Mean value theorems on symmetric spaces

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kæri Sigurður Helgason, til hamingju með 85 ára afmælið

ABSTRACT. Revisiting some mean value theorems by F. John, respectively S. Helgason, we study their extension to general Riemannian symmetric spaces, resp. their restatement in a more detailed form, with emphasis on their link with the infinitesimal structure of the symmetric space.

1. An old formula by Fritz John

In his inspiring 1955 book *Plane waves and spherical means* [7], Fritz John considers the mean value operator on spheres in the Euclidean space $\mathbb{R}^n$:

$$M^X u(p) := \int_K u(p + k \cdot X) dk = M^x u(p) \text{ with } x = \|X\| \tag{1.1}$$

where $X, p \in \mathbb{R}^n$, u is a continuous function on $\mathbb{R}^n$, dk is the normalized Haar measure on the orthogonal group $K = SO(n)$ and dot denotes the natural action of this group on $\mathbb{R}^n$. This average of u over the sphere with center p and radius $x = \|X\|$ (the Euclidean norm of $\mathbb{R}^n$) only depends on p and this radius; it may be written $M^x u(p)$ as well.

For $X, Y, p \in \mathbb{R}^n$ the **iterated spherical mean** is

$$M^X M^Y u(p) = \int_K M^{X+k\cdot Y} u(p) dk, \tag{1.2}$$

as easily checked. Taking $z = \|X + k \cdot Y\|$ as the new variable this transforms into

$$M^x M^y u(p) = \int_{|x-y|}^{x+y} M^z u(p) a(x, y, z) z^{n-1} dz \ , \tag{1.3}$$

$$a(x, y, z) = \frac{C_n}{(xyz)^{n-2}} \left(\left(\frac{x+y+z}{2}\right) \left(\frac{x+y-z}{2}\right) \left(\frac{x-y+z}{2}\right) \left(\frac{-x+y+z}{2}\right) \right)^{(n-3)/2}$$

2000 *Mathematics Subject Classification.* Primary 43A85, 53C35; Secondary 33C80, 43A90.

Key words and phrases. Symmetric space, mean value.

Many thanks to Jens Christensen, Fulton Gonzalez and Eric Todd Quinto, organizers of this AMS special session in Boston and the satellite workshop in Tufts, for inviting me to participate - and to Tufts University for its support.

for $x, y > 0$, a formula first proved by John; here $C_n = 2^{n-3}\Gamma\left(\frac{n}{2}\right)/\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)$. A nice proof is given in Chapter VI of Helgason's book [6].

Is there a similar result for symmetric spaces? The purpose of this note is to prove an analog of John's formula for general Riemannian symmetric spaces and, focusing on the formula replacing (1.2), to show some interesting relations with the mean value operator itself, leading to an explicit series expansion of this operator.

From now on we shall work on a **Riemannian symmetric space** G/K, where G is a connected Lie group and K a compact subgroup. We use the customary notation $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the decomposition of the Lie algebra of G given by the symmetry, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ identifies with the tangent space at the origin o of G/K. Let $\|.\|$ be the K-invariant norm on $\mathfrak{p}$ corresponding to the Riemannian structure of G/K. Let $X \mapsto e^X$ denote the exponential mapping of the Lie group G and $\mathrm{Exp} : \mathfrak{p} \to G/K$ the exponential mapping of the symmetric space at o. For $X \in \mathfrak{p}$ the natural generalization of M^X above should be averaging over Riemannian spheres of radius $\|X\|$ in G/K. However we shall rather use averages over K-orbits (which are included in spheres), easier to handle in our group-theoretic framework; both notions agree if G/K is **isotropic**, i.e. K acts transitively on the unit sphere of $\mathfrak{p}$. Thus let

$$M^X u(gK) := \int_K u(g \cdot \mathrm{Exp}\,(k \cdot X))dk = \int_K u(gk \cdot \mathrm{Exp}\, X)dk \tag{1.4}$$

where $u : G/K \to \mathbb{C}$ is continuous, $X \in \mathfrak{p}$, $g \in G$ and dots denote the natural action of G on G/K, respectively the adjoint action of K on $\mathfrak{p}$. Then (1.2) generalizes as follows. Since Exp is a diffeomorphism between neighborhoods of the origins in $\mathfrak{p}$ and G/K we may define $Z(X, Y) \in \mathfrak{p}$ (for X, Y near the origin of $\mathfrak{p}$ at least) by

$$\mathrm{Exp}\, Z(X, Y) = e^X \cdot \mathrm{Exp}\, Y, \tag{1.5}$$

that is $e^{Z(X,Y)}K = e^X e^Y K$, and we have

$$M^X M^Y u(gK) = \int_K M^{Z(X,k\cdot Y)} u(gK)dk. \tag{1.6}$$

Remark. From (1.5) and (1.6) an *elementary proof of the commutativity* $M^X M^Y = M^Y M^X$ *for symmetric spaces* is easily obtained[1]. Indeed, let $k(X, Y) \in K$ be defined by $k(X, Y) := e^{-Z(X,Y)}e^X e^Y$ hence, applying the involution of G, $k(X, Y) = e^{Z(X,Y)}e^{-X}e^{-Y}$ and, combining both expressions, $e^{2Z(X,Y)} = e^X e^{2Y} e^X$. Therefore

$$e^X e^Y K = e^{-Z(X,Y)} e^{2Z(X,Y)} K = \left(e^{-Z(X,Y)} e^X e^Y\right) e^Y e^X K = k(X, Y) e^Y e^X K$$

and it follows that

$$Z(X, Y) = k(X, Y) \cdot Z(Y, X).$$

Then, for any $k' \in K$,

$$Z(X, k' \cdot Y) = k(X, k' \cdot Y) \cdot Z(k' \cdot Y, X) = (k(X, k' \cdot Y)k') \cdot Z(Y, k'^{-1} \cdot X)$$

and the K-invariance of $X \mapsto M^X u(gK)$ gives $M^{Z(X,k'\cdot Y)} = M^{Z(Y,k'^{-1}\cdot X)}$ and $M^X M^Y = M^Y M^X$ by (1.6), as claimed.

This preliminary result (1.6) will now be improved in two directions:
- an analog of John's formula (1.3) for rank one spaces (Theorem 2.1)

[1] More "sophisticated" proofs are suggested by the last remarks of sections 3.1 and 3.2. More generally, this commutativity property holds true for all Gelfand pairs (G, K): see [5] p. 80.

- a proof that $M^X M^Y u(gK) = \int_K M^{X+k\cdot Y} u(gK)\ f(X, k\cdot Y)\ dk$ for general Riemannian symmetric spaces (Theorem 3.1), with a specific factor f which turns out to have independent interest and will play a part in an expansion of M^X itself (Theorem 3.2).

2. Mean values on rank one spaces

In this section let G/K denote a Riemannian symmetric space *of the noncompact type and of rank one*, that is one of the hyperbolic spaces (real, complex, quaternionic, or exceptional). This space is isotropic and $M^X u(gK)$, which is a K-invariant function of $X \in \mathfrak{p}$, only depends on the (K-invariant) norm $x = \|X\|$; as in the Euclidean case we may thus write $M^X u(gK) = M^x u(gK)$. Note that Exp is here a global diffeomorphism of $\mathfrak{p}$ onto G/K, so that $Z(.,.)$ is globally defined. Let $p \geq 1$ and $q \geq 0$ denote the mutiplicities of the positive roots, $n = p+q+1 = \dim G/K$ and let C_n be as in (1.3).

Theorem 2.1. *For $x, y > 0$ and u continuous on G/K*

$$M^x M^y u(gK) = \int_{|x-y|}^{x+y} M^z u(gK) b(x,y,z)\delta(z)dz\ ,$$

where $\delta(z) = (\operatorname{sh} z)^{n-1} (\operatorname{ch} z)^q$,

$$b(x,y,z) = C_n \frac{(\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z)^{(p/2)-1}}{(\operatorname{sh} x \operatorname{sh} y \operatorname{sh} z)^{n-2}} B^{(n-3)/2}\ {}_2F_1\left(1-\frac{q}{2}, \frac{q}{2}; \frac{n-1}{2}; B\right)$$

and

$$B = \frac{1}{\operatorname{ch} x \operatorname{ch} y \operatorname{ch} z} \operatorname{sh}\left(\frac{x+y+z}{2}\right) \operatorname{sh}\left(\frac{x+y-z}{2}\right) \operatorname{sh}\left(\frac{x-y+z}{2}\right) \operatorname{sh}\left(\frac{-x+y+z}{2}\right).$$

One has $b(x,y,z) > 0$ for $x, y > 0$ and $|x-y| < z < x+y$.

Nice function, isn't it?

Remark 1. The hypergeometric factor is identically 1 if $q = 0$ that is $G/K = H^n(\mathbb{R})$.

Remark 2. $tb(tx, ty, tz)\delta(tz)$ tends to the Euclidean factor $a(x,y,z)z^{n-1}$ as $t \to 0$. John's formula is thus a "flat limit" of Theorem 2.1.

Remark 3. In [7] John inverts (1.3), an Abel type integral equation, so as to express $M^z u$ by means of $M^x M^y u$ and finally, taking $z = 0$, u itself as a sum of derivatives of its iterated mean values. This seems unworkable here however, with the present kernel b.

Proof. The main point is to prove

$$\int_K \varphi\left(\|Z(X, k\cdot Y)\|\right) dk = \int_{|x-y|}^{x+y} \varphi(z) b(x,y,z)\delta(z)dz \tag{2.1}$$

for any continuous function $\varphi : [0, \infty[\to \mathbb{C}$. Then taking $\varphi(t) = M^t u(gK)$ (with gK fixed) our claim will follow in view of (1.6).

To prove (2.1) one may use a Cartan type decomposition of K, which reduces the problem to a 2- or 1- dimensional subgroup of K. The classical technique of $SU(2,1)$-reduction then allows an explicit computation of $z = \|Z(X, k\cdot Y)\|$ and (2.1) is finally obtained by taking z as the new variable in the integral. Euler's integral representation of the hypergeometric function shows that $b(x,y,z) > 0$ for

$x, y > 0$ and $|x-y| < z < x+y$. Full details, lengthy but easy, will appear in [9], Chapter 3 and Appendix B.

Remark 4. Formula (2.1) for spherical functions can be derived from the work [1] of Flensted-Jensen and Koornwinder (or [8] §7.1). Indeed the following product formula is proved in §4 of [1] (in the more general framework of Jacobi functions):

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_{|x-y|}^{x+y} \varphi_\lambda(z) b(x,y,z)\delta(z) dz \tag{2.2}$$

for $x, y > 0$, $\lambda \in \mathbb{C}$; see also the last pages of Chapter III in [5]. Here b and δ are the functions defined above, φ_λ is one of Harish-Chandra's spherical functions of G/K and $\varphi_\lambda(x)$ is, abusing notation, $\varphi_\lambda(\operatorname{Exp} X)$ for $X \in \mathfrak{p}$ and $\|X\| = x$. Actually, to deduce (2.2) from (4.2) and (4.19) in [1] (or (7.11) in [8]) some minor changes need to be made: the parameters (α, β) of general Jacobi functions are here $\alpha = (n-2)/2$, $\beta = (q-1)/2$ (corresponding to the group case), the function B of [1] is our $1-2B$ and the hypergeometric formula

$$ {}_2F_1(a, b; c; t) = (1-t)^{c-a-b} \ {}_2F_1(c-a, c-b; c; t)$$

is applied. Finally the left-hand side of (2.2) is $\int_K \varphi_\lambda(\|Z(X, k\cdot Y)\|)\, dk$, by the functional equation of spherical functions.

Similarly, in the Euclidean setting of $\mathfrak{p}$, John's formula (1.3) follows from

$$\int_K \varphi(\|X + k \cdot Y\|)\, dk = \int_{|x-y|}^{x+y} \varphi(z) a(x,y,z) z^{n-1} dz$$

for any continuous function $\varphi : [0, \infty[\to \mathbb{C}$. Combining this with (2.1) we have

$$\int_K \varphi(\|Z(X, k\cdot Y)\|)\, dk =$$
$$\int_K \varphi(\|X + k\cdot Y\|)\, \frac{b}{a}(\|X\|, \|Y\|, \|X + k\cdot Y\|)\, \frac{\delta(\|X + k\cdot Y\|)}{\|X + k\cdot Y\|^{n-1}} dk,$$

thus, for any continuous K-invariant function F on $\mathfrak{p}$,

$$\int_K F(Z(X, k\cdot Y))\, dk = \int_K F(X + k\cdot Y)\, f(X, k\cdot Y) dk \tag{2.3}$$
$$\text{with } f(X,Y) := \frac{b}{a}(x,y,z)\frac{\delta(z)}{z^{n-1}}\ ,\ x = \|X\|, y = \|Y\|, z = \|X+Y\|.$$

Applying this to the mean value $F(X) = M^X u(gK)$ we obtain from (1.6)

$$M^X M^Y u(gK) = \int_K M^{X+k\cdot Y} u(gK) f(X, k\cdot Y) dk.$$

The latter formula extends to all Riemannian symmetric spaces, as will be shown now.

3. Mean values on Riemannian symmetric spaces

Throughout this section G is a connected Lie group and K a compact subgroup, such that G/K is a Riemannian symmetric space.

3.1. Iterated mean values.

THEOREM 3.1. *(i) There exists a neighborhood U of the origin in $\mathfrak{p} \times \mathfrak{p}$ and an analytic map $(X,Y) \mapsto c(X,Y)$ from U into K such that $c(k \cdot X, k \cdot Y) = kc(X,Y)k^{-1}$ for all $k \in K$ and, letting $\gamma_X(Y) := c(X,Y) \cdot Y$,*

$$F(Z(X, \gamma_X(Y))) = F(X+Y)$$

for any K-invariant continuous function F on $\mathfrak{p}$.
(ii) Let $f(X,Y) := \det_{\mathfrak{p}} D\gamma_X(Y)$ be the Jacobian of γ_X with respect to Y. Then, for $(X,Y) \in U$, one has $f(k \cdot X, k \cdot Y) = f(X,Y) > 0$ and

$$\int_K F(Z(X, k \cdot Y))dk = \int_K F(X + k \cdot Y) f(X, k \cdot Y) dk \ . \tag{3.1}$$

In particular, for u continuous on G/K, $g \in G$ and $(X,Y) \in U$,

$$M^X M^Y u(gK) = \int_K M^{X+k\cdot Y} u(gK) f(X, k\cdot Y) dk \ . \tag{3.2}$$

Proof. We shall only sketch the main steps and refer to Chapter 4 of [9] for details.
(i) Let $Z_t(X,Y) := t^{-1} Z(tX, tY)$ for $0 < t \le 1$ and $Z_0(X,Y) := X + Y$. The map $(t, X, Y) \mapsto Z_t(X,Y)$ is analytic in an open subset of $\mathbb{R} \times \mathfrak{p} \times \mathfrak{p}$ containing $[0,1] \times U$, where U is a suitably chosen neighborhood of $(0,0)$ in $\mathfrak{p} \times \mathfrak{p}$. In order to relate $Z(X,Y) = Z_1(X,Y)$ to its flat analog $Z_0(X,Y)$ we shall solve a differential equation with respect to t.
One can construct two series of even Lie brackets

$$A(X,Y) = -\frac{1}{3}[X,Y] + \frac{1}{90}\left(7[X,[X,[X,Y]]] + 12[Y,[X,[X,Y]]] + 4[Y,[Y,[X,Y]]]\right) + \cdots$$

$$C(X,Y) = -\frac{1}{3}[X,Y] + \frac{1}{45}\left(2[X,[X,[X,Y]]] + 3[Y,[X,[X,Y]]] + 2[Y,[Y,[X,Y]]]\right) + \cdots$$

such that $C(Y,X) = -C(X,Y)$ and

$$\partial_t Z_t = [Z_t, A_t] + (\partial_Y Z_t)\,[Y, C_t] \tag{3.3}$$

with $A_t(X,Y) = t^{-1}A(tX,tY)$, $C_t(X,Y) = t^{-1}C(tX,tY)$. All functions in (3.3) are evaluated at $(X,Y) \in U$ and, for $V \in \mathfrak{p}$, $(\partial_Y Z_t)\,V$ means $\partial_s \left(Z_t(X, Y + sV)\right)\big|_{s=0}$. Since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ and $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, both A_t and C_t map U into $\mathfrak{k}$. Such series of Lie brackets are obtained by manipulating the Campbell-Hausdorff formula for the Lie algebra $\mathfrak{g}$, written in the specific form introduced by Kashiwara and Vergne.
Now let $c_t = c_t(X,Y) \in K$ denote the solution of the differential equation

$$\partial_t c_t = D_e R_{c_t}\left(C_t(X, c_t \cdot Y)\right)\ ,\ c_0 = e,$$

where R_c denotes the right translation by c in K and $D_e R_c$ its differential at the origin e of K. The K-invariance $c_t(k\cdot X, k \cdot Y) = kc_t(X,Y)k^{-1}$ follows from $C_t(k \cdot X, k\cdot Y) = k \cdot C_t(X,Y)$ and the uniqueness of the solution.
Setting $V_t = c_t \cdot Y$ we have

$$\partial_t V_t = [C_t(X, V_t), V_t]\ ,\ V_0 = Y \tag{3.4}$$

and, for any smooth function F on $\mathfrak{p}$,

$$\begin{aligned}\partial_t\left(F(Z_t(X,V_t)\right) &= DF(Z_t(X,V_t))\left\{(\partial_t Z_t)(X,V_t) + (\partial_Y Z_t)(X,V_t)\partial_t V_t\right\} \\ &= DF\left(Z_t(X,V_t)\right)[Z_t, A_t](X,V_t)\end{aligned}$$

by (3.3). But if F is K-invariant we have $F(e^{sA} \cdot X) = F(X)$ for any $s \in \mathbb{R}$, $A \in \mathfrak{k}$, $X \in \mathfrak{p}$, hence $\partial_s \left(F(e^{sA}\cdot X)\right)\big|_{s=0} = DF(X)\ [A,X] = 0$ and finally

$\partial_t\left(F(Z_t(X,V_t)\right)=0$ for all $t\in[0,1]$. Thus $F(Z_0(X,V_0))=F(Z(X,V_1))$. Since $V_0=Y$ this implies our claim (with $c(X,Y)=c_1$, $\gamma_X(Y)=V_1=c(X,Y)\cdot Y$) for smooth F, and the general case follows by approximation.
(ii) The K-invariance of the Jacobian f is an easy consequence of the corresponding property of c. Let $f_t=f_t(X,Y)$ denote the Jacobian of the map $Y\mapsto V_t=c_t(X,Y)\cdot Y$ constructed above. From (3.4) we infer

$$\partial_t f_t=-f_t\ \operatorname{tr}_{\mathfrak{p}}\left(\operatorname{ad}V_t\circ(\partial_Y C_t)(X,V_t)\right),\ f_0=1,$$

where $\operatorname{tr}_{\mathfrak{p}}(..)$ means the trace of $(..)$ restricted to $\mathfrak{p}$. It follows that $f_t>0$ for $0\le t\le 1$; in particular $f_1=f$ is positive on U.
In order to prove (3.1) let φ be any K-invariant continuous function on $\mathfrak{p}$, compactly supported near the origin and let X be fixed in $\mathfrak{p}$ (near 0). Then

$$\begin{aligned}\int_{\mathfrak{p}}\varphi(Y)dY\int_K F(Z(X,k\cdot Y))dk&=\int_{\mathfrak{p}}\varphi(Y)F(Z(X,Y))dY\\&=\int_{\mathfrak{p}}\varphi(\gamma_X(Y))F(X+Y)\det_{\mathfrak{p}}D\gamma_X(Y)dY\\&=\int_{\mathfrak{p}}\varphi(Y)F(X+Y)f(X,Y)dY,\end{aligned}$$

using the K-invariance of φ and the change $Y\mapsto\gamma_X(Y)$. Then (3.1) follows since φ is an arbitrary K-invariant function, hence (3.2) taking $F(X)=M^X u(gK)$ and remembering (1.6). ■
Remark 1. For any X,Y close enough to the origin of $\mathfrak{p}$ we have $f(X,0)=f(0,Y)=1$. Indeed, by (3.1) with $F=1$ and the K-invariance of f we have $\int_K f(X,k\cdot Y)dk=\int_K f(k^{-1}\cdot X,Y)dk=1$.
Remark 2. The construction of $\gamma_X(Y)$ in *(i)* is universal and only depends on the infinitesimal structure of the symmetric space (the structure of the corresponding Lie triple system). The factor f introduced in Theorem 3.1 turns out to have deeper relations with analysis on G/K; some of them will appear during the proof of Theorem 3.2. Actually f is related to the "e-function" of [9] by $e(X,Y)=(J(X)J(Y)/J(X+Y))^{1/2}f(X,Y)$ where $J(X)=\det_{\mathfrak{p}}(\operatorname{sh}(\operatorname{ad}X)/\operatorname{ad}X)$ is the Jacobian of Exp. In particular, if G is a complex Lie group and K a compact real form of G it can be shown that $e(X,Y)=1$ hence[2] $f(X,Y)=(J(X+Y)/J(X)J(Y))^{1/2}$.
A direct proof of Theorem 3.1 could be given for spherical functions, under the further assumption that G is a complex semisimple Lie group with finite center and K a maximal compact subgroup. Indeed, by Helgason's Theorem 4.7 in [4], Chapter IV, the spherical functions of G/K are then given by

$$J(X)^{1/2}\varphi_\lambda(\operatorname{Exp}X)=\int_K e^{i\langle\lambda,k\cdot X\rangle}dk,$$

[2] By [9] again the same holds for a compact Lie group G viewed as the symmetric space $(G\times G)/K$ where K is the diagonal subgroup.

where λ is an arbitrary linear form on $\mathfrak{p}$. Therefore

$$\begin{aligned}\int_K \varphi_\lambda(\operatorname{Exp} Z(X,k\cdot Y))dk &= \int_K \varphi_\lambda\left(e^X k\cdot \operatorname{Exp} Y\right)dk = \varphi_\lambda(\operatorname{Exp} X)\varphi_\lambda(\operatorname{Exp} Y)\\ &= \int_{K\times K} e^{i\left\langle \lambda, k\cdot X+k'\cdot Y\right\rangle}\left(J(X)J(Y)\right)^{-1/2}dkdk'\\ &= \int_{K\times K} e^{i\left\langle \lambda, k\cdot (X+k'\cdot Y)\right\rangle}\left(J(X)J(Y)\right)^{-1/2}dkdk'\\ &= \int_K \varphi_\lambda(\operatorname{Exp}(X+k'\cdot Y))\left(\frac{J(X+k'\cdot Y)}{J(X)J(k'\cdot Y)}\right)^{1/2}dk',\end{aligned}$$

hence (3.1) with $F=\varphi_\lambda\circ\operatorname{Exp}$ and $f(X,Y)=\left(J(X+Y)/J(X)J(Y)\right)^{1/2}$.

Remark 3. A close look at the construction of c shows that $f(X,Y)=f(Y,X)$ ([9], Chapter 4); in the rank one case this was clear from the explicit expression of f. Thus Theorem 3.1 gives another proof of $M^XM^Y=M^YM^X$.

3.2. Expansion of the mean value operator. In the isotropic case a K-invariant analytic function F on $\mathfrak{p}$ can be expanded as $F(X)=\sum_{m=0}^\infty a_m\left\|X\right\|^{2m}$ and the coefficients are given, for $m\geq 1$, by

$$\Delta^m F(0)=a_m c_m \text{ with } c_m=\Delta^m\left\|X\right\|^{2m}=2^m m! n(n+2)\cdots(n+2m-2),$$

where Δ is the Euclidean Laplace operator on $\mathfrak{p}$ and $n=\dim\mathfrak{p}$.

A similar expansion is valid in an arbitrary Riemannian symmetric space G/K, as we shall now see. Applying it to $F(X)=M^Xu(gK)$ we shall obtain an expansion of the mean value operator in terms of differential operators on $\mathfrak{p}$. It is more interesting however to replace them by elements of $\mathbb{D}(G/K)$, the algebra of G-invariant differential operators on G/K. This is the goal of the next theorem, where the function f of Theorem 3.1 will play again a significant role.

Let $\mathbb{D}(\mathfrak{p})^K=S(\mathfrak{p})^K$ denote the algebra of K-invariant differential operators on $\mathfrak{p}$ with constant (real) coefficients, canonically isomorphic to the algebra of K-invariant elements in the symmetric algebra of $\mathfrak{p}$, that is K-invariant polynomials on its dual $\mathfrak{p}^*$. This algebra is finitely generated ([4], Chapter III, Theorem 1.10). Thus let $P_1,...,P_l$ be a system of homogeneous generators of this algebra: from the set of all

$$P^\alpha := P_1^{\alpha_1}\cdots P_l^{\alpha_l}\ ,\ \alpha=(\alpha_1,...,\alpha_l)\in\mathbb{N}^l,$$

we can thus extract a basis of $\mathbb{D}(\mathfrak{p})^K$, say $\left(P^\alpha\right)_{\alpha\in B}$ where B is some subset of $\mathbb{N}^l$. We shall denote by $(P^*_\alpha)_{\alpha\in B}$ the dual basis of $S(\mathfrak{p}^*)^K$ (the K-invariant polynomials on $\mathfrak{p}$) with respect to the *Fischer product*:

$$\left\langle P^\alpha | P^*_\beta\right\rangle = P^\alpha(\partial_X)P^*_\beta(0)=\delta_{\alpha\beta}.$$

Each P^α is homogeneous of degree $\alpha\cdot d=\alpha_1d_1+\cdots+\alpha_ld_l$ (with $d_j=\deg P_j$) and it follows easily that P^*_α is homogeneous of the same degree.

Example 1. Trivial case: $K=\{e\}$, $B=\mathbb{N}^n$, $P^\alpha=\partial_X^\alpha$, $P^*_\alpha=X^\alpha/\alpha!$ in multi-index notation.

Example 2. Isotropic case: $\alpha=m\in B=\mathbb{N}$, $P^\alpha=\Delta^m$, $P^*_\alpha=\left\|X\right\|^{2m}/c_m$.

Any K-invariant analytic function on $\mathfrak{p}$ can now be expanded by means of the P^*_α's. In particular, we have the following K-invariant Taylor expansion of the mean value of an analytic function u on G/K

$$M^X u(gK) = \sum_{\alpha \in B} a_\alpha(gK) P_\alpha^*(X)$$

(for X near the origin of $\mathfrak{p}$), with coefficients given by

$$\begin{aligned} a_\alpha(gK) &= P^\alpha(\partial_X)\left(M^X u(gK)\right)_{X=0} = P^\alpha(\partial_X)\left(\int_K u(g \cdot \operatorname{Exp}(k \cdot X))dk\right)_{X=0} \\ &= \widetilde{P^\alpha} u(gK), \end{aligned}$$

introducing $\widetilde{P} \in \mathbb{D}(G/K)$, the G-invariant differential operator on G/K corresponding to $P \in \mathbb{D}(\mathfrak{p})^K$ under

$$\widetilde{P} u(gK) := P(\partial_X)\left(u(g \cdot \operatorname{Exp} X)\right)_{X=0}. \tag{3.5}$$

The map $P \mapsto \widetilde{P}$ is a linear bijection of $\mathbb{D}(\mathfrak{p})^K$ onto $\mathbb{D}(G/K)$ (not in general an isomorphism of algebras). Thus

$$M^X u(gK) = \sum_{\alpha \in B} \widetilde{P^\alpha} u(gK) P_\alpha^*(X). \tag{3.6}$$

Our final step is to express this by means of the generators $\widetilde{P_1}, ..., \widetilde{P_l}$ of $\mathbb{D}(G/K)$. Let

$$f(X,Y) = \sum_{\sigma,\tau \in \mathbb{N}^n} f_{\sigma\tau} X^\sigma Y^\tau = 1 + \sum_{|\sigma| \geq 1, |\tau| \geq 1} f_{\sigma\tau} X^\sigma Y^\tau$$

denote the Taylor series of f at the origin (with respect to some basis of $\mathfrak{p}$); the latter expression of this series follows from Remark 1 in 3.1.

Theorem 3.2. *Let u be an analytic function on a neighborhood of a point $g_0 K$ in G/K. Then there exist a neighborhood V of g_0 in G, a radius $R > 0$ and a sequence of polynomials $(A_\alpha)_{\alpha \in B}$ such that, for $g \in V$, $X \in \mathfrak{p}$ and $|X| < R$,*

$$M^X u(gK) = \sum_{\alpha \in B} P_\alpha^*(X) A_\alpha\left(f; \widetilde{P_1}, ..., \widetilde{P_l}\right) u(gK)\ .$$

Each $A_\alpha(f; \lambda_1, ..., \lambda_l)$ is a polynomial in the λ_j (with $1 \leq j \leq l$) and the $f_{\sigma\tau}$ (with $|\sigma| + |\tau| \leq \alpha \cdot d$), homogeneous of degree $\alpha \cdot d$ if one assigns $\deg \lambda_j = \deg P_j = d_j$ and $\deg f_{\sigma\tau} = |\sigma| + |\tau|$. Furthermore

$$A_\alpha(f; \lambda_1, ..., \lambda_l) = \lambda_1^{\alpha_1} \cdots \lambda_l^{\alpha_l} + \textit{lower degree in } \lambda\ .$$

The coefficients of A_α only depend on the structure of the algebra $\mathbb{D}(\mathfrak{p})^K$ and the choice of generators P_j.

This theorem provides a precise form of a result proved in 1959 by Helgason ([3], Chapter IV, or [5], Chapter II, Theorem 2.7). Helgason's theorem states that, for arbitrary Riemannian homogeneous spaces,

$$M^X u(gK) = \sum_{n=0}^{\infty} p_n(D_1, ..., D_l)\, u(gK)$$

where the p_n's are polynomials (with $p_0 = 1$) and the D_j's are generators of the algebra $\mathbb{D}(G/K)$. Here, restricting to the case of a symmetric space, we obtain a more explicit expression of p_n, showing its dependence on X, and an inductive procedure to compute the A_α's, relating them to the infinitesimal structure of the space (the coefficients $f_{\sigma\tau}$).

For a spherical function φ we have $M^X\varphi(gK) = \varphi(\operatorname{Exp} X)\varphi(gK)$ by the functional equation and we immediately infer the following expansion. Again, a more general but less precise form of this expansion is given in Helgason [5], Chapter II, Corollary 2.8.

Corollary 1. *Let φ be a spherical function on G/K and let λ_j denote the eigenvalues corresponding to our generators of $\mathbb{D}(G/K)$: $\widetilde{P_j}\varphi = \lambda_j\varphi$ for $j = 1, ..., l$. Then there exists $R > 0$ such that, for $|X| < R$,*

$$\varphi(\operatorname{Exp} X) = \sum_{\alpha\in B} A_\alpha(f; \lambda_1, ..., \lambda_l)P^*_\alpha(X) \ .$$

Proof of Theorem 3.2. We shall prove the equality

$$\widetilde{P^\alpha} = A_\alpha\left(f; \widetilde{P_1}, ..., \widetilde{P_l}\right) \tag{3.7}$$

for all $\alpha \in \mathbb{N}^l$ by induction on the degree $\alpha\cdot d$ of P^α. First $\widetilde{P^0} = 1$ and our claim holds true with $A_0 = 1$. The main step is to compare $\widetilde{P^\alpha P_j}$ with $\widetilde{P^\alpha}\circ\widetilde{P_j}$ and this is where f enters the picture.
Indeed let u be an arbitrary smooth function on G/K. By Theorem 3.1 applied to the K-invariant function $F(X) = \int_K u(\operatorname{Exp}(k\cdot X))dk$ we have

$$\int_{K\times K} u\left(e^{k\cdot X}\cdot \operatorname{Exp}(k'\cdot Y)\right)dkdk' = \int_{K\times K} u(\operatorname{Exp}(k\cdot X + k'\cdot Y))f(k\cdot X, k'\cdot Y)dkdk'.$$

Now let $P, Q \in \mathbb{D}(\mathfrak{p})^K$. Applying to both sides of this equality the differential operator $P(\partial_X)Q(\partial_Y)$ we obtain, at $X = Y = 0$,

$$P(\partial_X)Q(\partial_Y)\left(u\left(e^X\cdot \operatorname{Exp} Y\right)\right)_{X=Y=0} = P(\partial_X)Q(\partial_Y)\left(u\left(\operatorname{Exp}(X+Y)\right)f(X,Y)\right)_{X=Y=0}$$

in view of the K-invariance of P and Q. In other words, remembering the definition (3.5) and the expansion of f,

$$\begin{aligned}\widetilde{P}\circ\widetilde{Q}u(o) &= P(\partial_X)Q(\partial_Y)\left(u\left(\operatorname{Exp}(X+Y)\right)f(X,Y)\right)_{X=Y=0}\\ &= P(\partial_X)Q(\partial_X)\left(u\operatorname{Exp}(X)\right)_{X=0} + \\ &+ \sum_{|\sigma|\geq 1, |\tau|\geq 1} f_{\sigma\tau}\ P^{(\sigma)}(\partial_X)Q^{(\tau)}(\partial_X)(u(\operatorname{Exp} X))_{X=0}\end{aligned} \tag{3.8}$$

by Leibniz' formula, where $P^{(\sigma)}(\xi) = \partial^\sigma_\xi P(\xi)$ for $\xi \in \mathfrak{p}^*$. To put it briefly we have $\widetilde{PQ}u(o) = \widetilde{P}\circ\widetilde{Q}u(o) - \widetilde{R}u(o)$, where $R = \sum_{|\sigma|\geq 1, |\tau|\geq 1} f_{\sigma\tau}\ P^{(\sigma)}(\partial_X)Q^{(\tau)}(\partial_X)$ (finite sum) belongs to $\mathbb{D}(\mathfrak{p})^K$ and has lower order than PQ. The G-invariance of all operators allows replacing the origin o by any point in G/K, hence

$$\widetilde{PQ} = \widetilde{P}\circ\widetilde{Q} - \widetilde{R}, \tag{3.9}$$

an equality in $\mathbb{D}(G/K)$.
Now let $\alpha \in \mathbb{N}^l$ with $\alpha_j \geq 1$ for some j. We may write $P^\alpha = P^\beta P_j$ (composition of operators in $\mathbb{D}(\mathfrak{p})^K$) with $\beta \in \mathbb{N}^n$ and (3.9) gives $\widetilde{P^\beta P_j} = \widetilde{P^\beta}\circ\widetilde{P_j} - \widetilde{R}$. In $\mathbb{D}(\mathfrak{p})^K$ the operator R decomposes as $R = \sum_\gamma r_\gamma P^\gamma$, where the r_γ's are scalars and $\sum$ runs over $\gamma \in B$, $\gamma\cdot d \leq \beta\cdot d + d_j - 2 = \alpha\cdot d - 2$. Therefore, assuming (3.7) for all β with $\beta\cdot d < \alpha\cdot d$ we obtain

$$\widetilde{P^\alpha} = \widetilde{P^\beta P_j} = A_\beta\left(f; \widetilde{P_1}, ..., \widetilde{P_l}\right)\circ\widetilde{P_j} - \sum_\gamma r_\gamma A_\gamma\left(f; \widetilde{P_1}, ..., \widetilde{P_l}\right),$$

and the result follows by induction on $\alpha \cdot d$. ■

Remark 1. In [2] Gray and Willmore proved the following mean value expansion for an arbitrary Riemannian manifold M. Let u be an analytic function on a neighborhood of a point $a \in M$ and let $M^r u(a)$ denote the mean value of u over the (Riemannian) sphere $S(a,r)$ with center a and radius r. Besides let J be the Jacobian of the exponential map Exp_a, let Δ be (as above) the Euclidean Laplace operator on the tangent space to M at a, and Δ_a the differential operator on a neighborhood of a in M defined by $(\Delta_a u) \circ \mathrm{Exp}_a = \Delta\,(u \circ \mathrm{Exp}_a)$. Then

$$M^r u(a) = \frac{j_{(n/2)-1}\left(r\sqrt{-\Delta_a}\right)(Ju)(a)}{j_{(n/2)-1}\left(r\sqrt{-\Delta_a}\right)(J)(a)}, \tag{3.10}$$

where j is the (modified) Bessel function

$$j_{(n/2)-1}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{c_m} x^{2m}\ ,\ c_m = 2^m m! n(n+2)\cdots(n+2m-2).$$

This "*generalized Pizzetti's formula*" follows easily from the corresponding formula in Euclidean space, since the Riemannian sphere $S(a,r)$ is the image under Exp_a of the Euclidean sphere with center 0 and radius r. For isotropic Riemannian symmetric spaces $M = G/K$ the spheres are K-orbits and we may compare (3.10) with our Theorem 3.2: the Gray-Willmore expansion uses the same simple power series as in the Euclidean case, but in general their differential operator Δ_a is not G-invariant on M. Thus no simple expansion of the spherical functions seems to come out of (3.10).

Remark 2. The symmetry $f(X,Y) = f(Y,X)$ (Remark 3 of 3.1) together with (3.8) imply the commutativity of the algebra $\mathbb{D}(G/K)$. Thus Theorem 3.2 gives yet another proof of $M^X M^Y = M^Y M^X$.

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