,9] that $\mathbf{curl}\,\boldsymbol{e}$ is the sum of a function in $H^{1/2}(\Omega)^3$ and of the gradient of a solution p of the Laplace equation with Neumann boundary conditions on $\Gamma_{\mathrm{i}}\cup\Gamma_{\mathrm{c}}$ and Dirichlet boundary conditions on Γ_{t}. The right-hand side of the Laplace equation, the Neumann data and the Dirichlet data belong, respectively, to $H^{-1/2}(\Omega)$, to $L^2(\Gamma_{\mathrm{i}}\cup\Gamma_{\mathrm{c}})$ and to $H^1(\Gamma_{\mathrm{t}})$. Then it follows from an extension of Reference [10, Section II.4.c] combined with [22, Chapter 5] that the function p belongs to $H^{s+1}(\Omega)$ for all $s\leqslant\frac{1}{2}$ such that $s<\pi/2\alpha$, where α denotes the angle between Γ_{c} and Γ_{t}. Since the domain Ω is contained in the half-space $z<z_{\mathrm{t}}$ and Γ_{t} is the intersection of the $\overline{\Omega}$ with the plane $z=z_{\mathrm{t}}$, the angle α is $<\pi$, whence the desired result. □

4. FOURIER EXPANSION, AND THE TWO-DIMENSIONAL PROBLEMS

We now write system (1) in cylindrical co-ordinates. Indeed, denoting by (r,θ,z) these co-ordinates defined by

$$x=r\cos\theta,\quad y=r\sin\theta\quad\text{with}\quad r\geqslant 0,\quad -\pi<\theta\leqslant\pi$$

we observe that Ω is the set of points with co-ordinates (r,θ,z) where θ runs through $]-\pi,\pi]$ and (r,z) runs through the meridian domain ω. From the previous assumptions on Ω, ω is a polygon in the half-plane $\mathbb{R}_+\times\mathbb{R}$. Denoting by γ_{i}, γ_{c} and γ_{t} the parts of the boundary $\partial\omega$ such that

$$\Gamma_{\mathrm{i}}=\{(r,\theta,z);\ (r,z)\in\gamma_{\mathrm{i}}\text{ and }-\pi<\theta\leqslant\pi\}$$

$$\Gamma_{\mathrm{c}}=\{(r,\theta,z);\ (r,z)\in\gamma_{\mathrm{c}}\text{ and }-\pi<\theta\leqslant\pi\}$$

$$\Gamma_{\mathrm{t}}=\{(r,\theta,z);\ (r,z)\in\gamma_{\mathrm{t}}\text{ and }-\pi<\theta\leqslant\pi\}$$

we see that γ_{i} and γ_{t} coincide with the segments $[0,R_{\mathrm{i}}[\times\{z_{\mathrm{i}}\}$ and $[0,R_{\mathrm{t}}[\times\{z_{\mathrm{t}}\}$. Finally we denote by γ_0 the interior of the intersection of $\partial\omega$ with the axis $r=0$. It coincides either with the whole interval $\{0\}\times]z_{\mathrm{i}},z_{\mathrm{t}}[$ or with this interval minus some possible open intervals delimited by the intersection of some $\Gamma_{\mathrm{c}\ell}$ with the rotation axis. This notation is illustrated in Figure 2.

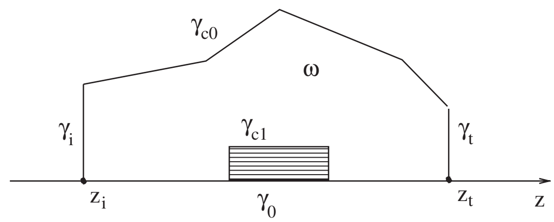

Figure 2. The meridian domain ω.

From now on, the radial, angular and axial components of a vector field $\boldsymbol{v}$ are denoted by v_r, v_θ and v_z. We also recall that the **curl** of a vector field $\boldsymbol{v}$ is given by

$$(\mathbf{curl}\,\boldsymbol{v})_r = r^{-1}\partial_\theta v_z - \partial_z v_\theta, \quad (\mathbf{curl}\,\boldsymbol{v})_\theta = \partial_z v_r - \partial_r v_z,$$
$$(\mathbf{curl}\,\boldsymbol{v})_z = \partial_r v_\theta + r^{-1}v_\theta - r^{-1}\partial_\theta v_r \tag{28}$$

Since the coefficients in this formula are independent of θ, for any solution $\boldsymbol{e}$ of problem (1) and for any η in $]0,\pi[$, the function: $(r,\theta,z)\mapsto \boldsymbol{e}(r,\theta+\eta,z)$ is a solution of problem (1) with data $(E_{\mathrm{i}r}(r,\theta+\eta), E_{\mathrm{i}\theta}(r,\theta+\eta))$. So the next result is an immediate consequence of Lemma 3.2.

Proposition 4.1
If the data $(E_{\mathrm{i}r}, E_{\mathrm{i}\theta})$ are independent of θ, so is the solution $\boldsymbol{e}=(e_r,e_\theta,e_z)$ of problem (7).

We now write the Fourier expansion of the data and the solution of system (1) with respect to the angular variable θ and the two-dimensional partial differential equation satisfied by each Fourier coefficient of the solution. Next, we introduce the weighted Sobolev spaces on ω associated with the measure $r\,\mathrm{d}r\,\mathrm{d}z$. Finally, we write the variational formulation of the previous equations and recall their well-posedness.

4.1. Fourier expansion

Let us write the Fourier expansion of the data $\boldsymbol{E}_\mathrm{i}$

$$\boldsymbol{E}_\mathrm{i}(r,\theta) = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \boldsymbol{E}_\mathrm{i}^k(r)\mathrm{e}^{\mathrm{i}k\theta}, \quad \text{with} \quad \boldsymbol{E}_\mathrm{i}^k(r) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \boldsymbol{E}_\mathrm{i}(r,\theta)\mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\theta \tag{29}$$

and also of the solution $\boldsymbol{e}$

$$\boldsymbol{e}(r,\theta,z) = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}} \boldsymbol{e}^k(r,z)\mathrm{e}^{\mathrm{i}k\theta}, \quad \text{with} \quad \boldsymbol{e}^k(r,z) = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} \boldsymbol{e}(r,\theta,z)\mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\theta \tag{30}$$

It is readily checked from (28) that the **curl** of any vector field $\boldsymbol{v}(r,z)\mathrm{e}^{\mathrm{i}k\theta}$ is equal to $(\mathbf{curl}_k\boldsymbol{v})(r,z)\mathrm{e}^{\mathrm{i}k\theta}$, where the operator $\mathbf{curl}_k$ is given by

$$(\mathbf{curl}_k\,\boldsymbol{v})_r = \mathrm{i}kr^{-1}v_z - \partial_z v_\theta, \quad (\mathbf{curl}_k\,\boldsymbol{v})_\theta = \partial_z v_r - \partial_r v_z$$
$$(\mathbf{curl}_k\,\boldsymbol{v})_z = \partial_r v_\theta + r^{-1}v_\theta - \mathrm{i}kr^{-1}v_r \tag{31}$$

So the first line of problem (1) is equivalent to

$$\mathbf{curl}_k\,\mathbf{curl}_k\,\boldsymbol{e}^k - \kappa^2\boldsymbol{e}^k = \mathbf{0}, \quad k \in \mathbb{Z}$$

On the other hand, the angular component n_θ of the unit outward normal vector $\boldsymbol{n}$ is zero on $\partial\Omega$ and the vector $\tilde{\boldsymbol{n}}$ with radial component n_r and axial component n_z is the unit outward normal vector to ω on $\overline{\gamma}_{\mathrm{i}} \cup \overline{\gamma}_{\mathrm{c}} \cup \overline{\gamma}_{\mathrm{t}}$. For any vector field $\boldsymbol{v}$, we agree to denote by $\boldsymbol{v} \times \tilde{\boldsymbol{n}}$ the vector with components

$$(\boldsymbol{v} \times \tilde{\boldsymbol{n}})_r = v_\theta n_z, \quad (\boldsymbol{v} \times \tilde{\boldsymbol{n}})_\theta = v_z n_r - v_r n_z, \quad (\boldsymbol{v} \times \tilde{\boldsymbol{n}})_z = -v_\theta n_r \tag{32}$$

The boundary conditions satisfied by each $\boldsymbol{e}^k$ are then obviously derived from the last three lines of problem (1). Note moreover that, on γ_{i}, n_r is equal to 0 and n_z to -1.

It follows from all this that problem (1) is equivalent to the following system, where k runs through $\mathbb{Z}$:

$$\begin{cases} \mathbf{curl}_k\,\mathbf{curl}_k\,\boldsymbol{e}^k - \kappa^2\boldsymbol{e}^k = \mathbf{0} & \text{in } \omega \\ (e_r^k, e_\theta^k) = (E_{\mathrm{i}\theta}^k, -E_{\mathrm{i}r}^k) & \text{on } \gamma_{\mathrm{i}} \\ \boldsymbol{e}^k \times \tilde{\boldsymbol{n}} = \mathbf{0} & \text{on } \gamma_{\mathrm{c}} \\ (\mathbf{curl}_k\,\boldsymbol{e}^k \times \tilde{\boldsymbol{n}} - \mathrm{i}\kappa\boldsymbol{e}^k) \times \tilde{\boldsymbol{n}} = \mathbf{0} & \text{on } \gamma_{\mathrm{t}} \end{cases} \tag{33}$$

As standard for axisymmetric geometries, it can be noted that there is no boundary condition on γ_0.

4.2. The weighted Sobolev spaces

The variational formulation of problems set in an axisymmetric domain relies on the basic spaces, see Reference [10, Section II.1]

$$\begin{aligned} L_1^2(\omega) &= \left\{ v : \omega \to \mathbb{C} \text{ measurable;} \quad \int_\omega |v(r,z)|^2 r\,\mathrm{d}r\,\mathrm{d}z < +\infty \right\} \\ L_{-1}^2(\omega) &= \left\{ v : \omega \to \mathbb{C} \text{ measurable;} \quad \int_\omega |v(r,z)|^2 r^{-1}\,\mathrm{d}r\,\mathrm{d}z < +\infty \right\} \end{aligned} \tag{34}$$

We define the complete scale of Sobolev spaces $H_1^s(\omega)$:

- When s is an integer, $H_1^s(\omega)$ is the space of functions in $L_1^2(\omega)$ such that all their partial derivatives of order $\leqslant s$ belong to $L_1^2(\omega)$
- When s is not an integer, $H_1^s(\omega)$ is defined by Hilbertian interpolation between $H_1^{[s]+1}(\omega)$ and $H_1^{[s]}(\omega)$, where $[s]$ stands for the integral part of s.

We also introduce the space of 'flat' functions

$$V_1^1(\omega) = H_1^1(\omega) \cap L_{-1}^2(\omega) \tag{35}$$

All these spaces are provided with the norms which result from their definitions. We refer to Reference [10, Section II.1] for their main properties, such as the density of smooth functions or the existence of traces.

For each k in $\mathbb{Z}$, the space $H(\mathbf{curl}_k, \omega)$ is now defined as the domain of the operator $\mathbf{curl}_k$ in $L^2_1(\omega)^3$, more precisely

$$H(\mathbf{curl}_k, \omega) = \{\boldsymbol{v} \in L^2_1(\omega)^3; \mathbf{curl}_k\, \boldsymbol{v} \in L^2_1(\omega)^3\} \tag{36}$$

We now introduce the space

$$L^2_1(\gamma_{\mathrm{t}}) = \left\{ v : \gamma_{\mathrm{t}} \to \mathbb{C} \text{ measurable; } \int_0^{R_{\mathrm{t}}} |v(r)|^2 r\, \mathrm{d}r < +\infty \right\}, \tag{37}$$

there also provided with the natural norm. The space $X^k(\omega)$ is the space of functions $\boldsymbol{v}$ in $H(\mathbf{curl}_k, \omega)$ such that $(\boldsymbol{v} \times \tilde{\boldsymbol{n}})_{|\gamma_{\mathrm{t}}}$ belongs to $L^2_1(\gamma_{\mathrm{t}})^3$. Its subspace $X^k_0(\omega)$ is made of functions in $X^k(\omega)$ such that

$$\boldsymbol{v} \times \tilde{\boldsymbol{n}} = \mathbf{0} \quad \text{on } \gamma_{\mathrm{i}} \cup \gamma_{\mathrm{c}} \tag{38}$$

and its subspace $X^k_{\boldsymbol{E}_{\mathrm{i}}}(\omega)$ is made of functions in $X^k(\omega)$ such that

$$(v_r, v_\theta) = (E^k_{\mathrm{i}\theta}, -E^k_{\mathrm{i}r}) \quad \text{on } \gamma_{\mathrm{i}} \quad \text{and} \quad \boldsymbol{v} \times \tilde{\boldsymbol{n}} = \mathbf{0} \quad \text{on } \gamma_{\mathrm{c}} \tag{39}$$

These three spaces are equipped with the norm

$$\|\boldsymbol{v}\|_{X^k(\omega)} = (\|\boldsymbol{v}\|^2_{L^2_1(\omega)^3} + \|\mathbf{curl}_k\, \boldsymbol{v}\|^2_{L^2_1(\omega)^3} + \|\boldsymbol{v} \times \tilde{\boldsymbol{n}}\|^2_{L^2_1(\gamma_{\mathrm{t}})^3})^{1/2} \tag{40}$$

Finally, the space $L^2_1(\gamma_{\mathrm{i}})$ is defined as $L^2_1(\gamma_{\mathrm{t}})$, with all indices t replaced by i, and the full scale of spaces $H^s_1(\gamma_{\mathrm{t}})$ can be constructed as previously. However, the characterization of the spaces on γ_{i} to which each $E^k_{\mathrm{i}r}$ and $E^k_{\mathrm{i}\theta}$ must belong in order that conditions (8) are satisfied is rather complex, so we will work with slightly smoother data $\boldsymbol{E}_{\mathrm{i}}$ without restriction (we have no applications for less regular data).

4.3. *Variational formulation of the two-dimensional problems*

For each k in $\mathbb{Z}$, we consider the following variational problem:
Find $\boldsymbol{e}^k$ in $X^k_{\boldsymbol{E}_{\mathrm{i}}}(\omega)$ such that

$$\begin{aligned}\forall \boldsymbol{v} \in X^k_0(\omega), \quad & \int_\omega (\mathbf{curl}_k\, \boldsymbol{e}^k \cdot \mathbf{curl}_{-k}\, \bar{\boldsymbol{v}} - \kappa^2 \boldsymbol{e}^k \cdot \bar{\boldsymbol{v}})\, r\, \mathrm{d}r\, \mathrm{d}z \\ & + \mathrm{i}\kappa \int_0^{R_{\mathrm{t}}} (\boldsymbol{e}^k \times \tilde{\boldsymbol{n}})(r, z_{\mathrm{t}}) \cdot (\bar{\boldsymbol{v}} \times \tilde{\boldsymbol{n}})(r, z_{\mathrm{t}})\, r\, \mathrm{d}r = 0\end{aligned} \tag{41}$$

The next results are now easily derived from their three-dimensional analogues, as established in Sections 2 and 3.

Proposition 4.2
A vector field $\boldsymbol{e}$ in $X(\Omega)$ is a solution of system (1) if and only if each Fourier coefficient $\boldsymbol{e}^k$ of $\boldsymbol{e}$, $k \in \mathbb{Z}$, is a solution of problem (41).

Proof
If a vector field $\boldsymbol{e}$ is a solution of system (1), it is a solution of problem (7) thanks to Proposition 2.2. So each $\boldsymbol{e}^k$ belongs to $X^k_{\boldsymbol{E}_\mathrm{i}}(\omega)$ and taking the test function $\boldsymbol{v}$ in (7) of the form $\boldsymbol{w}(r,z)\mathrm{e}^{\mathrm{i}k\theta}$ yields (41). Conversely, any solution $\boldsymbol{e}^k$ of (41) is a solution of problem (33) (this follows by taking first $\boldsymbol{v}$ in $\mathscr{D}(\omega)^3$, second $\boldsymbol{v}$ in $\mathscr{D}(\omega \cup \gamma_\mathrm{t})^3$), and the system made of all problems (33), $k \in \mathbb{Z}$, is clearly equivalent to (1). □

The next result follows from Proposition 4.2 combined with Proposition 2.2 and Theorem 3.8.

Theorem 4.3
For each k in $\mathbb{Z}$ and for any $\boldsymbol{E}^k_\mathrm{i}$ satisfying

$$(E^k_{\mathrm{i}r}, E^k_{\mathrm{i}\theta}) \in L^2_1(\gamma_\mathrm{i})^2, \quad \partial_r E^k_{\mathrm{i}r} + r^{-1}E^k_{\mathrm{i}r} + \mathrm{i}kr^{-1}E^k_{\mathrm{i}\theta} \in L^2_1(\gamma_\mathrm{i}), \quad E^k_{\mathrm{i}r}(R_\mathrm{i}, z_\mathrm{i}) = 0 \tag{42}$$

problem (41) admits a unique solution $\boldsymbol{e}^k$ in $X^k_{\boldsymbol{E}_\mathrm{i}}(\omega)$.

A consequence of Proposition 3.9 is that, if the data $\boldsymbol{E}^k_\mathrm{i}$ satisfy (42), both the solution $\boldsymbol{e}^k$ of problem (41) and $\mathbf{curl}_k\, \boldsymbol{e}^k$ belong to $H^{1/2}_1(\omega)^3$. We refer to Reference [10, Theorem II. 3.6] for a slightly more precise result.

5. ANALYSIS OF THE TRUNCATION ERROR OF THE FOURIER EXPANSION

The main idea for the discretization of system (1) consists in handling only a finite, if possible small, number of problems (33). This relies on Fourier truncation, as explained in the following lines.

Let $\mathscr{K}$ be a fixed integer $\geqslant 2$. We set

$$\boldsymbol{e}_{\mathscr{K}}(r,\theta,z) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\mathscr{K}}^{\mathscr{K}} \boldsymbol{e}^k(r,z)\mathrm{e}^{\mathrm{i}k\theta} \tag{43}$$

where each $\boldsymbol{e}^k$ is the solution of problem (33).

We also introduce the function

$$\boldsymbol{E}_{\mathrm{i}\mathscr{K}}(r,\theta,z) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\mathscr{K}}^{\mathscr{K}} \boldsymbol{E}^k_\mathrm{i}(r,z)\,\mathrm{e}^{\mathrm{i}k\theta} \tag{44}$$

Since problem (1) is linear, it can be noted that $\boldsymbol{e} - \boldsymbol{e}_{\mathscr{K}}$ is the solution of problem (1) or (7) with data $\boldsymbol{E}_\mathrm{i}$ replaced by $\boldsymbol{E}_\mathrm{i} - \boldsymbol{E}_{\mathrm{i}\mathscr{K}}$. Applying estimate (25) to this new problem leads to

$$\|\boldsymbol{e} - \boldsymbol{e}_{\mathscr{K}}\|_{X(\Omega)} \leqslant c\|\boldsymbol{E}_\mathrm{i} - \boldsymbol{E}_{\mathrm{i}\mathscr{K}}\|_{W(\Gamma_\mathrm{i})} \tag{45}$$

As standard, in order to evaluate the right-hand side, we introduce some anisotropic weighted Sobolev spaces on Γ_i.

We recall that $W(\Gamma_\mathrm{i})$ is the space of functions $(E_{\mathrm{i}x}, E_{\mathrm{i}y})$ satisfying (8), provided with the corresponding norm. The family of Sobolev spaces $H^{s,W}(\Gamma_\mathrm{i})$, $s \geqslant 0$, is defined as follows:

- When s is an integer, $H^{s,W}(\Gamma_\mathrm{i})$ is the space of functions in $W(\Gamma_\mathrm{i})$ such that all their partial derivatives with respect to θ and of order $\leqslant s$ belong to $W(\Gamma_\mathrm{i})$.

- When s is not an integer, $H^{s,W}(\Gamma_{\rm i})$ is defined by Hilbertian interpolation between $H^{[s]+1,W}(\Gamma_{\rm i})$ and $H^{[s],W}(\Gamma_{\rm i})$, where $[s]$ stands for the integral part of s.

The next result is now a straightforward consequence of (45), see Reference [23, Theorem 1.1].

Theorem 5.1
Assume that the data $\boldsymbol{E}_{\rm i}$ belong to $H^{s,W}(\Gamma_{\rm i})$, $s \geqslant 0$. Then, the following error estimate holds between the solution $\boldsymbol{e}$ of problem (1) and the function $\boldsymbol{e}_{\mathscr{K}}$ introduced in (43)

$$\|\boldsymbol{e} - \boldsymbol{e}_{\mathscr{K}}\|_{X(\Omega)} \leqslant c\mathscr{K}^{-s}\|\boldsymbol{E}_{\rm i}\|_{H^{s,W}(\Gamma_{\rm i})} \tag{46}$$

However, in many practical situations, the Fourier coefficients of $\boldsymbol{E}_{\rm i}$ cannot be computed exactly. So the idea is to replace the integrals which define them, see (29), by quadrature formulas. We introduce the nodes $\theta_m = 2m\pi/2\mathscr{K}+1$, $-\mathscr{K} \leqslant m \leqslant \mathscr{K}$, and we define approximate Fourier coefficients by the formula, for $-\mathscr{K} \leqslant k \leqslant \mathscr{K}$,

$$\boldsymbol{E}^k_{{\rm i}\bigstar}(r) = \frac{\sqrt{2\pi}}{2\mathscr{K}+1}\sum_{m=-\mathscr{K}}^{\mathscr{K}} \boldsymbol{E}_{\rm i}(r,\theta_m){\rm e}^{-{\rm i}k\theta_m} \tag{47}$$

Let now $\boldsymbol{e}^k_{\bigstar}$, $-\mathscr{K} \leqslant k \leqslant \mathscr{K}$, denote the solution of problem (33) with data $\boldsymbol{E}^k_{\rm i}$ replaced by $\boldsymbol{E}^k_{{\rm i}\bigstar}$. We set

$$\boldsymbol{e}_{\mathscr{K}\bigstar}(r,\theta,z) = \frac{1}{\sqrt{2\pi}}\sum_{k=-\mathscr{K}}^{\mathscr{K}} \boldsymbol{e}^k_{\bigstar}(r,z){\rm e}^{{\rm i}k\theta} \tag{48}$$

The next estimate follows from a starred version of (45), combined with the formula

$$\boldsymbol{E}^k_{{\rm i}\bigstar} = \sum_{\ell\in\mathbb{Z}} \boldsymbol{E}_{\rm i}^{k+(2\mathscr{K}+1)\ell} \tag{49}$$

see Reference [23, Theorem 1.2].

Theorem 5.2
Assume that the data $\boldsymbol{E}_{\rm i}$ belong to $H^{s,W}(\Gamma_{\rm i})$, $s > \frac{1}{2}$. Then, the following error estimate holds between the solution $\boldsymbol{e}$ of problem (1) and the function $\boldsymbol{e}_{\mathscr{K}\bigstar}$ introduced in (48)

$$\|\boldsymbol{e} - \boldsymbol{e}_{\mathscr{K}\bigstar}\|_{X(\Omega)} \leqslant c\mathscr{K}^{-s}\|\boldsymbol{E}_{\rm i}\|_{H^{s,W}(\Gamma_{\rm i})} \tag{50}$$

As a conclusion, note that the truncation error is very small for smooth data $\boldsymbol{E}_{\rm i}$ (and even for data $\boldsymbol{E}_{\rm i}$ smoothly depending only on θ). Working with a small value of $\mathscr{K}$ provides a very efficient technique for discretizing the three-dimensional problem (1) since only $2\mathscr{K}+1$ two-dimensional problems (33) must now be approximated.

Remark
From a practical point of view, the choice of $\mathscr{K}$ only depends on the regularity of the data $\boldsymbol{E}_{\rm i}$ and can be made as follows: We first take a high value $\widetilde{\mathscr{K}}$, and compute the

corresponding $\widetilde{\boldsymbol{E}}_{\mathrm{i}\star}^{k}$ thanks to formula (47) with $\mathscr{K}$ replaced by $\widetilde{\mathscr{K}}$. When $\boldsymbol{E}_{\mathrm{i}}$ is smooth enough, these coefficients decrease when $|k|$ increases. So we take $\mathscr{K}$ as the smallest $k>0$ such that the ratio

$$\|\widetilde{\boldsymbol{E}}_{\mathrm{i}\star}^{k}\|_{L_1^2(0,R_{\mathrm{i}})^2}/\|\widetilde{\boldsymbol{E}}_{\mathrm{i}\star}^{0}\|_{L_1^2(0,R_{\mathrm{i}})^2}$$

is smaller than a given tolerance.

APPENDIX A

The aim of this appendix is to prove the density of $\mathscr{D}(\Omega\cup\Gamma_{\mathrm{t}})^3$ in $X_0(\Omega)$, as stated in Lemma 2.1. The proof follows the same steps as in Reference [14], however it requires a further argument.

Let us first observe that we can work with a simplified geometry. Indeed, thanks to the assumptions on the geometry of Ω, there exists a $z_{\mathrm{t}}^{-}<z_{\mathrm{t}}$ such that the intersection of Ω with the half-space $z>z_{\mathrm{t}}^{-}$ is equal to the trapezoid Ω_{t} with two parallel faces, the first one equal to Γ_{t} and the second one, denoted by Γ_{t}^{-}, equal to another disk contained in the plane $z=z_{\mathrm{t}}^{-}$, with centre $(0,0,z_{\mathrm{t}}^{-})$ and radius R_{t}^{-} (see Figure A1). By using a partition of unity and the density of $\mathscr{D}(\Omega^*)^3$ into $H_0(\mathbf{curl},\Omega^*)$ for any domain Ω^* with a Lipschitz-continuous boundary (see Reference [11, Chapter I, Lemma 2.4]), it suffices to prove the density result in the trapezoid Ω_{t} instead of Ω.

Let $H(\Omega_{\mathrm{t}})$ denote the space of functions in $H^1(\Omega_{\mathrm{t}})$ which vanish on $\partial\Omega_{\mathrm{t}}\setminus\Gamma_{\mathrm{t}}$ and such that the restrictions of their traces to Γ_{t} belong to $H^1(\Gamma_{\mathrm{t}})$, provided with the corresponding norm. Note that the trace on Γ_{t} of any function in $H(\Omega_{\mathrm{t}})$ belongs in fact to $H_0^1(\Gamma_{\mathrm{t}})$.

Lemma A.1
The space $\mathscr{D}(\Omega_{\mathrm{t}}\cup\Gamma_{\mathrm{t}})$ is dense in $H(\Omega_{\mathrm{t}})$.

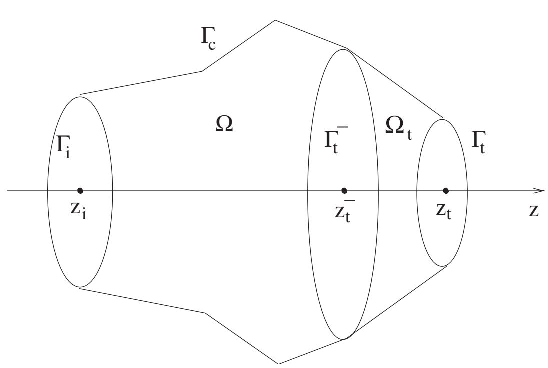

Figure A1. The trapezoid Ω_{t}.

Proof
Let q be a function of $H(\Omega_t)$ and ε be any positive real number. We prove the result in two steps.

(1) We use Reference [11, Chapter IV, Lemma 2.4] in a very simple case: for each $\alpha > 0$, there exists a function φ_α in $\mathscr{C}^\infty(0,1)$ with values in $[0,1]$ such that

$$\varphi_\alpha(r) = \begin{cases} 1 & \text{if } r \leqslant 1 - e^{-1/\alpha}, \\ 0 & \text{if } r \geqslant 1 - e^{-2/\alpha}, \end{cases} \quad \text{and} \quad |\varphi'_\alpha(r)| \leqslant \frac{\alpha}{1-r}, \quad 1 - e^{-1/\alpha} \leqslant r \leqslant 1 - e^{-2/\alpha}$$

Next, we observe that, for $z_t^- \leqslant \tilde{z} \leqslant z_t$, the intersection of Ω_t with the plane $z = \tilde{z}$ coincides with the open disk with radius

$$\psi(\tilde{z}) = R_t - (R_t - R_t^-)\frac{z_t - \tilde{z}}{z_t - z_t^-}$$

and we define the functions, by using cylindrical co-ordinates on Ω_t,

$$\varphi_\alpha^*(r,z) = \varphi_\alpha\left(\frac{r}{\psi(z)}\right), \quad q_1(r,\theta,z) = \varphi_\alpha^*(r,z)\,q(r,\theta,z)$$

To evaluate the distance of q to q_1, we observe that, thanks to the choice of φ_α and when denoting by S_α the support of $1 - \varphi_\alpha^*$ in Ω_t,

$$\|(1-\varphi_\alpha^*)\,q\|_{H^1(\Omega_t)} \leqslant \|q\|_{H^1(S_\alpha)} + c\,\alpha\,\|q\,(\psi(z) - r)^{-1}\|_{L^2(\Omega_t)}$$

Since the measure of S_α tends to zero when α tends to zero, the first term in the right-hand side of this inequality also tends to zero. Concerning the second term, for almost each z, $z_t^- \leqslant z \leqslant z_t^+$, the function q vanishes at $(\psi(z),\theta,z)$, so that the standard Hardy's inequality yields

$$\|q\,(\psi(z) - r)^{-1}\|_{L^2(\Omega_t)} \leqslant \|q\|_{H^1(\Omega_t)}$$

When multiplied by $c\,\alpha$, this quantity also tends to zero when α tends to zero. Since the trace of q on Γ_t belongs to $H_0^1(\Gamma_t)$, exactly the same arguments allow for estimating $\|q - q_1\|_{H^1(\Gamma_t)}$. So there exists a positive real number α such that

$$\|q - q_1\|_{H(\Omega_t)} \leqslant \frac{\varepsilon}{2} \tag{A1}$$

(2) Let χ be a function of $\mathscr{D}(\Omega_t \cup \Gamma_t \cup \Gamma_{t\_})$ with values in $[0,1]$, equal to 1 on the support of φ_α^*. For a fixed function q_2 in $\mathscr{C}^\infty(\overline{\Omega}_t)$, we define q_3 as the product $\chi\,q_2$. Since q_1 is zero when χ is not equal to 1, $\chi\,q_1$ is equal to q_1. So we have

$$\|q_1 - q_3\|_{H^1(\Omega_t)} \leqslant \|\chi(q_1 - q_2)\|_{H^1(\Omega_t)} \leqslant \|q_1 - q_2\|_{H^1(\Omega_t)} + \|\mathbf{grad}\,\chi\|_{L^\infty(\Omega_t)^3}\,\|q_1 - q_2\|_{L^2(\Omega_t)}$$

and the same inequality holds for $\|q_1 - q_3\|_{H^1(\Gamma_t)}$. Noting that the function q_1 belongs to $H^1(\Omega_t)$ and has a trace in $H^1(\partial\Omega_t)$, we derive from Reference [14] that there exists a function q_2 in $\mathscr{C}^\infty(\overline{\Omega}_t)$ such that

$$\|q_1 - q_2\|_{H^1(\Omega_t)} + \|q_1 - q_2\|_{H^1(\Gamma_t)} \leqslant \frac{\varepsilon}{2\,(1 + \|\mathbf{grad}\,\chi\|_{L^\infty(\Omega_t)^3})}$$

whence

$$\|q_1 - q_3\|_{H(\Omega_t)} \leqslant \frac{\varepsilon}{2} \tag{A2}$$

By combining (A1) and (A2), we prove the density of $\mathscr{D}(\Omega \cup \Gamma_t \cup \Gamma_t^-)$ in $H(\Omega_t)$. The density of $\mathscr{D}(\Omega \cup \Gamma_t)$ in $H(\Omega_t)$ is then easily derived by introducing a partition of the unity with respect to the z-variable and using the density of $\mathscr{D}(\Omega_t)^3$ into $H_0(\mathbf{curl}, \Omega_t)$. $\square$

Remark
Of course, the result of Lemma A.1 still holds for more general geometries. However, in the previous proof, the quantity $R_t - r$ must be replaced by the distance to $\partial\Gamma_t$, which is not regular enough when $\partial\Gamma_t$ is not Lipschitz-continuous. So the result seems difficult to establish when $\partial\Gamma_t$ is not a Lipschitz-continuous submanifold of Γ_t.

A similar but simpler argument leads to the next result (see Reference [24] for a different proof in general domains).

Lemma A.2
The space $\mathscr{D}(\Omega_t \cup \Gamma_t)$ is dense in the space of functions in $H^1(\Omega_t)$ which vanish on $\partial\Omega_t \setminus \Gamma_t$.

Let us define $X_0(\Omega_t)$ as the space of restrictions to Ω_t of functions in $X_0(\Omega)$ which have a null tangential trace on Γ_t^- and also $X_{00}(\Omega_t)$ as the space of functions $\boldsymbol{v}$ in $X_0(\Omega_t)$ such that $\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n}$ vanishes on $\partial\Omega_t$.

Lemma A.3
The space $\mathscr{D}(\Omega_t \cup \Gamma_t)^3$ is dense in $X_{00}(\Omega_t)$.

Proof
Let $\boldsymbol{v}$ be any function in $X_{00}(\Omega_t)$. We introduce the spaces

$$X_T(\Omega_t) = \{\boldsymbol{v} \in L^2(\Omega_t)^3; \mathbf{curl}\,\boldsymbol{v} \in L^2(\Omega_t)^3,\ \operatorname{div}\boldsymbol{v} \in L^2(\Omega_t) \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega_t\}$$

$$K_T(\Omega_t) = \{\boldsymbol{v} \in X_T(\Omega_t); \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega_t\}$$

The problem: find $\boldsymbol{\zeta}$ in $K_T(\Omega_t)$ such that

$$\forall\boldsymbol{\varphi} \in K_T(\Omega_t), \quad \int_{\Omega_t} \mathbf{curl}\,\boldsymbol{\zeta}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = \int_{\Omega_t} (\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\varphi} - \mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} \tag{A3}$$

has a unique solution since the ellipticity of the left-hand member can be derived from Reference [18, Corollary 3.16] (this relies on the fact that Ω_t is simply-connected). Next, for any function $\boldsymbol{\psi}$ in $\mathscr{D}(\Omega_t)^3$ with support $\Omega(\boldsymbol{\psi})$, let σ denote the unique solution of the problem

$$\begin{cases} -\Delta\sigma = -\operatorname{div}\boldsymbol{\psi} & \text{in } \Omega(\boldsymbol{\psi}) \\ \sigma = 0 & \text{on } \partial\Omega(\boldsymbol{\psi}) \end{cases}$$

and σ^* denote an extension of σ in $H_0^2(\Omega_t)$. Thus the function $\boldsymbol{\psi} - \mathbf{grad}\,\sigma^*$ belongs to $K_T(\Omega)$. So, by choosing $\boldsymbol{\varphi}$ equal to $\boldsymbol{\psi} - \mathbf{grad}\,\sigma^*$ in (A3) and noting that the integral of the product

$\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{grad}\,\sigma^*$ vanishes, we obtain the equation

$$\forall\boldsymbol{\psi}\in\mathscr{D}(\Omega_{\mathrm{t}})^3,\quad \int_{\Omega_{\mathrm{t}}}\mathbf{curl}\,\boldsymbol{\zeta}\cdot\mathbf{curl}\,\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}=\int_{\Omega_{\mathrm{t}}}(\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\psi}-\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{\psi})\,\mathrm{d}\boldsymbol{x}$$

whence the equation $\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{\zeta})=\mathbf{0}$ in the distribution sense. This leads to the boundary condition

$$\forall\boldsymbol{\varphi}\in K_T(\Omega_{\mathrm{t}}),\quad \int_{\partial\Omega_{\mathrm{t}}}((\mathbf{curl}\,\boldsymbol{\zeta}-\boldsymbol{v})\times\boldsymbol{n})\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{\tau}=0 \tag{A4}$$

Let now $\boldsymbol{\psi}$ be any function in $H(\mathbf{curl},\Omega_{\mathrm{t}})$. It follows from Reference [18, Section 3.e] that it can be written as $\boldsymbol{\varphi}+\mathbf{grad}\,q$, where the function $\boldsymbol{\varphi}$, equal to the curl of a vector potential, belongs to $K_T(\Omega)$ and the function q is in $H^1(\Omega)$. A consequence of the definition of $X_{00}(\Omega_{\mathrm{t}})$ is that Equation (A3) and consequently Equation (A4) also hold with $\boldsymbol{\varphi}$ replaced by $\mathbf{grad}\,q$. Thus we have

$$\forall\boldsymbol{\psi}\in H(\mathbf{curl},\Omega_{\mathrm{t}}),\quad \int_{\partial\Omega_{\mathrm{t}}}((\mathbf{curl}\,\boldsymbol{\zeta}-\boldsymbol{v})\times\boldsymbol{n})\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{\tau}=0$$

whence the equation $(\mathbf{curl}\,\boldsymbol{\zeta})\times\boldsymbol{n}=\boldsymbol{v}\times\boldsymbol{n}$ on $\partial\Omega_{\mathrm{t}}$. So, equivalently $\boldsymbol{\zeta}$ is the solution of the system

$$\begin{cases}\mathbf{curl}\,(\mathbf{curl}\,\boldsymbol{\zeta})=\mathbf{0} & \text{in } \Omega_{\mathrm{t}}\\ \operatorname{div}\boldsymbol{\zeta}=0 & \text{in } \Omega_{\mathrm{t}}\\ \boldsymbol{\zeta}\cdot\boldsymbol{n}=0 \quad\text{and}\quad (\mathbf{curl}\,\boldsymbol{\zeta})\times\boldsymbol{n}=\boldsymbol{v}\times\boldsymbol{n} & \text{on } \partial\Omega_{\mathrm{t}}\end{cases}$$

We use the expansion $\boldsymbol{v}=\mathbf{curl}\,\boldsymbol{\zeta}+(\boldsymbol{v}-\mathbf{curl}\,\boldsymbol{\zeta})$, indeed:

(1) Since $\mathbf{curl}\,\boldsymbol{\zeta}$ has a zero curl, it is the gradient of a function q which is defined up to an additive constant. Moreover, since $\boldsymbol{v}\times\boldsymbol{n}$ vanishes on $\partial\Omega_{\mathrm{t}}\setminus\Gamma_{\mathrm{t}}$, the function q can be chosen in $H(\Omega_{\mathrm{t}})$. It follows from Lemma A.1 that there exists a sequence $(q_n)_n$ of $\mathscr{D}(\Omega_{\mathrm{t}}\cup\Gamma_{\mathrm{t}})$ which converges to q in $H(\Omega_{\mathrm{t}})$. So the sequence $(\mathbf{grad}\,q_n)_n$ belongs to $\mathscr{D}(\Omega\cup\Gamma_{\mathrm{t}})^3$ and converges to $\mathbf{grad}\,q$ dans $X_0(\Omega_{\mathrm{t}})$.

(2) The function $\boldsymbol{v}-\mathbf{curl}\,\boldsymbol{\zeta}$ belongs to $H_0(\mathbf{curl},\Omega_{\mathrm{t}})$, hence it is the limit in $H(\mathbf{curl},\Omega_{\mathrm{t}})$ of a sequence $(\boldsymbol{w}_n)_n$ of $\mathscr{D}(\Omega_{\mathrm{t}})^3$, see Reference [11, Chapter I, Lemma 2.4].

Thus, the sequence $(\boldsymbol{w}_n+\mathbf{grad}\,\boldsymbol{q}_n)_n$ belongs to $\mathscr{D}(\Omega_{\mathrm{t}}\cup\Gamma_{\mathrm{t}})^3$ and converges to $\boldsymbol{v}$ in $X_{00}(\Omega)$, which concludes the proof.

Lemma A.4

The space $\mathscr{D}(\Omega_{\mathrm{t}}\cup\Gamma_{\mathrm{t}})^3$ is dense in $X_0(\Omega_{\mathrm{t}})$.

Proof

Let $\boldsymbol{v}$ be any function in $X_0(\Omega_{\mathrm{t}})$. The problem

$$\begin{cases}-\Delta p=0 & \text{in } \Omega_{\mathrm{t}}\\ \partial_n p=\mathbf{curl}\,\boldsymbol{v}\cdot\boldsymbol{n} & \text{on } \partial\Omega_{\mathrm{t}}\end{cases} \tag{A5}$$

has a unique solution, up to an additive constant. Moreover, since $\mathbf{grad}\,p$ is divergence-free, there exists a function $\boldsymbol{w}$ such that $\mathbf{grad}\,p=\mathbf{curl}\,\boldsymbol{w}$. In order to choose this $\boldsymbol{w}$ in an appropriate

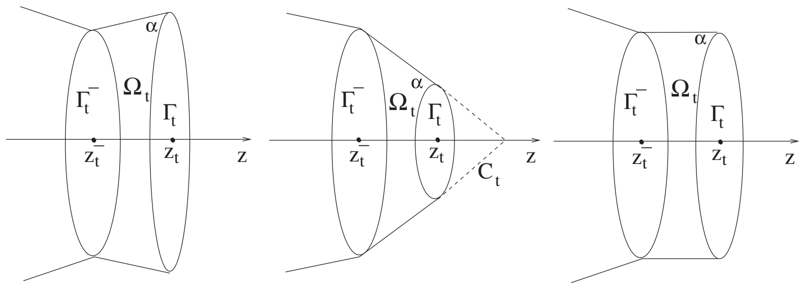

Figure A2. The three cases of angle α.

way, we first note that, since $\boldsymbol{v} \times \boldsymbol{n}$ vanishes on $\partial\Omega_t \setminus \Gamma_t$, $\partial_n p = \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n}$ is zero on $\partial\Omega_t \setminus \Gamma_t$. We now distinguish three types of geometry, according to the aperture α of the angle between Γ_t and $\partial\Omega_t \setminus \Gamma_t$, as illustrated in Figure A2.

- If α is $< \pi/2$, it follows from Reference [18, form. (3.40)] that a first divergence-free potential vector $\boldsymbol{w}_0$ such that $\mathbf{grad}\, p = \mathbf{curl}\, \boldsymbol{w}_0$ can be chosen such that $\boldsymbol{w}_0 \times \boldsymbol{n}$ vanishes on $\partial\Omega_t \setminus \Gamma_t$ and $\boldsymbol{w}_0 \cdot \boldsymbol{n}$ vanishes on Γ_t. Moreover, it follows from Reference [9] and the regularity properties of the Laplace equation with mixed boundary conditions that $\boldsymbol{w}_0$ belongs to $H^1(\Omega)^3$.
- If α is $> \pi/2$, we extend Ω_t by a cone C_t in the half-plane $z > z_t$, such that the plane face of ∂C_t coincides with Γ_t and that the interior Ω_t^c of $\overline{\Omega}_t \cup \overline{C}_t$ is convex. Solving the equation

$$\begin{cases} -\Delta p^c = 0 & \text{in } C_t \\ \partial_n p^c = \mathbf{curl}\, \boldsymbol{v} \cdot \boldsymbol{n} & \text{on } \Gamma_t \\ \partial_n p^c = 0 & \text{on } \partial C_t \setminus \Gamma_t \end{cases}$$

  we observe that the function equal to $\mathbf{grad}\, p$ on Ω_t and to $\mathbf{grad}\, p^c$ on C_t belongs to $H(\mathrm{div}, \Omega_t^c)$ and is divergence-free on Ω_t^c, hence is equal to $\mathbf{curl}\, \boldsymbol{w}_0$ for a vector potential $\boldsymbol{w}_0$ in $H^1(\Omega_t^c)^3$ such that $\boldsymbol{w}_0 \times \boldsymbol{n}$ vanishes on $\partial\Omega_t^c$, see Reference [18, Theorems 2.17 and 3.17].
- If α is equal to $\pi/2$, we use exactly the same arguments as in the previous case, with the cone C_t replaced by a bounded cylinder C_t such that one of its plane faces coincides with Γ_t.

Finally, we define $\boldsymbol{w}$ as equal to $\boldsymbol{w}_0 - \mathbf{grad}\, q$, where q belongs to $H^2(\Omega_t)$ and satisfies $q = 0$ and $\partial_n q = \boldsymbol{w}_0 \cdot \boldsymbol{n}$ on $\partial\Omega_t \setminus \Gamma_t$.

(1) Since the function $\boldsymbol{v} - \boldsymbol{w}$ belongs to $X_{00}(\Omega_t)$, applying Lemma A.3 yields that it is the limit in $X(\Omega)$ of a sequence in $\mathscr{D}(\Omega_t \cup \Gamma_t)^3$.
(2) Since the function $\boldsymbol{w}$ belongs to $H^1(\Omega_t)^3$ and vanishes on $\partial\Omega_t \setminus \Gamma_t$, the existence of a sequence in $\mathscr{D}(\Omega_t \cup \Gamma_t)^3$ which converges to $\boldsymbol{w}$ in $H^1(\Omega_t)^3$, hence in $X(\Omega)$, is a consequence of Lemma A.2. □

Thanks to the arguments given at the beginning of the appendix, Lemma 2.1 is a direct consequence of Lemma A.4.

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