On the rational invariants of a Lie algebra

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The purpose of this work is to set down some basic tools towards the computation of the algebraic and rational invariants of a Lie algebra $\underset{\sim}{G}$, that is to say the polynomial and rational functions on $\underset{\sim}{G}^*$ which are invariant under the coadjoint action of the associated group, or to put it another way the centers of the universal enveloping algebra of $\underset{\sim}{G}$ and of its field of fractions.

New ideas introduced here are those of the <u>soul</u> of a Lie algebra, and of its <u>rational soul</u>: they are invariant ideals characterized as the smallest whose enveloping algebra (resp. field) contains all the algebraic (resp. rational) invariants. To each invariant is also attached an ideal, its <u>carrier</u>, and the largest of them is shown to be the soul.

These ideas apply only trivially to the case of reductive algebras, where the invariants are well known anyway. They are more developed in the case of algebraic solvable Lie algebras, and specially nilpotent ones, where souls are looked at as limits of sequences of invariant commutator ideals (called here <u>reducing</u> ideals). Also in case $\underset{\sim}{G}$ is algebraic, we give an effective way to compute the rational soul.

The souls are themselves Lie algebras of a very particular type, including all reductive algebras but not so much more, and their classification seems less hopeless than the general one.

Many examples of explicitly computed rings of invariants have been spread in the text, illustrating (and using) the notions introduced.

Part of this was developed in collaboration with Anne Fenard (see [12]).

§1- The setting and notations

1.1 $\underset{\sim}{G}$ is an n-dimensional Lie algebra over the field $\underline{k}$ of characteristic zero, and throughout the text we assume one of the two following hypotheses to be satisfied:

- either $\underline{k} = \mathbb{R}$ or $\mathbb{C}$ (the fields of real or complex numbers)
- or $\underset{\sim}{G}$ is algebraic.

In either case we call $\underline{G}$ the connected and simply connected Lie group (resp. the adjoint algebraic group) whose Lie algebra is $\underset{\sim}{G}$. $\underline{G}$ acts on $\underset{\sim}{G}$ via the adjoint representation Ad, and on the dual $\underset{\sim}{G}^*$ of $\underset{\sim}{G}$ via the coadjoint representation Ad^* :

$$\forall x \in \underline{G},\ X \in \underset{\sim}{G},\ f \in \underset{\sim}{G}^*, \qquad \langle \mathrm{Ad}^*(x)f\ ,\ X\rangle = \langle f\ ,\ \mathrm{Ad}(x^{-1})X\rangle \ .$$

For $f \in G^*$, $X \in G$ we define $X.f = \varphi_f(X) \in \underset{\sim}{G}^*$ by

$$\forall\ Y \in \underset{\sim}{G} \qquad (X.f)(Y) = -f([X,Y])$$

and write as usual $\underset{\sim}{G}(f) = \mathrm{Ker}\ \varphi_f$. Clearly $\underset{\sim}{G}(f)^{\perp} = \mathrm{Im}\ \varphi_f$, and

$$X.f = \frac{d}{dt}\Big|_{t=o} \mathrm{Ad}^*(\exp tX)f \quad .$$

1.2 If V is a vector space over $\underline{k}$, S(V) is the symmetric algebra of V, and R(V) its field of fractions. If E is any subset of V, we write $\langle E\rangle$ for the linear envelope of E in V.

In particular $S(\underset{\sim}{G})$ and $R(\underset{\sim}{G})$ are here identified with the ring of polynomial functions and the field of rational functions on $\underset{\sim}{G}^*$.

If $P: G^* \longrightarrow \underline{k}$ is a rational function, we call $r(P)$ the Zariski open subset of G^* where P is regular, and for $f \in r(P)$, dP(f) is the differential of P at the point f .

1.3 $U(\underset{\sim}{G})$ is the universal enveloping algebra of $\underset{\sim}{G}$, with its usual filtration by the subspaces $U_m(\underset{\sim}{G}) = \left\{ P \in U(\underset{\sim}{G}) \,\middle|\, \deg P \leqslant m \right\}$ $(m \in \mathbb{N})$, and $\lambda : S(\underset{\sim}{G}) \longrightarrow U(\underset{\sim}{G})$ the symmetrisation mapping.

If $\underset{\sim}{H}$ is a subalgebra of $\underset{\sim}{G}$, we identify $U(\underset{\sim}{H})$ and $S(\underset{\sim}{H})$ with subalgebras of $U(\underset{\sim}{G})$ and $S(\underset{\sim}{G})$ respectively, and $R(\underset{\sim}{H})$ with a subfield of $R(\underset{\sim}{G})$. This is compatible with the bijection λ.

1.4 If δ is a set of derivations of $\underset{\sim}{G}$, we call $U(\underset{\sim}{G})^{\delta}$, $S(\underset{\sim}{G})^{\delta}$, $R(\underset{\sim}{G})^{\delta}$ the rings of δ-invariant elements of $U(\underset{\sim}{G})$, $S(\underset{\sim}{G})$, $R(\underset{\sim}{G})$ respectively. If $\underset{\sim}{H}$ is a subalgebra of $\underset{\sim}{G}$, we will write $U(\underset{\sim}{G})^{\underset{\sim}{H}}$, $S(\underset{\sim}{G})^{\underset{\sim}{H}}$, $R(\underset{\sim}{G})^{\underset{\sim}{H}}$ instead of $U(\underset{\sim}{G})^{\mathrm{ad}\underset{\sim}{H}}$, $S(\underset{\sim}{G})^{\mathrm{ad}\underset{\sim}{H}}$, $R(\underset{\sim}{G})^{\mathrm{ad}\underset{\sim}{H}}$. We will also write $Z(\underset{\sim}{G})$ for the center $U(\underset{\sim}{G})^{\underset{\sim}{G}}$ of $U(\underset{\sim}{G})$.

1.5 The symmetrisation mapping λ is bijective from $S(\underset{\sim}{G})^{\underset{\sim}{G}}$ onto $Z(\underset{\sim}{G})$, and it is an algebra isomorphism if $\underset{\sim}{G}$ is nilpotent ([6],prop. 4.8.12). Actually for any $\underset{\sim}{G}$ it could be modified into an algebra isomorphism between the two (see [9], théorème 2). If $\underset{\sim}{G}$ is nilpotent or reductive, or $\underset{\sim}{G} = [\underset{\sim}{G},\underset{\sim}{G}]$, $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ is the field of fractions of $S(\underset{\sim}{G})^{\underset{\sim}{G}}$ (see the proof of lemma 1 in [5], or in the nilpotent case [2], lemma 10).

<u>1.6 Proposition</u>: <u>$S(\underset{\sim}{G})^{\underset{\sim}{G}}$ and $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ are engendered by their homogeneous elements.</u>

<u>Proof</u>: If $\{X_1,\dots,X_n\}$ is a basis of $\underset{\sim}{G}$, and

$$[X_i,X_j] = \sum_{k=1}^{n} c_{ij}^k X_k \qquad (1 \leqslant i,j,k \leqslant n \ ; \ c_{ij}^k \in \underline{k})$$

$S(\underset{\sim}{G})^{\underset{\sim}{G}}$ and $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ are the polynomial and rational solutions P of the system of differential equations on $\underset{\sim}{G}^*$:

$$(*) \qquad \sum_{j=1}^{n} \left(\sum_{k=1}^{n} c_{ij}^{k} \, x_k \right) \partial_j P = 0 \qquad (i=1,\dots,n)$$

where the x_k are the coordinate functions on $\underset{\sim}{G}^*$ inthe dual basis of $\{X_1,\dots,X_n\}$, and $\partial_j = \frac{\partial}{\partial x_j}$.

As these equations are homogeneous of degree zero, $S(\underset{\sim}{G})^{\underset{\sim}{G}}$ is the direct sum of its homogeneous subspaces. Now if $P = AB^{-1} \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$, A and B being relatively prime in $S(\underset{\sim}{G})$, the system (*) is equivalent to the existence of $\chi \in \underset{\sim}{G}^*$ such that, for $i=1,\dots,n$

$$\sum_{j=1}^{n} \left(\sum_{k=1}^{n} c_{ij}^{k} \, x_k \right) \partial_j A - \exp \chi(X_i).A = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} c_{ij}^{k} \, x_k \right) \partial_j B - \exp \chi(X_i).B =$$

(reasoning as in [5] lemma 1). As these equations are again homogeneous of degree zero, the conclusion follows for $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ too. ■

We will often use the following basic result of C. Chevalley and J. Dixmier ([2], lemmas 7 and 8) :

1.7 Theorem: Let $\underset{\sim}{G}$ be the Lie algebra of an n-dimensional algebraic group G of automorphisms of a vector space V of dimension p (all over k). Take a basis $\{X_1,\dots,X_n\}$ of $\underset{\sim}{G}$, a basis $\{V_1,\dots,V_n\}$ of V , and call $B = (b_{ij})$ the $n \times p$ matrix with entries $b_{ij} = X_i(V_j)$ in $V \subset R(V)$. Then

(i) the transcendental degree of $R(V)^{\underset{\sim}{G}}$ over k is the dimension r of Ker B : $R(V)^p \longrightarrow R(V)^n$

(ii) $\forall\, P \in R(V) \qquad P \in R(V)^{\underset{\sim}{G}} \Longleftrightarrow dP \in \operatorname{Ker} B$

(iii) $\left\{ dP \;\middle|\; P \in R(V)^{\underset{\sim}{G}} \right\}$ engenders Ker B .

§2- Algebraic definition of the soul

2.1 Definition: We call soul $\underset{\sim}{A} = \underset{\sim}{A}(\underset{\sim}{G})$ of a Lie algebra $\underset{\sim}{G}$ the intersection of all subalgebras $\underset{\sim}{H}$ of $\underset{\sim}{G}$ such that $U(\underset{\sim}{H}) \supset Z(\underset{\sim}{G})$.

2.2 Proposition: $\underset{\sim}{A}(\underset{\sim}{G})$ is the smallest subalgebra $\underset{\sim}{H}$ of $\underset{\sim}{G}$ such that $U(\underset{\sim}{H}) \supset Z(\underset{\sim}{G})$, and it is an (Aut $\underset{\sim}{G}$ -)invariant ideal of $\underset{\sim}{G}$.

Proof: The family $\underline{\underline{F}}$ of subalgebras $\underset{\sim}{H}$ of $\underset{\sim}{G}$ such that $U(\underset{\sim}{H}) \supset Z(\underset{\sim}{G})$ is obviously stable under finite intersections and not empty. So any element of $\underline{\underline{F}}$ of minimal dimension is $\underset{\sim}{A}(\underset{\sim}{G})$. If $\varphi \in \mathrm{Aut}\,\underset{\sim}{G}$, $\varphi(\underset{\sim}{A})$ is also in $\underline{\underline{F}}$, and of the same dimension as $\underset{\sim}{A}$, so $\varphi(\underset{\sim}{A}) = \underset{\sim}{A}$. ■

2.3 Proposition: (a) The soul of a direct sum of algebras is the direct sum of their souls.

(b) A reductive algebra is its own soul.

Proof: (a) Clearly $\underset{\sim}{A}(\underset{\sim}{G}_1 \oplus \underset{\sim}{G}_2) \subset \underset{\sim}{A}(\underset{\sim}{G}_1) \oplus \underset{\sim}{A}(\underset{\sim}{G}_2)$, since $Z(\underset{\sim}{G}_1 \oplus \underset{\sim}{G}_2) = Z(\underset{\sim}{G}_1) \otimes Z(\underset{\sim}{G}_2)$.

But for j=1,2 $Z(\underset{\sim}{G}_j) \subset Z(\underset{\sim}{G}_1 \oplus \underset{\sim}{G}_2) \subset U(\underset{\sim}{A}(\underset{\sim}{G}_1 \oplus \underset{\sim}{G}_2))$ and thus $\underset{\sim}{A}(\underset{\sim}{G}_j) \subset \underset{\sim}{A}(\underset{\sim}{G}_1 \oplus \underset{\sim}{G}_2)$.

(b) Using (a) we can assume that $\underset{\sim}{G}$ is simple or abelian. If $\underset{\sim}{G}$ is simple, $\underset{\sim}{A}(\underset{\sim}{G})$ is either $\{o\}$ or $\underset{\sim}{G}$, by 2.2. But $\underset{\sim}{A}(\underset{\sim}{G}) = \{o\}$ is clearly equivalent to $Z(\underset{\sim}{G}) = \underline{k}$. As the Casimir element of $\underset{\sim}{G}$ is a second degree element of $Z(\underset{\sim}{G})$, we must have $\underset{\sim}{A}(\underset{\sim}{G}) = \underset{\sim}{G}$ in this case, as obviously in the other case. ■

2.4 As the above shows, the notion of soul is empty for a reductive algebra. On the other side we have $\underset{\sim}{A}(\underset{\sim}{G}) = \{o\}$ as soon as $Z(\underset{\sim}{G}) = \underline{k}$, and this is already the case, for instance, for the 2-dimensional non-abelian Lie algebra. But "in general" the soul of a Lie algebra is a proper invariant ideal, which does not even belong to any of the three classical (central descending, central ascending, and derived, here written $\underline{\underline{C}}_{\bullet}\underset{\sim}{G}$, $\underline{\underline{C}}^{\bullet}\underset{\sim}{G}$, and $\underset{\sim}{G}^{(\bullet)}$) series of invariant ideals, as the following example shows:

$\underset{\sim}{G}$ is the 6-dimensional nilpotent Lie algebra defined in a basis $\{X_1, \ldots, X_6\}$ by the brackets

$$[X_1, X_2] = X_4 \ , \ [X_1, X_4] = X_5 \ , \ [X_1, X_5] = [X_2, X_3] = [X_2, X_4] = X_6$$

Clearly $\underline{\underline{C}}^4\underset{\sim}{G} = \underset{\sim}{G}'' = \{0\}$; $\underline{\underline{C}}^3\underset{\sim}{G} = \underline{\underline{C}}_1\underset{\sim}{G} = \langle X_6\rangle$; $\underline{\underline{C}}^2\underset{\sim}{G} = \langle X_5, X_6\rangle$; $\underline{\underline{C}}_2\underset{\sim}{G} = \langle X_3, X_5, X_6\rangle$; $\underline{\underline{C}}^1\underset{\sim}{G} = \underset{\sim}{G}' = \langle X_4, X_5, X_6\rangle$; $\underline{\underline{C}}_3\underset{\sim}{G} = \langle X_3, X_4, X_5, X_6\rangle$; and $\underline{\underline{C}}_4\underset{\sim}{G} = \underset{\sim}{G}$. But $Z(\underset{\sim}{G}) = \underline{k}\,[X_6 \ , \ X_5^2 - 2X_4X_6 + 2X_3X_6]$, and so $\underset{\sim}{A}(\underset{\sim}{G}) = \langle X_3 - X_4, X_5, X_6\rangle$.

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§3- Geometric definitions of the soul

3.1 Proposition(M. Raïs): The soul of $\underset{\sim}{G}$ is the subspace $\underset{\sim}{A}(\underset{\sim}{G}) = \sum_{P \in S(\underset{\sim}{G})^{\underset{\sim}{G}}, f \in \underset{\sim}{G}^*} dP(f)$

It is the smallest subspace V of $\underset{\sim}{G}$ such that $S(V) \supset S(\underset{\sim}{G})^{\underset{\sim}{G}}$.

Proof: If V is a subspace of $\underset{\sim}{G}$ and $S(V) \supset S(\underset{\sim}{G})^{\underset{\sim}{G}}$, then for any $P \in S(\underset{\sim}{G})^{\underset{\sim}{G}}$, $f \in \underset{\sim}{G}^*, f' \in V^{\perp}$, we have $P(f+f') = P(f)$, so that $\langle dP(f), f' \rangle = o$. Hence V contains the space $V_o = \sum_{P \in S(\underset{\sim}{G})^{\underset{\sim}{G}}, f \in \underset{\sim}{G}^*} dP(f)$.

Reciprocally if $P \in S(\underset{\sim}{G})^{\underset{\sim}{G}}$, $f \in \underset{\sim}{G}^*$, $f' \in V_o^{\perp}$ and we put $\varphi(t) = P(f+tf')$ $(t \in \underline{k})$, we have $\varphi'(t) = \langle dP(f+tf'), f' \rangle = o$, so $\varphi(t) = \varphi(o)$. Thus $S(\underset{\sim}{G})^{\underset{\sim}{G}} \subset S(V_o)$.

For any $x \in \underline{G}$ and $f \in \underset{\sim}{G}^*$, the invariance of P implies $Ad(x)dP(f) = dP(Ad^*(x)f)$.

Hence V_o is an ideal of $\underset{\sim}{G}$, and by the symmetrisation mapping $U(V_o) \supset Z(\underset{\sim}{G})$.

If now $\underset{\sim}{H}$ is a subalgebra such that $U(\underset{\sim}{H}) \supset Z(\underset{\sim}{G})$, we have again by symmetrisation $S(\underset{\sim}{H}) \supset S(\underset{\sim}{G})^{\underset{\sim}{G}}$, thus $\underset{\sim}{H} \supset V_o$. By 2.2, $V_o = \underset{\sim}{A}(\underset{\sim}{G})$. ■

3.2 Proposition (M. Raïs): Assume $k=\mathbb{R}$ or $\mathbb{C}$ and there exists a dense G-invariant open subset Ω of G^* such that $S(\underset{\sim}{G})^{\underline{G}}$ separates the G-orbits in Ω. Then for any subset Ω' of Ω which is Zariski-dense in $\underset{\sim}{G}^*$, one has

$$\underset{\sim}{A}(\underset{\sim}{G}) = \sum_{f \in \Omega'} \underset{\sim}{G}(f) .$$

Proof: Take $f' \in \bigcap_{f \in \Omega'} \underset{\sim}{G}(f)^{\perp}$ and $P \in S(\underset{\sim}{G})^{\underset{\sim}{G}}$. For each $f' \in \Omega'$ we can choose $X \in \underset{\sim}{G}$ such that $f' = X.f$. Since P is $\underline{G}$-invariant,

$$\langle dP(f), f' \rangle = \langle dP(f), X.f \rangle = o$$

and the polynomial function $f \longmapsto \langle dP(f), f' \rangle$ vanishes on Ω', thus everywhere on $\underset{\sim}{G}^*$: $f' \in \bigcap_{f \in \underset{\sim}{G}^*} (dP(f))^{\perp}$. Finally

$$\bigcap_{f\in\Omega'} \underset{\sim}{G}(f)^{\perp} \subset \bigcap_{P\in S(\underset{\sim}{G})^{\underset{\sim}{G}},\, f\in \underset{\sim}{G}^*} (dP(f))^{\perp} = \underset{\sim}{A}(\underset{\sim}{G})^{\perp} \quad \text{by 3.2 .}$$

Reciprocally if $f'\in \underset{\sim}{A}(\underset{\sim}{G})^{\perp}$, for any $f\in\Omega$ we can find $\varepsilon>0$ such that for any $t\in \underline{k}$, $|t|<\varepsilon$ implies $f+tf'\in\Omega$. If $P\in S(\underset{\sim}{G})^{\underset{\sim}{G}}$, we have by 3.1 $P(f+tf') = P(f)$, and using our assumption it follows that $f+tf'\in Ad^*(\underset{\sim}{G})f$ for $|t|<\varepsilon$. Differentiating with respect to t, there exists $X\in \underset{\sim}{G}$ such that $f' = X.f$, hence $f'\in \underset{\sim}{G}(f)^{\perp}$. Thus

$$\underset{\sim}{A}(\underset{\sim}{G})^{\perp} \subset \bigcap_{f\in\Omega} \underset{\sim}{G}(f)^{\perp} \subset \bigcap_{f\in\Omega'} \underset{\sim}{G}(f)^{\perp} \subset \underset{\sim}{A}(\underset{\sim}{G})^{\perp} . \blacksquare$$

<u>3.3 Definition</u>: Let $\underline{\underline{O}}$ be an orbit of the coadjoint representation. We call <u>saturation subspace</u> $D(\underline{\underline{O}})$ of $\underline{\underline{O}}$ the set of all $f'\in\underset{\sim}{G}^*$ such that for any $f\in\underline{\underline{O}}$, $\{t\in\underline{k} \mid f+tf'\in\underline{\underline{O}}\}$ is open. Clearly $D(\underline{\underline{O}}) = \bigcap_{f\in\underline{\underline{O}}} \underset{\sim}{G}(f)^{\perp}$, that is $D(\underline{\underline{O}})^{\perp} \supset \underset{\sim}{G}(\underline{\underline{O}})$, where $\underset{\sim}{G}(\underline{\underline{O}}) = \sum_{f\in\underline{\underline{O}}} \underset{\sim}{G}(f)$ is the smallest ideal in $\underset{\sim}{G}$ containing any one of the $\underset{\sim}{G}(f)$ for $f\in\underline{\underline{O}}$.

<u>3.4 Corollary</u>: <u>Under the same assumption as in proposition 3.2, we have</u>

$$\underset{\sim}{A}(\underset{\sim}{G})^{\perp} = \bigcap_{\underline{\underline{O}}\subset\Omega} D(\underline{\underline{O}}) .$$

<u>Proof</u>: By 3.3, $\underset{\sim}{A}(\underset{\sim}{G})^{\perp} = \bigcap_{f\in\Omega} \underset{\sim}{G}(f)^{\perp} = \bigcap_{\underline{\underline{O}}\subset\Omega} D(\underline{\underline{O}})$. $\blacksquare$

§4- The soul of a nilpotent Lie algebra

4.1 We assume in this paragraph that $\underset{\sim}{G}$ is nilpotent, and $\underline{k} = \mathbb{R}$ or $\mathbb{C}$. The field of fractions of $Z(\underset{\sim}{G})$ can be identified, via the symmetrisation, with the field of fractions of $S(\underset{\sim}{G})^{\underset{\sim}{G}}$, and this one, since $\underset{\sim}{G}$ is nilpotent, with $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ (see 1.5). Furthermore $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ is a purely transcendental extension of $\underline{k}$ of degree $r = n-2d$, where d is the commutativity defect of $\underset{\sim}{G}$ ([2], definition 2), and one can find $P_o, P_1, \ldots, P_r$ in $Z(\underset{\sim}{G})$ such that $P_o \neq o$; $\overline{P}_j = \lambda^{-1}(P_j) \in S(\underset{\sim}{G})^{\underset{\sim}{G}}$ is homogeneous for $j = o,1,\ldots,r$; the natural morphism $\underline{k}(\overline{P}_1,\ldots,\overline{P}_r) \longrightarrow R(\underset{\sim}{G})^{\underset{\sim}{G}}$ is an isomorphism ; and the $\underline{G}$-orbits in $\underset{\sim}{G}^*$ which are contained in the $\underline{G}$-stable Zariski open subset $\Omega = \left\{ x \in \underset{\sim}{G}^* \mid \overline{P}_o(x) \neq o \right\}$ are exactly the algebraic subvarieties defined by the equations

$$\overline{P}_j(x) = a_j \qquad (j=1,\ldots,r \ ; \ a_j \in \underline{k} \ )$$

(See for all this [2], and [14], proposition 2.2). Thus 3.4 applies here :

4.2 Corollary: <u>If $\underset{\sim}{G}$ is nilpotent, the orthogonal of its soul is the intersection of the saturation subspaces of all the orbits contained in any open subset Ω of $\underset{\sim}{G}^*$ defined as in 4.1</u> .

4.3 Let $\underset{\sim}{G}_1$ be a 1-codimensional ideal of $\underset{\sim}{G}$, and $\pi : \underset{\sim}{G}^* \longrightarrow \underset{\sim}{G}_1^*$ the canonical projection. It is well known ([14]) that only one of the two following situations can occur :

- <u>either $Z(\underset{\sim}{G}) \subset Z(\underset{\sim}{G}_1)$</u> , hence $\underset{\sim}{A}(\underset{\sim}{G}) \subset \underset{\sim}{G}_1$, $\operatorname{Ker} \pi \subset \underset{\sim}{A}(\underset{\sim}{G})^{\perp}$, and by 4.2 each orbit $\underset{=}{O} \subset \Omega$ contains with any point x the entire fibre $\pi^{-1}(\pi(x))$.

We shall then say that $\underset{\sim}{G}_1$ is a <u>vertical</u> ideal of $\underset{\sim}{G}$.

- or $Z(\underset{\sim}{G}) \not\subset Z(\underset{\sim}{G}_1)$, but then $Z(\underset{\sim}{G}_1) = U(\underset{\sim}{G}_1) \cap Z(\underset{\sim}{G})$, $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ is a transcendental extension of degree one of $R(\underset{\sim}{G}_1)^{\underset{\sim}{G}_1} = R(\underset{\sim}{G}_1)^{\underset{\sim}{G}}$, and one can find $P_o, \dots, P_r \in Z(\underset{\sim}{G})$ such that $P_1, \dots, P_{r-1}$ generate $R(\underset{\sim}{G}_1)^{\underset{\sim}{G}_1}$, $P_r \notin U(\underset{\sim}{G}_1)$ (see [2] lemmas 9, 10, 11 and proposition 3) , and $\{P_o, \dots, P_r\}$ has all the properties described in 4.1 . In particular if $\underline{\underline{O}}$ is an orbit in Ω and l a fibre of π meeting $\underline{\underline{O}}$, $\overline{P}_r - a_r\big|_l$ is not identically zero, and $l \cap \underline{\underline{O}}$ is thus finite. As $l \cap \underline{\underline{O}}$ is also connected ([14] lemma 6.1), it is a single point. We shall then say that $\underset{\sim}{G}_1$ is a <u>transversal</u> ideal of $\underset{\sim}{G}$.

Furthermore we shall say that $\underset{\sim}{G}$ is <u>completely transversal</u> (resp. <u>completely vertical</u>) if all its 1-codimensional ideals are transversal (resp. vertical).

<u>4.4 Proposition: Let $\underset{\sim}{G}$ be a nilpotent algebra over $\mathbb{R}$ or $\mathbb{C}$, $\underset{\sim}{G}'$ its derived algebra, $\underset{\sim}{A}$ its soul, and $\underset{\sim}{G}_1$ a 1-codimensional ideal of $\underset{\sim}{G}$.</u>

(a) $\underset{\sim}{G}_1$ <u>is vertical if and only if</u> $\underset{\sim}{G}_1 \supset \underset{\sim}{A}$, <u>transversal otherwise</u>.

(b) $\underset{\sim}{G}$ <u>is completely vertical if and only if</u> $\underset{\sim}{A} \subset \underset{\sim}{G}'$.

(c) $\underset{\sim}{G}$ <u>is completely transversal if and only if</u> $\underset{\sim}{A} = \underset{\sim}{G}$.

(d) <u>The intersection of vertical (1-codimensional) ideals of</u> $\underset{\sim}{G}$ <u>is the invariant ideal</u> $\underset{\sim}{A}+\underset{\sim}{G}'$.

<u>Proof</u>: (a) comes from the equivalences

$$\underset{\sim}{A} \subset \underset{\sim}{G}_1 \Longleftrightarrow Z(\underset{\sim}{G}) \subset U(\underset{\sim}{G}_1) \Longleftrightarrow Z(\underset{\sim}{G}) \subset Z(\underset{\sim}{G}_1)$$

(b) The family $\underline{\underline{F}}$ of 1-codimensional ideals of $\underset{\sim}{G}$ is precisely the family of 1-codimensional subspaces of $\underset{\sim}{G}$ containing $\underset{\sim}{G}'$; thus

$$\underset{\sim}{A} \subset \underset{\sim}{G}' \Longleftrightarrow \underset{\sim}{A} \subset \bigcap_{\underset{\sim}{G}_1 \in \underline{\underline{F}}} \underset{\sim}{G}_1$$ and (b) follows from (a).

(c) As soon as $\underset{\sim}{A} \neq \underset{\sim}{G}$ one can find a 1-codimensional ideal $\underset{\sim}{G}_1$ containing $\underset{\sim}{A}$, and $\underset{\sim}{G}_1$ is then vertical by (a).

(d) follows from (a) and the proof of (b).■

4.5 Let us call $\underset{\sim}{Z}$ the center of $\underset{\sim}{G}$. Clearly $\underset{\sim}{A} \supset \underset{\sim}{Z}$. So if $\underset{\sim}{G}$ is completely vertical, necessarily $\underset{\sim}{Z} \subset \underset{\sim}{G}'$. But the converse is not true: if $\underset{\sim}{G}$ is the only 4-dimensional nilpotent Lie algebra which cannot be split into a direct sum, it can be defined in a basis $\{X_1,\dots,X_4\}$ by the brackets

$$[X_1,X_2] = X_3 \ , \quad [X_1,X_3] = X_4 \ .$$

Then $\underset{\sim}{G}_1 = \langle X_2,X_3,X_4\rangle$ is vertical while $\underset{\sim}{G}_2 = \langle X_1,X_3,X_4\rangle$ is transversal, since $Z(\underset{\sim}{G}) = \underline{k}\,[X_4, X_3^2-2X_2X_4]$ and thus $\underset{\sim}{A}(\underset{\sim}{G}) = \langle X_2,X_3,X_4\rangle = \underset{\sim}{G}_1 \not\subset \underset{\sim}{G}_2$.

4.6 We will call unsplitable a Lie algebra $\underset{\sim}{G}$ which cannot be written $\underset{\sim}{G}_1 \oplus \underset{\sim}{G}_2$, where the dimensions of $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$ are strictly less than that of $\underset{\sim}{G}$.

4.7 Proposition: Let $\underset{\sim}{G}$ be nilpotent, unsplitable, of dimension > 1. Then if $Z(\underset{\sim}{G}) \cap \underset{\sim}{G}$ engenders $Z(\underset{\sim}{G})$, $\underset{\sim}{G}$ is completely vertical.

Proof: $Z(\underset{\sim}{G}) \cap \underset{\sim}{G} = \underset{\sim}{Z}$ is the center of $\underset{\sim}{G}$. So $Z(\underset{\sim}{G}) \subset U(\underset{\sim}{Z})$, and clearly $\underset{\sim}{A}(\underset{\sim}{G}) = \underset{\sim}{Z}$. If $\{X_1,\dots,X_p\}$, $\{X_1,\dots,X_p,X_{p+1},\dots,X_q\}$, $\{X_1,\dots,X_p,X_{q+1},\dots,X_r\}$, $\{X_1,\dots,X_r,X_{r+1},\dots,X_n\}$ are bases of $\underset{\sim}{Z} \cap \underset{\sim}{G}'$, $\underset{\sim}{Z}$, $\underset{\sim}{G}'$, $\underset{\sim}{G}$ respectively, and if $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$ are the subspaces engendered by $\{X_{p+1},\dots,X_q\}$ and $\{X_1,\dots,X_p,X_{q+1},\dots,X_n\}$ respectively, we have $\underset{\sim}{G} = \underset{\sim}{G}_1 \oplus \underset{\sim}{G}_2$. As $\underset{\sim}{G}_2 = \{0\}$ would imply $n=q$, so $\underset{\sim}{G} = \underset{\sim}{Z}$, and an unsplitable abelian algebra is of dimension one, we have $\underset{\sim}{G}_1 = \{0\}$, so $p=q$, and $\underset{\sim}{A}(\underset{\sim}{G}) = \underset{\sim}{Z} \subset \underset{\sim}{G}'$. The conclusion follows by 4.4(b).■

4.8 The completely vertical nilpotent Lie algebras are not rare. For instance there are 6 classes of isomorphism of unsplitable nilpotent algebras of dimension 5 (they are given and called $\Gamma_{5,j}$, $1 \leq j \leq 6$ in [3]), and 24 of dimension 6 over $\mathbb{R}$, 20 over $\mathbb{C}$ (given and called $G_{6,k}$, $1 \leq k \leq 24$ in [15]).

One can check that the completely vertical ones are the $\Gamma_{5,j}$ for j=1,3,6 and the $G_{6,k}$ for k=2,8,9,11,13,16,17,19,20,21,22,23,24. All but the first five $G_{6,k}$ cited here satisfy the hypothesis of proposition 4.7 .

4.9 The completely transversal nilpotent Lie algebras, that is the algebras which are equal to their soul, are much less common, as the following table shows:

(see table on next page)

Apart from k itself, the seven unsplitable completely transversal algebras of dimension $\leqslant 7$ (over $\mathbb{R}$ or $\mathbb{C}$) are defined below by their brackets in a basis $\{X_1,\ldots,X_n\}$, and we give for each one the center Z of its enveloping algebra:

(a) (= $\Gamma_{5,4}$ in [3]) $[X_1,X_2] = X_3$, $[X_1,X_3] = X_4$, $[X_2,X_3] = X_5$

$$Z = \underline{k}\left[X_4,X_5,X_3^2-2X_2X_4+2X_1X_5\right]$$

(b) (= $G_{6,15}$ in [15]) $[X_1,X_2] = X_4$, $[X_1,X_3] = X_5$, $[X_2,X_3] = X_6$

$$Z = \underline{k}\left[X_4,X_5,X_6,X_3X_4-X_2X_5+X_1X_6\right]$$

(c) $[X_1,X_2] = X_3$, $[X_1,X_3] = X_4$, $[X_1,X_4] = X_5$,

$[X_1,X_5] = X_6$, $[X_2,X_5] = X_7$, $[X_3,X_4] = -X_7$

$$Z = \underline{k}\left[X_6,X_7,\ X_4^2-2X_3X_5+2X_2X_6-2X_1X_7\right]$$

(d) $[X_1,X_2] = X_3$, $[X_1,X_3] = X_4$, $[X_1,X_4] = X_5$, $[X_1,X_5] = X_6$

$[X_2,X_3] = X_6$, $[X_2,X_5] = X_7$, $[X_3,X_4] = -X_7$

$$Z = \underline{k}\left[X_6,X_7,\ X_5^2X_6+X_4^2X_7-2X_4X_6^2-2X_3X_5X_7+2X_2X_6X_7-2X_1X_7^2\right]$$

dimension (over $\mathbb{R}$ or $\mathbb{C}$)	1	2	3	4	5	6	7	>7
number of isomorphism classes of unsplitable nilpotent Lie algebras	1	0	1	1	6	24	∞	∞
number of completely transversal ones	1	0	0	0	1	1	5	?

(e) $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_1, X_4] = X_5$, $[X_1, X_5] = X_6$

$[X_2, X_3] = X_5$, $[X_2, X_4] = X_6$, $[X_2, X_5] = X_7$, $[X_3, X_4] = -X_7$

$$Z = \underline{k}\left[X_6, X_7, 2X_5^3+3X_4^2X_7-6X_4X_5X_6+6X_3X_6^2-6X_3X_5X_7+6X_2X_6X_7-6X_1X_7^2\right]$$

(f) $[X_1, X_2] = X_4$, $[X_1, X_3] = X_5$, $[X_1, X_5] = X_6$

$[X_2, X_4] = X_6$, $[X_2, X_5] = X_7$, $[X_3, X_4] = X_7$

$$Z = \underline{k}\left[X_6, X_7, X_5^2X_6-2X_4X_5X_7-2X_3X_6^2+2X_2X_6X_7-2X_1X_7^2\right]$$

(g) $[X_1, X_2] = X_4$, $[X_1, X_3] = X_5$, $[X_1, X_4] = X_6$

$[X_1, X_5] = X_7$, $[X_2, X_4] = X_7$, $[X_3, X_5] = X_6$

$$Z = \underline{k}\left[X_6, X_7, X_5^2X_7+X_4^2X_6-2X_3X_7^2-2X_2X_6^2+2X_1X_6X_7\right]$$

That the last five are the only completely transversal nilpotent algebras of dimension 7 can be checked on the list of all 7-dimensional nilpotent Lie algebras over $\mathbb{R}$ or $\mathbb{C}$ given in [18].

§5- Reducing ideals and the soul

5.1 Definition: If $\underset{\sim}{J}$ is a subalgebra of $\underset{\sim}{G}$, we say that $Q \in U(\underset{\sim}{G})$ is $\underset{\sim}{J}$-reducing if $\operatorname{ad} Q : \underset{\sim}{J} \longrightarrow U(\underset{\sim}{G})$ is of rank one, and we call $c(Q)$ the kernel of this mapping, that is to say the commutator of Q in $\underset{\sim}{J}$.

5.2 Lemma: Let $Q \in U(\underset{\sim}{G})$ be $\underset{\sim}{J}$-reducing, and $P \in U(\underset{\sim}{J})$.
If $[P,Q] = o$, then $P \in U(c(Q))$.

Proof: Let $\{X_1,\dots,X_m\}$ be a basis of $\underset{\sim}{J}$ such that $\{X_2,\dots,X_m\}$ is a basis of $c(Q)$. If q and p are the degrees of Q and $P = \sum a_\alpha X_1^{\alpha_1}\dots X_m^{\alpha_m}$ $(\alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{N}^m)$, we have

$$o = [P,Q] = \sum_{j=1}^{m} \frac{\partial P}{\partial x_j}[X_j,Q] \qquad \text{modulo } U_{p+q-2}(\underset{\sim}{G})$$

$$= \Big(\sum_{\alpha_1+\dots+\alpha_m=p} \alpha_1 a_\alpha X_1^{\alpha_1-1} X_2^{\alpha_2}\dots X_m^{\alpha_m}\Big)[X_1,Q] \qquad \text{modulo } U_{p+q-2}(\underset{\sim}{G})$$

As $[X_1,Q] \neq o$ is of degree q, this implies $\alpha_1 a_\alpha = o$ as soon as $\alpha_1+\dots+\alpha_m = p$, that is to say $P = P_p + P'$ with $P_p \in U(c(Q))$, $P' \in U_{p-1}(\underset{\sim}{J})$, and $[P',Q] = [P_p+P',Q] = o$. The proof follows by induction on p. ∎

5.3 For $q \in \mathbb{N}$, let $R_q(\underset{\sim}{J})$ be the intersection of the commutators $c(Q)$ of all $\underset{\sim}{J}$-reducing $Q \in U_q(\underset{\sim}{G})$.

Lemma: If $\underset{\sim}{J}$ is an invariant ideal of $\underset{\sim}{G}$, so is $R_q(\underset{\sim}{J})$.

Proof: If $\varphi \in \operatorname{Aut} \underset{\sim}{G}$, $\varphi|_{\underset{\sim}{J}} \in \operatorname{Aut} \underset{\sim}{J}$ and if $Q \in U_q(\underset{\sim}{G})$ is $\underset{\sim}{J}$-reducing, $\varphi(Q) \in U_q(\underset{\sim}{G})$ is thus also $\underset{\sim}{J}$-reducing, and $c(\varphi(Q)) = \varphi(c(Q))$. So $\varphi(R_q(\underset{\sim}{J})) = \varphi(\bigcap_Q c(Q)) = \bigcap_{\varphi(Q)} c(\varphi(Q)) = R_q(\underset{\sim}{J})$. ∎

5.4 Definition: Write $R(J) = \bigcap_{q \in \mathbb{N}} R_q(J)$, and define a sequence $R^j(J)$ by $R^{j+1}(J) = R(R^j(J))$ and $R^0(J) = J$. We will call the $R^j(G)$ the reducing ideals of G , and finally we put $R^\infty(J) = \bigcap_{j \in \mathbb{N}} R^j(J)$. The following proposition follows immediately from lemma 5.3 :

5.5 Proposition: Let J be an invariant ideal of G.

(a) For any finite sequence $q_1,\dots,q_j$ of integers, $R_{q_1}(R_{q_2}(\dots(R_{q_j}(J))\dots))$ is an invariant ideal of G .

(b) The $R^j(J)$ $(j \in \mathbb{N})$ are a decreasing (thus stationary) sequence of invariant ideals of G . In particular $R^\infty(J)$ is an invariant ideal of G .

5.6 The notion of an J-reducing $Q \in U(G)$ is well known in the case where $\deg Q = 1$, $J = G$, and $[Q,G] \subset Z(G)$ (see [14] lemma 4.1; [17] II Chap.II, §3 ; [6] 4.7.7,8,11,12 ; [11] Satz 1.5), but it may happen that $U(G)$ has no G-reducing element of degree one, while having many of higher degrees: for instance if G is the 6-dimensional algebra defined by the brackets

$$[X_1,X_2] = X_5 , \quad [X_1,X_3] = X_6 , \quad [X_2,X_4] = X_6 , \quad [X_3,X_4] = -X_5$$

one checks easily that $\operatorname{ad} X : G \longrightarrow G$ is of rank 2 or 0 for any $X \in G$. But $X_2X_5+X_3X_6$, $X_1X_5-X_4X_6$, $X_1X_6+X_3X_5$ and $X_2X_6-X_3X_5$ are G-reducing, and $R(G)$ is the center $\langle X_5,X_6\rangle$ of G .

5.7 Lemma: Let G be nilpotent, J a subalgebra, $Q \in U(G)$ be J-reducing. Assume $[Q,J] = k.Q_1$, with $Q_1 \in U(G)$. Then either Q_1 is J-reducing and $c(Q_1) = c(Q)$, or Q_1 commutes with J .

Proof: First note that $c(Q)$ is not only a subalgebra of $\underset{\sim}{G}$, but an ideal of $\underset{\sim}{J}$: take $X \in \underset{\sim}{J} - c(Q)$, so that $[Q,X] = Q_1 \neq o$ and $J = \underline{k}.X \oplus c(Q)$; if $Y \in c(Q)$, we have $[X,Y] = aX$ modulo $c(Q)$, and hence

$$aQ_1 = [Q,aX] = [Q,[X,Y]] = [[Q,X],Y] = [Q_1,Y]$$

So a is an eigenvalue of $\mathrm{ad}\, Y : U_q(\underset{\sim}{G}) \longrightarrow U_q(\underset{\sim}{G})$, where q is the degree of Q and Q_1. As $\underset{\sim}{G}$ is nilpotent, $a = o$, and $[\underset{\sim}{J},c(Q)] \subset c(Q)$. Now

$$[Q_1,Y] = [[Q,X],Y] = [Q,[X,Y]] + [[Q,Y],X] = o . \blacksquare$$

5.8 Proposition: If $\underset{\sim}{G}$ is nilpotent, $\underset{\sim}{J}$ is a subalgebra, and $\underset{\sim}{Q}$ is $\underset{\sim}{J}$-reducing, there is another $\underset{\sim}{J}$-reducing Q' of the same degree, such that $c(Q') = c(Q)$, and $[Q',\underset{\sim}{J}] \subset U(\underset{\sim}{G})^{\underset{\sim}{J}}$.

Proof: Choose $X \in \underset{\sim}{J} - c(Q)$, and put $Q_o = Q$, $Q_1 = [Q,X] \neq o$, $Q_j = [Q_{j-1},X]$ for $j > 1$. By the preceding lemma, all the non-zero Q_j are of the same degree q , and either in $U(\underset{\sim}{G})^{\underset{\sim}{J}}$ or $\underset{\sim}{J}$-reducing and such that $c(Q_j) = c(Q)$. As $\mathrm{ad}\, X$ is nilpotent in $U_q(\underset{\sim}{G})$, there is a smallest integer j_o such that $(\mathrm{ad}\, X)^{j_o}(Q) = o$. We have $j_o \geqslant 2$ and choose $Q' = Q_{j_o - 2}$. $\blacksquare$

5.9 Lemma: Let $\underset{\sim}{A}$ be the soul of $\underset{\sim}{G}$, $\underset{\sim}{J}$ a subalgebra, and $Q \in U(\underset{\sim}{G})$ be $\underset{\sim}{J}$-reducing. If $\underset{\sim}{A} \subset \underset{\sim}{J}$, then $\underset{\sim}{A} \subset c(Q)$.

Proof: To any $P \in Z(\underset{\sim}{G}) \subset U(\underset{\sim}{A}) \subset U(\underset{\sim}{J})$ we can apply lemma 5.2. Thus $Z(\underset{\sim}{G}) \subset U(c(Q))$, and hence $\underset{\sim}{A} \subset c(Q)$. $\blacksquare$

5.10 Corollary: All the reducing ideals $R_{q_1}(\ldots(R_{q_j}(\underset{\sim}{G}))\ldots)$, $R^j(\underset{\sim}{G})$, $R^{\infty}(\underset{\sim}{G})$ contain the soul of $\underset{\sim}{G}$.

Proof: Apply lemma 5.9 as many times as necessary. $\blacksquare$

5.11 Proposition: Let $\underset{\sim}{G}$ be an algebraic Lie algebra, $\underset{\sim}{J}$ and $\underset{\sim}{J}_1$ ideals of $\underset{\sim}{G}$, $\underset{\sim}{J}_1$ of codimension one in $\underset{\sim}{J}$. If $R(\underset{\sim}{J})^{\underset{\sim}{G}} \subset R(\underset{\sim}{J}_1)$, then $R(\underset{\sim}{G})^{\underset{\sim}{J}_1}$ is a transcendental extension of degree one of $R(\underset{\sim}{G})^{\underset{\sim}{J}}$.

Proof: Let $\{X_{r+2},\dots,X_n\}$, $\{X_{r+1},\dots,X_n\}$, $\{X_1,\dots,X_n\}$ be bases of $\underset{\sim}{J}_1$, $\underset{\sim}{J}$, and $\underset{\sim}{G}$ respectively. Then for $P \in R(\underset{\sim}{J})$ we have

$$P \in R(\underset{\sim}{J})^{\underset{\sim}{G}} \Longleftrightarrow \forall\, i=1,\dots,n \qquad \sum_{j=r+1}^{n} [X_i,X_j]\frac{\partial P}{\partial x_j} = 0$$

and the inclusion of $R(\underset{\sim}{J})^{\underset{\sim}{G}}$ in $R(\underset{\sim}{J}_1)$ means :

$$(*) \quad \forall P \in R(\underset{\sim}{J}) \quad \left\{\forall i=1,\dots,n \quad \sum_{j=r+1}^{n} [X_i,X_j]\frac{\partial P}{\partial x_j} = 0\right\} \Longrightarrow \frac{\partial P}{\partial X_{r+1}} = 0 .$$

By 1.7 applied with $V = \underset{\sim}{J}$, we know that the solutions $\{Q_{r+1},\dots,Q_n\}$ in $R(\underset{\sim}{J})^{n-r}$ of the system of equations

$$\forall\, i=1,\dots,n \quad \sum_{j=r+1}^{n} [X_i,X_j]Q_j = 0$$

are linearly generated over $R(\underset{\sim}{J})$ by the solutions $\left\{\frac{\partial P}{\partial X_{r+1}},\dots,\frac{\partial P}{\partial X_n}\right\}$, where $P \in R(\underset{\sim}{J})^{\underset{\sim}{G}}$. Thus $(*)$ implies

$$(**) \quad \forall \{Q_{r+1},\dots,Q_n\} \in R(\underset{\sim}{J})^{n-r} \quad \left\{\forall\, i=1,\dots,n \quad \sum_{j=r+1}^{n} [X_i,X_j]\,Q_j = 0\right\} \Longrightarrow Q_{r+1} = 0 .$$

Put $a_{ij} = [X_i,X_j] \in S(\underset{\sim}{J})$ for $1 \leq i \leq n$, $r+1 \leq j \leq n$. $A = (a_{ij})$ defines a $R(\underset{\sim}{J})$-linear mapping $R(\underset{\sim}{J})^{n-r} \longrightarrow R(\underset{\sim}{J})^n$ and $\operatorname{rank} A = (n-r) - \dim(\operatorname{Ker} A)$. Condition $(**)$ means $\operatorname{Ker} A \subset W = \left\{\{Q_{r+1},\dots,Q_n\} \in R(\underset{\sim}{J})^{n-r} \mid Q_{r+1} = 0\right\}$. Thus $\operatorname{rank} A|_W = \operatorname{rank} A - 1$. Let $B = -{}^tA$ and B_1 be the matrix obtained by deleting the first line of B. As $a_{ij} \in \underset{\sim}{J}_1 \subset R(\underset{\sim}{J}_1)$ for $1 \leq i \leq n$ and $r+2 \leq j \leq n$, the rank of B_1 over $R(\underset{\sim}{J}_1)$ is the rank of B over $R(\underset{\sim}{J})$ minus one, and for any $R \in R(\underset{\sim}{G})$ we have

$$R \in R(\underset{\sim}{G})^{\underset{\sim}{J}} \Longleftrightarrow \forall j=r+1,\ldots,n \quad \sum_{i=1}^{n} [X_j, X_i] \frac{\partial R}{\partial X_i} = 0$$

$$R \in R(\underset{\sim}{G})^{\underset{\sim}{J}_1} \Longleftrightarrow \forall j=r+2,\ldots,n \quad \sum_{i=1}^{n} [X_j, X_i] \frac{\partial R}{\partial X_i} = 0$$

By 1.7 again, the transcendental degrees of $R(\underset{\sim}{G})^{\underset{\sim}{J}}$ and $R(\underset{\sim}{G})^{\underset{\sim}{J}_1}$ over $\underline{k}$ are n - rank B and n - rank B_1 respectively . ■

5.12 Corollary: *Assume that $\underset{\sim}{G}$ is nilpotent. Then, with the same notations, if $U(\underset{\sim}{J})^{\underset{\sim}{G}} \subset U(\underset{\sim}{J}_1)$, $U(\underset{\sim}{G})^{\underset{\sim}{J}}$ is strictly included in $U(\underset{\sim}{G})^{\underset{\sim}{J}_1}$.*

Proof: By 1.5 we have $U(\underset{\sim}{J})^{\underset{\sim}{G}} = \lambda^{-1}(S(\underset{\sim}{J})^{\underset{\sim}{G}})$ and $U(\underset{\sim}{J}_1)^{\underset{\sim}{G}} = \lambda^{-1}(S(\underset{\sim}{J}_1)^{\underset{\sim}{G}})$, and their respective fields of fractions are isomorphic to $R(\underset{\sim}{J})^{\underset{\sim}{G}}$ and $R(\underset{\sim}{J}_1)^{\underset{\sim}{G}}$ by [2] lemma 10 . Thus our assumption implies $R(\underset{\sim}{J})^{\underset{\sim}{G}} \subset R(\underset{\sim}{J}_1)^{\underset{\sim}{G}}$. As $\underset{\sim}{G}$ is algebraic, we can find by the last proposition $R_o \in R(\underset{\sim}{G})^{\underset{\sim}{J}_1} - R(\underset{\sim}{G})^{\underset{\sim}{J}}$, and again by [2] lemma 10, $R_o = P_o Q_o^{-1}$, with $P_o, Q_o \in S(\underset{\sim}{G})^{\underset{\sim}{J}_1}$. As $R_o \notin R(\underset{\sim}{G})^{\underset{\sim}{J}}$, there exists $X \in \underset{\sim}{J} - \underset{\sim}{J}_1$ such that

$$o \neq [R_o, X] = ([P_o, X] - P_o Q_o^{-1} [Q_o, X]) Q_o^{-1}$$

Thus $[P_o, X] \neq P_o Q_o^{-1} [Q_o, X]$, and either $[P_o, X] \neq o$ or $[Q_o, X] \neq o$. Hence either P_o or Q_o belongs to $S(\underset{\sim}{G})^{\underset{\sim}{J}_1} - S(\underset{\sim}{G})^{\underset{\sim}{J}}$, and its image by λ to $U(\underset{\sim}{G})^{\underset{\sim}{J}_1} - U(\underset{\sim}{G})^{\underset{\sim}{J}}$. ■

5.13 Theorem: *If $\underset{\sim}{G}$ is nilpotent,* $R^{\infty}(\underset{\sim}{G}) = A(\underset{\sim}{G})$.

Proof: By 5.5 the sequence $R^j(\underset{\sim}{G})$ is stationary, say at $R^{j_o}(\underset{\sim}{G}) = R^{\infty}(\underset{\sim}{G})$, and by 5.10, $\underset{\sim}{A} = A(\underset{\sim}{G}) \subset R^{j_o}(\underset{\sim}{G}) = \underset{\sim}{J}$. If the last inclusion was strict one could find an ideal $\underset{\sim}{J}_1$ of $\underset{\sim}{G}$ of codimension 1 in $\underset{\sim}{J}$ and containing $\underset{\sim}{A}$, and

$$Z(\underset{\sim}{G}) \subset U(\underset{\sim}{A})^{\underset{\sim}{G}} \subset U(\underset{\sim}{J}_1)^{\underset{\sim}{G}} \subset U(\underset{\sim}{J})^{\underset{\sim}{G}} \subset U(\underset{\sim}{G})^{\underset{\sim}{G}} = Z(\underset{\sim}{G})$$

would imply $U(\underset{\sim}{J})^{\underset{\sim}{G}} = U(\underset{\sim}{J}_1)^{\underset{\sim}{G}}$. By corollary 5.12 there would exist $Q \in U(\underset{\sim}{G})$

commuting to $\underset{\sim}{J}_1$ but not to $\underset{\sim}{J}$, thus $\underset{\sim}{J}$-reducing. But then $R^{j_o+1}(\underset{\sim}{G}) \subset c(Q) = \underset{\sim}{J}_1$ would be stictly included in $\underset{\sim}{J} = R^{j_o}(\underset{\sim}{G})$, in contradiction with the definition of j_o. ∎

5.14 The necessity of considering reducing elements of all degrees to get a statement like 5.13 will be shown on the example of the triangular algebras in the next paragraph. The necessity of several successive reductions (the sequence R^j) follows already from the remark that in any case $R(\underset{\sim}{G}) \supset \underset{\sim}{G}'$ when $\underset{\sim}{G}$ is nilpotent, since all the $c(Q)$ are 1-codimensional ideals of $\underset{\sim}{G}$ (see the proof of lemma 5.7), thus contain $\underset{\sim}{G}'$.

5.15 Example: Take $\underset{\sim}{G} = \Gamma_{5,3}$ (notation of [3]), that is to say the 5-dimensional algebra defined by the brackets

$$[X_1,X_2] = X_4 , \quad [X_1,X_4] = X_5 , \quad [X_2,X_3] = X_5$$

One can check that X_4, X_3, and $X_4^2-2X_2X_5$ are $\underset{\sim}{G}$-reducing, and

$$c(X_4) = \langle X_2,X_3,X_4,X_5\rangle, \; c(X_3) = \langle X_1,X_3,X_4,X_5\rangle, \; c(X_4^2-2X_2X_5) = \langle X_1,X_2,X_4,X_5\rangle .$$

So $R(\underset{\sim}{G}) \subset \langle X_4,X_5\rangle = \underset{\sim}{G}'$, thus $R(\underset{\sim}{G}) = \underset{\sim}{G}'$ by 5.14.

But X_1 is $\underset{\sim}{G}'$-reducing and $c(X_1) = \langle X_5\rangle$. Finally :

$$\underset{\sim}{A}(\underset{\sim}{G}) = R^{\infty}(\underset{\sim}{G}) = R^2(\underset{\sim}{G}) = \langle X_5\rangle \neq \langle X_4,X_5\rangle = R(\underset{\sim}{G}) = \underset{\sim}{G}' .$$

5.16 Example: $\underset{\sim}{G}$ is the 6-dimensional nilpotent algebra (called $G_{6,23}$ in [15]) defined by the brackets

$$[X_1,X_2] = X_4 , \quad [X_1,X_3] = X_5 , \quad [X_1,X_4] = X_6 , \quad [X_2,X_3] = X_6 , \quad [X_2,X_4] = -X_5$$

$Q_1 = X_3X_5+X_4X_6$, $Q_2 = X_4X_5-X_3X_6$ and $Q_3 = X_4^2-2X_1X_5-2X_2X_6$ are the only $\underset{\sim}{G}$-reducing elements in $U_2(\underset{\sim}{G})$ and more precisely :

$c(Q_1) = \langle X_2, X_3, G' \rangle$, $c(Q_2) = \langle X_1, X_3, G' \rangle$, $c(Q_3) = \langle X_1, X_2, G' \rangle$

and $-[Q_1, X_1] = [Q_2, X_2] = [Q_3, X_3] = X_5^2 + X_6^2 \in Z(G)$.

So $R_2(G) = R(G) = \langle X_4, X_5, X_6 \rangle = G'$. Further X_1 is G'-reducing, and

$c(X_1) = \langle X_5, X_6 \rangle = R^2(G) = A(G)$.

5.17 Example: G is the 7-dimensional algebra defined by the brackets

$[X_1, X_2] = [X_3, X_4] = X_7$, $[X_1, X_3] = [X_2, X_4] = X_6$, $[X_1, X_4] = [X_2, X_3] = X_5$

The only G-reducing elements in $U_2(G)$ are

$Q_1 = -X_2X_7 + X_3X_6 - X_4X_5$, $Q_2 = X_1X_7 - X_3X_5 + X_4X_6$, $Q_3 = -X_1X_6 + X_2X_5 - X_4X_7$

and $Q_4 = X_1X_5 - X_2X_6 + X_3X_7$, and $[Q_j, X_j] = X_5^2 - X_6^2 + X_7^2$ $(j=1,2,3,4)$.

Thus $R_2(G) = R(G) = A(G) = \langle X_5, X_6, X_7 \rangle$.

5.18 Corollary: *For a nilpotent algebra G, the following conditions are equivalent:*

(a) *G is the soul of a Lie algebra over k*

(b) *G is its own soul*

(c) *There is no G-reducing element in $U(G)$*

(d) *G is completely transversal.*

Proof: If $G = A(H)$, there is no G-reducing element in $U(H)$ by 5.9. In particular there is none in $U(G)$. Thus (a) $\Longrightarrow$ (c) . (c) $\Longrightarrow$ (b) by 5.15, and (b) $\Longrightarrow$ (a) trivially. Finally (b) $\Longleftrightarrow$ (d) by 4.4(c) . ∎

§6- The example of strictly lower triangular matrices

6.1 We call $\underset{\sim}{T}_n$ the nilpotent Lie algebra of strictly lower triangular $n\times n$ matrices with entries in $\underset{\sim}{k}$, and $\{X_{ij}\}_{1\leq j<i\leq n}$ its canonical basis : all the entries of X_{ij} are zero, but for the (i,j)th, equal to one. Clearly

$$(*)\quad [X_{ij},X_{kl}] = \delta_{jk}X_{il} - \delta_{li}X_{kj} \quad \text{for } i>j,k>l ,$$

with $\delta_{jk} = o$ if $j\neq k$, 1 if $j=k$.

we will use the notation $|A|$ for the formal determinant of an $m\times m$ matrix $A = (a_{ij})$ with entries in a not necessarily abelian ring, meaning :

$$|A| = \sum_{\sigma\in\mathfrak{S}_m} \varepsilon(\sigma) a_{\sigma(1),1}\cdots a_{\sigma(m),m}$$

where $\varepsilon(\sigma)$ is the signature of the permutation σ of $\{1,2,\ldots,m\}$.

Let us write $D_1 = |X_{n,1}|$, $D_2 = \begin{vmatrix} X_{n-1,1} & X_{n-1,2} \\ X_{n,1} & X_{n,2}\end{vmatrix}$, ... ,

$$D_{\left[\frac{n}{2}\right]} = \begin{vmatrix} X_{\left[\frac{n+1}{2}\right]+1,1} & \cdots & X_{\left[\frac{n+1}{2}\right]+1,\frac{n}{2}} \\ \vdots & & \\ X_{n,1} & \cdots & X_{n,\frac{n}{2}} \end{vmatrix}$$

where $[\cdot]$ means the integral part of a rational number.

It is known that $R(\underset{\sim}{T}_n)^{\underset{\sim}{T}_n}$ is the field $\underset{\sim}{k}(\bar{D}_1,\ldots,\bar{D}_{\left[\frac{n}{2}\right]})$ of the rational functions of the $\bar{D}_j$, $1\leq j\leq\left[\frac{n}{2}\right]$ ([4], th.1 ; we again write $\bar{P} = \lambda^{-1}(P)$), and more precisely $S(\underset{\sim}{T}_n)^{\underset{\sim}{T}_n}$ is the ring $\underset{\sim}{k}\left[\bar{D}_1,\ldots,\bar{D}_{\left[\frac{n}{2}\right]}\right]$ of polynomial functions of these variables ([4], lemma 2). Thus

$$Z(\underset{\sim}{T}_n) = \underset{\sim}{k}\left[D_1,\ldots,D_{\left[\frac{n}{2}\right]}\right] \; ; \quad \text{by 1.5 , and we conclude}$$

$$\underset{\sim}{A}(\underset{\sim}{T}_n) = \left\langle \left\{X_{ij} \;\middle|\; 1\leq j\leq\left[\tfrac{n}{2}\right],\ 1+\left[\tfrac{n+1}{2}\right]\leq i\leq n\right\}\right\rangle .$$

For any $q < \left[\frac{n+1}{2}\right]$, we call A_q the matrix $(X_{ij})_{n-q+1 \leq i \leq n, 1 \leq j \leq q}$, and for $1 \leq l \leq q$, we call $A_{q,l}$ the matrix obtained by replacing in A_q the l-th column by the column $\left\{X_{i,q+1} \mid n-q+1 \leq i \leq n\right\}$, and $A'_{q,l}$ the one obtained by replacing in A_q the (n-l+1)-th row by the row $\left\{X_{n-q,j} \mid 1 \leq j \leq q\right\}$. We put $Q_{q,l} = |A_{q,l}|$, $Q'_{q,l} = |A'_{q,l}|$, and we call $\underset{\sim}{J}_q$ and $\underset{\sim}{C}_q$ the subspaces of $\underset{\sim}{T}_n$ engendered by the X_{ij} for $1 \leq j \leq n-q$, $q+1 \leq i \leq n$ $(i > j)$, and for $1 \leq j \leq q$, $n-q+1 \leq i \leq n$ respectively (they are ideals of $\underset{\sim}{T}_n$) :

6.2 Proposition: (a) $Q_{q,l}$ is $\underset{\sim}{J}_q$-reducing and more precisely

$$X_{i,j} \in \underset{\sim}{J}_q \Longrightarrow \left[Q_{q,l}, X_{i,j}\right] = \begin{cases} D_q & \text{if } (i,j) = (q+1,l) \\ o & \text{otherwise}. \end{cases}$$

(b) $Q'_{q,l}$ is $\underset{\sim}{J}_q$-reducing and more precisely

$$X_{ij} \in \underset{\sim}{J}_q \Longrightarrow \left[Q'_{q,l}, X_{j,j}\right] = \begin{cases} D_q & \text{if } (i,j) = (n-l+1, n-q) \\ o & \text{otherwise}. \end{cases}$$

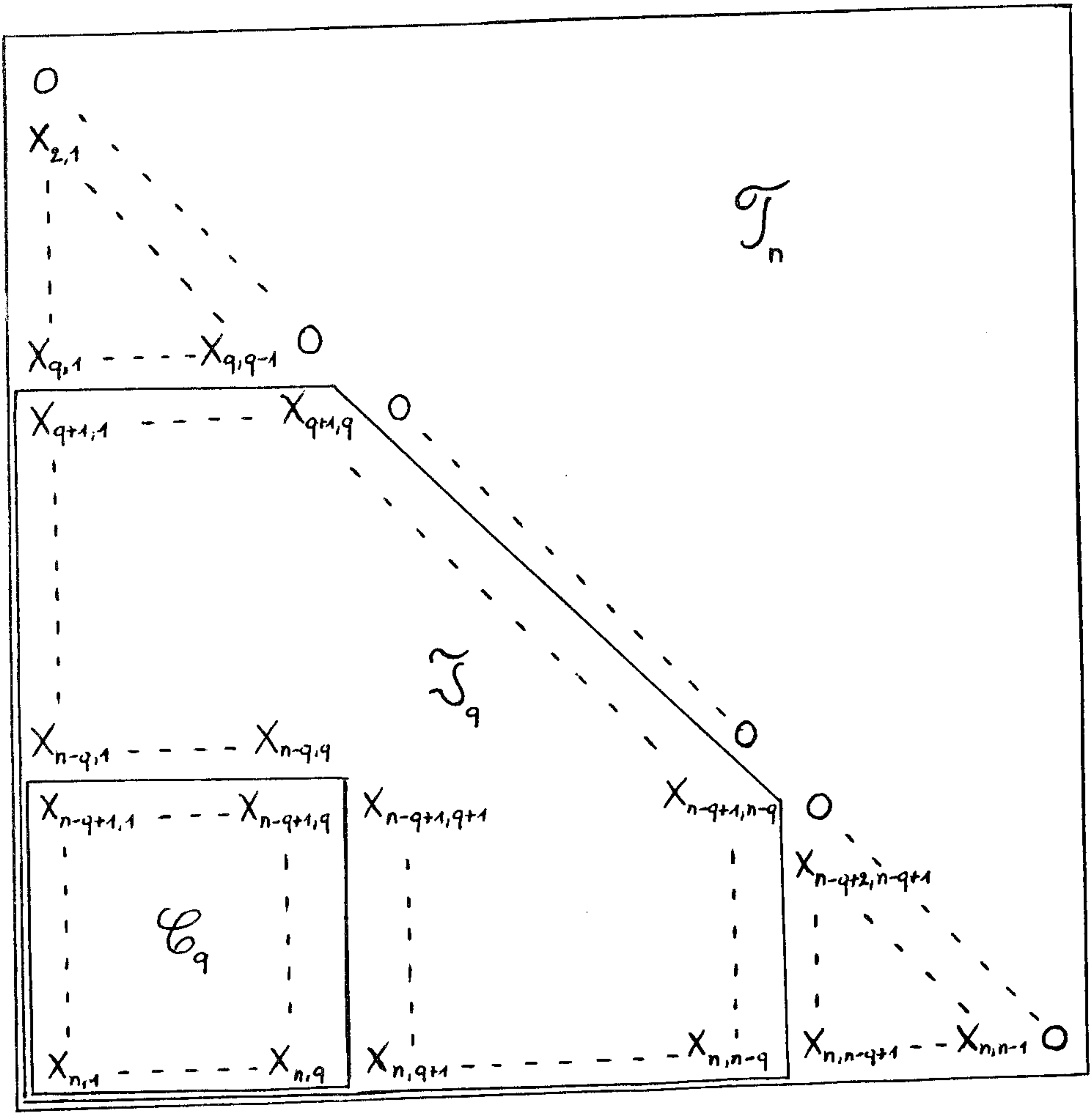
Proof: A direct computation, using the relations (*) . ∎

By induction on q one can deduce easily :

6.3 Corollary: $R^q(\underset{\sim}{T}_n) \subset \underset{\sim}{J}_{q+1}$ for $o \leq q < \left[\frac{n+1}{2}\right]$.

And $R^{\left[\frac{n+1}{2}\right]-1}(\underset{\sim}{T}_n) = \underset{\sim}{J}_{\left[\frac{n+1}{2}\right]} = \underset{\sim}{A}(\underset{\sim}{T}_n) = R^{\infty}(\underset{\sim}{T}_n)$.

The necessity of considering reducing elements of arbitrarily high degree for obtaining the soul of an algebra as the limit of reducing ideals is shown on this example by the following proposition :



0
$X_{2,1}$
$\mathcal{T}_n$
$X_{q,1}$ - - - $X_{q,q-1}$ 0
$X_{q+1,1}$ - - - $X_{q+1,q}$ 0
$\mathcal{T}_q$
0
$X_{n-q,1}$ - - - $X_{n-q,q}$
$X_{n-q+1,1}$ - - - $X_{n-q+1,q}$
$X_{n-q+1,q+1}$
$X_{n-q+1,n-q}$
0
$X_{n-q+2,n-q+1}$
$\mathcal{C}_q$
$X_{n,1}$ - - - $X_{n,q}$
$X_{n,q+1}$ - - - $X_{n,n-q}$
$X_{n,n-q+1}$ - - $X_{n,n-1}$ 0

6.4 Proposition: $R_q(J_q) = J_q$ for $o \leq q \leq \left[\frac{n}{2}\right]$.

Proof: The proof uses the next statement. Suppose there exists an J_q-reducing $Q \in U_q(T_n)$. By 5.8 we can assume $[Q, J_q] \subset U(T_n)^{J_q}$.

For any l, $q \leq l \leq n-q$, $Q_l = [Q, X_{l+1,l}]$ is, by theorem 6.5, a polnomial of the X_{ij} $(n-q+1 \leq i \leq n, 1 \leq j \leq q)$ and the $D_m(q+1 \leq m \leq \left[\frac{n}{2}\right])$. As $\deg Q_l \leq q$, Q_l is a polynomial of the X_{ij} alone, that is $Q_l \in U(C_q)$. But

$$Q_l = \sum_{r=l+2}^{n} \frac{\partial Q}{\partial X_{r,l+1}} . X_{rl} \quad \text{modulo terms of a lower degree,}$$

and since no $X_{r,l+1}$ belongs to C_q, we conclude $Q_l = o$.

Hence $c(Q)$ is an ideal of J_q (since T_n is nilpotent, see the proof of lemma 5.7) containing all the $X_{l+1,l}$ $(q \leq l \leq n-q)$; but this implies $c(Q) = J_q$, contrary to the definition of a reducing element. ∎

6.5 Theorem: The commutator of J_q in $U(T_n)$ is the abelian algebra of the polynomials of the X_{ij} $(n-q+1 \leq i \leq n, 1 \leq j \leq q)$ and of the D_l $(q+1 \leq l \leq \left[\frac{n}{2}\right])$.

For $q = \left[\frac{n}{2}\right]$, this is nothing but theorems 1 and 4 of [4], and the proof we give here generalizes the arguments of [4]. We divide it into five lemmas.

For $p \in \mathbb{N}$ we call $M_{p,p'}$ the space of $p \times p'$ matrices with entries in $\underline{k}$, $M_p = M_{p,p}$, and if $X = (x_{ij}) \in M_p$ and $1 \leq q \leq p$, we put

$$\Delta_q(X) = \begin{vmatrix} x_{1,q} & \cdots\cdots\cdots & x_{1,p} \\ \vdots & & \vdots \\ x_{p-q+1,q} & \cdots\cdots & x_{p-q+1,p} \end{vmatrix}$$

and $N_{p,q} = \left\{ X \in M_p \;\middle|\; \Delta_q(X).\Delta_{q+1}(X).\cdots.\Delta_p(X) \neq o \right\}$.

6.6 Lemma: Any $X \in N_{p,q}$ can be written in only one way $X = YEZ$, with

$Y = \begin{pmatrix} I & 0 \\ C & D \end{pmatrix}$, $Z = \begin{pmatrix} A & 0 \\ B & I \end{pmatrix}$, I <u>being the unit-matrix of</u> M_q , A <u>and</u> D <u>lower triangular matrices of</u> M_{p-q} <u>with entries</u> 1 <u>on the diagonal,</u> $B \in M_{q,p-q}$, $C \in M_{p-q,q}$, <u>and</u> $E = (e_{ij})$ <u>with</u>

$$e_{ij} = \begin{cases} x_{ij} & \text{if } 1 \leq i \leq q \text{ and } p-q+1 \leq j \leq p \\ 0 & \text{if } i > q \text{ or } j \leq p-q \text{, and } i+j \neq p+1 \\ (-1)^{i-1} \cdot \dfrac{\Delta_i(X)}{\Delta_{i-1}(X)} & \text{if } q+1 \leq i \leq p \text{ and } j = p-i+1 \end{cases} .$$

<u>Proof</u>: Note that Y^{-1} has the same form as Y and put $Y^{-1} = (y_{ij})$, $Z = (z_{ij})$.

The equation $X = YEZ$ becomes :

$$(**) \quad \forall i,j \quad \sum_{k \geq i} y_{ik} x_{kj} = \sum_{k \geq j} e_{ik} z_{kj} .$$

<u>For $q+1 \leq i \leq p$ and $j > p-i+1$</u> , (**) means $\sum_{k \leq i-1} y_{ik} x_{kj} = -x_{ij}$, and this determines the y_{ik} entirely, the determinant of the system being $\Delta_{i-1}(X) \neq 0$. Hence Y is uniquely determined.

<u>For $q+1 \leq i \leq p$ and $j = p-i+1$</u> . (**) gives $e_{i,p-i+1} = \sum_{k \leq i} y_{ik} x_{k,p-i+1}$, and by the Cramer formulae giving the y_{ik} :

$$e_{i,p-i+1} = (-1)^{i-1} \cdot \frac{\Delta_i(X)}{\Delta_{i-1}(X)} . \text{ In particular } e_{i,p-i+1} \neq 0 .$$

<u>For $q+1 \leq i \leq p$ and $j < p-i+1$</u> , we get $\sum_{k \leq i-1} y_{ik} x_{kj} = e_{i,p-i+1} z_{p-i+1,j} - x_{ij}$ which determines the $z_{p-i+1,j}$ for $1 \leq j \leq p-i$, $q+1 \leq i \leq p$, that is to say the matrix A .

<u>For $1 \leq i \leq q$</u> . (**) becomes $x_{ij} = \sum_{k \geq j} e_{ik} z_{kj}$,

and we get $x_{ij} = e_{ij}$ for $p-q+1 \leq j \leq p$, and for $1 \leq j \leq p-q$

$$x_{ij} = \sum_{k \geq p-q+1} e_{ik} z_{kj} .$$

The determinant of this last system with unknown z_{kj} ($p-q+1 \leq k \leq p$, j fixed) is $\Delta_q(X) \neq 0$, and the matrix B is thus entirely determined . ■

We owe the proofs of the two following lemmas to J. Briançon.

6.7 Lemma: Let $F = \sum F_\alpha \Delta^\alpha$ be a polynomial function on M_p, where $\Delta^\alpha = \Delta_{q+1}^{\alpha_{q+1}} \ldots \Delta_p^{\alpha_p}$, and for each $\alpha = (\alpha_{q+1}, \ldots, \alpha_p) \in \mathbb{N}^{p-q}$, F is a polynomial of the x_{ij} for $1 \leq i \leq l$, $p-l+1 \leq j \leq p$, where $l = \sup \{k \mid \alpha_k \neq 0\}$. Then if Δ_q divides F, it divides all the F_α.

Proof: By induction on l, $q \leq l \leq p$. (It is clear for $l=q$). If $F = \sum F_\alpha \Delta^\alpha$ is a polynomial on M_l $(l > q)$, $\Delta^\alpha = \Delta_{q+1}^{\alpha_{q+1}} \ldots \Delta_l^{\alpha_l}$, and for each $\alpha = (\alpha_{q+1}, \ldots, \alpha_l) \in \mathbb{N}^{l-q}$ F_α is a polynomial of the x_{ij} for $1 \leq i \leq l'$, $p-l'+1 \leq j \leq p$, where $l' = \sup \{k \mid \alpha_k \neq 0\}$, and if Δ_q divides F, we can write $F = G + \bar{F}$ where G is a multiple of Δ_q and $\bar{F} = \sum_{\alpha \in A} F_\alpha \Delta^\alpha$, A being the set of the $\alpha \in \mathbb{N}^{l-q}$ suchthat Δ_q does not divide F. The restriction of $\bar{F}$ to the subspace of M_l defined by the equations $x_{i,p-l+1} = 0$ $(1 \leq i \leq l)$ is $\sum_{\alpha \in A,\, \alpha_l = 0} F_\alpha \Delta^\alpha$, and it is again a multiple of Δ_q. By induction we have $\{\alpha \in A \mid \alpha_l = 0\} = \emptyset$, and thus

$$\bar{F} = \Delta_l \left(\sum_{\alpha \in A} F_\alpha \Delta^{\alpha - \varepsilon_l} \right)$$

where $\alpha - \varepsilon_l \in \mathbb{N}^{l-q}$ and $\varepsilon_l = (0, \ldots, 0, 1)$.

As Δ_l and Δ_q are relatively prime, the bracket is itself a multiple of Δ_q, and the proof follows by induction on $|\alpha| = \alpha_{q+1} + \ldots + \alpha_l$. ■

6.8 Lemma: Every polynomial function on M_p which is a polynomial of the x_{ij} $(1 \leq i \leq q$, $p-q+1 \leq j \leq p)$ and of $\frac{\Delta_{q+1}}{\Delta_q}, \ldots, \frac{\Delta_p}{\Delta_q}$, is actually a polynomial of the same x_{ij} and of $\Delta_{q+1}, \ldots, \Delta_p$.

Proof: Let

$$P = \sum_{\alpha \in \Lambda} P_\alpha \left(\frac{\Delta_{q+1}}{\Delta_q} \right)^{\alpha_{q+1}} \ldots \left(\frac{\Delta_p}{\Delta_q} \right)^{\alpha_p}$$

be a polynomial function on M_p, where $\Lambda \subset \mathbb{N}^{p-q}$ is finite, and for $\alpha = (\alpha_{q+1}, \ldots, \alpha_p)$ P is a non-zero polynomial of the x_{ij} $(1 \leq i \leq q, p-q+1 \leq j \leq p)$. For $\alpha \in \Lambda$, put

$P_\alpha = Q_\alpha (\Delta_q)^{r_\alpha}$ where Δ_q does not divide Q_α, and $\sigma = \sup_{\alpha \in \Lambda} \{|\alpha| - r_\alpha\}$.

Then $P = \sum_{\alpha \in \Lambda} Q_\alpha \Delta_q^{r_\alpha - |\alpha|} \Delta_{q+1}^{\alpha_{q+1}} \cdots \Delta_p^{\alpha_p}$ is a polynomial on M_p, and

$$\Delta_q^\sigma P = \sum_{\alpha \in \Lambda, |\alpha| - r_\alpha = \sigma} Q_\alpha \Delta_{q+1}^{\alpha_{q+1}} \cdots \Delta_p^{\alpha_p} \quad \text{modulo a multiple of } \Delta_q .$$

If one had $\sigma > o$, the right term would be a multiple of Δ_q, and this would imply by lemma 6.7 $\{\alpha \in \Lambda \mid |\alpha| - r_\alpha = \sigma\} = \emptyset$, an absurd statement, since Λ is finite. Thus for all α, $|\alpha| \leq r_\alpha$. ■

<u>6.9</u> <u>Lemma</u>: <u>With the notations of lemma 6.6, for any polynomial function</u> $f : M_p \longrightarrow \underline{k}$, <u>the following are equivalent</u> :

(a) $\forall X \in M_p$, $\forall Y, Z$ <u>as in lemma 6.6</u>, $f(YXZ) = f(X)$

(b) f <u>is a polynomial of the</u> x_{ij} $(1 \leq i \leq q, p-q+1 \leq j \leq p)$ <u>and of</u> $\Delta_{q+1}, \ldots, \Delta_p$.

<u>Proof</u>: Put $(x'_{ij}) = X' = YXZ$. Then $x'_{ij} = x_{ij}$ for $1 \leq i \leq q$, $p-q+1 \leq j \leq p$, and for any r $(q+1 \leq r \leq p)$,

$$(x'_{ij})_{\substack{1 \leq i \leq r \\ p-r+1 \leq j \leq p}} = (y_{ij})_{\substack{1 \leq i \leq r \\ 1 \leq j \leq r}} \cdot (x_{ij})_{\substack{1 \leq i \leq r \\ p-r+1 \leq j \leq p}} \cdot (z_{ij})_{\substack{p-r+1 \leq i \leq p \\ p-r+1 \leq j \leq p}},$$

so that $\left| x'_{ij} \right|_{\substack{1 \leq i \leq r \\ p-r+1 \leq j \leq p}} = \left| x_{ij} \right|_{\substack{1 \leq i \leq r \\ p-r+1 \leq j \leq p}}$, and (b) implies (a).

Reciprocally, if f satisfies (a), let g be its restriction to the subspace of M_p defined by the equations $x_{ij} = o$ ($i > q$ or $j \leq p-q$, and $i+j \neq p+1$). Then g is a polynomial of the x_{ij} $(1 \leq i \leq q,\ p-q+1 \leq j \leq p)$ and of $x_{q+1,p-q}, \ldots, x_{p,1}$. If $X \in N_{p,q}$, write $X = YEZ$, using lemma 6.6. Then

$$\begin{aligned} f(X) &= f(YEZ) = f(E) \\ &= g\left\{e_{ij}(1 \leq i \leq q, p-q+1 \leq j \leq p), e_{q+1,p-q}, \ldots, e_{p,1}\right\} \\ &= g\left\{x_{ij}(1 \leq i \leq q, p-q+1 \leq j \leq p), (-1)^q \frac{\Delta_{q+1}(X)}{\Delta_q(X)}, \ldots, (-1)^{p-1} \frac{\Delta_p(X)}{\Delta_{p-1}(X)}\right\}. \end{aligned}$$

Let h be the restriction of f to the subspace of the matrices of the form

$$\begin{pmatrix} 0 & \cdots & 0 & x_{1,p-q+1} & \cdots & x_{1,p} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & x_{q,p-q+1} & \cdots & x_{q,p} \\ x_{q+1,1} & \cdots & x_{q+1,p-q} & 0 & \cdots & 0 \\ 0 & \cdots 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Then h is still a polynomial of the remaining variables, and its restriction to the open subset $\left\{x_{q+1,2} \neq 0 , \dots , x_{q+1,p-q} \neq 0\right\}$ is

$$h\left\{x_{ij}(1\leq i\leq q, p-q+1\leq j\leq p), x_{q+1,p-q}, x_{q+1,p-q-1}, x_{q+1,1}\right\}$$
$$= g\left\{x_{ij}(1\leq i\leq q, p-q+1\leq j\leq p), -x_{q+1,p-q}, (-1)^{q+1}\frac{x_{q+1,p-q+1}}{x_{q+1,p-q}}, \dots, (-1)^{p-1}\frac{x_{q+1,1}}{x_{q+1,2}}\right\} .$$

Comparing the two last expressions we have of g, we get

$$f(X) = h\left\{x_{ij}(1\leq i\leq q, p-q+1\leq j\leq p), (-1)^{q+1}\frac{\Delta_{q+1}}{\Delta_q} , (-1)^{q+1}\frac{\Delta_{q+2}}{\Delta_q} , \dots, (-1)^{q+1}\frac{\Delta_p}{\Delta_q}\right\}.$$

The conclusion follows then from lemma 6.8 . ∎

<u>6.10</u> <u>Lemma</u>: <u>For</u> $0\leq q\leq\left[\frac{n}{2}\right]$, $S(\underset{\sim}{T}_n)^{\underset{\sim}{J}_q} \subset S(\underset{\sim}{A}(\underset{\sim}{T}_n))$.

<u>Proof</u>: $P\in S(\underset{\sim}{T}_n)$ belongs to $S(\underset{\sim}{T}_n)^{\underset{\sim}{J}_q}$ if and only if

$$(***) \quad 0 = \sum_{k>l}\left[X_{ij}, X_{kl}\right]\frac{\partial P}{\partial X_{kl}} \qquad \text{for all } (i,j) \text{ such that } i>j, i\geq q+1, j\leq n-q$$

$$= \sum_{l<j} X_{il}\frac{\partial P}{\partial X_{jl}} - \sum_{k>i} X_{kj}\frac{\partial P}{\partial X_{ki}} \qquad \text{by the relations } (*) .$$

For instance for $(i,j) = (n-1,1)$ and $(n,2)$, one gets $\dfrac{\partial P}{\partial X_{n,n-1}} = \dfrac{\partial P}{\partial X_{2,1}} = 0$.

Assume $P\in S(\underset{\sim}{T}_n)^{\underset{\sim}{J}_q}$ and does not depend on the X_{ij} for $i\leq p$ or $j\geq n-p+1$, for a given integer $p<\left[\frac{n+1}{2}\right]$. Then, as $p<n-p$, (***) applied with $i=n-p$ and $1\leq j\leq p$, gives

$$\sum_{k>n-p} X_{k,p}\frac{\partial P}{\partial X_{k,n-p}} = \sum_{1<j\leq p} X_{n-p,1}\frac{\partial P}{\partial X_{j,1}} = 0 \quad , \text{ hence } \frac{\partial P}{\partial X_{k,n-p}} = 0$$

for $k>n-p$. Inthe same way, for $j = p+1$ and $n-p+1\leq i\leq n$, (***) gives

$$\sum_{1<p+1} X_{i1}\frac{\partial P}{\partial X_{p+1,1}} = \sum_{k>i\geq n-p+1} X_{k,p+1}\frac{\partial P}{\partial X_{k,i}} = 0 \text{ , hence } \frac{\partial P}{\partial X_{p+1,1}} = 0$$

for $1<p+1$. By induction on p, we conclude that P does not depend on any X_{ij} such that $i\leq\left[\frac{n+1}{2}\right]$ or $j\geq\left[\frac{n}{2}\right]+1$. ∎

6.11 Proof of Theorem 6.5: For any q ($0\leq q\leq\left[\frac{n}{2}\right]$), each $x\in \exp \underset{\sim}{J}_q$ can be written

$$x = \begin{pmatrix} I & 0 & 0 & 0 \\ B & I+E & 0 & 0 \\ C & F & I+K & 0 \\ D & G & L & I \end{pmatrix} \qquad \text{or} \qquad x = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ B & I+E & 0 & 0 & 0 \\ a & b & 1 & 0 & 0 \\ C & F & c & I+K & 0 \\ D & G & d & L & I \end{pmatrix}$$

(if $n=2p$) (if $n=2p+1$)

with $E,F,K\in M_{p-q,p-q}$; $D\in M_{q,q}$; $B,C\in M_{p-q,q}$; $G,L\in M_{q,p-q}$; $a\in M_{1,q}$; $b\in M_{1,p-q}$; $c\in M_{p-q,1}$; $d\in M_{q,1}$; E and K being strictly lower triangular, and I denoting the unit matrix.

In any case Ad x is an automorphism of the invariant ideal $\underset{\sim}{A}(\underset{\sim}{T}_n) = \underset{\approx}{C}_p$ of $\underset{\sim}{T}_n$, and if $W = \begin{pmatrix} C_0 & F_0 \\ D_0 & G_0 \end{pmatrix} \in \underset{\approx}{C}_p \simeq M_p$, with $F_0\in M_{p-q,p-q}$, $D_0\in M_{q,q}$, $C_0\in M_{p-q,q}$, $G_0\in M_{q,p-q}$, we get, whether n is even or odd,

$$\mathrm{Ad}(x)W = \begin{pmatrix} (I+K)C_0 - (I+K)F_0(I+E)^{-1}B & (I+K)F_0(I+E)^{-1} \\ LC_0 + D_0 - (LF_0+G_0)(I+E)^{-1}B & (LF_0+G_0)(I+E)^{-1} \end{pmatrix}$$

$$= ZWY^{-1} \quad \text{with } Z = \begin{pmatrix} I+K & 0 \\ L & I \end{pmatrix} \quad \text{and } Y = \begin{pmatrix} I & 0 \\ B & I+E \end{pmatrix}$$

Let us identify $\underset{\approx}{C}_p \simeq M_p$ with its dual by means of the canonical bilinear form $(W,W')\longmapsto \mathrm{tr}\, WW'$. By lemma 6.10, if $P\in S(\underset{\sim}{T}_n)^{\underset{\sim}{J}_q}$, we have

$P \in S(\underset{\approx}{C}_p) = S(M_p)$, and since $\quad tr\ (W.Ad(x)W') = tr\ (WZW'Y^{-1}) = tr\ (Y^{-1}WZW')$,

any $P \in S(M_p)$ belongs to $S(\underset{\sim}{T}_n)^{\underset{\sim}{J}_q}$ if and only if the associated polynomial function on $M_p^* = M_p$ is invariant under the automorphisms of M_p

$$W \longmapsto Y^{-1}WZ \quad .$$

By lemma 6.9, this means that P is a polynomial of the x_{ij} $(1 \leq i \leq q, p-q+1 \leq j \leq p)$ and of the Δ_l $(q+1 \leq l \leq p = \left[\frac{n}{2}\right])$. The conclusion follows by transposing again this result by means of the same bilinear form. ∎

§7- The rational soul

7.1 Proposition: There is one and only one structure of Lie algebra on $R(\underset{\sim}{G})$ prolonging that of $\underset{\sim}{G}$ and such that

(*) $\forall R_1, R_2, R_3 \in R(\underset{\sim}{G}) \qquad [R_1 R_2, R_3] = [R_1, R_3] R_2 + R_1 [R_2, R_3]$

This structure is defined and studied in [19] (lemmas 2.3, 2.4) and called the Poisson structure on $R(\underset{\sim}{G})$. By a straightforward computation based on (*), one gets

7.2 Lemma: If $\{X_1, \ldots, X_n\}$ is any basis of $\underset{\sim}{G}$, and R_1, $R_2 \in R(\underset{\sim}{G})$, the Poisson bracket of R_1 and R_2 is :

$$[R_1, R_2] = \sum_{i,j=1}^{n} \frac{\partial R_1}{\partial X_i} \frac{\partial R_2}{\partial X_j} [X_i, X_j]$$

It is thus clear that the center of $R(\underset{\sim}{G})$ for its Poisson structure is precisely the subfield $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ of the rational invariants of G .

7.3 Definition: We call rational soul $\bar{\underset{\sim}{A}} = \bar{\underset{\sim}{A}}(\underset{\sim}{G})$ of $\underset{\sim}{G}$ the intersection of all subalgebras $\underset{\sim}{H}$ of $\underset{\sim}{G}$ such that $R(\underset{\sim}{H}) \supset R(\underset{\sim}{G})^{\underset{\sim}{G}}$.

Most of the statements of §2,3 and 5 on the soul can be adapted to the rational soul, and this is what we do in §7 and 8 . Many proofs, analoguous to those of the corresponding statements on the soul, will thus be omitted.

7.4 Proposition: $\bar{\underset{\sim}{A}}(\underset{\sim}{G})$ is the smallest subalgebra $\underset{\sim}{H}$ of $\underset{\sim}{G}$ such that $R(\underset{\sim}{H}) \supset R(\underset{\sim}{G})^{\underset{\sim}{G}}$, and it is an invariant ideal of $\underset{\sim}{G}$.

Proof: as in proposition 2.2 . ∎

7.5 Proposition: The rational soul of $\underset{\sim}{G}$ is the subspace

$$\overline{\underset{\sim}{A}}(\underset{\sim}{G}) = \sum_{P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}, f \in r(P)} dP(f) ,$$

and it is the smallest subspace V of $\underset{\sim}{G}$ such that $R(V) \supset R(\underset{\sim}{G})^{\underset{\sim}{G}}$.

Proof: as in proposition 3.1 . ∎

7.6 Proposition: The rational soul of a direct sum is the direct sum of the rational souls.

Proof: Assume $\underset{\sim}{G}_o = \underset{\sim}{G}_1 \oplus \underset{\sim}{G}_2$ is the direct sum of two algebraic Lie algebras, then for j = 1,2 :

$$R(\underset{\sim}{G}_j)^{\underset{\sim}{G}_j} \subset R(\underset{\sim}{G}_o)^{\underset{\sim}{G}_o} \subset R(\overline{\underset{\sim}{A}}(\underset{\sim}{G}_o)) \Longrightarrow \overline{\underset{\sim}{A}}(\underset{\sim}{G}_j) \subset \overline{\underset{\sim}{A}}(\underset{\sim}{G}_o)$$

Hence $\overline{\underset{\sim}{A}}(\underset{\sim}{G}_o) \supset \overline{\underset{\sim}{A}}(\underset{\sim}{G}_1) \oplus \overline{\underset{\sim}{A}}(\underset{\sim}{G}_2)$. In order to prove the converse, complete bases $\{X_1,\dots,X_{p_1}\}$ and $\{X_{n_1+1},\dots,X_{n_1+p_2}\}$ of $\overline{\underset{\sim}{A}}(\underset{\sim}{G}_1)$ and $\overline{\underset{\sim}{A}}(\underset{\sim}{G}_2)$ respectively into bases $\{X_1,\dots,X_{n_1}\}$ and $\{X_{n_1+1},\dots,X_{n_1+n_2}\}$ of $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$.

If $n_o=n_1+n_2$, $a_{ij} = [X_i,X_j]$ $(1 \leq i,j \leq n_o)$, $A_1 = (a_{ij})_{1 \leq i,j \leq n_1}$, $A_2 = (a_{ij})_{n_1+1 \leq i,j \leq n_1+n_2}$, rank $A_l = n_l - q_l$ (l=o,1,2) , the rank being taken over $R(\underset{\sim}{G})$, then we have $A_o = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = (a_{ij})_{1 \leq i,j \leq n_o}$, and

by 1.7 , the degree of $R(\underset{\sim}{G}_l)^{\underset{\sim}{G}_l}$ over $\underline{k}$ is q_l (l=o,1,2).

Take systems $\{Q_j(X_1,\dots,X_{n_1}) \mid j=1,\dots,q_1\}$ and $\{Q'_j(X_{n_1+1},\dots,X_{n_1+n_2}) \mid j=1,\dots,q_2\}$ of algebraically independent elements of $R(\underset{\sim}{G}_1)^{\underset{\sim}{G}_1}$ and $R(\underset{\sim}{G}_2)^{\underset{\sim}{G}_2}$ respectively.

If $R(\overline{\underset{\sim}{A}}(\underset{\sim}{G}_1) \oplus \overline{\underset{\sim}{A}}(\underset{\sim}{G}_2))$ did not contain $R(\underset{\sim}{G}_o)^{\underset{\sim}{G}_o}$, we could find

$Q(X_1,\dots,X_{n_o}) \in R(\underset{\sim}{G}_o)^{\underset{\sim}{G}_o}$ such that $\frac{\partial Q}{\partial X_i} \neq o$ for some

$i \in \{p_1+1,\dots,n_1,n_1+p_2+1,\dots,n_o\}$. But then Q would be algebraically independent of the Q_j and Q'_j , so that $R(\underset{\sim}{G}_o)^{\underset{\sim}{G}_o}$, containing $\underline{k}(Q,Q_1,\dots,Q_{q_1},Q'_1,\dots,Q'_{q_2})$ would have at least degree q_1+q_2+1 over $\underline{k}$, and this is absurd, since $q_o=q_1+q_2$. Hence $R(\underset{\sim}{G}_o)^{\underset{\sim}{G}_o} \subset R(\overline{A}(\underset{\sim}{G}_1)\oplus\overline{A}(\underset{\sim}{G}_2))$, and finally $\overline{A}(\underset{\sim}{G}_o) \subset \overline{A}(\underset{\sim}{G}_1)\oplus\overline{A}(\underset{\sim}{G}_2)$.

Now if $\underset{\sim}{G}_1$ and $\underset{\sim}{G}_2$ are not necessarily algebraic, but $\underline{k} = \mathbb{R}$ or $\mathbb{C}$, we still have $\overline{A}(\underset{\sim}{G}_1\oplus\underset{\sim}{G}_2) \subset \overline{A}(\underset{\sim}{G}_1)\oplus\overline{A}(\underset{\sim}{G}_2)$ by the same argument as above. As the rational invariants of $\underset{\sim}{G}_1$, $\underset{\sim}{G}_2$ and $\underset{\sim}{G}_o = \underset{\sim}{G}_1\oplus\underset{\sim}{G}_2$ are the rational solutions of the corresponding systems of differential equations (*) considered in 1.6 , the inverse inclusion follows easily from proposition 7.5 and the classical Frobenius theory of linear differential systems homogeneous of order one . ∎

7.7 $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ always contains the field of fractions of $S(\underset{\sim}{G})^{\underset{\sim}{G}}$, hence $\overline{A}(\underset{\sim}{G}) \supset A(\underset{\sim}{G})$, and they are equal if and only if $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ is the field of fractions of $S(\underset{\sim}{G})^{\underset{\sim}{G}}$. This happens for instance whenever $\underset{\sim}{G}$ is reductive (in this case $A(\underset{\sim}{G}) = \overline{A}(\underset{\sim}{G}) = \underset{\sim}{G}$ by 2.3(b)) , or nilpotent (see 1.5) .

7.8 Example: $\underset{\sim}{G}$ is the 2-dimensional non-abelian Lie algebra : we can find a basis $\{X,Y\}$ of it such that $[X,Y] = Y$. If $P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$, we have $\forall a,b \in \underline{k} \quad o = [P,aX+bY] = Y(b\frac{\partial P}{\partial X} - a\frac{\partial P}{\partial Y})$ by 7.2 , and so P is constant. Thus $R(\underset{\sim}{G})^{\underset{\sim}{G}} = \underline{k}$, and $\overline{A}(\underset{\sim}{G}) = A(\underset{\sim}{G}) = \{o\}$.

7.9 Example: Let $\underset{\sim}{G} = \underset{\sim}{G}_\lambda$ $(\lambda \in R)$ be the solvable 3-dimensional Lie algebra over $\mathbb{R}$ defined on a basis $\{X,Y,Z\}$ by the brackets

$$[X,Y] = Y \quad , \quad [X,Z] = \lambda Z \ .$$

$$(*) \quad \forall P \in \mathbb{R}(G) \quad [X,P] = \frac{\partial P}{\partial Y} Y + \lambda \frac{\partial P}{\partial Z} Z \ , \quad [Y,P] = -\frac{\partial P}{\partial X} Y \ , \quad [Z,P] = -\lambda\frac{\partial P}{\partial X} Z \ .$$

So $P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$ if and only if it is a rational function of Y and Z only, and a function of $Y^{-\lambda}Z$ only .

- if λ is rational negative, writing $\lambda = -\frac{a}{b}$ $(a,b \in \mathbb{N}$, $a \wedge b = 1)$, we get

  $S(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}[Y^aZ^b]$, $R(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}(Y^aZ^b)$, and $\underset{\sim}{A}(\underset{\sim}{G}) = \overline{\underset{\sim}{A}}(\underset{\sim}{G}) = \langle Y,Z \rangle$.

- if $\lambda = 0$, $S(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}[Z]$, $R(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}(Z)$, and $\underset{\sim}{A}(\underset{\sim}{G}) = \overline{\underset{\sim}{A}}(\underset{\sim}{G}) = \langle Z \rangle$.

- if λ is rational positive, writing $\lambda = \frac{a}{b}$ $(a,b \in \mathbb{N}$, $a \wedge b = 1)$, we get

  $S(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}$, $R(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}(Y^{-a}Z^b)$, and $\underset{\sim}{A}(\underset{\sim}{G}) = \{0\}$, $\overline{\underset{\sim}{A}}(\underset{\sim}{G}) = \langle Y,Z \rangle$.

- if λ is irrational (that is if $\underset{\sim}{G}$ is not algebraic), $S(\underset{\sim}{G})^{\underset{\sim}{G}} = R(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}$,

  and $\underset{\sim}{A}(\underset{\sim}{G}) = \overline{\underset{\sim}{A}}(\underset{\sim}{G}) = \{0\}$.

7.10 Proposition: Assume $\underline{k} = \mathbb{R}$ or $\mathbb{C}$ and there exists a dense $\underline{G}$-invariant open subset Ω of $\underline{G}^*$ such that $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ separates the orbits in Ω. Then for any subset Ω' of Ω which is Zariski-dense in $\underset{\sim}{G}^*$, we have

$$\overline{\underset{\sim}{A}}(\underset{\sim}{G}) = \sum_{f \in \Omega'} \underset{\sim}{G}(f)$$

Proof: analoguous to that of proposition 3.2 .■

7.11 Corollary: Under the same assumption as in 7.10 : $\overline{\underset{\sim}{A}}(\underset{\sim}{G})^{\perp} = \bigcap_{\underline{\underline{O}} \subset \Omega} D(\underline{\underline{O}})$

Proof: as in corollary 3.4 .■

7.12 If $\underline{k}$ is algebraically closed, and $\underset{\sim}{G}$ is algebraic, there exists a dense $\underline{G}$-invariant Zariski open subset Ω of $\underset{\sim}{G}^*$ such that $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ separates the $\underline{G}$-orbits in Ω. ([8], proposition 1 even defines such an Ω which is canonical).

§8- Reducing ideals and the rational soul

8.1 Definition: If $\underset{\sim}{J}$ is a subalgebra of $\underset{\sim}{G}$ we say that $Q \in R(\underset{\sim}{G})$ is $\underset{\sim}{J}$-reducing if $\operatorname{ad} Q : \underset{\sim}{J} \longrightarrow R(\underset{\sim}{G})$ is of rank one, and we note $c(Q)$ the kernel of this mapping, that is to say the commutator of Q in $\underset{\sim}{J}$. It is a subalgebra of $\underset{\sim}{J}$.

8.2 Lemma: Let $Q \in R(\underset{\sim}{G})$ be $\underset{\sim}{J}$-reducing, and $P \in R(\underset{\sim}{J})$. If $[P,Q] = o$, then $P \in R(c(Q))$.

Proof: Write $P = AB^{-1}$, with A and B relatively prime in $S(\underset{\sim}{J})$. From 7.1

we deduce $\qquad [AB^{-1}B\ ,\ Q] = [AB^{-1},\ Q]B + [B\ ,\ Q]AB^{-1}$

and thus $\qquad [AB^{-1}\ ,\ Q] = [A\ ,\ Q]B^{-1} - [B,Q]AB^{-2}$

so that $\qquad [AB^{-1}\ ,\ Q] = o \Longleftrightarrow [A,Q]B = [B,Q]A$.

Complete a basis $\{X_2,\dots,X_m\}$ of $c(Q)$ into a basis $\{X_1,\dots,X_m\}$ of $\underset{\sim}{J}$.

By lemma 7.2, $[A,Q] = [X_1,Q]\dfrac{\partial A}{\partial X_1}$ and $[B,Q] = [X_1,Q]\dfrac{\partial B}{\partial X_1}$, hence

$$[P,Q] = o \Longleftrightarrow \frac{\partial A}{\partial X_1}\cdot B = \frac{\partial B}{\partial X_1}\cdot A$$

and this implies $\dfrac{\partial A}{\partial X_1} = \dfrac{\partial B}{\partial X_1} = o$, since A and B are relatively prime. ■

8.3 In complete analogy to §5, we call $\overline{R}(\underset{\sim}{J})$ the intersection of the commutators $c(Q)$ of all $\underset{\sim}{J}$-reducing $Q \in R(\underset{\sim}{G})$, $\overline{R}^{j+1}(\underset{\sim}{J}) = \overline{R}(\overline{R}^{j}(\underset{\sim}{J}))$, $\overline{R}^{\infty}(\underset{\sim}{J}) = \bigcap_{j \in \mathbb{N}} \overline{R}^{j}(\underset{\sim}{J})$, and we prove that they all are invariant ideals of $\underset{\sim}{G}$ as soon as $\underset{\sim}{J}$ is one.

8.4 Lemma: $\bar{R}^{\infty}(G)$ contains the rational soul of G .

Proof: Assume $\bar{A}(G) \subset \bar{R}^{j}(G)$. To any $P \in R(G)^{G} \subset R(\bar{A}(G)) \subset R(\bar{R}^{j}(G))$ and to any $\bar{R}^{j}(G)$-reducing $Q \in R(G)$ we can apply lemma 8.2 with $J = \bar{R}^{j}(G)$. Thus $R(G)^{G} \subset R(\bar{R}^{j+1}(G))$, and $\bar{A}(G) \subset \bar{R}^{j+1}(G)$. ∎

8.5 Theorem: If G is algebraic and solvable, then $\bar{R}^{\infty}(G) = \bar{A}(G)$.

Proof: By lemma 8.4, $\bar{A}(G) \subset \bar{R}^{\infty}(G) = \bar{R}^{j_o}(G) = J$. If the inclusion was strict one could find an ideal J_1 of G of codimension one in J and containing $\bar{A} = \bar{A}(G)$, and thus

$$R(G)^{G} \subset R(\bar{A})^{G} \subset R(J_1)^{G} \subset R(J)^{G} \subset R(G)^{G} .$$

Hence $R(J)^{G} = R(J_1)^{G}$, and by proposition 5.11, $R(G)^{J_1}$ would be of degree one over $R(G)^{J}$. But then any $Q \in R(G)^{J_1} - R(G)^{J}$ would be J-reducing , and $\bar{R}^{j_o+1}(G) \subset c(Q) = J_1$ would be strictly included in $\bar{R}^{j_o}(G)$. ∎

8.6 Corollary: For an algebraic solvable Lie algebra G over a field of characteristic zero, the following conditions are equivalent :

(a) G is the rational soul of a Lie algebra over k

(b) G is its own rational soul

(c) There is no G-reducing element in $R(G)$.

Proof: If $G = \bar{A}(H)$, there is no G-reducing Q in $R(G)$, otherwise lemma 8.2 applied to any $P \in R(G)^{G}$ would imply $\bar{A}(H) \subset c(Q) \subsetneq G$. Thus (a) $\Longrightarrow$ (c).

But (c) $\Longrightarrow$ (b) by theorem 8.5, and (b) $\Longrightarrow$ (a) trivially . ∎

8.7 Example: G is the Lie algebra of upper triangular matrices of order 2 :

$$\begin{pmatrix} x_1 & x_3 \\ o & x_2 \end{pmatrix} = x_1X_1 + x_2X_2 + x_3X_3 \text{ , with the brackets}$$

$$[X_1,X_3] = X_3 , \quad [X_2,X_3] = -X_3$$

One checks easily $Z(\underset{\sim}{G}) = \underline{k}\,[X_1+X_2]$, $R(\underset{\sim}{G})^{\underset{\sim}{G}} = \underline{k}(X_1+X_2)$

so $\underset{\sim}{A}(\underset{\sim}{G}) = \bar{\underset{\sim}{A}}(\underset{\sim}{G}) = \langle X_1+X_2\rangle = R^2(\underset{\sim}{G}) = \bar{R}^{\infty}(\underset{\sim}{G})$

(X_1 and X_2 are $\underset{\sim}{G}$-reducing, and X_3 is $\langle X_1,X_2\rangle$-reducing).

This is in contrast with the next and higher dimensions:

8.8 Example: $\underset{\sim}{G}$ is the Lie algebra of upper triangular matrices of order 3

$$\begin{pmatrix} x_1 & x_4 & x_6 \\ o & x_2 & x_5 \\ o & o & x_3 \end{pmatrix} = \sum_{j=1}^{6} x_j X_j \quad \text{, with brackets}$$

$$[X_1,X_4] = X_4 , \quad [X_1,X_6] = X_6 , \quad [X_2,X_4] = -X_4 , \quad [X_2,X_5] = X_5 ,$$

$$[X_3,X_5] = -X_5 , \quad [X_3,X_6] = -X_6 , \quad [X_4,X_5] = X_6 .$$

$$S(\underset{\sim}{G})^{\underset{\sim}{G}} = \underline{k}[X_1+X_2+X_3] \text{ , but } R(\underset{\sim}{G})^{\underset{\sim}{G}} = \underline{k}(X_1+X_2+X_3 , X_2 - \frac{X_4X_5}{X_6}) ,$$

so $\underset{\sim}{A}(\underset{\sim}{G}) = \langle X_1+X_2+X_3\rangle \underset{\neq}{\subset} \bar{\underset{\sim}{A}}(\underset{\sim}{G}) = \langle X_1+X_3, X_2, X_4, X_5, X_6\rangle = R(G) = \bar{R}^{\infty}(G)$

(X_6 is $\underset{\sim}{G}$-reducing .) .

8.9 The general case of the algebra $\underset{\sim}{G}_n$ of upper triangular matrices of order n has been studied in [12] , where one can find explicit computations of the reducing ideals, the soul, the rational soul, and the algebraic and rational invariants. We only note here that the rational soul of $\underset{\sim}{G}_n$ is much bigger than its soul, for $n \geq 3$: if we put

$$\begin{pmatrix} x_1 & & (y_{ij}) \\ & \ddots & \\ (o) & & x_n \end{pmatrix} = \sum_{j=1}^{n} x_j X_j + \sum_{1 \leq i < j \leq n} y_{ij} Y_{ij}$$

and $\underset{\sim}{J}_q = \langle X_1+X_n, X_2+X_{n-1}, \ldots, X_q+X_{n-q+1} \rangle \coprod \langle Y_{ij} \mid 1 \leq i < j \leq n \rangle$

for $0 \leq q < \left[\frac{n+1}{2}\right]$, we have

$$\bar{\underset{\sim}{A}}(\underset{\sim}{G}_n) = \underset{\sim}{J}_{\left[\frac{n+1}{2}\right]+(-1)^n} \qquad \text{while} \qquad \underset{\sim}{A}(\underset{\sim}{G}_n) = \langle X_1+X_2+\ldots+X_n \rangle .$$

The first explicit description of $R(\underset{\sim}{G}_n)^{\underset{\sim}{G}_n}$ is to be found in [16].

8.10 Going through the list of solvable Lie algebras over $\mathbb{R}$ of dimension ≤ 4 given in [1], one finds that the only one of them which is equal to its rational soul is equal to its soul, and it is the "diamond" algebra :

$$[X_1, X_2] = X_3 , \quad [X_3, X_1] = X_2 , \quad [X_2, X_3] = X_4$$

for which $Z(\underset{\sim}{G}) = \underline{k}\left[X_4, X_2^2+X_3^2+2X_1X_4\right]$, $R(\underset{\sim}{G})^{\underset{\sim}{G}} = \underline{k}(X_4, X_2^2+X_3^2+2X_1X_4)$.

§9- The carrier of an invariant

9.1 Definition: We call *carrier* $\mathbf{A}(P)$ of a rational invariant $P \in R(\mathbf{G})^{\mathbf{G}}$ the intersection of all subalgebras $\mathbf{H}$ of $\mathbf{G}$ such that $P \in R(\mathbf{H})$. Clearly if $P \in S(\mathbf{G})^{\mathbf{G}}$ we have $P \in R(\mathbf{H})$ if and only if $P \in S(\mathbf{H})$.

9.2 Proposition: *$\mathbf{A}(P)$ is the smallest subalgebra $\mathbf{H}$ of $\mathbf{G}$ such that $P \in R(\mathbf{H})$, and it is an ideal of $\mathbf{G}$.*

Proof: The family $\underline{\underline{F}}$ of subalgebras $\mathbf{H}$ such that $P \in R(\mathbf{H})$ is stable under finite intersections. An inner automorphism of $\mathbf{G}$ extends to an automorphism of $R(\mathbf{G})$ which preserves $P \in R(\mathbf{G})^{\mathbf{G}}$, thus also $\underline{\underline{F}}$ globally, and finally $\mathbf{A}(P)$. ∎

9.3 Proposition: (a) *If* $P \in S(\mathbf{G})^{\mathbf{G}}$, $\mathbf{A}(\mathbf{A}(P)) = \mathbf{A}(P)$

(b) *If* $P \in R(\mathbf{G})^{\mathbf{G}}$, $\overline{\mathbf{A}}(\mathbf{A}(P)) = \mathbf{A}(P)$.

Proof: (a) Write $\mathbf{B} = \mathbf{A}(P)$. We have $P \in S(\mathbf{B})^{\mathbf{G}} \subset S(\mathbf{B})^{\mathbf{B}}$. If $\mathbf{H}$ is a subalgebra of $\mathbf{B}$, $P \in R(\mathbf{H})$ implies $\mathbf{H} \supset \mathbf{B}$ by 9.2, and thus $\mathbf{H} = \mathbf{B}$. But this means $\mathbf{A}(\mathbf{B}) = \mathbf{B}$.

(b) is proved in the same way. ∎

9.4 Remark: Obviously we have

$$\mathbf{A}(\mathbf{G}) = \sum_{P \in S(\mathbf{G})^{\mathbf{G}}} \mathbf{A}(P) \quad \text{and} \quad \overline{\mathbf{A}}(\mathbf{G}) = \sum_{P \in R(\mathbf{G})^{\mathbf{G}}} \mathbf{A}(P) .$$

9.5 Proposition: For $P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$, $\underset{\sim}{A}(P) = \sum_{f \in r(P)} dP(f)$, and it is the smallest subspace V of $\underset{\sim}{G}$ such that $P \in R(V)$.

Proof: If V is a subspace of $\underset{\sim}{G}$ and $P \in R(V)$, then for any $f \in r(P)$, $f' \in V^{\perp}$, we have $\langle dP(f), f' \rangle = o$. Hence V contains $V_o = \sum_{f \in r(P)} dP(f)$. In particular $\underset{\sim}{A}(P) \supset V_o$. Reciprocally if $f \in r(P)$, $f' \in V_o^{\perp}$ and we put $\varphi(t) = P(f+tf')$, we have $\varphi'(t) = \langle dP(f+tf'), f' \rangle = o$, and thus $P(f+tf') = P(f)$, so $P \in R(V_o)$.

For any $x \in \underset{\sim}{G}$, we have $Ad(x)dP(f) = dP(Ad^*(x)f)$ since P is invariant, hence V_o is an ideal of $\underset{\sim}{G}$, and thus $V_o \supset \underset{\sim}{A}(P)$ by 9.2. ∎

9.6 Remark: Clearly we also have $\underset{\sim}{A}(P) = \sum_{f \in r(P) \cap \Omega} dP(f)$, whenever Ω is a dense open subset of $\underset{\sim}{G}^*$.

9.7 Proposition: If $P_o, P_1, \ldots, P_r \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$ and P_o is algebraically related to $P_1, \ldots, P_r$, then

$$\underset{\sim}{A}(P_o) \subset \sum_{j=1}^{r} \underset{\sim}{A}(P_j) .$$

Proof: Take $F \in \underline{k}[Y_o, Y_1, \ldots, Y_r]$ such that $F(P_o, P_1, \ldots, P_r) = o$ and $\frac{\partial F}{\partial Y_o} \neq o$.

For any $f \in \bigcap_{j=o}^{r} r(P_j)$, we have $\frac{\partial F}{\partial Y_o}(f) dP_o(f) = -\sum_{j=1}^{r} \frac{\partial F}{\partial Y_j}(f) dP_j(f)$

and the conclusion follows from 9.5 and 9.6. ∎

9.8 Corollary: If $P_1, P_2 \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$ are algebraically related, $\underset{\sim}{A}(P_1) = \underset{\sim}{A}(P_2)$.

9.9 Proposition: If $P_1, P_2 \in R(\underset{\sim}{G})^{\underset{\sim}{G}} - \{o\}$, one can find integers α, β such that

$$\underset{\sim}{A}(P_1^{\alpha} P_2^{\beta}) = \underset{\sim}{A}(P_1) + \underset{\sim}{A}(P_2) .$$

Proof: By 9.7 we have $\underset{\sim}{A}(P_1^{\alpha} P_2^{\beta}) \subset \underset{\sim}{A}(P_1) + \underset{\sim}{A}(P_2)$. Let $\{X_1, \ldots, X_r\}$ and $\{X_1, \ldots, X_n\}$ be bases of $\underset{\sim}{A}(P_1) + \underset{\sim}{A}(P_2)$ and $\underset{\sim}{G}$ respectively. Now assume $r > o$, and for instance $\underset{\sim}{A}(P_1^{\alpha} P_2^{\beta}) \subset \{X_2, \ldots, X_r\}$. We have

$$o = P_1^{-\alpha} P_2^{-\beta} \frac{\partial}{\partial X_1}(P_1^{\alpha} P_2^{\beta}) = \alpha P_1^{-1} \frac{\partial P_1}{\partial X_1} + \beta P_2^{-1} \frac{\partial P_2}{\partial X_1}$$

and since $\frac{\partial P_1}{\partial X_1}$ and $\frac{\partial P_2}{\partial X_1}$ are not both zero, this can only happen when (α, β) belongs to a straight line in $\mathbb{N}^2$, say L_1. Reasoning in the same way for each X_j $(j=2, \ldots, r)$ and choosing (α, β) outside $L = \bigcup_{j=1,\ldots,r} L_j$, we conclude $\underset{\sim}{A}(P_1^{\alpha} P_2^{\beta}) \supset \{X_1, \ldots, X_r\} = \underset{\sim}{A}(P_1) + \underset{\sim}{A}(P_2)$. ■

9.10 The product $P_1^{\alpha} P_2^{\beta}$ in proposition 9.9 is cumbersome, but it may happen on the other hand that $\underset{\sim}{A}(aP_1 + bP_2) \subsetneq \underset{\sim}{A}(P_1) + \underset{\sim}{A}(P_2)$ for all pairs $(a,b) \in \underline{k}^2$, as in the following example :

$\underset{\sim}{G}$ is the 6-dimensional nilpotent algebra (isomorphic to $G_{6,18}$ of [15]) defined by the brackets

$$[X_1, X_2] = X_4 , \quad [X_1, X_3] = X_5 , \quad [X_1, X_4] = X_6$$

One can check that $R(\underset{\sim}{G})^{\underset{\sim}{G}} = \underline{k}(X_5, X_6, P_1, P_2)$ with $P_1 = X_4^2 - 2X_2X_6$ and $P_2 = X_4X_5 - X_3X_6$. (Note that $S(\underset{\sim}{G})^{\underset{\sim}{G}}$ is not a free algebra : $(P_2^2 - X_5^2 P_1)X_6^{-1} \in S(\underset{\sim}{G})^{\underset{\sim}{G}}$). Clearly $\underset{\sim}{A}(P_1) = \langle X_2, X_4, X_6 \rangle$, $\underset{\sim}{A}(P_2) = \langle X_3, X_4, X_5, X_6 \rangle$, $\underset{\sim}{A}(P_1) + \underset{\sim}{A}(P_2) = \underset{\sim}{A}(G) = \overline{\underset{\sim}{A}}(G) = \langle X_2, X_3, X_4, X_5, X_6 \rangle$. But

$$\underset{\sim}{A}(aP_1 + bP_2) = \begin{cases} \langle 2aX_2 + bX_3 , X_4, X_5, X_6 \rangle & \text{if } b \neq o \\ \langle aX_2, aX_4, aX_6 \rangle & \text{if } b = o . \end{cases}$$

9.11 Theorem: (a) There exists $P \in S(G)^G$ such that $A(P) = A(G)$

(b) There exists $P \in R(G)^G$ such that $A(P) = \bar{A}(G)$

(c) In both cases one can choose such a P homogeneous.

Proof: Take a maximal system of algebraically independent elements in $S(G)^G$ (resp. $R(G)^G$), say $\{P_1, \dots, P_r\}$. By proposition 9.9 and an induction on r we can find $P \in S(G)^G$ (resp. $R(G)^G$) such that

$$A(P) = A(P_1) + \dots + A(P_r) .$$

Any $Q \in S(G)^G$ (resp. $R(G)^G$) is algebraically related to $P_1, \dots, P_r$. Hence

$$\begin{aligned} A(P) \subset A(G) \quad & (\text{resp. } \bar{A}(G)) \\ = \sum_{Q \in S(G)^G} A(Q) \quad & (\text{resp. } \sum_{Q \in R(G)^G} A(Q)) \qquad \text{by } 9.4 \\ \subset A(P_1) + \dots + A(P_r) = A(P) & \qquad \text{by } 9.7 . \end{aligned}$$

So we have (a) and (b), and (c) follows from the construction of P by 9.9 and the fact that we can take $P_1, \dots, P_r$ homogeneous, since $S(G)^G$ and $R(G)^G$ are engendered by their homogeneous elements (proposition 1.6). ∎

9.12 Example: G is the 6-dimensional nilpotent Lie algebra (isomorphic to $G_{6,7}$ of [15]) defined by the brackets

$$[X_1, X_2] = X_4 , \quad [X_1, X_3] = X_5 , \quad [X_2, X_4] = X_5 , \quad [X_2, X_5] = X_6 , \quad [X_3, X_4] = X_6$$

$A(G) = \bar{A}(G) = \langle X_1, X_4, X_5, X_6 \rangle$, and $Z(G) = k[X_6, P]$, $R(G)^G = k(X_6, P)$, with $P = X_5^3 - 3X_4X_5X_6 + 3X_1X_6^2$, so that $A(P) = A(G)$, and there is no $Q \in Z(G)$ of smaller degree whose carrier is $A(G)$.

§10- Souls and quadratic Lie algebras

10.1 Definition: We shall call *soul* (resp. *rational soul*) a Lie algebra which is equal to its soul (resp. rational soul). Reductive algebras are souls. Nilpotent algebras are souls if and only if they are rational souls (cf. remark 7.7). Also recall the characterisations 5.18 and 8.6 .

As an immediate consequence of theorem 9.11 we have

10.2 Corollary: (a) $\underset{\sim}{G}$ *is a soul if and only if there exists* $P \in S(\underset{\sim}{G})^{\underset{\sim}{G}}$ *such that* $\underset{\sim}{A}(P) = \underset{\sim}{G}$.

(b) $\underset{\sim}{G}$ *is a rational soul if and only if there exists* $P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$ *such that* $\underset{\sim}{A}(P) = \underset{\sim}{G}$.

(c) *In both cases one can choose such a* P *homogeneous*.

10.3 Definition: We will say that $\underset{\sim}{G}$ is a soul of degree m if $\underset{\sim}{G}$ is a soul and m is the smallest degree of a homogeneous $P \in S(\underset{\sim}{G})^{\underset{\sim}{G}}$ such that $\underset{\sim}{A}(P) = \underset{\sim}{G}$.

10.4 Let $\underset{\sim}{G}$ be a soul of degree two, and $P \in S(\underset{\sim}{G})^{\underset{\sim}{G}}$, homogeneous of degree two, such that $\underset{\sim}{A}(P) = \underset{\sim}{G}$. Then P is an $Ad^*(G)$-invariant quadratic form on $\underset{\sim}{G}^*$, which is non-degenerate by proposition 3.1. Identifying $\underset{\sim}{G}$ and $\underset{\sim}{G}^*$ by means of this form and transposing P on $\underset{\sim}{G}$, we get a non-degenerate Ad(G)-invariant quadratic form on $\underset{\sim}{G}$, that is to say $\underset{\sim}{G}$ is a *quadratic Lie algebra* (cf. [13] §2.9, Ex.2.10, and [10])

For instance reductive Lie algebras $\underset{\sim}{G}$ are all quadratic Lie algebras (souls of degree two), since $\underset{\sim}{G} = \underset{\sim}{A}(P)$ with $P = C + D$, where C is the Casimir element of their semi-simple part, and D is any non-degenerate quadratic form on the dual of their center .

<u>10.5</u> <u>Example</u> (of a soul of higher degree than two)
There are only five souls among the unsplitable nilpotent Lie algebras of dimension 7 over $\mathbb{R}$ or $\mathbb{C}$, and we gave their definition in 4.9. Only one of them is quadratic and the four others are of degree three. Here is another example, of physical interest:

$\underset{\sim}{G}$ is the Lie algebra tangent to the group of affine isometries of $\mathbb{R}^4$.
In the basis of infinitesimal rotations R_{ij} and infinitesimal translations T_j along the axes $(1 \leq i < j \leq 4)$, $\underset{\sim}{G}$ is defined by the brackets

$$[R_{ij}, R_{ik}] = -R_{jk} \qquad \text{and} \qquad [R_{ij}, T_i] = -T_j$$

(all the brackets that cannot be deduced from these by antisymmetry are zero).
It is well known that the only invariant quadratic form on $\underset{\sim}{G}^*$ is, up to a multiple, the Laplace element $\Delta = T_1^2 + T_2^2 + T_3^2 + T_4^2$

but one can check that

$$\begin{aligned}
\Box = {} & R_{12}^2(T_3^2 + T_4^2) + R_{13}^2(T_2^2 + T_4^2) + R_{14}^2(T_2^2 + T_3^2) \\
& + R_{23}^2(T_1^2 + T_4^2) + R_{24}^2(T_1^2 + T_3^2) + R_{34}^2(T_1^2 + T_2^2) \\
& - 2R_{12}R_{13}T_2T_3 - 2R_{12}R_{14}T_2T_4 + 2R_{12}R_{13}T_1T_3 \\
& + 2R_{12}R_{24}T_1T_4 - 2R_{13}R_{14}T_3T_4 - 2R_{13}R_{23}T_1T_2 \\
& + 2R_{13}R_{34}T_1T_4 - 2R_{14}R_{24}T_1T_2 - 2R_{14}R_{34}T_1T_3 \\
& - 2R_{23}R_{24}T_3T_4 + 2R_{23}R_{34}T_2T_4 - 2R_{24}R_{34}T_2T_3
\end{aligned}$$

belongs to $Z(\underset{\sim}{G})$, and more precisely $Z(\underset{\sim}{G}) = S(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}[\Delta, \Box]$ and $R(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}(\Delta, \Box)$.
In particular $\underset{\sim}{A}(\Box) = \underset{\sim}{G}$ and $\underset{\sim}{G}$ is a soul of degree four, no less.

<u>10.6</u> Souls of degree two, that is to say quadratic Lie algebras, have been studied in [10], where it is proved that any unsplitable quadratic Lie algebra with non-trivial center is a certain central extension of another quadratic Lie algebra of dimension two less. This procedure, following an idea of V. Kac ([13], loc. cit.) is enough to construct many examples of

(non-reductive) quadratic Lie algebras, and in particular one can describe in this way, by induction on the dimension, all solvable quadratic Lie algebras. But there are also non-reductive quadratic Lie algebras with trivial center :

10.7 Example: $\underset{\sim}{G}$ is the Lie algebra tangent to the group of affine isometries of $\mathbb{R}^3$. In the basis of infinitesimal rotations R_j and translations T_j (j=1,2,3) along the axes, it is defined by the brackets

$$\left[R_i, R_j\right] = -\varepsilon R_k \quad , \quad \left[R_i, T_j\right] = -\varepsilon T_k \quad , \quad \left[T_i, T_j\right] = 0$$

where ε is the signature of the permutation $\{i,j,k\}$ of $\{1,2,3\}$.

It is easily checked that

$$Z(\underset{\sim}{G}) = S(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}\left[\Delta, \Box\right] \quad \text{and} \quad R(\underset{\sim}{G})^{\underset{\sim}{G}} = \mathbb{R}(\Delta, \Box) \quad \text{, with}$$

$$\Delta = T_1^2 + T_2^2 + T_3^2 \qquad \text{and} \qquad \Box = R_1T_1 + R_2T_2 + R_3T_3$$

Hence the center of $\underset{\sim}{G}$ is trivial, $\underset{\sim}{A}(\Box) = \underset{\sim}{G}$, and $\underset{\sim}{G}$ is a quadratic Lie algebra .

§11- Souls and the defining form B

11.1 We recall here some of the ideas and notations of [7]. Call B the bilinear antisymmetric mapping $\underset{\sim}{G} \times \underset{\sim}{G} \longrightarrow \underset{\sim}{G}$ defining the Lie algebra structure of $\underset{\sim}{G}$: $\forall X, Y \in \underset{\sim}{G} \quad B(X,Y) = [X,Y]$.

The tensor product $R(\underset{\sim}{G}) \underset{\underline{k}}{\otimes} \underset{\sim}{G}$ has a canonical structure of n-dimensional Lie algebra over $R(\underset{\sim}{G})$, and according to [6],1.11.1, there is one and only one $R(\underset{\sim}{G})$-bilinear antisymmetric form on $R(\underset{\sim}{G}) \otimes \underset{\sim}{G}$ (with values in $R(\underset{\sim}{G})$), which we will again call B, such that

$$\forall X, Y \in \underset{\sim}{G} \qquad B(1 \otimes X, 1 \otimes Y) = [X,Y]$$

If $\beta = \{X_1, \dots, X_n\}$ is any basis of $\underset{\sim}{G}$, the matrix of B in the basis $\{1 \otimes X_1, \dots, 1 \otimes X_n\}$ of $R(\underset{\sim}{G}) \otimes \underset{\sim}{G}$ is $B^{\beta} = (b_{ij}) = ([X_i, X_j])_{1 \leq i, j \leq n}$.

The differential $P \longmapsto dP$ is a natural linear mapping from $R(\underset{\sim}{G})$ into $R(\underset{\sim}{G}) \otimes \underset{\sim}{G}$, and the following is a particular case of 1.7 :

<u>11.2 Corollary</u>: <u>If $\underset{\sim}{G}$ is algebraic, the kernel of</u> B <u>in</u> $R(\underset{\sim}{G}) \otimes \underset{\sim}{G}$ <u>is (linearly) engendered by</u> $\{dP \mid P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}\}$ <u>over</u> $R(\underset{\sim}{G})$, <u>and the transcendental degree of</u> $R(\underset{\sim}{G})^{\underset{\sim}{G}}$ <u>over</u> $\underline{k}$ <u>is precisely the dimension</u> r <u>of</u> Ker B <u>over</u> $R(\underset{\sim}{G})$.

11.3 For any $f \in \underset{\sim}{G}^*$ consider the $\underline{k}$-bilinear form $B_f = B \circ f$ on $\underset{\sim}{G}$ defined by $\forall X, Y \in \underset{\sim}{G} \qquad B_f(X,Y) = \langle f, [X,Y] \rangle$

The matrix of B_f in the basis β is $B_f^{\beta} = (\langle f, b_{ij} \rangle) = (\langle f, [X_i, X_j] \rangle)$

Hence the rank (n-r) of B over $R(\underset{\sim}{G})$ is none other than the maximal rank of B_f over $\underline{k}$ for $f \in \underset{\sim}{G}^*$; it is an even integer $2d = n - r$, and it is the maximal dimension of the $Ad^*(\underline{G})$-orbits in $\underset{\sim}{G}^*$. In particular we have the

11.4 Proposition: For an algebraic $\underset{\sim}{G}$ over an algebraically closed field $\underline{k}$, the three following properties are equivalent :

(i) B is non-degenerate over $R(\underset{\sim}{G})$

(ii) There is an open $Ad^*(\underset{\sim}{G})$-orbit in $\underset{\sim}{G}^*$.

(iii) $R(\underset{\sim}{G})^{\underset{\sim}{G}} = \underline{k}$

Proof: (i) $\Longrightarrow$(ii) By 11.3 one can find $f \in \underset{\sim}{G}^*$ such that B_f is non-degenerate. So for any $X \in \underset{\sim}{G} - \{o\}$, $X.f \neq o$ (notation defined in 1.1). As $\{X.f \mid X \in \underset{\sim}{G}\}$ is the tangent space to the $Ad^*(\underset{\sim}{G})$-orbit $\underline{\underline{O}}_f$ of f , which is a smooth subvariety of $\underset{\sim}{G}^*$, $\underline{\underline{O}}_f$ is thus of dimension n .

(ii) $\Longrightarrow$ (iii) for instance by [8], proposition 1.

(iii) $\Longrightarrow$ (ii) by 11.2.

11.5 Call $\underline{\underline{i}}$ the $(R(\underset{\sim}{G}),\underline{k})$-bilinear mapping $(R(\underset{\sim}{G}) \otimes \underset{\sim}{G}) \times \underset{\sim}{G}^* \longrightarrow R(\underset{\sim}{G})$ which satisfies

$$\forall Q \in R(\underset{\sim}{G}),\ X \in \underset{\sim}{G},\ f \in \underset{\sim}{G}^* \qquad i(Q \otimes X, f) = \langle f, X \rangle Q$$

As a consequence of corollary 11.2, we get an effective way of computing the rational soul of any algebraic Lie algebra :

11.6 Corollary: If $\underset{\sim}{G}$ is algebraic, the orthogonal of $\bar{A}(\underset{\sim}{G})$ in $\underset{\sim}{G}^*$ is the orthogonal of Ker B for the bilinear mapping $\underline{\underline{i}}$.

Proof:
$$\bar{A}(\underset{\sim}{G})^{\perp} = \Big(\sum_{P \in R(\underset{\sim}{G})^{\underset{\sim}{G}},\, f \in r(P)} dP(f) \Big)^{\perp} \quad \text{by 7.5}$$
$$= \{ f' \in \underset{\sim}{G}^* \mid \forall P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}, \forall f \in r(P) \quad \langle dP(f), f' \rangle = o \}$$
$$= \{ f' \in \underset{\sim}{G}^* \mid \forall P \in R(\underset{\sim}{G})^{\underset{\sim}{G}} \quad \underline{\underline{i}}(dP, f') = o \}$$
$$= \{ f' \in \underset{\sim}{G}^* \mid \forall Q \in \operatorname{Ker} B \quad \underline{\underline{i}}(Q, f') = o \} \quad \text{by 11.2 .} \blacksquare$$

By elementary methods of linear algebra over $R(\underset{\sim}{G})$, one can easily calculate a basis of Ker B for any algebra $\underset{\sim}{G}$ defined by explicit brackets, and this gives a simple and effective algorithm for computing the rational soul .

11.7 Definition: We shall say that $\underset{\sim}{G}$ is balanced if for any basis β of $\underset{\sim}{G}$ each row (or column) of the matrix B^{β} is a linear combination of the others with coefficients in $R(\underset{\sim}{G})$.

Multiplying each one of these relations by the common denominator, separating homogeneous parts and combining the relations thus obtained, one checks easily :

11.8 Lemma: *$\underset{\sim}{G}$ is balanced if and only if for any basis β of $\underset{\sim}{G}$, there is a linear relation beteween the rows (or columns) of the matrix B^{β} with all non-zero coefficients, homogeneous of the same degree in $S(\underset{\sim}{G})$* .

11.9 If $\beta' = {}^tP\beta$, with $P \in GL(n,\underset{\sim}{k})$ is another basis of $\underset{\sim}{G}$, we have

$$B^{\beta'} = {}^tPB^{\beta}P \quad .$$

It may well happen that each row (or column) of B^{β} is a linear combination of the others, while this is not true of $B^{\beta'}$, as the following example shows:

$\underset{\sim}{G}$ is the 7-dimensional nilpotent Lie algebra defined by the brackets

$$[X_1,X_2] = X_5 \ , \quad [X_1,X_3] = X_6 \ , \quad [X_1,X_4] = X_7 \ , \quad [X_2,X_4] = X_5 \ , \quad [X_3,X_4] = X_6$$

In the basis $\beta' = \{X'_1,\dots,X'_7\}$ defined by

$$X'_1 = \tfrac{1}{2}(X_1+X_4) \ , \ X'_4 = \tfrac{1}{2}(X_1-X_4) \ , \ X'_2 = X_2 \ , \ X'_3 = X_3 \ , \ X'_j = -X_j \quad (j \geqslant 5)$$

the non-zero brackets are : $[X'_1,X'_4] = X'_7 \ , \ [X'_2,X'_4] = X'_5 \ , \ [X'_3,X'_4] = X'_6$

Thus

$$B^{\beta} = \begin{pmatrix} 0 & X_5 & X_6 & X_7 & 0 & 0 & 0 \\ -X_5 & 0 & 0 & X_5 & 0 & 0 & 0 \\ -X_6 & 0 & 0 & X_6 & 0 & 0 & 0 \\ -X_7 & -X_5 & -X_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{, but } B^{\beta'} = \begin{pmatrix} 0 & 0 & 0 & X'_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & X'_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & X'_6 & 0 & 0 & 0 \\ -X'_7 & -X'_5 & -X'_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now X_5, X_6, X_7, $Z_1 = X_1X_5 + X_4X_5 + 2X_3X_7$, $Z_2 = X_1X_6 + X_4X_6 + 2X_3X_7$, and $Z_3 = X_2X_6 - X_3X_5$ engender $Z(\underset{\sim}{G})$, and more precisely :

$$Z(\underset{\sim}{G}) = S(\underset{\sim}{G})^{\underset{\sim}{G}} = \underline{k}[X_5,X_6,X_7,Z_1,Z_2,Z_3] \diagup X_7Z_3 - X_6Z_1 + X_5Z_2 \quad ,$$

$$R(\underset{\sim}{G})^{\underset{\sim}{G}} = \underline{k}(X_5,X_6,X_7,Z_1,Z_2) \text{ , and } \overline{A}(\underset{\sim}{G}) = \langle X_1+X_4,X_2,X_3,X_5,X_6,X_7 \rangle \neq \underset{\sim}{G} .$$

<u>11.10 Lemma</u>: <u>Let $\underset{\sim}{G}$ be algebraic (of dimension n), and</u> p <u>the dimension of</u> $\overline{A}(\underset{\sim}{G})$. <u>Then</u> p <u>is the smallest integer</u> q <u>such that there exists a basis</u> β <u>of $\underset{\sim}{G}$ satisfying the three conditions</u> :

(i) <u>Each one of the</u> q <u>last columns of</u> B^{β} <u>is a linear combination of the</u> q-1 <u>others</u>.

(ii) <u>The</u> n-q <u>first columns of</u> B^{β} <u>are linearly independent</u>.

(iii) <u>The subspaces engendered by the</u> n-q <u>first columns and by the</u> q <u>last ones intersect trivially</u>.

<u>Furthermore the bases satisfying these conditions with</u> q = p <u>are those completed from a basis</u> $\{X_{n-p+1},\dots,X_n\}$ <u>of</u> $\overline{A}(\underset{\sim}{G})$.

<u>Proof</u>: Assume β is a basis $\{X_1,\dots,X_n\}$ of $\underset{\sim}{G}$ satisfying (i),(ii),(iii). Then the first n-q coefficients of any $R(\underset{\sim}{G})$-linear relation between the columns of B^{β} must be zero. Hence Ker $B^{\beta}: R(\underset{\sim}{G})^n \longrightarrow R(\underset{\sim}{G})^n$ is included in the subspace $W = \{(Q_1,\dots,Q_n) \in R(\underset{\sim}{G})^n \mid Q_1 = \dots = Q_{n-q} = 0\}$. In view of 11.2 we have in particular for any $P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$:

$$\frac{\partial P}{\partial X_1} = \dots = \frac{\partial P}{\partial X_{n-q}} = 0 \quad \text{, and thus by 7.5 } \overline{A}(\underset{\sim}{G}) \subset \langle X_{n-q+1},\dots,X_n \rangle .$$

So $q \geqslant p$, and if $q = p$, $\overline{A}(\underset{\sim}{G}) = \langle X_{n-p+1},\dots,X_n \rangle$.

Reciprocally, if β is any basis of $\underset{\sim}{G}$ completed from a basis $\{X_{n-p+1},\dots,X_n\}$ of $\overline{A}(\underset{\sim}{G})$,

$$\forall P \in R(\underset{\sim}{G})^{\underset{\sim}{G}} \qquad \frac{\partial P}{\partial X_1} = \dots = \frac{\partial P}{\partial X_{n-p}} = 0 \qquad \text{since} \quad P \in R(\overline{A}(\underset{\sim}{G}))$$

and by 11.2 $\quad \operatorname{Ker} B^{\beta} \subset V = \left\{(Q_1,\dots,Q_n) \in R(\underset{\sim}{G})^n \;\middle|\; Q_1 = \dots = Q_{n-p} = 0\right\}$.

If B^{β}_{*} is the matrix obtained by deleting the first $n-p$ columns of B^{β}, we have $\operatorname{rank} B^{\beta}_{*} = \operatorname{rank} B^{\beta} - p$, and so none of the first $n-p$ columns of B^{β} depends on the others . ∎

<u>11.11</u> <u>Theorem</u>: <u>An algebraic Lie algebra $\underset{\sim}{G}$ is a rational soul if and only if it is balanced</u> .

<u>11.12</u> Theorem 11.11 is an immediate application of lemma 11.10, for $p = n$. But we can make it more precise. If $\underset{\sim}{G}$ is a rational soul, then by theorem 9.11 we can find $P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$ such that $\underset{\sim}{A}(P) = \underset{\sim}{G}$. In any basis $\beta = \{X_1,\dots,X_n\}$ of $\underset{\sim}{G}$ we thus have $\frac{\partial P}{\partial X_i} \neq 0 \quad (i=1,\dots,n)$

and calling c_i $(i=1,\dots,n)$ the columns of B^{β} we get the linear relation

$$\sum_{i=1}^{n} \frac{\partial P}{\partial X_i} c_i = 0 \quad .$$

For instance if $\underset{\sim}{G}$ is reductive, $\{X_1,\dots,X_p\}$ is a basis of $[\underset{\sim}{G},\underset{\sim}{G}]$ in which the Killing form is diagonal, and we complete it into a basis $\{X_1,\dots,X_n\}$ of $\underset{\sim}{G}$ by adding elements of the center of $\underset{\sim}{G}$, we have

$$P = \sum_{i=1}^{n} X_i^2 \in Z(\underset{\sim}{G}) , \qquad \text{and thus} \qquad \sum_{i=1}^{n} X_i c_i = 0 \quad ,$$

where actually the last $n-p$ columns c_i are zero, and this generalizes in an obvious way to any quadratic Lie algebra (cf. §10) .

We will end by giving a "geometric" characterisation of the ideals of an algebra that can carry a rational invariant, similar to the characterisation 11.11 of rational souls .

11.13 Definition: We shall say that an ideal $\underset{\sim}{J}$ of $\underset{\sim}{G}$ is a balanced ideal of $\underset{\sim}{G}$ if for any basis $\beta = \{X_1, \ldots, X_n\}$ of $\underset{\sim}{G}$ completed from a basis $\{X_{n-p+1}, \ldots, X_n\}$ of $\underset{\sim}{J}$, each one of the p last rows (or columns) of the matrix B^β is a linear combination of the $p-1$ remaining last ones over $R(\underset{\sim}{G})$.

In particular balanced ideals are themselves balanced, but an ideal of $\underset{\sim}{G}$ which is a balanced algebra is not always a balanced ideal of $\underset{\sim}{G}$.

11.14 Proposition: An ideal $\underset{\sim}{J}$ of an algebraic Lie algebra $\underset{\sim}{G}$ is the carrier of some rational invariant of $\underset{\sim}{G}$ if and only if it is a balanced ideal of $\underset{\sim}{G}$.

Proof: Assume $\underset{\sim}{J} = \underset{\sim}{A}(P)$, with $P \in R(\underset{\sim}{G})^{\underset{\sim}{G}}$, and choose a basis β of $\underset{\sim}{G}$ as in 11.13. Then $dP = (o, \ldots, o, \frac{\partial P}{\partial X_{n-p+1}}, \ldots, \frac{\partial P}{\partial X_n}) \in \operatorname{Ker} B^\beta$,

and calling c_j $(j=1,\ldots,n)$ the columns of B^β, we have

$$\sum_{j=n-p+1}^{n} \frac{\partial P}{\partial X_j} c_j = o \quad , \quad \text{and} \quad \frac{\partial P}{\partial X_j} \neq o \quad \text{for } j = n-p+1, \ldots, n \; .$$

Reciprocally if $\underset{\sim}{J}$ is a balanced ideal of $\underset{\sim}{G}$ and β a basis chosen as in 11.13, call B^β_* the matrix of the p last columns of B^β.

If $$c_j = \sum_{\substack{k=n-p+1 \\ k \neq j}}^{n} Q_{jk}(X_1, \ldots, X_n)\, c_k \qquad \text{for } j = n-p+1, \ldots, n$$

with all $Q_{jk} \neq o$, substituting appropriate scalars $a_1, \ldots, a_{n-p}$ to $X_1, \ldots, X_{n-p}$, we get

$$c_j = \sum_{\substack{k=n-p+1 \\ k \neq j}}^{n} Q_{jk}(a_1, \ldots, a_{n-p}, X_{n-p+1}, \ldots, X_n)\, c_k$$

with all $Q_{jk}(a_1, \ldots, a_{n-p}, \bullet, \bullet \ldots, \bullet) \neq o$

since the entries of the columns c_k belong to $R(\underset{\sim}{J})$, $\underset{\sim}{J}$ being an ideal .

Hence each column of B_*^β is a linear combination of the others with coefficients in $R(\underset{\sim}{J})$, and $\operatorname{Ker} B_*^\beta : R(\underset{\sim}{J})^p \longrightarrow R(\underset{\sim}{J})^n$ contains a vector $(Q_{n-p+1},\ldots,Q_n)$ whose entries Q_j are all non-zero .
Applying theorem 1.7 with $V = \underset{\sim}{J}$ and the natural action of $\underset{\sim}{G}$ on $\underset{\sim}{J}$, we conclude that in any basis $\{X_{n-p+1},\ldots,X_n\}$ of $\underset{\sim}{J}$, there exists $P_{o,j} \in R(\underset{\sim}{J})^{\underset{\sim}{G}}$ such that $\frac{\partial P_{o,j}}{\partial X_j} \neq o$ for each $j = n-p+1,\ldots,n$.
Now $R(\underset{\sim}{J})^{\underset{\sim}{G}}$ has over $\underline{k}$ the transcendental degree $r = \dim \operatorname{Ker} B_*^\beta$ (over $R(\underset{\sim}{J})$) and if we take $P_1,\ldots,P_r$ algebraically independent in $R(\underset{\sim}{J})^{\underset{\sim}{G}}$, and put $\underset{\sim}{J}' = \underset{\sim}{A}(P_1) + \ldots + \underset{\sim}{A}(P_r) \subset \underset{\sim}{J}$, we have $dP_j \in R(\underset{\sim}{J}')$ for $j=1,\ldots,r$, and thus $dP \in R(\underset{\sim}{J}')$ for any $P \in R(\underset{\sim}{J})^G$. If $\underset{\sim}{J}' \neq \underset{\sim}{J}$, this contradicts the existence of the $P_{o,j}$ in any basis completed from a basis of $\underset{\sim}{J}'$.
So $\underset{\sim}{J}' = \underset{\sim}{J}$, and choosing P as in the proof of 9.11, we get

$$\underset{\sim}{A}(P) = \underset{\sim}{A}(P_1) + \ldots + \underset{\sim}{A}(P_r) = \underset{\sim}{J}' = \underset{\sim}{J} \ . \blacksquare$$

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