Hypergeometric integral transforms

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The following integral formula plays an essential role in Cormack's 1981 study [1] of the line Radon transform in the plane:

$$2\int_r^s \frac{sT_n(p/s)}{\sqrt{s^2-p^2}} \frac{rT_n(p/r)}{\sqrt{p^2-r^2}} \frac{dp}{p} = \pi \ , \ n \in \mathbb{Z}, 0 < r < s, \tag{1}$$

where T_n denotes the Chebyshev polynomial defined by $T_n(\cos\theta) = \cos n\theta$. A few years later, Cormack [2] extended his method to the Radon transform on certain hypersurfaces in $\mathbb{R}^n$, using a similar formula with Chebyshev polynomials replaced by Gegenbauer's.

All these polynomials may be viewed as hypergeometric functions. Here we shall prove in Section 2 a more general integral formula (Proposition 3) and an inversion formula for a general hypergeometric integral transform (Theorem 4). These results are applied to Chebyshev polynomials in Section 3.

For the reader's convenience we give self-contained proofs, only sketched in the appendices to Cormack's papers [1] and [2]. No previous knowledge of hypergeometric functions is assumed; the basics are given in Section 1.

1 Hypergeometric functions

Let a, b, c, z denote complex numbers. For $k \in \mathbb{N}$, let

$$(a)_k := a(a+1)\cdots(a+k-1) \text{ if } k \geq 1 \ , \ (a)_0 := 1.$$

For instance the classical binomial series may be written as

$$(1-z)^{-a} = \sum_{k\in\mathbb{N}} (a)_k \frac{z^k}{k!} \ , \ |z| < 1, \tag{2}$$

where $(1-z)^{-a}$ denotes the principal value. It is useful to note that $(a)_k = \Gamma(a+k)/\Gamma(a)$ if $a \notin -\mathbb{N}$, where $\Gamma(s) := \int_0^\infty e^{-x}x^{s-1}dx$ is Euler's Gamma function.

Assuming $c \notin -\mathbb{N}$ from now on, the **hypergeometric function** is defined by the series

$$F(a, b; c; z) := \sum_{k\in\mathbb{N}} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}, \tag{3}$$

absolutely convergent for $|z| < 1$. When dealing with more general hypergeometric functions, it is customary to denote by ${}_2F_1(a, b; c; z)$ our $F(a, b; c; z)$. Some properties of F, such as the symmetry

$$F(a, b; c; z) = F(b, a; c; z), \tag{4}$$

are obvious from this definition. Some others follow from Euler's integral representation

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt, \qquad (5)$$

valid for $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$. Indeed, let us assume $|z| < 1$ to begin with. Expanding $(1-tz)^{-a}$ by (2) and remembering Euler's Beta function

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \ , \ \mathrm{Re}\, x > 0, \mathrm{Re}\, y > 0, \qquad (6)$$

it is readily checked that the right-hand side of (5) equals the series (3) for $|z| < 1$. Then (5) allows extending F to an analytic function of z on the cut plane $\mathbb{C} \setminus D$ where $D := [1, \infty[$.

We shall need the following formula, due to Pfaff, with $(1-z)(1-z') = 1$ that is $z' := z/(z-1)$:

$$F\left(a,b;c;z'\right) = (1-z)^a F(a, c-b; c; z) \ , \ \mathrm{Re}\, c > \mathrm{Re}\, b > 0, z \in \mathbb{C} \setminus D. \qquad (7)$$

To prove (7) note that z' belongs to $\mathbb{C} \setminus D$ if and only if z belongs to this set and, setting $t = 1-s$ in (5), we obtain

$$\begin{aligned} F\left(a,b;c;z'\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}(1-z)^a \int_0^1 s^{c-b-1}(1-s)^{b-1}(1-sz)^{-a}ds \\ &= (1-z)^a F(a, c-b; c; z). \end{aligned}$$

For $|z| < 1$ small enough both sides of (7) can be expanded as convergent power series in z; the identification of their coefficients gives a sequence of equalities between rational functions of a, b, c. It follows that (7) remains valid for all a, b, c (with $c \notin -\mathbb{N}$), provided that $|z| < 1$ and $|z'| < 1$ (meaning that 0 is closer to z than 1, i.e. $\mathrm{Re}\, z < 1/2$).

Switching z for z' in (7) we have $(z')' = z$ and, remembering $(1-z)(1-z') = 1$ and the symmetry (4),

$$F(a,b;c;z) = (1-z')^a F(a, c-b; c; z') = (1-z)^{-a} F(c-b, a; c; z'),$$

for $|z| < 1$ and $\mathrm{Re}\, z < 1/2$ at least. By (7) again

$$F(c-b, a; c; z') = (1-z)^{c-b} F(c-b, c-a; c; z),$$

therefore, using the symmetry,

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z). \qquad (8)$$

The identity (8), due to Euler, thus holds for $|z| < 1$, $\mathrm{Re}\, z < 1/2$, and any a, b, c (with $c \notin -\mathbb{N}$). It extends to $|z| < 1$ and, if $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$, to all $z \in \mathbb{C} \setminus D$.
Variant. For $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$, (8) may also be proved directly from (5): assuming $z \in \mathbb{C} \setminus D$ is real, that is $z < 1$, the result follows from the change $t = (1-s)/(1-sz)$, $0 < s < 1$, then extends analytically to all $z \in \mathbb{C} \setminus D$.

2 Some hypergeometric integral formulas

We begin with an extension of Euler's formula (5) (obtained for $b = c$).

Lemma 1 *Let* $a, b, c, d \in \mathbb{C}$ *with* $\operatorname{Re} d > \operatorname{Re} c > 0$ *and* $|z| < 1$*. Then*

$$\int_0^1 t^{c-1}(1-t)^{d-c-1} F(a, b; c; tz)dt = \frac{\Gamma(c)\Gamma(d-c)}{\Gamma(d)} F(a, b; d; z).$$

Proof. Expanding F by (3) the integral becomes

$$\sum_{m\in\mathbb{N}} \frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}\int_0^1 t^{c+m-1}(1-t)^{d-c-1}dt = \sum_{m\in\mathbb{N}} \frac{(a)_m(b)_m}{(c)_m}\frac{\Gamma(c+m)\Gamma(d-c)}{\Gamma(d+m)}\frac{z^m}{m!}$$

by (6). But $\Gamma(c+m) = (c)_m\Gamma(c)$, $\Gamma(d+m) = (d)_m\Gamma(d)$ and the lemma follows. ■

Lemma 2 *Let* $\alpha, \beta, \gamma, k, z \in \mathbb{C}$ *with* $\operatorname{Re} k > \operatorname{Re} \gamma > 0$*. Let* $z' = z/(z-1)$*. Then, for* $|z| < 1$ *and* $\operatorname{Re} z < 1/2$*,*

$$\int_0^1 t^{k-\gamma-1}(1-t)^{\gamma-1} F(\alpha, \beta; \gamma; (1-t)z')F(k-\alpha, k-\beta; k-\gamma; tz)dt =$$

$$= \frac{\Gamma(\gamma)\Gamma(k-\gamma)}{\Gamma(k)}(1-z)^{\alpha+\beta-k}.$$

Proof. Since $|z'| < 1$ we can expand the first hypergeometric factor by (3) and the integral becomes

$$\sum_{m\in\mathbb{N}} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{z'^m}{m!}\int_0^1 t^{k-\gamma-1}(1-t)^{\gamma+m-1}F(k-\alpha, k-\beta; k-\gamma; tz)dt =$$

$$= \sum_{m\in\mathbb{N}} \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{z'^m}{m!}\frac{\Gamma(k-\gamma)\Gamma(\gamma+m)}{\Gamma(k+m)}F(k-\alpha, k-\beta; k+m; z),$$

by Lemma 1 with $a = k-\alpha$, $b = k-\beta$, $c = k-\gamma$, $d = k+m$. But $\Gamma(\gamma+m) = (\gamma)_m\Gamma(\gamma)$, $\Gamma(k+m) = (k)_m\Gamma(k)$ and, by (8),

$$F(k-\alpha, k-\beta; k+m; z) = (1-z)^{\alpha+\beta+m-k}F(\alpha+m, \beta+m; k+m; z).$$

Since $z'(1-z) = -z$ our integral is therefore

$$\frac{\Gamma(\gamma)\Gamma(k-\gamma)}{\Gamma(k)}(1-z)^{\alpha+\beta-k}\sum_{m\in\mathbb{N}}\frac{(\alpha)_m(\beta)_m}{(k)_m}F(\alpha+m, \beta+m; k+m; z)\frac{(-z)^m}{m!}.$$

To conclude we observe that term by term derivation of $F(\alpha, \beta; k; z)$ gives

$$\frac{(\alpha)_m(\beta)_m}{(k)_m}F(\alpha+m, \beta+m; k+m; z) = \left(\frac{d}{dz}\right)^m F(\alpha, \beta; k; z).$$

The above sum $\sum_m$ is now, by Taylor's formula,

$$\sum_{m\in\mathbb{N}}\left(\frac{d}{dz}\right)^m F(\alpha, \beta; k; z)\frac{(-z)^m}{m!} = F(\alpha, \beta; k; z-z) = 1.$$

This implies our claim. ■

Proposition 3 *Assuming* $0 < x < y < 2x$ *and* $a, b, c, k \in \mathbb{C}$ *with* $\operatorname{Re} k > \operatorname{Re} c > 0$*, we have*

$$\int_x^y (y-u)^{c-1}(u-x)^{k-c-1} F\left(a, b; c; 1-\frac{u}{y}\right) F\left(-a, -b; k-c; 1-\frac{u}{x}\right) u^{-k} du = \\ = \frac{\Gamma(c)\Gamma(k-c)}{\Gamma(k)} x^{-c} y^{c-k} (y-x)^{k-1}.$$

If $F(-a, -b; k-c; z)$ *is an analytic function of* z *in a domain containing* $]-\infty, 0]$ *(for instance if* $\operatorname{Re} k > \operatorname{Re}(c-b) > \operatorname{Re} c > 0$*), the result remains valid whenever* $0 < x < y$.

Proof. *(i)* Let us assume first $0 < x < y < 2x$, so that $|1 - u/x| < 1$ and $|1 - u/y| < 1$ in the integral. Thanks to Euler's identity (8) applied to both hypergeometric functions:

$$F\left(a, b; c; 1-\frac{u}{y}\right) F\left(-a, -b; k-c; 1-\frac{u}{x}\right) = \\ = \left(\frac{u}{y}\right)^{c-a-b} F\left(c-a, c-b; c; 1-\frac{u}{y}\right) \left(\frac{u}{x}\right)^{k-c+a+b} F\left(k+a-c, k+b-c; k-c; 1-\frac{u}{x}\right),$$

the factor u^{-k} disappears in the integral, which becomes

$$\left(\frac{y}{x}\right)^{a+b-c} x^{-k} \int_x^y (y-u)^{c-1}(u-x)^{k-c-1} F\left(c-a, c-b; c; 1-\frac{u}{y}\right) \times \\ \times F\left(k-c+a, k-c+b; k-c; 1-\frac{u}{x}\right) du.$$

Setting $u = (1-t)x + ty$ with $0 \le t \le 1$ and $z := 1 - \frac{y}{x} \in]-1, 0[$ we have $z' = \frac{z}{z-1} = 1 - \frac{x}{y} \in]0, 1/2[$. Since $1 - \frac{u}{y} = (1-t)z'$ and $1 - \frac{u}{x} = tz$ the integral is now

$$\left(\frac{y}{x}\right)^{a+b-c} x^{-k}(y-x)^{k-1} \int_0^1 t^{k-c-1}(1-t)^{c-1} F\left(c-a, c-b; c; (1-t)z'\right) \times \\ \times F\left(k-c+a, k-c+b; k-c; tz\right) dt,$$

and the result follows by Lemma 2, applied with $\alpha = c-a$, $\beta = c-b$, $\gamma = c$.
(ii) Only assuming $0 < x < y$ we have $1 - u/x < 0$ and $1 - u/y \in]0, 1[$. If $F(-a, -b; k-c; z)$ is analytic in a domain containing $]-\infty, 0]$ (e.g. if $\operatorname{Re}(k-c) > \operatorname{Re}(-b) > 0$), the integral is an analytic function of (x, y) in this domain and we can extend the result of *(i)*. ■

Theorem 4 *Let* $\mathcal{D}^+$ *denote the space of functions* $f \in C^\infty(]0, \infty[)$ *which vanish identically on a neighborhood of* $+\infty$*. For* $a, b, c \in \mathbb{C}$ *with* $\operatorname{Re} c > 0$*, the integral tranform* $f \mapsto \varphi$ *defined by*

$$\varphi(y) := \int_y^\infty (x-y)^{c-1} F\left(a, b; c; 1-\frac{y}{x}\right) f(x) dx \ , \ y > 0,$$

maps $\mathcal{D}^+$ *into itself and is inverted by*

$$f(x) = \frac{(-1)^k}{\Gamma(c)\Gamma(k-c)} \int_x^\infty (y-x)^{k-c-1} F\left(-a, -b; k-c; 1-\frac{y}{x}\right) \varphi^{(k)}(y) dy \ , \ x > 0,$$

where $\varphi^{(k)}$ is the k-th derivative of φ. This inversion formula holds true if k is any integer such that $k > \mathrm{Re}\, c > 0$ and $F(-a, -b; k-c; z)$ is an analytic function of z in a domain containing $]-\infty, 0]$.

Proof. Let $f \in \mathcal{D}^+$ with $f(x) = 0$ for $x \geq A > 0$. Then $\varphi(u) = 0$ for $u \geq A$ and, for $0 < u \leq A$,

$$\varphi(u) = \int_u^A (v-u)^{c-1} F\left(a, b; c; 1 - \frac{u}{v}\right) f(v) dv.$$

Since $0 \leq 1 - \frac{u}{v} \leq 1 - \frac{u}{A} < 1$ in the integral, it follows that $\varphi \in \mathcal{D}^+$.
Let $F_1(z) := F(a, b; c; z)$ and $F_2(z) := F(-a, -b; k-c; z)$ for short. Multiplying by $(u-x)^{k-c-1} u^{-k} F_2\left(1 - \frac{u}{x}\right)$ for $x > 0$ and integrating on $u \in [x, \infty[$ we obtain

$$\begin{aligned}
\int_x^\infty (u-x)^{k-c-1} u^{-k} F_2\left(1 - \frac{u}{x}\right) \varphi(u) du = \\
= \int_x^\infty f(v) dv \int_x^v (v-u)^{c-1} (u-x)^{k-c-1} F_1\left(1 - \frac{u}{v}\right) F_2\left(1 - \frac{u}{x}\right) u^{-k} du \\
= \frac{\Gamma(c)\Gamma(k-c)}{\Gamma(k)} x^{-c} \int_x^\infty (v-x)^{k-1} v^{c-k} f(v) dv \quad (9)
\end{aligned}$$

by Proposition 3.
Let $g(x) := \int_x^\infty (v-x)^{k-1} v^{c-k} f(v) dv$. It is now easily checked that $g^{(k)}(x) = (-1)^k (k-1)! x^{c-k} f(x)$, so that the above integral allows reconstructing f from φ. Indeed (9) is, with $u = tx$,

$$\begin{aligned}
g(x) &= \frac{\Gamma(k)}{\Gamma(c)\Gamma(k-c)} x^c \int_x^\infty (u-x)^{k-c-1} u^{-k} F_2\left(1 - \frac{u}{x}\right) \varphi(u) du \\
&= \frac{\Gamma(k)}{\Gamma(c)\Gamma(k-c)} \int_1^\infty (t-1)^{k-c-1} t^{-k} F_2(1-t) \varphi(tx) dt,
\end{aligned}$$

therefore

$$(-1)^k x^{c-k} f(x) = \frac{1}{\Gamma(c)\Gamma(k-c)} \int_1^\infty (t-1)^{k-c-1} F_2(1-t) \varphi^{(k)}(tx) dt.$$

The result follows, changing again the variable t for $y = tx$. ∎

3 Application to Chebyshev polynomials

The Chebyshev polynomials T_n (defined by $T_n(\cos\theta) = \cos n\theta$) may be written as hypergeometric functions. Lemma 5 will serve as a preparation for Lemma 6.

Lemma 5 *For $n \in \mathbb{N}$, $n \geq 1$, and $z \in \mathbb{C}$*

$$T_n(z) = F\left(n, -n; \frac{1}{2}; \frac{1-z}{2}\right).$$

Proof. The Chebyshev polynomials are characterized by the recurrence relation (easily checked for $z = \cos\theta$)

$$T_{n+1}(z) + T_{n-1}(z) = 2zT_n(z) \text{ for } n \geq 1,\ T_0(z) = 1\ ,\ T_1(z) = z.$$

Let

$$F_n(x) := F\left(n, -n; \frac{1}{2}; x\right) = 1 + \sum_{k\geq 1} \frac{(n)_k(-n)_k}{(1/2)_k}\frac{x^k}{k!},$$

a finite sum actually since $(-n)_k = 0$ for $k > n$. Then $F_0(x) = 1$, $F_1(x) = 1-2x$ and (setting $z = 1 - 2x$) we wish to prove that, for $n \geq 1$,

$$2F_n(x) - F_{n+1}(x) - F_{n-1}(x) = 4xF_n(x).$$

Setting $z = 1 - 2x$ it will follow that $F_n((1-z)/2) = F_n(x)$ satisfies the same recurrence relation as $T_n(z)$, hence $T_n(z) = F_n((1-z)/2)$ as claimed.
For $n \geq 1$,

$$\begin{aligned}&2F_n(x) - F_{n+1}(x) - F_{n-1}(x) = \\ &\qquad = \sum_{k=1}^{n}\left\{\frac{2(n)_k(-n)_k - (n+1)_k(-n-1)_k - (n-1)_k(-n+1)_k}{(1/2)_k}\right\}\frac{x^k}{k!}.\end{aligned}$$

But

$$(a+1)_k = (a)_k\frac{a+k}{a}\ ,\ (a-1)_k = (a)_k\frac{a-1}{a+k-1}$$

and the coefficient of $x^k/k!$ becomes, after some elementary computations,

$$\{\cdots\} = 4\frac{(n)_k}{n+k-1}\frac{(-n)_k}{n-k+1}\frac{\frac{1}{2}-k}{(1/2)_k}k = 4\frac{(n)_{k-1}(-n)_{k-1}}{(1/2)_{k-1}}k.$$

Thus

$$2F_n(x) - F_{n+1}(x) - F_{n-1}(x) = 4\sum_{k=1}^{n}\frac{(n)_{k-1}(-n)_{k-1}}{(1/2)_{k-1}}\frac{x^k}{(k-1)!} = 4xF_n(x).$$

■

Lemma 6 *For $n \in \mathbb{N}$,*

$$\begin{aligned}T_n(t) &= F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; 1-t^2\right)\ ,\ t \in \mathbb{R} \\ &= tF\left(\frac{1+n}{2}, \frac{1-n}{2}; \frac{1}{2}; 1-t^2\right)\ ,\ t > 0.\end{aligned}$$

Proof. This result might be inferred from the previous lemma by means of a quadratic transformation of the hypergeometric function ([3], p. 111, formula (2)). Here is an elementary proof, based on $(1-\cos 2\theta)/2 = 1 - \cos^2\theta$.
(i) For $n = 2p$, $p \in \mathbb{N}$, Lemma 5 with $z = \cos 2\theta$ gives

$$T_{2p}(\cos\theta) = \cos 2p\theta = T_p(\cos 2\theta) = F\left(p, -p; \frac{1}{2}; 1-\cos^2\theta\right),$$

implying the identity of these polynomials in $t = \cos\theta$.
(ii) For $n = 2p-1$, $p \geq 1$, the equality $\cos 2p\theta = T_p(\cos 2\theta)$ implies $-2p \sin 2p\theta = -2T_p'(\cos 2\theta)\sin 2\theta$. Thus

$$\cos(2p-1)\theta = \cos\theta\cos 2p\theta + \sin\theta\sin 2p\theta$$

may be written as

$$\begin{aligned} T_{2p-1}(\cos\theta) &= \cos\theta\ T_p(\cos 2\theta) + \frac{1}{p}\sin\theta\sin 2\theta\ T_p'(\cos 2\theta) \\ &= \cos\theta\ \left[T_p(\cos 2\theta) + \frac{1}{p}(1-\cos 2\theta)T_p'(\cos 2\theta)\right]. \end{aligned}$$

Considering the factor $[\cdots]$ we note that, by Lemma 5,

$$T_p(z) + \frac{1}{p}(1-z)T_p'(z) = F\left(p, -p; \frac{1}{2}; \frac{1-z}{2}\right) - \frac{1}{p}\frac{1-z}{2}F'\left(p, -p; \frac{1}{2}; \frac{1-z}{2}\right).$$

But, for a general hypergeometric series (3),

$$bF(a,b;c;x) + xF'(a,b;c;x) = bF(a,b+1;c;x),$$

an immediate consequence of $b(b)_k + k(b)_k = b(b+1)_k$. With $a = p$, $b = -p$, $c = 1/2$ we infer the following polynomial identity:

$$T_p(z) + \frac{1}{p}(1-z)T_p'(z) = F\left(p, 1-p; \frac{1}{2}; \frac{1-z}{2}\right).$$

With $z = \cos 2\theta$ we conclude that

$$T_{2p-1}(\cos\theta) = \cos\theta\ F\left(p, 1-p; \frac{1}{2}; 1-\cos^2\theta\right).$$

Replacing $\cos\theta$ by t, the right-hand side therefore extends to an analytic function of $t \in \mathbb{R}$ (actually a polynomial).
(iii) We have thus proved the first result of the lemma for n even, resp. the second for n odd. The second, resp. first, then follows by Euler's formula (8).
■

Application of Proposition 3. Let $0 < r < s$. Setting $x = r^2$, $y = s^2$ and $u = p^2$ in Proposition 3 with $a = n/2$, $b = -n/2$, $c = 1/2$, $k = 1$, Lemma 6 gives $F(a,b;c;z) = F(-a,-b;k-c;z) = T_n\left(\sqrt{1-z}\right)$, analytic in $\mathbb{C} \setminus [1,\infty[$, and we obtain formula (1):

$$\begin{aligned} \int_r^s \left(s^2-p^2\right)^{-1/2}\left(p^2-r^2\right)^{-1/2} T_n\left(\frac{p}{r}\right) T_n\left(\frac{p}{s}\right) 2p^{-1}dp &= \left(\Gamma(1/2)\right)^2 r^{-1}s^{-1} \\ &= \pi/rs. \end{aligned}$$

See also Gorenflo and Vessella [4] p. 119 for a direct elementary proof of this equality.

Application of Theorem 4. Cormack's study of the line Radon transform in the plane led him to consider the transform $f \mapsto \varphi$ defined by

$$\varphi(p) = \int_p^\infty f(s)T_n\left(\frac{p}{s}\right)\frac{2sds}{\sqrt{s^2-p^2}}\ ,\ p > 0,$$

where f, resp. φ, is the n-th coefficient in the Fourier series expansion of a function on $\mathbb{R}^2$ in polar coordinates, resp. of its Radon transform on lines. Let $g(x) := f(\sqrt{x})$ and $\psi(y) := \varphi(\sqrt{y})$. Changing s for $t = s^2$ we obtain, in view of Lemma 6,

$$\psi(y) = \int_y^\infty F\left(\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}; 1 - \frac{y}{x}\right) g(x) \frac{dx}{\sqrt{x-y}}.$$

Theorem 4, with $a = n/2$, $b = -n/2$, $c = 1/2$, $k = 1$ yields the inversion formula

$$g(x) = -\frac{1}{\pi} \int_t^\infty F\left(-\frac{n}{2}, \frac{n}{2}; \frac{1}{2}; 1 - \frac{y}{x}\right) \frac{\psi'(y)dy}{\sqrt{y-x}},$$

that is, with $x = r^2$ and $y = p^2$,

$$f(r) = g(r^2) = -\frac{1}{\pi} \int_r^\infty T_n\left(\frac{p}{r}\right) \frac{\varphi'(p)dp}{\sqrt{p^2 - r^2}}.$$

More generally, Theorem 4 yields inversion formulas for the Radon transform on certain families of hypersurfaces in $\mathbb{R}^n$, with Fourier series replaced by spherical harmonics expansions and Chebyshev's polynomials by Gegenbauer's; see Cormack [2].

References

[1] CORMACK, A.M., The Radon problem on a family of curves in the plane, *Proc. Amer. Math. Soc.* 83 (1981), 325-330 and 86 (1982), 293-298.

[2] CORMACK, A.M., Radon's problem for some surfaces in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.* 99 (1987), 305-312.

[3] ERDÉLYI, A., *Higher transcendental functions, volume I*, McGraw-Hill 1953.

[4] GORENFLO, R. and VESSELLA, S., *Abel integral equations*, Lecture Notes in Maths. 1461, Springer-Verlag 1991.