RADON TRANSFORM ON THE TORUS

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Abstract. We consider the Radon transform on the (flat) torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ defined by integrating a function over all closed geodesics. We prove an inversion formula for this transform and we give a characterization of the image of the space of smooth functions on $\mathbb{T}^n$.

1. Introduction

Trying to reconstruct a function on a manifold knowing its integrals over a certain family of submanifolds is one of the main problems of integral geometry. In the framework of Riemannian manifolds a natural choice is the family of all geodesics. The simple example of lines in Euclidean space has suggested naming *X-ray transform* the corresponding integral operator, associating to a function f its integrals $Rf(l)$ along all geodesics l of the manifold.

Few explicit formulas are known to invert the X-ray transform. With no attempt to give an exhaustive list, let us quote Helgason [5] for Euclidean spaces, hyperbolic spaces and spheres, Berenstein and Casadio Tarabusi [4] for hyperbolic spaces, Helgason [6] or the second author [7] for more general symmetric spaces, [8] for Damek-Ricci spaces etc.

We consider here the n-dimensional (flat) torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and the Radon transform defined by integrating f along all *closed geodesics* of $\mathbb{T}^n$, that is all lines with rational slopes. Arithmetic properties will thus enter the picture, as in the case of Radon transforms on $\mathbb{Z}^n$ already studied by the first author and collaborators (see [1, 2, 3]). Our present problem was introduced by Strichartz [9], who gave a solution for $n = 2$. But, relying on a special property of the two-dimensional case (see Remark 1 at the end of Section 3 below), his method does not extend in an obvious way to the n-dimensional torus. The inversion formula proved here for $\mathbb{T}^n$ (Theorem 1) makes use of a weighted dual Radon transform R^*_φ, with a weight function φ to ensure convergence.

In Section 2 we describe a suitable set of parameters for the closed geodesics on the torus. Our main result (Theorem 1) is proved in Section 3. Section 4 is devoted to a range theorem (Theorem 2), characterizing the space of Radon transforms of all functions in $C^\infty(\mathbb{T}^n)$.

2. Closed geodesics of the torus

The following notation will be used throughout.

Notation. Let $x \cdot y = x_1y_1 + \cdots + x_ny_n$ be the canonical scalar product of $x, y \in \mathbb{R}^n$. For $p = (p_1, ..., p_n) \in \mathbb{Z}^n\backslash 0$ the set $I(p) = \{k \cdot p\ ,\ k \in \mathbb{Z}^n\}$ is an ideal of $\mathbb{Z}$, not $\{0\}$, and we shall denote by $d(p) = d(p_1, ..., p_n)$ its smallest strictly positive element. Thus $d(p)$ is the highest common divisor of $p_1, ..., p_n$ and $I(p) = d(p)\mathbb{Z}$. Let

$$\mathcal{P} = \{p = (p_1, ..., p_n) \in \mathbb{Z}^n \backslash \{0\}\ \mid d(p_1, ..., p_n) = 1\}$$

and, for $k \in \mathbb{Z}^n$,

$$\mathcal{P}_k = \{p \in \mathcal{P} \mid k \cdot p = 0\}\ .$$

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For $n \geq 2$ be the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is equipped with the (flat) Riemannian metric induced by the canonical Euclidean structure of $\mathbb{R}^n$ and the corresponding (translation invariant) measure dx; thus $\int_{\mathbb{T}^n} dx = 1$. We denote by $\mathrm{pr} : \mathbb{R}^n \to \mathbb{T}^n$ the canonical projection.

All functions considered here are complex-valued.

In $\mathbb{T}^n$ the geodesic from $x = (x_1, ..., x_n) \in \mathbb{T}^n$ with (non zero) initial speed $v = (v_1, ..., v_n) \in \mathbb{R}^n$ is the line

$$\ell = \{x + \mathrm{pr}(tv), t \in \mathbb{R}\} \ .$$

Since $v \neq 0$ the set of all t such that tv belongs to $\mathbb{Z}^n$ is a discrete subgroup of $\mathbb{R}$, that is $T\mathbb{Z}$ for some $T \geq 0$. Thus the geodesic ℓ is closed if and only if $T > 0$. In this case we set $Tv = p = (p_1, ..., p_n) \in \mathbb{Z}^n \setminus \{0\}$, and the p_j's are relatively prime: a nontrivial common divisor would contradict the definition of T. Thus p belongs to $\mathcal{P}$.

Now for $p \in \mathcal{P}$ and $t \in \mathbb{R}$ we have

$$tp \in \mathbb{Z}^n \Longleftrightarrow t \in \mathbb{Z} \ .$$

Indeed the set of such t is a discrete subgroup $\tau\mathbb{Z}$ of $\mathbb{R}$ (for some $\tau \geq 0$), obviously containing 1. Thus $\tau = 1/m$ for some integer $m > 0$, so that m is a common divisor to all p_j's, which implies $m = 1$ and $\tau = 1$.

For $x \in \mathbb{T}^n$ and $p \in \mathcal{P}$ we denote by

$$\ell(x, p) = \{x + \mathrm{pr}(tp), t \in \mathbb{R}\}$$

the corresponding closed geodesic from x. By the above remark the map $t \mapsto x + \mathrm{pr}(tp)$ induces a bijection from $\mathbb{R}/\mathbb{Z}$ onto $\ell(x, p)$, since $\mathrm{pr}(tp) = \mathrm{pr}(t'p)$ if and only if $(t-t')p \in \mathbb{Z}^n$ that is $t - t' \in \mathbb{Z}$. We shall therefore let t run over $[0, 1[$ only in the sequel. The length of $\ell(x, p)$ is $\|p\| = \left(\sum_{j=1}^n p_j^2\right)^{1/2}$.

Lemma 1. *Let $x, y \in \mathbb{T}^n$ and $p, q \in \mathcal{P}$. The following are equivalent:*
(i) $\ell(x, p) = \ell(y, q)$
(ii) $q = \pm p$ and there exists $s \in \mathbb{R}$ such that $y = x + \mathrm{pr}(sp)$.

The set of closed geodesics from x is therefore in one-to-one correspondence with $\mathcal{P}/\{\pm 1\}$.
Proof. *(ii) implies (i)* since, for any s,

$$\ell(x, p) = \ell(x, -p) \ , \ \ell(x + \mathrm{pr}(sp), p) = \ell(x, p) \ . \tag{1}$$

(i) implies (ii). Conversely, assume $\ell(x, p) = \ell(y, q)$. Since y belongs to $\ell(x, p)$ there exists $s_0 \in \mathbb{R}$ such that $y = x + \mathrm{pr}(s_0 p)$. More generally, for any $t \in \mathbb{R}$ there exists $s \in \mathbb{R}$ such that $y + \mathrm{pr}(tq) = x + \mathrm{pr}(sp)$. Replacing s by $s + s_0$ we have: for any $t \in \mathbb{R}$ there exist $s \in \mathbb{R}$ and $z \in \mathbb{Z}^n$ such that

$$tq = sp + z \ .$$

Let us fix $j \in \{1, 2, ..., n\}$ so that $p_j \neq 0$. Fixing $l \neq j$ let

$$k = (k_1, ..., k_n) \text{ with } k_j = p_l \ , \ k_l = -p_j \text{ and } k_m = 0 \text{ if } m \neq j, l \ .$$

Then $k \cdot p = 0$, therefore $t(k \cdot q) = k \cdot z$ is an integer for any $t \in \mathbb{R}$. It follows that $k \cdot q = 0$, that is

$$q_j p_m = p_j q_m \tag{2}$$

for any $m = 1, ..., n$ (including $m = j$). Applying d we get $d(q_j p) = d(p_j q)$ whence $|q_j| d(p) = |p_j| d(q)$, that is $p_j = \pm q_j$ since $p, q \in \mathcal{P}$. In view of $p_j \neq 0$, (2) gives $p = \pm q$ as claimed. ■

3. An inversion formula

3.1. Let f be a continuous function on $\mathbb{T}^n$. We define its *X-ray transform* Rf as the integral of f over closed geodesics of $\mathbb{T}^n$, namely

$$Rf(\ell(x,p)) = \int_0^1 f(x + \mathrm{pr}(tp))\ dt\ , \tag{3}$$

with $x \in \mathbb{T}^n$ and $p \in \mathcal{P}$. As noted in the previous section $x + \mathrm{pr}(tp)$ runs over the whole geodesic $\ell(x,p)$ when t varies from 0 to 1.

The natural *dual transform* R^* is obtained by summing over all closed geodesics through a given point, that is

$$R^*F(x) = \frac{1}{2}\sum_{p\in\mathcal{P}} F(\ell(x,p))\ ,$$

where F is a function on the set of all closed geodesics and the factor $1/2$ is introduced because of (1). However such an operator is not even defined on constant functions and we shall rather replace it by a weighted dual transform as follows. By *weight function* on $\mathcal{P}$ we mean

$$\varphi : \mathcal{P} \to]0,\infty[\text{ such that } \varphi(-p) = \varphi(p) \text{ and } \sum_{p\in\mathcal{P}} \varphi(p) < \infty\ , \tag{4}$$

for instance the restriction to $\mathcal{P}$ of any strictly positive even function in $l^1(\mathbb{Z}^n)$ such as $\varphi(p) = e^{-\|p\|}$ or $\varphi(p) = (1+\|p\|)^{-n-1}$. The *weighted dual transform* R^*_φ is then defined as

$$R^*_\varphi F(x) = \frac{1}{2}\sum_{p\in\mathcal{P}} \varphi(p) F(\ell(x,p)) \tag{5}$$

and the series converges whenever F is a bounded function on the set of all closed geodesics. The transform R^*_φ is dual to R in the following sense

$$\int_{\mathbb{T}^n} R^*_\varphi F(x) f(x) dx = \frac{1}{2}\sum_{p\in\mathcal{P}} \varphi(p) \int_{\mathbb{T}^n} F(\ell(x,p)) Rf(\ell(x,p)) dx\ , \tag{6}$$

valid if f is continuous on $\mathbb{T}^n$ and F is bounded. Indeed, by (1) and (5), $R^*_\varphi F(x) = \frac{1}{2}\sum_p \varphi(p) F(\ell(x - \mathrm{pr}(tp), p))$ for any $t \in \mathbb{R}$ and the left-hand side of (6) is

$$\begin{aligned}\int_{\mathbb{T}^n} R^*_\varphi F(x) f(x) dx &= \frac{1}{2}\sum_p \varphi(p) \int_{\mathbb{T}^n} F(\ell(x - \mathrm{pr}(tp), p)) f(x) dx \\ &= \frac{1}{2}\sum_p \varphi(p) \int_{\mathbb{T}^n} F(\ell(x,p)) f(x + \mathrm{pr}(tp)) dx\ .\end{aligned}$$

Then (6) follows by integration with respect to $t \in [0,1]$. The calculations are valid since, for any t,

$$\sum_p \varphi(p) \int_{\mathbb{T}^n} |F(\ell(x,p))||f(x + \mathrm{pr}(tp))| dx \le \sum_p \varphi(p) \sup|F| \sup|f| < \infty\ .$$

3.2. Several classical inversion formulas for Radon transforms involve R^*Rf. As noted before the sum defining it does not converge here in general (not even for a constant function f) and we shall use $R^*_\varphi Rf$ instead with R^*_φ defined by (5). As usual we denote by[1]

$$\widehat{f}(k) = \int_{\mathbb{T}^n} f(x)\ e^{-2i\pi k\cdot x}\ dx$$

with $k \in \mathbb{Z}^n$ the Fourier coefficients of f. Let us recall the notation $\mathcal{P}_k = \{p \in \mathcal{P} | k\cdot p = 0\}$.

[1]The exponential $e^{-2i\pi k\cdot x}$ is of course unambiguously defined, with $x \in \mathbb{R}^n/\mathbb{Z}^n$ replaced by any representative in $\mathbb{R}^n$.

Theorem 1. *Let φ be a weight function on $\mathcal{P}$ satisfying (4) and, for $k \in \mathbb{Z}^n$,*

$$\psi(k) = \frac{1}{2} \sum_{p \in \mathcal{P}_k} \varphi(p) \ .$$

Then ψ is strictly positive on $\mathbb{Z}^n$, the operator $R_\varphi^* R$ is a convolution operator on $\mathbb{T}^n$ ($n \geq 2$) and, for any continuous function f on $\mathbb{T}^n$ such that $\widehat{f} \in l^1(\mathbb{Z}^n)$, the X-ray transform R is inverted by

$$f(x) = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{T}^n} \frac{e^{2i\pi k \cdot (x-y)}}{\psi(k)} \left(R_\varphi^* R f \right)(y) \ dy \ , \ x \in \mathbb{T}^n \ .$$

This inversion formula applies in particular to any function $f \in C^n(\mathbb{T}^n)$.

Formally $\sum_k e^{2i\pi k \cdot x} \psi(k)^{-1}$ is thus a convolution inverse for $R_\varphi^* R$. However the natural assumption to justify this by permutation of series and integral, namely $\sum_k \psi(k)^{-1} < \infty$, is never true since $\psi(lk) = \psi(k)$ for any strictly positive integer l.

Proof. *(i)* Our definitions imply, for any continuous f,

$$\begin{aligned} R_\varphi^* R f(y) &= \frac{1}{2} \sum_{p \in \mathcal{P}} \varphi(p) R f(\ell(y,p)) = \frac{1}{2} \sum_{p \in \mathcal{P}} \varphi(p) \int_0^1 f(y + \mathrm{pr}(tp)) dt \\ &= < S(x), f(y-x) > \ , \end{aligned}$$

where S is the distribution on $\mathbb{T}^n$ defined by

$$< S, f > = \frac{1}{2} \sum_{p \in \mathcal{P}} \varphi(p) \int_0^1 f(-\mathrm{pr}(tp)) dt \ .$$

Indeed the estimate $| < S, f > | \leq \frac{1}{2} \left(\sum_{p \in \mathcal{P}} \varphi(p) \right) \sup |f|$ shows that S is actually a measure on $\mathbb{T}^n$. Thus

$$R_\varphi^* R f = S * f \tag{7}$$

(convolution on $\mathbb{T}^n$), and this convolution equation can be easily inverted by means of Fourier coefficients. From (7) we have

$$\widehat{R^* \varphi R} f(k) = \widehat{S}(k) \widehat{f}(k) \ ,$$

with

$$\widehat{S}(k) = < S(x), e^{-2i\pi k \cdot x} > = \frac{1}{2} \sum_{p \in \mathcal{P}} \varphi(p) \int_0^1 e^{2i\pi t k \cdot p} dt \ .$$

The integral vanishes whenever $k \cdot p \neq 0$, therefore

$$\widehat{S}(k) = \frac{1}{2} \sum_{p \in \mathcal{P}_k} \varphi(p) = \psi(k) \ .$$

(ii) Given an arbitrary $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ we claim that $\psi(k) > 0$. Indeed $\varphi > 0$ and the set $\mathcal{P}_k$ is nonempty:

- if $k_j = 0$ for all j, then $\mathcal{P}_k = \mathcal{P}$.
- if $k_j \neq 0$ for some j and $k_l = 0$ for all $l \neq j$, then $\mathcal{P}_k$ is the set of all $p \in \mathcal{P}$ such that $p_j = 0$.
- if $k_j \neq 0$ and $k_l \neq 0$ for some j, l with $j \neq l$, then $\mathcal{P}_k$ contains $p = (p_1, ..., p_n)$ with

$$p_j = \frac{k_l}{d(k_j, k_l)} \ , \ p_l = -\frac{k_j}{d(k_j, k_l)} \ , \ p_m = 0 \text{ if } m \neq j, l \ .$$

Finally, in view of the assumptions $f \in C(\mathbb{T}^n)$ and $\widehat{f} \in l^1(\mathbb{Z}^n)$, the Fourier inversion applies to f whence, by *(i)*,

$$f(x) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k)\ e^{2i\pi k\cdot x} = \sum_{k} \frac{1}{\psi(k)} \widehat{R_\varphi^* Rf}(k)\ e^{2i\pi k\cdot x}$$

for all $x \in \mathbb{T}^n$ and the inversion formula follows.
(iii) If f belongs to $C^n(\mathbb{T}^n)$ we have $\widehat{\partial_j^n f}(k) = (2i\pi k_j)^n\, \widehat{f}(k)$ (with $\partial_j = \partial/\partial x_j$) and $\sum_{k\in\mathbb{Z}^n} k_j^{2n} |\widehat{f}(k)|^2 < \infty$ by Parseval's formula applied to $\partial_j^n f \in L^2(\mathbb{T}^n)$. Therefore

$$\left(\sum_{k\in\mathbb{Z}^n, k\neq 0} |\widehat{f}(k)| \right)^2 \leq \sum_{k\in\mathbb{Z}^n, k\neq 0} \left(k_1^{2n} + \cdots + k_n^{2n}\right)^{-1} \sum_{k\in\mathbb{Z}^n} \left(k_1^{2n} + \cdots + k_n^{2n}\right) |\widehat{f}(k)|^2 < \infty$$

by Cauchy-Schwarz inequality, thus $\widehat{f}$ belongs to $l^1(\mathbb{Z}^n)$. ■

Variant of the proof. Though natural, the distribution S can be skipped in the first part of the proof of Theorem 1. Given a function F on $\mathbb{T}^n \times \mathcal{P}$ let us write

$$\widehat{F}(k,p) = \int_{\mathbb{T}^n} F(x,p) e^{-2i\pi k\cdot x} dx\ . \tag{8}$$

We then have the following "*Fourier slice theorem*", with f continuous on $\mathbb{T}^n$ (and a slight abuse of notation),

$$\widehat{Rf}(k,p) = \begin{cases} \widehat{f}(k) \text{ if } p \in \mathcal{P}_k \\ 0 \quad \text{otherwise,} \end{cases} \tag{9}$$

emphasizing the important rôle in our problem of the set $\mathcal{P}_k$ of all $(k,p) \in \mathbb{Z}^n \times \mathcal{P}$ such that $k \cdot p = 0$. Indeed

$$\begin{aligned} \widehat{Rf}(k,p) &= \int_0^1 dt \int_{\mathbb{T}^n} f(x + \mathrm{pr}(tp)) e^{-2i\pi k\cdot x} dx \\ &= \int_{\mathbb{T}^n} f(x) e^{-2i\pi k\cdot x} dx \int_0^1 e^{2i\pi t k\cdot p} dt \end{aligned}$$

and (9) follows.
Since $\sum_{p\in\mathcal{P}} \varphi(p) < \infty$ and $|Rf(\ell(x,p))| \leq \sup|f|$ the series

$$R_\varphi^* Rf(x) = \frac{1}{2} \sum_{p\in\mathcal{P}} \varphi(p) Rf(\ell(x,p))$$

converges uniformly on $\mathbb{T}^n$, therefore

$$\widehat{R_\varphi^* Rf}(k) = \frac{1}{2} \sum_{p\in\mathcal{P}} \varphi(p) \widehat{Rf}(k,p) = \frac{1}{2} \sum_{p\in\mathcal{P}_k} \varphi(p) \widehat{f}(k) = \psi(k) \widehat{f}(k)\ ,$$

and the proof ends as before.

Remark 1. For $n = 2$ and $k \neq 0$ the set $\mathcal{P}_k$ only has two elements:

$$\mathcal{P}_k = \{p(k), -p(k)\} \text{ with } p(k) = \left(\frac{k_2}{d(k_1,k_2)}, -\frac{k_1}{d(k_1,k_2)} \right)\ .$$

Indeed $k_1 p_1 = -k_2 p_2$ with $d(p_1,p_2) = 1$ implies $k_1 = l p_2$ and $k_2 = -l p_1$ for some $l \in \mathbb{Z}$, whence $d(k_1,k_2) = |l|$ and $p = \pm p(k)$.

The finiteness of $\mathcal{P}_k$ in this case was the key to Strichartz' inversion formula for $n = 2$ in [9] p. 422. Writing $e_k(x) = e^{2i\pi k\cdot x}$ he observed that, for $k \neq 0$, $Re_k(\ell(x,p)) = e_k(x)$ if $p \in \mathcal{P}_k$, $Re_k(\ell(x,p)) = 0$ if $p \notin \mathcal{P}_k$ (cf. (9) above), therefore

$$e_k(x) = \frac{1}{2}\sum_{p\in\mathcal{P}} Re_k(\ell(x,p))$$

where the sum only contains two (equal) terms. Multiplying by the Fourier coefficient $\widehat{f}(k)$ and summing over all $k \neq 0$ he obtained

$$\sum_{k\neq 0} \widehat{f}(k)e_k(x) = \frac{1}{2}\sum_{p\in\mathcal{P}} R\left(\sum_{k\neq 0}\widehat{f}(k)e_k\right)(x,p)\ ,$$

that is

$$f(x) - \widehat{f}(0) = \frac{1}{2}\sum_{p\in\mathcal{P}}\left(Rf(\ell(x,p)) - \widehat{f}(0)\right)$$

since $Rc = c$ obviously for any constant c. This is actually an inversion formula for R because

$$\widehat{f}(0) = \int_0^1\int_0^1 f(x_1,x_2)dx_1dx_2 = \int_0^1 Rf(\ell(x(s),p_0))ds$$

with $x(s) = (s,0)$ and $p_0 = (0,1)$ for instance.

Remark 2. Strichartz' method does not extend in an obvious way to $\mathbb{T}^n$ for $n > 2$, the sets $\mathcal{P}_k$ being infinite. Indeed let $n \geq 3$ and $k \in \mathbb{Z}^n$. If $(k_1,k_2) = (0,0)$, $\mathcal{P}_k$ contains $(l,1,0,...,0)$ for all $l \in \mathbb{Z}$. If $(k_1,k_2) \neq (0,0)$ let $k'_j = k_j/d(k_1,k_2)$. By Bezout's theorem there exist $q_1,q_2 \in \mathbb{Z}$ such that $k'_1q_1 + k'_2q_2 + k_3 = 0$, therefore

$$\left(q_1 + lk'_2, q_2 - lk'_1, d(k_1,k_2), 0, ..., 0\right)$$

is orthogonal to k for all $l \in \mathbb{Z}$. Dividing the first three components by their highest common divisor we obtain elements of $\mathcal{P}_k$, easily seen to be distinct when l runs over $\mathbb{Z}$.

Remark 3. As noted above we can pick, for any $k \in \mathbb{Z}^n$, an element $p(k)$ of $\mathcal{P}$ such that $k\cdot p(k) = 0$. By (9) we have $\widehat{f}(k) = \widehat{Rf}(k,p(k))$ therefore

$$\int_{\mathbb{T}^n}|f(x)|^2dx = \sum_{k\in\mathbb{Z}^n}\left|\widehat{f}(k)\right|^2 = \sum_{k\in\mathbb{Z}^n}\left|\widehat{Rf}(k,p(k))\right|^2\ .$$

This may be viewed as a Plancherel type theorem, expressing the L^2 norm of f by means of its Radon transform.

4. A range theorem

In order to state the next theorem we shall denote by $\mathcal{V}$ the space of all functions F on $\mathbb{T}^n \times \mathcal{P}$ satisfying the following three conditions:
(i) for any $p \in \mathcal{P}$ the map $x \mapsto F(x,p)$ belongs to $C^\infty(\mathbb{T}^n)$ and, for any multi-index $\alpha \in \mathbb{N}^n$, there exists a constant C_α such that $|\partial_x^\alpha F(x,p)| \leq C_\alpha$ for all $(x,p) \in \mathbb{T}^n \times \mathcal{P}$
(ii) $\widehat{F}(k,p) = 0$ whenever $k \in \mathbb{Z}^n$, $p \in \mathcal{P}$ and $p \notin \mathcal{P}_k$
(iii) $\widehat{F}(k,p) = \widehat{F}(k,q)$ whenever $k \in \mathbb{Z}^n$ and $p,q \in \mathcal{P}_k$.
Properties *(ii)* and *(iii)* of $\widehat{F}$ defined by (8) are the "*moment conditions*" relevant to our problem.

Theorem 2. *The X-ray transform $f \mapsto F$, with $F(x,p) = Rf(\ell(x,p))$, is a bijection of $C^\infty(\mathbb{T}^n)$ onto $\mathcal{V}$.*

Proof. Given $f \in C^\infty(\mathbb{T}^n)$ the function $F(x,p) = Rf(\ell(x,p)) = \int_0^1 f(x+\mathrm{pr}(tp))dt$ clearly satisfies *(i)*. And *(ii)*, *(iii)* follow from (9).

By Theorem 1 the map $f \mapsto F$ is injective ; only the surjectivity remains to be proved. Given $F \in \mathcal{V}$ let

$$g(k) = \widehat{F}(k,p) \text{ for any } p \in \mathcal{P}_k \ , \tag{10}$$

well defined by assumption *(iii)*, and let us consider the Fourier series

$$f(x) = \sum_{k \in \mathbb{Z}^n} g(k)\ e^{2i\pi k \cdot x} \ . \tag{11}$$

By *(i)* for any $l \in \mathbb{N}$ there exists a constant C_l such that $|\widehat{F}(k,p)| \leq C_l(1 + \|k\|)^{-l}$ for all $k \in \mathbb{Z}^n$, $p \in \mathcal{P}$. The series (11) therefore defines a C^∞ function on the torus. Finally let G be the function defined by $G(x,p) = Rf(\ell(x,p))$. Using (9), (11) and (10) successively we have, for $k \cdot p = 0$,

$$\widehat{G}(k,p) = \widehat{f}(k) = g(k) = \widehat{F}(k,p) \ .$$

Besides, by (9) again and *(ii)*, $\widehat{G}(k,p) = 0 = \widehat{F}(k,p)$ for $k \cdot p \neq 0$. Thus $\widehat{G}$ and $\widehat{F}$ coincide and it follows that $Rf(\ell(x,p)) = F(x,p)$, which completes the proof. ■

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