Nonlinear Radon and Fourier Transforms

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May 16, 2015

Abstract

In this note we explain a generalization, due to Leon Ehrenpreis, of the classical Radon transform on hyperplanes. A function f on $\mathbb{R}^n$ can be reconstructed from nonlinear Radon transforms, obtained by integrating f and a finite number of multiples $x^\alpha f$ over a family of algebraic hypersurfaces of degree m. This follows by solving a Cauchy problem for the nonlinear Fourier transform of f. We also give an inversion formula for this Radon transform.

1 Introduction

This expository note is an attempt at explaining the pages from Ehrenpreis' treatise [5] in which he develops the nonlinear Radon and Fourier transforms he had introduced in his previous papers [1][2][3][4]. The goal is to extend the classical hyperplane Radon transform $R_0 f$ (integrals of a function f over all hyperplanes in $\mathbb{R}^n$) to a family of algebraic submanifolds defined by higher degree polynomial equations. Is the generalized transform R still injective? Can we give an inversion formula? Unfortunately it is readily seen that R is no more injective (in general): reconstructing f from Radon transforms needs more than Rf alone.

We shall explain here several results of the following type: there exists a finite number of low-degree polynomial functions a_k (with $a_1 = 1$) such that f is determined by the Radon transforms $R(a_k f)$. Besides, the restriction of the $R(a_k f)$'s to a certain subfamily of algebraic manifolds may even be sufficient, provided one increases the number of polynomials a_k.

After a brief reminder of the classical hyperplane transform (this Section) we shall introduce Ehrenpreis' nonlinear Radon transform and the related nonlinear Fourier transform, so as to get a *projection slice theorem* which plays a crucial role in this study (Section 2). The reconstruction problem boils down to a Cauchy problem for a system of partial differential equations, solved in a naive way in Section 3 then, in Section 4, by the more sophisticated tools of harmonic polynomials. In Section 5 we discuss an inversion formula for the nonlinear Radon transform.

In order to motivate the forthcoming construction, let us briefly recall a few facts about the classical Radon transform R_0. In the Euclidean space $\mathbb{R}^n$ it is given by integration of a compactly supported smooth function $f \in \mathcal{D}(\mathbb{R}^n)$ over the family of all hyperplanes. A hyperplane being defined by the equation $\omega \cdot x = t$ where ω is a unit vector, t a real number and $\cdot$ denotes the scalar product, we consider

$$R_0 f(t,\omega) := \int_{\omega\cdot x = t} f,$$

an integral with respect to the measure induced on the hyperplane by the Euclidean measure dx of $\mathbb{R}^n$. Note that (t,ω) and $(-t,-\omega)$ define the same hyperplane, thus $R_0 f(t,\omega) = R_0 f(-t,-\omega)$. For any $\tau \in \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} e^{i\tau\omega\cdot x} f(x)dx = \int_{\mathbb{R}} dt \int_{\omega\cdot x=t} e^{i\tau\omega\cdot x} f(x) = \int_{\mathbb{R}} e^{i\tau t} R_0 f(t,\omega)dt.$$

This gives the *projection slice theorem*

$$\widehat{f}(\tau\omega) = \widehat{R_0 f}(\tau,\omega) \tag{1}$$

for $\tau \in \mathbb{R}$, $\omega \in \mathbb{R}^n$ and $\|\omega\| = 1$.
Caution: on the left-hand side of (1) the hat denotes the n-dimensional Fourier transform on x but on the right-hand side it denotes the 1-dimensional Fourier transform on t. Both sides are smooth functions on $\mathbb{R} \times \mathbb{S}^{n-1}$, rapidly decreasing with respect to τ.

Knowing the integrals of f over all hyperplanes, i.e. $R_0 f$, the Fourier transform $\widehat{f}$ is therefore known and R_0 is easily inverted as follows. Writing the Fourier inversion formula for f in spherical coordinates we have

$$f(x) = (2\pi)^{-n} \int_{\|\omega\|=1} d\omega \int_0^\infty e^{-i\tau\omega\cdot x} \widehat{R_0 f}(\tau,\omega)\tau^{n-1} d\tau$$

where $d\omega$ is the Euclidean measure on the unit sphere of $\mathbb{R}^n$. In order to use Fourier analysis in one variable we can replace $\int_0^\infty$ by $\int_{\mathbb{R}}$: indeed $\widehat{R_0 f}(\tau,\omega) = \widehat{R_0 f}(-\tau,-\omega)$ and, changing τ into $-\tau$ then ω into $-\omega$, we obtain

$$f(x) = C \int_{\|\omega\|=1} d\omega \int_{\mathbb{R}} e^{-i\tau\omega\cdot x} \widehat{R_0 f}(\tau,\omega)|\tau|^{n-1} d\tau$$

with $C := \frac{1}{2}(2\pi)^{-n}$. Let $F(t,\omega)$ be a smooth function on $\mathbb{R}\times\mathbb{S}^{n-1}$, rapidly decreasing with respect to t, and let the operator $|\partial_t|^{n-1}$ be defined by

$$\left(|\partial_t|^{n-1} F\right)\widehat{\ }(\tau,\omega) = \widehat{F}(\tau,\omega)|\tau|^{n-1}.$$

Thus $|\partial_t|^{n-1} = (-1)^k \partial_t^{n-1}$ if $n = 2k+1$ is odd; if n is even $|\partial_t|^{n-1}$ is the composition of ∂_t^{n-1} and a Hilbert integral operator (see Helgason [7] p. 22). We infer the following inversion formula

$$f = C R_0^* |\partial_t|^{n-1} R_0 f \tag{2}$$

where the *dual transform* R_0^* is defined by

$$R_0^* F(x) := \int_{\|\omega\|=1} F(\omega\cdot x,\omega)d\omega$$

(integration over the set of all hyperplanes containing x).

2 A Nonlinear Radon Transform

2.1 Integration on Hypersurfaces

Let $\varphi : \Omega \to \mathbb{R}$ be a smooth function on an open subset Ω of the Euclidean space $\mathbb{R}^n$. A convenient way to introduce our Radon transform is to consider first, for $f \in \mathcal{D}(\Omega)$ (a smooth function with compact support contained in Ω) and $t \in \mathbb{R}$,

$$f_\varphi(t) := \int_{\varphi(x)<t} f(x)dx$$

where dx is the Lebesgue measure of $\mathbb{R}^n$. Let m and M denote the lower and upper bounds of $\varphi(x)$ for $x \in \operatorname{supp} f$; then $f_\varphi(t) = 0$ for $t \leq m$ and $f_\varphi(t) = \int_\Omega f(x)dx$ for $t \geq M$.

The example $\Omega = \mathbb{R}$ and $\varphi(x) = x^3$ gives $f_\varphi(t) = F(t^{1/3})$ with $F(u) = \int_{-\infty}^u f(x)dx$; thus f_φ is not necessarily smooth. However the following result holds true.

Proposition 1 *Assume the gradient φ' of φ never vanishes on Ω. For $f \in \mathcal{D}(\Omega)$, f_φ is then a smooth function on $\mathbb{R}$ and we may define*

$$R_\varphi f(t) := (f_\varphi)'(t) = \partial_t \int_{\varphi(x)<t} f(x)dx. \tag{3}$$

(i) $R_\varphi f$ is a smooth function on $\mathbb{R}$ and $\operatorname{supp} R_\varphi f \subset [m, M]$.
(ii) For any $u \in C^\infty(\mathbb{R})$

$$\int_{\mathbb{R}^n} u(\varphi(x))f(x)dx = \int_{\mathbb{R}} u(t)R_\varphi f(t)dt. \tag{4}$$

(iii) Let dS_t be the Euclidean measure on the hypersurface $S_t := \{x \in \Omega | \varphi(x) = t\}$. Then

$$R_\varphi f(t) = \int_{S_t} f(x)\frac{1}{\|\varphi'(x)\|}dS_t(x). \tag{5}$$

Formula (5) gives the geometrical meaning of $R_\varphi f$ as an integral of f over the level hypersurface $\varphi(x) = t$; we may write it for short as

$$R_\varphi f(t) = \int_{\varphi(x)=t} f. \tag{6}$$

According to (4) it may also be viewed as $R_\varphi f(t) = \langle \varphi^* \delta_t, f\rangle$ where $\varphi^*\delta_t$ is the pullback by φ of the Dirac measure δ_t of $\mathbb{R}$ at t (see Friedlander [6] Section 7.2 or Hörmander [8] Section 6.1).

Proof. *(i)* and *(iii)* Given $a \in \Omega$ we have $\varphi'(a) \neq 0$ thus (for instance) $\partial_n\varphi(a) \neq 0$. By the inverse function theorem there exists an open neighborhood U of a such that the map $x = (x', x_n) \mapsto y = (x', \varphi(x))$ is a diffeomorphism of U onto $V \times I$, where $x' = (x_1, ..., x_{n-1})$, V is an open neighborhood of $(a_1, ..., a_{n-1})$ in $\mathbb{R}^{n-1}$ and I is an open interval containing $\varphi(a)$. Let $y = (y', y_n) \mapsto x = (y', \psi(y', y_n))$ denote the inverse map. Then $dy = |\partial_n\varphi(x)|dx$ and, assuming $\operatorname{supp} f \subset U$, we have

$$f_\varphi(t) = \int_{\varphi(x)<t} f(x)dx = \int_{y_n<t} \frac{f}{|\partial_n\varphi|}(y', \psi(y', y_n))dy'dy_n.$$

The y_n integral actually runs over $[a,b]\cap]-\infty,t[$ where $[a,b]$ is compact and contained in I. Thus f_φ is a smooth function of $t\in\mathbb{R}$ and

$$\begin{aligned} R_\varphi f(t) &= (f_\varphi)'(t) = \int_V \frac{f}{|\partial_n\varphi|}(y',\psi(y',t))dy' \text{ for } t\in I \\ &= 0 \text{ for } t\notin I \end{aligned}$$

is smooth on $\mathbb{R}$.
Besides, $\varphi(y',\psi(y',t))=t$ for $y'\in V$ and $t\in I$ therefore

$$\partial_i\varphi(y',\psi(y',t))+\partial_n\varphi(y',\psi(y',t))\partial_i\psi(y',t)=0$$

for $i=1,...,n-1$. It follows that $\|\varphi'\|=|\partial_n\varphi|\left(1+\sum_1^{n-1}(\partial_i\psi)^2\right)^{1/2}$ and, for $t\in I$,

$$\begin{aligned} R_\varphi f(t) &= \int_V \frac{f}{\|\varphi'\|}(y',\psi(y',t))\left(1+\sum_1^{n-1}\left(\partial_i\psi(y',t)\right)^2\right)^{1/2} dy' \\ &= \int_{S_t}\frac{f}{\|\varphi'\|}(x)dS_t(x), \end{aligned}$$

the hypersurface integral being computed by means of the parameters y'. The latter equality also holds for $t\notin I$ (both sides vanish) and this proves *(i)* and *(iii)* for $\operatorname{supp} f\subset U$. The general case follows by partition of unity.
(ii) Since $\operatorname{supp} R_\varphi f\subset[m,M]$ we have

$$\begin{aligned} \int_{\mathbb{R}} u(t)R_\varphi f(t)dt &= \int_m^M u(t)\left(f_\varphi\right)'(t)dt = [u(t)f_\varphi(t)]_m^M - \int_m^M u'(t)f_\varphi(t)dt \\ &= u(M)\int_\Omega f(x)dx - \int_{\varphi(x)<t<M} u'(t)f(x)dtdx. \end{aligned}$$

The latter integral is

$$\int_\Omega f(x)dx\int_{\varphi(x)}^M u'(t)dt = \int_\Omega f(x)(u(M)-u(\varphi(x)))dx$$

and (4) follows. ■

2.2 Nonlinear Radon and Fourier Transforms

We now wish to extend the classical Radon transform of Section 1, replacing the hyperplanes $\omega\cdot x=t$ by level hypersurfaces of homogeneous polynomials of given degree $m\geq 1$ in $\mathbb{R}^n$. We write such polynomials as

$$\lambda\cdot p(x):=\sum_{|\alpha|=m}\lambda_\alpha x^\alpha$$

where $x\in\mathbb{R}^n$ and, in multi-index notation, $\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{N}^n$, $|\alpha|=\sum_1^n\alpha_i$, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ and $\lambda_\alpha\in\mathbb{R}$.

It is easily checked that the number of terms in $\sum_{|\alpha|=m}$ is the binomial coefficient $N = N(m,n) = \frac{(m+n-1)!}{m!(n-1)!}$. Indeed let us consider

$$\prod_{j=1}^{n} \frac{1}{1-tx_j} = \prod_{j=1}^{n} \left(1 + tx_j + t^2x_j^2 + \cdots\right).$$

Expanding the product we see that the coefficient of t^m is $\sum_{|\alpha|=m} x^\alpha$, therefore equals $N(m,n)$ when all x_i's are 1. Thus $N(m,n)$ is the coefficient of t^m in the expansion of $(1-t)^{-n}$ and the result follows. Note that $N > n$ for $n \geq 2$ and $m \geq 2$.

Let $\lambda \in \mathbb{R}^N$, $\lambda \neq 0$, and $\Omega := \{x | \lambda \cdot p(x) \neq 0\}$. By Euler's identity for the homogeneous function $\varphi(x) = \lambda \cdot p(x)$ on $\mathbb{R}^n$ the gradient φ' does not vanish on Ω. The level surface $\lambda \cdot p(x) = t$ is thus a smooth hypersurface of $\mathbb{R}^n$ for $t \in \mathbb{R}$, $t \neq 0$. The **nonlinear Radon transform** of a test function $f \in \mathcal{D}(\Omega)$ is then defined, in the notation of (6), by

$$Rf(t,\lambda) := R_\varphi f(t) = \int_{\lambda \cdot p(x)=t} f. \tag{7}$$

For $m = 1$ we have $N = n$ and R is the classical hyperplane Radon transform R_0.

Properties of R.
(i) By Proposition 1, for $f \in \mathcal{D}(\Omega)$ and $\lambda \neq 0$, $Rf(.,\lambda)$ is a compactly supported smooth function of t on $\mathbb{R}$. By (4)

$$\int_{\mathbb{R}^n} F\left(\lambda \cdot p(x), \lambda\right) f(x)dx = \int_{\mathbb{R}} F(t,\lambda) Rf(t,\lambda)dt$$

for $\lambda \neq 0$ and any F continuous on $\mathbb{R} \times \mathbb{R}^N$. In particular, for $\tau \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} e^{i\tau\lambda\cdot p(x)} f(x)dx = \int_{\mathbb{R}} e^{i\tau t} Rf(t,\lambda)dt = \widehat{Rf}(\tau,\lambda) = \widehat{Rf}(1,\tau\lambda) \tag{8}$$

is the one-dimensional Fourier transform of Rf with respect to the variable t. This extends the *projection slice theorem* (1).
(ii) The left-hand side of (8) is well-defined for all $f \in \mathcal{D}(\mathbb{R}^n)$ (without assuming $\operatorname{supp} f \subset \Omega$), and extends to an entire function of (τ,λ) on $\mathbb{C} \times \mathbb{C}^N$. This suggests defining $\widehat{Rf}(\tau, 0) = \int f$, that is $Rf(t,0) = \left(\int_{\mathbb{R}^n} f(x)dx\right)\delta(t)$ where δ is the Dirac measure at the origin of $\mathbb{R}$.
Actually, the restrictive assumptions $\operatorname{supp} f \subset \Omega$, $t \neq 0$, $\lambda \neq 0$ may be left out in the sequel, as we shall work with $\widehat{Rf}$ rather than Rf.
(iii) From (8) it follows that

$$\partial_{\lambda_\alpha} \widehat{Rf}(\tau,\lambda) = i\tau \int_{\mathbb{R}^n} e^{i\tau\lambda\cdot p(x)} x^\alpha f(x)dx = i\tau \widehat{R\left(x^\alpha f\right)}(\tau,\lambda), \tag{9}$$

therefore

$$\partial_{\lambda_\alpha} Rf(t,\lambda) = -\,\partial_t R\left(x^\alpha f\right)(t,\lambda) \tag{10}$$

for $f \in \mathcal{D}(\Omega)$, $\lambda \neq 0$ and $\alpha \in \mathbb{N}^n$, $|\alpha| = m$.

(iv) Note that, for m even, $Rf = 0$ whenever f is an odd function: R is not an injective map and, in this case, f cannot be reconstructed from Rf alone. We shall see in the next sections how to circumvent this difficulty.

Let us introduce the **nonlinear Fourier transform of** f defined, for all $f \in \mathcal{D}(\mathbb{R}^n)$, by

$$\widetilde{f}(\xi, \lambda) := \int_{\mathbb{R}^n} e^{i(\xi \cdot x + \lambda \cdot p(x))} f(x) dx \ , \ \xi \in \mathbb{R}^n \ , \ \lambda \in \mathbb{R}^N. \tag{11}$$

It extends to an entire function of $(\xi, \lambda) \in \mathbb{C}^n \times \mathbb{C}^N$. As a function on $\mathbb{R}^n \times \mathbb{R}^N$ it is bounded by $\int_{\mathbb{R}^n} |f(x)| dx$ and, for fixed λ, it is rapidly decreasing with respect to ξ.

On the one hand $\widetilde{f}(\xi, 0) = \widehat{f}(\xi)$ is the classical n-dimensional Fourier transform of f; on the other hand $\widetilde{f}(0, \tau\lambda) = \widehat{Rf}(\tau, \lambda)$ is the 1-dimensional Fourier transform of Rf:

$$\begin{array}{ccc} & \widetilde{f}(\xi, \lambda) & \\ \swarrow & & \searrow \\ \widetilde{f}(\xi, 0) = \widehat{f}(\xi) & & \widetilde{f}(0, \lambda) = \widehat{Rf}(1, \lambda). \end{array}$$

Reconstructing $\widetilde{f}(\xi, \lambda)$ from $\widetilde{f}(0, \lambda)$ would therefore allow to reconstruct f from Rf. For this we shall consider partial differential equations satisfied by $\widetilde{f}$.

2.3 Partial Differential Equations

Taking derivatives of (11) under the integral sign we get, for $j = 1, ..., n$ and $\alpha \in \mathbb{N}^n$, $|\alpha| = m$,

$$\partial_{\xi_j} \widetilde{f}(\xi, \lambda) = i \int_{\mathbb{R}^n} e^{i(\xi \cdot x + \lambda \cdot p(x))} x_j f(x) dx = i \widetilde{(x_j f)}(\xi, \lambda) \tag{12}$$

$$\partial_{\lambda_\alpha} \widetilde{f}(\xi, \lambda) = i \int_{\mathbb{R}^n} e^{i(\xi \cdot x + \lambda \cdot p(x))} x^\alpha f(x) dx = i \widetilde{(x^\alpha f)}(\xi, \lambda). \tag{13}$$

Thus $\widetilde{f}$ satisfies the system of N linear partial differential equations on $\mathbb{R}^n \times \mathbb{R}^N$

$$\boxed{i^{m-1} \partial_{\lambda_\alpha} \widetilde{f} = \partial_\xi^\alpha \widetilde{f}} \text{ for } \alpha \in \mathbb{N}^n, |\alpha| = m. \tag{14}$$

For any $\alpha, \beta, \gamma, \delta \in \mathbb{N}^n$ of length m such that $x^\alpha x^\beta = x^\gamma x^\delta$ we infer that, as a function of λ, $\widetilde{f}$ satisfies the *Plücker equations*

$$\left(\partial_{\lambda_\alpha} \partial_{\lambda_\beta} - \partial_{\lambda_\gamma} \partial_{\lambda_\delta} \right) \widetilde{f} = 0. \tag{15}$$

Given α, β, all such multi-indices γ, δ are obtained as $\gamma = \alpha - \varepsilon$, $\delta = \beta + \varepsilon$, where $\varepsilon = (\varepsilon_1, ..., \varepsilon_n) \in \mathbb{Z}^n$ satisfies $-\beta_j \leq \varepsilon_j \leq \alpha_j$ for $j = 1, ..., n$ and $\sum_1^n \varepsilon_j = 0$.

Example. For $m = n = 2$ we have $\lambda \cdot p(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_1 x_2$ (here $N = 3$) and

$$i \partial_{\lambda_1} \widetilde{f} = \partial_{\xi_1}^2 \widetilde{f} \ , \ i \partial_{\lambda_2} \widetilde{f} = \partial_{\xi_2}^2 \widetilde{f} \ , \ i \partial_{\lambda_3} \widetilde{f} = \partial_{\xi_1} \partial_{\xi_2} \widetilde{f}.$$

The identity $(x_1 x_2)^2 = x_1^2 x_2^2$ leads to he hyperbolic equation $\partial_{\lambda_3}^2 \widetilde{f} = \partial_{\lambda_1} \partial_{\lambda_2} \widetilde{f}$.

3 A Cauchy Problem

Given $f \in \mathcal{D}(\mathbb{R}^n)$ let us now try to reconstruct $\widetilde{f}(\xi, \lambda)$ from $\widetilde{f}(0, \lambda) = \widehat{Rf}(1, \lambda)$ by solving a Cauchy problem for the system (14) with data on $\xi = 0$. In order to achieve this goal we shal l of course need more than $\widehat{Rf}(1, \lambda)$: let us recall that $\widetilde{f}(0, \lambda) = 0$ for m even and f odd, though $\widetilde{f}$ may be not identically zero. It should be noted that $\widetilde{f}(0, \lambda)$ satisfies the Plücker equations (15), but this fact will not be taken into account here (see Remark below however).

Since $\widetilde{f}$ is an entire function we have

$$\widetilde{f}(\xi, \lambda) = \sum_{\alpha \in \mathbb{N}^n} \partial_\xi^\alpha \widetilde{f}(0, \lambda) \frac{\xi^\alpha}{\alpha!},$$

an absolutely convergent series for all $\xi \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}^N$.

To work it out we shall only need the derivatives $\partial_\xi^\alpha \widetilde{f}(0, \lambda)$ for $|\alpha| < m$; the higher order derivatives will be given by (14). More precisely, $\partial_\xi^\alpha \widetilde{f} = i^{|\alpha|} \widetilde{x^\alpha f}$ for all α by (12), and equals $i^{m-1} \partial_{\lambda_\alpha} \widetilde{f}$ by (14) if $|\alpha| = m$. For any $\alpha \in \mathbb{N}^n$ we may write $|\alpha| = qm + r$ with $q, r \in \mathbb{N}$, $0 \leq r < m$, and factorize ∂_ξ^α as

$$\partial_\xi^\alpha = \partial_\xi^{\beta_1} \cdots \partial_\xi^{\beta_q} \partial_\xi^\gamma$$

with $\beta_1, ...\beta_q, \gamma \in \mathbb{N}^n$, $|\beta_1| = \cdots = |\beta_q| = m$ and $|\gamma| = r$; this factorization is not unique. It follows that

$$\partial_\xi^\alpha \widetilde{f} = i^{|\alpha| - q} \partial_{\lambda_{\beta_1}} \cdots \partial_{\lambda_{\beta_q}} \widetilde{(x^\gamma f)}$$

and

$$\widetilde{f}(\xi, \lambda) = \sum_{\alpha \in \mathbb{N}^n} i^{|\alpha| - q} \partial_{\lambda_{\beta_1}} \cdots \partial_{\lambda_{\beta_q}} \widetilde{(x^\gamma f)}(0, \lambda) \frac{\xi^\alpha}{\alpha!}$$

(with $q, \beta_1, ..., \beta_q, \gamma$ depending on α in the sum).
Remembering $\widetilde{(x^\gamma f)}(0, \lambda) = \widehat{R(x^\gamma f)}(1, \lambda)$ for $\lambda \neq 0$, we see that $\widetilde{f}$ *is determined by the nonlinear Radon transforms of all functions* $x^\gamma f$ *for* $\gamma \in \mathbb{N}^n$ *and* $|\gamma| < m$. Their number is $\sum_{k=0}^{m-1} N(k, n) = N(m-1, n+1) = \frac{m}{n} N(m, n)$ (induction on m). In particular if $R(x^\gamma f) = 0$ for all γ with $|\gamma| < m$, then $f = 0$.

Example. For $m = n = 2$ (Section 2.3), $\partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2}$ factorizes as powers of $\partial_{\xi_1}^2$ and $\partial_{\xi_2}^2$, possibly composed with ∂_{ξ_1} or ∂_{ξ_2} or $\partial_{\xi_1} \partial_{\xi_2}$ according to the parity of α_1 and α_2. Gathering together similar terms the above result reads

$$\widetilde{f}(\xi, \lambda) = C(D_1) C(D_2) \widetilde{f} + S(D_1) S(D_2) D_3 \widetilde{f} + \\ + i\xi_1 S(D_1) C(D_2) \widetilde{(x_1 f)} + i\xi_2 C(D_1) S(D_2) \widetilde{(x_2 f)} \quad (16)$$

where

$$D_1 = i\xi_1^2 \partial_{\lambda_1} \ , \ D_2 = i\xi_2^2 \partial_{\lambda_2} \ , \ D_3 = i\xi_1 \xi_2 \partial_{\lambda_3}$$

$$C(z) = \sum_{k=0}^{\infty} \frac{z^k}{(2k)!} \ , \ S(z) = \sum_{k=0}^{\infty} \frac{z^k}{(2k+1)!}$$

and, in the right-hand side of (16), $\widetilde{f}$, $\widetilde{(x_1 f)}$, $\widetilde{(x_2 f)}$ are evaluated at $(0, \lambda)$. Thus the knowledge of the three Radon transforms Rf, $R(x_1 f)$ and $R(x_2 f)$ determines $\widetilde{f}$.
Remark. The Plücker equations (15), here $\partial^2_{\lambda_3} \widetilde{f} = \partial_{\lambda_1} \partial_{\lambda_2} \widetilde{f}$, haven't been taken into account. They imply $\partial^{2k}_{\lambda_3} \widetilde{f} = (\partial_{\lambda_1} \partial_{\lambda_2})^k \widetilde{f}$, $\partial^{2k+1}_{\lambda_3} \widetilde{f} = (\partial_{\lambda_1} \partial_{\lambda_2})^k \partial_{\lambda_3} \widetilde{f}$ for $k \in \mathbb{N}$, hence the Taylor expansion

$$\begin{aligned}\widetilde{f}(0, \lambda_1, \lambda_2, \lambda_3) &= \sum_{k \in \mathbb{N}} \partial^k_{\lambda_3} \widetilde{f}(0, \lambda_1, \lambda_2, 0) \frac{\lambda_3^k}{k!} \\ &= C(E)\, \widetilde{f}(0, \lambda_1, \lambda_2, 0) + \lambda_3 S(E)\, (\partial_{\lambda_3} \widetilde{f})(0, \lambda_1, \lambda_2, 0) \end{aligned} \tag{17}$$

where $E = \lambda_3^2 \partial_{\lambda_1} \partial_{\lambda_2}$, and similarly

$$\partial_{\lambda_3} \widetilde{f}(0, \lambda_1, \lambda_2, \lambda_3) = \lambda_3 \partial_{\lambda_1} \partial_{\lambda_2} S(E)\, \widetilde{f}(0, \lambda_1, \lambda_2, 0) + C(E)\, (\partial_{\lambda_3} \widetilde{f})(0, \lambda_1, \lambda_2, 0). \tag{18}$$

Combining (16) (17) and (18) it follows that $\widetilde{f}$ can be reconstructed from $\widetilde{f}$, $\partial_{\lambda_3} \widetilde{f}$, $\widetilde{(x_1 f)}$, $\partial_{\lambda_3} \widetilde{(x_1 f)}$, $\widetilde{(x_2 f)}$ and $\partial_{\lambda_3} \widetilde{(x_2 f)}$ at $(0, \lambda_1, \lambda_2, 0)$ only.
Remembering (13) $\partial_{\lambda_3} \widetilde{f} = i \widetilde{(x_1 x_2 f)}$, these 6 functions can be replaced by $\widetilde{f}$, $\widetilde{(x_1 f)}$, $\widetilde{(x_2 f)}$, $\widetilde{(x_1 x_2 f)}$, $\widetilde{(x_1^2 x_2 f)}$ and $\widetilde{(x_1 x_2^2 f)}$, that is $\widehat{Rf}$, $\widehat{R(x_1 f)}$,..., $\widehat{R(x_1 x_2^2 f)}$ evaluated at $(1; \lambda_1, \lambda_2, 0)$. In other words the integrals of f, $x_1 f$,..., $x_1 x_2^2 f$ over the conics $\lambda_1 x_1^2 + \lambda_2 x_2^2 = t$ will determine f. A stronger (and more general) result is given in the next section.

4 Harmonic Polynomials and the Cauchy Problem

Two chapters of [5] are devoted to a general theory of harmonic polynomials which, when applied to nonlinear Radon transforms, leads to a refined version of the results of Section 3. We shall only present here a simplified approach to the harmonic polynomials relevant to our problem.
Notation. All polynomials considered here have complex coefficients. Let us order the N monomials $(x^\alpha)_{|\alpha|=m}$ as $x_1^m, ..., x_n^m$ first, then $(x^\beta)_{\beta \in B}$ where B is the set of the $N-n$ remaining multi-indices of length m. In accordance with this we replace our previous notation $\lambda = (\lambda_\alpha)_{|\alpha|=m} \in \mathbb{R}^N$ by $(\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^{N-n}$ with $\lambda = (\lambda_1, ..., \lambda_n)$ and $\mu = (\mu_\beta)_{\beta \in B}$; the former $\sum_\alpha \lambda_\alpha x^\alpha$ is replaced by $\sum_{j=1}^n \lambda_j x_j^m + \sum_{\beta \in B} \mu_\beta x^\beta$. Let $(x, p, q) \in \mathbb{R}^{n+N}$ denote dual variables to (ξ, λ, μ), with $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $p = (p_1, ..., p_n) \in \mathbb{R}^n$ and $q = (q_\beta)_{\beta \in B} \in \mathbb{R}^{N-n}$.

In this new notation the partial differential equations (14) become

$$\left(-i\partial_{\xi_j}\right)^m \widetilde{f} = -i\partial_{\lambda_j} \widetilde{f}\ ,\ (-i\partial_\xi)^\beta\, \widetilde{f} = -i\partial_{\mu_\beta} \widetilde{f} \text{ for } j = 1, ..., n \text{ and } \mu \in B. \tag{19}$$

They are dual to

$$x_j^m F = p_j F\ ,\ (x^\beta - q_\beta) F = 0 \text{ for } j = 1, ..., n \text{ and } \mu \in B, \tag{20}$$

where F is the tempered distribution on $\mathbb{R}^{n+N}$ corresponding to $\widetilde{f}$ via the Fourier transform on $\mathbb{R}^{n+N}$ (being smooth and bounded, $\widetilde{f}$ is tempered on $\mathbb{R}^{n+N}$).

Let us introduce the following N polynomials on $\mathbb{R}^n \times \mathbb{R}^{N-n} = \mathbb{R}^N$:

$$u_j(x, q) := x_j^m \ , \ u_\beta(x, q) := x^\beta - q_\beta \text{ for } j = 1, ..., n \text{ and } \beta \in B. \tag{21}$$

The system (20) implies that the support of F is contained in the closed set V of $\mathbb{R}^{n+N}$ defined by the N equations

$$V = \left\{(x, p, q) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{N-n} | u_j(x, q) = p_j \ , u_\beta(x, q) = 0 \ , 1 \leq j \leq n, \beta \in B\right\}.$$

Being the graph of a map $x \mapsto (p, q)$, V is a n-dimensional submanifold of $\mathbb{R}^{n+N}$.

Definition 2 *A polynomial function $h(x, q)$ on $\mathbb{R}^n \times \mathbb{R}^{N-n}$ is called **harmonic** if*

$$u_j(\partial_x, \partial_q)h = 0 \ , \ u_\beta(\partial_x, \partial_q)h = 0 \text{ for } j = 1, ..., n \text{ and } \beta \in B.$$

*It is called **homogeneous of degree** d if $h(tx, t^m q) = t^d h(x, q)$ for all $t \in \mathbb{R}$ (thus each x_j has degree 1 and each q_β has degree m).*

Proposition 3 *Let $D := \sum_{\beta \in B} q_\beta \partial_x^\beta$. Then $u_\beta(\partial_x, \partial_q) = - \ e^D \circ \partial_{q_\beta} \circ e^{-D}$. The space of harmonic polynomials is m^n-dimensional. Its elements are given by*

$$h = e^D f$$

where f is an arbitrary polynomial of the following form

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \text{ with } 0 \leq \alpha_j \leq m - 1 \text{ for } j = 1, ..., n \text{ and } a_\alpha \in \mathbb{C}.$$

Besides $h = e^D f$ is homogeneous of degree d (in the sense of Definition 4) if and only if f is homogeneous of degree d.

Proof. Since $u_\beta(\partial_x, \partial_q) = \partial_x^\beta - \partial_{q_\beta}$ we have $[D, u_\beta(\partial_x, \partial_q)] = \partial_x^\beta$ and $\left[D, \partial_x^\beta\right] = 0$, thus $(\operatorname{ad} D)^2 u_\beta(\partial_x, \partial_q) = 0$ and

$$\begin{aligned} e^{-D} u_\beta(\partial_x, \partial_q) e^D &= e^{-\operatorname{ad} D} u_\beta(\partial_x, \partial_q) = (1 - \operatorname{ad} D)\, u_\beta(\partial_x, \partial_q) \\ &= u_\beta(\partial_x, \partial_q) - \partial_x^\beta = - \ \partial_{q_\beta}. \end{aligned}$$

[This proof may also be written without any Lie formalism, by computing the derivative with respect to t of $e^{-tD} u_\beta(\partial_x, \partial_q) e^{tD}$.]
Since e^D is a linear isomorphism of the space of polynomials onto itself, a polynomial $h(x, q)$ is harmonic if and only if

$$\partial_{x_j}^m h = 0 \ , \ \partial_{q_\beta}\left(e^{-D} h\right) = 0 \text{ for } j = 1, ..., n \text{ and } \beta \in B.$$

The latter equations imply $h = e^D f$ for some polynomial f in the x variables. Since $\left[D, \partial_{x_j}^m\right] = 0$ the former equations imply $\partial_{x_j}^m f = 0$ for $j = 1, ..., n$ whence our claim about f.
The operator D preserves homogeneity in (x, q) and the last statement follows. ■

Examples. Let us write down, as an example, a basis of homogeneous harmonic polynomials for $n = 2$ and $m = 4$. Here $N = 5$, $\beta = (\beta_1, \beta_2)$ with $0 \le \beta_j \le 3$, $\beta_1 + \beta_2 = 4$, $q = (q_{13}, q_{22}, q_{31})$ and $D = \sum q_{\beta_1\beta_2} \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2}$. The 16 monomials $f(x) = x_1^a x_2^b$, $0 \le a \le 3$, $0 \le b \le 3$, make up a basis of the relevant polynomials f. Since the degree of f is 6 at most we have $D^2 f = 0$ and the 16 corresponding harmonic polynomials are $h = f + Df$, that is[1]

$$\begin{aligned}
&1 \ , \ x_1 \ , \ x_2 \ , \ x_1^2 \ , \ x_1x_2 \ , \ x_2^2 \ , \ x_1^3 \ , \ x_1^2x_2 \ , \ x_1x_2^2 \ , \ x_2^3 \ , \\
&x_1^3x_2 + 6q_{31} \ , \ x_1^2x_2^2 + 4q_{22} \ , \ x_1x_2^3 + 6q_{13} \ , \\
&x_1^3x_2^2 + 12q_{22}x_1 + 12q_{31}x_2 \ , \ x_1^2x_2^3 + 12q_{13}x_1 + 12q_{22}x_2 \ , \\
&x_1^3x_2^3 + 18q_{13}x_1^2 + 36q_{22}x_1x_2 + 18q_{31}x_2^2 \ .
\end{aligned}$$

For $m = n = 2$ (already considered) we have $N = 3$, $q \in \mathbb{R}$, and the corresponding basis of harmonic polynomials is

$$1 \ , \ x_1 \ , \ x_2 \ , \ x_1x_2 + q.$$

More generally, let A denote the set of all $\alpha \in \mathbb{N}^n$ such that $0 \le \alpha_j \le m - 1$ for $j = 1, ..., n$. By Proposition 5 the $h_\alpha := e^D x^\alpha$, $\alpha \in A$, make up a basis of the space of harmonic polynomials.

Proposition 4 *For any polynomial $P(x, q)$ on $\mathbb{R}^n \times \mathbb{R}^{N-n}$ there exists a family of m^n polynomials Q_α, $\alpha \in A$, on $\mathbb{R}^N$ such that*

$$P(x, q) = \sum_{\alpha \in A} Q_\alpha(u_1(x, q), ..., u_N(x, q)) h_\alpha(x, q),$$

where $u_1, ..., u_N$ denote the polynomials defined by (21).

Proof. Let $\langle a, b\rangle = a(\partial)\bar{b}(0)$ be the Fischer inner product on the space of polynomials on $\mathbb{R}^n \times \mathbb{R}^{N-n}$. Then h is harmonic if and only if $u_k(\partial_x, \partial_q)\overline{h} = 0$ for $k = 1, ..., N$, i.e. $\langle au_k, h\rangle = 0$ for all polynomials a. The space of harmonic polynomials is thus the orthogonal complement of the ideal $\left\{\sum_{k=1}^N a_k(x, q)u_k(x, q)\right\}$ generated by the u_k's (where the a_k's are arbitrary polynomials).
A given $P(x, q)$ now has a unique decomposition as

$$P = h + \sum_{k=1}^N a_k u_k$$

with h harmonic. Separating homogeneous components we may assume P is homogeneous of degree d (in the sense of Definition 2). Since u_k is homogeneous, each homogeneous component of a harmonic polynomial is harmonic. We may therefore assume h and all $a_k u_k$ homogeneous of degree d, therefore a_k is homogeneous of degree $d - m$. Writing similar decompositions for each a_k the result easily follows. ■

[1] Cf. [5] p. 312, where the coefficients 16 should be replaced, I think, by 18.

Example. For $m = n = 2$ the generators and harmonic polynomials are respectively

$$u_1 = x_1^2\ ,\ u_2 = x_2^2\ ,\ u_3 = x_1x_2 - q$$
$$h_0 = 1\ ,\ h_1 = x_1\ ,\ h_2 = x_2\ ,\ h_3 = x_1x_2 + q$$

and the first non-trivial examples of decomposition in Proposition 4 are:

$$2x_1x_2 = u_3h_0 + h_3\ ,\ 2q = -u_3h_0 + h_3$$
$$x_1q = -u_3h_1 + u_1h_2\ ,\ x_2q = -u_3h_2 + u_2h_1$$
$$q^2 = u_1u_2h_0 - u_3h_3\ ,\ 2x_1x_2q = (2u_1u_2 - u_3^2)h_0 - u_3h_3.$$

Replacing x_j by $-i\partial_{\xi_j}$ and q_β by $-i\partial_{\mu_\beta}$ we infer from Proposition 4 an equality of differential operators. Applying them to $\widetilde{f}$ we obtain

$$\begin{aligned} P(-i\partial_\xi, -i\partial_\mu)\widetilde{f} &= \sum_{\alpha\in A} Q_\alpha\left(\left(-i\partial_{\xi_j}\right)^m, (-i\partial_\xi)^\beta - (-i\partial_{\mu_\beta})\right) h_\alpha\left(-i\partial_\xi, -i\partial_\mu\right)\widetilde{f} \\ &= \sum_{\alpha\in A} Q_\alpha\left(-i\partial_\lambda, 0\right) h_\alpha\left(-i\partial_\xi, -i\partial_\mu\right)\widetilde{f} \end{aligned}$$

in view of (19) and the commutativity of differential operators. In particular all derivatives $\partial_\xi^\rho\partial_\mu^\sigma\widetilde{f}$ may be written in this form with polynomials Q_α depending on ρ, σ whence, by Taylor's formula on the variables (ξ, μ),

$$\widetilde{f}(\xi, \lambda, \mu) = \sum Q_{\alpha\rho\sigma}(-i\partial_\lambda, 0)h_\alpha\left(-i\partial_\xi, -i\partial_\mu\right)\widetilde{f}(0, \lambda, 0)\frac{\xi^\rho}{\rho!}\frac{\mu^\sigma}{\sigma!} \tag{22}$$

where $\sum$ runs over all $\rho \in \mathbb{N}^n$, $\sigma \in \mathbb{N}^{N-n}$ and $\alpha \in A$. Remembering (12)(13) $-i\partial_{\xi_j}\widetilde{f} = \widetilde{x_jf}$, $-i\partial_{\mu_\beta}\widetilde{f} = \widetilde{x^\beta f}$ we have $h_\alpha\left(-i\partial_\xi, -i\partial_\mu\right)\widetilde{f} = (h_\alpha(x, q)f)^\sim$ with $q_\beta = x^\beta$ for $\beta \in B$.

Lemma 5 *For all α there exists a positive integer C_α such that, when replacing each q_β by x^β for $\beta \in B$,*

$$h_\alpha(x, q) = h_\alpha(x, (x^\beta)_{\beta\in B}) = C_\alpha x^\alpha.$$

Proof. For $\alpha \in \mathbb{N}^n$ we have

$$Dx^\alpha = \sum_{\beta\in B} q_\beta\partial_x^\beta x^\alpha = \sum_{\beta\in B}\frac{\alpha!}{(\alpha-\beta)!}q_\beta x^{\alpha-\beta}$$
$$D^2x^\alpha = \sum_{\beta,\gamma\in B}\frac{\alpha!}{(\alpha-\beta-\gamma)!}q_\beta q_\gamma x^{\alpha-\beta-\gamma}$$

etc (the coefficients being 0 unless $\beta \leq \alpha$, resp. $\beta + \gamma \leq \alpha$). When replacing q_β by x^β, q_γ by x^γ etc, the polynomials Dx^α, D^2x^α etc thus become x^α times a positive integer coefficient. The same holds for $h_\alpha = e^Dx^\alpha$, whence the lemma. ■

Going back to (22) we have $h_\alpha\left(-i\partial_\xi, -i\partial_\mu\right)\widetilde{f} = C_\alpha\ \widetilde{x^\alpha f}$ and we conclude that, for $(\xi, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{N-n}$,

$$\widetilde{f}(\xi, \lambda, \mu) = \sum_{\rho,\sigma,\alpha} C_\alpha Q_{\alpha\rho\sigma}(-i\partial_\lambda, 0)\widetilde{(x^\alpha f)}(0, \lambda, 0)\frac{\xi^\rho}{\rho!}\frac{\mu^\sigma}{\sigma!}.$$

Therefore *the restriction to all* $(0, \lambda, 0)$ *of the* m^n *functions* $\widetilde{x^\alpha f}$, $\alpha \in A$, *determines* f. In other words, the Cauchy problem for (19) is well-posed with the Cauchy data $h_\alpha\left(-i\partial_\xi, -i\partial_\mu\right)\widetilde{f} = C_\alpha\ \widetilde{x^\alpha f}$ on the n-plane of $\mathbb{R}^{n+N}$ defined by $\xi = \mu = 0$.

In terms of Radon transforms we obtain the following result.

Theorem 6 *A function* $f \in \mathcal{D}(\mathbb{R}^n)$ *is uniquely determined by the* m^n *nonlinear Radon transforms* $R\left(x^\alpha f\right)(t, \lambda, 0)$ *(with* $\alpha \in \mathbb{N}^n$, $0 \le \alpha_j < m$, $t \in \mathbb{R}$, $\lambda \in \mathbb{R}^n \setminus \{0\}$*), that is by the integrals of each* $x^\alpha f$ *on the hypersurfaces*

$$\lambda_1 x_1^m + \cdots + \lambda_n x_n^m = t.$$

5 Inversion Formulas

Let us now look for an inversion formula for the nonlinear Radon transform. The nonlinear Fourier transform $\widetilde{f}$ is greatly overdetermined, with $n + N$ variables (ξ, λ) instead of n for f. As in Section 3 we shall restrict $\widetilde{f}$ to $\xi = 0$ and, assuming the monomials x^α are ordered as $x_1^m, ..., x_n^m$ first, followed by the other x^β's, it turns out that (as in the final remark of Section 3) we can also restrict to $\lambda = (\lambda_1, ..., \lambda_n, 0, ..., 0)$, written as $\lambda \in \mathbb{R}^n$ for short. Then

$$\widetilde{f}(0, \tau\lambda) = \int_{\mathbb{R}^n} e^{i\tau \sum_1^n \lambda_j x_j^m} f(x)dx = \widehat{Rf}(\tau, \lambda) \text{ with } \tau \in \mathbb{R}, \lambda \in \mathbb{R}^n. \tag{23}$$

5.1 First Case: m odd

Let U denote the dense open subset of $\mathbb{R}^n$ defined by $x_j \neq 0$ for all j. For m odd the map $\psi : x \mapsto y = x^m := (x_1^m, ..., x_n^m)$ is a diffeomorphism of U onto itself. Then

$$\widehat{Rf}(\tau, \lambda) = \int_{\mathbb{R}^n} e^{i\tau\lambda\cdot y} g(y)dy = \widehat{g}(\tau\lambda) \tag{24}$$

with $\lambda \cdot y = \sum_1^n \lambda_j y_j$ and

$$g(y) := m^{-n}|y_1 \cdots y_n|^{(1/m)-1} f(y^{1/m})\ ,\ y \in \mathbb{R}^n.$$

As above $\widehat{g}$ denotes the classical n-dimensional Fourier transform and $\widehat{Rf}$ is the 1-dimensional Fourier transform with respect to t.

The change $x \mapsto y$ thus reduces the nonlinear Radon transform R to the linear one considered in the introduction: $Rf(t, \lambda) = R_0 g(t, \lambda)$. But g is not necessarily smooth, $\widehat{g}(\lambda) = \widehat{Rf}(1, \lambda)$ is not necessarily rapidly decreasing and the inversion formula (2) may become invalid here. However g is integrable on $\mathbb{R}^n$ and vanishes outside a compact set, therefore defines a tempered distribution. Denoting by $\mathcal{F}$ the inverse Fourier transform for tempered distributions on $\mathbb{R}^n$ we have $g = \mathcal{F}\widehat{g}$ hence, for any $u \in \mathcal{D}(U)$,

$$\begin{aligned}\int_U f(x)u(x^m)dx = \int_U g(y)u(y)dy &= \langle \mathcal{F}\widehat{g}(y), u(y)\rangle \\ &= \left\langle \left(\psi^*\mathcal{F}\widehat{g}\right)(x), |\det\psi'(x)|u(\psi(x)\right\rangle \\ &= \left\langle m^n(x_1\cdots x_n)^{m-1}\left(\psi^*\mathcal{F}\widehat{g}\right)(x), u(x^m)\right\rangle,\end{aligned}$$

using the pullback by ψ of the distribution $\mathcal{F}\widehat{g}$ on U (cf. [6] p. 80). The absolute value may be skipped here since $m-1$ is even and $\det\psi' > 0$. Therefore, for $f \in \mathcal{D}(\mathbb{R}^n)$,

$$f(x) = m^n(x_1\cdots x_n)^{m-1}(\psi^*\mathcal{F}\widehat{Rf}(1,\cdot))(x), \tag{25}$$

an equality of distributions on U.

5.2 Second Case: m even

The above map $\psi : x \mapsto y$ is no more a bijection: given y with all $y_j > 0$, the equations $y = x^m$ now have 2^n solutions $x = \left(\pm y_1^{1/m}, ..., \pm y_n^{1/m}\right)$.

For $x, y \in \mathbb{R}^n$ we write $xy := (x_1y_1, ..., x_ny_n)$. Let $E := \{1, -1\}^n$ denote the set of all $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)$ with $\varepsilon_j = \pm 1$ and

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_j > 0 \text{ for } 1 \le j \le n\}.$$

Viewing the integral (23) as a sum of integrals over the quadrants $\varepsilon\mathbb{R}^n_+$, $\varepsilon \in E$, we obtain, by the change of variables $x \mapsto y$ with $x_j = \varepsilon_j y_j^{1/m}$, $y_j > 0$, on $\varepsilon\mathbb{R}^n_+$,

$$\widehat{Rf}(\tau, \lambda) = \widetilde{f}(0, \tau\lambda) = \int_{\mathbb{R}^n_+} e^{i\tau\lambda\cdot y} g(y)dy$$

with $\tau \in \mathbb{R}$, $\lambda \in \mathbb{R}^n$ and, for $y \in \mathbb{R}^n_+$,

$$g(y) := m^{-n}\left(y_1\cdots y_n\right)^{(1/m)-1}\sum_{\varepsilon\in E} f(\varepsilon y^{1/m}).$$

Let H denote the Heaviside function $H(y) = 1$ if $y \in \mathbb{R}^n_+$, $H(y) = 0$ otherwise. Equation (24) is now replaced by

$$\widehat{Rf}(\tau, \lambda) = \int_{\mathbb{R}^n} e^{i\tau\lambda\cdot y} H(y)g(y)dy = \widehat{H}g(\tau\lambda).$$

Again Hg is integrable and vanishes outside a compact set, hence tempered on $\mathbb{R}^n$, and as above the Fourier inversion $Hg = \mathcal{F}\widehat{Hg}$ implies the following equality of distributions on $\mathbb{R}^n_+$

$$\sum_{\varepsilon\in E} f(\varepsilon x) = m^n\left(x_1\cdots x_n\right)^{m-1}(\psi^*\mathcal{F}\widehat{Rf}(1,\cdot))(x). \tag{26}$$

This gives f if its support is contained in some quadrant $\varepsilon\mathbb{R}^n_+$. Otherwise we must separate the components $f(\varepsilon x)$, which can be achieved by replacing f with $x^\alpha f$ for suitably chosen α's as follows.

With each $\varepsilon = (\varepsilon_1, ..., \varepsilon_n) \in E$ we associate the monomial

$$p_\varepsilon(x) := x_{i_1} \cdots x_{i_k}$$

where $1 \leq i_1 < \cdots < i_k \leq n$ is the (ordered) set of indices i such that $\varepsilon_i = -1$; for instance, $n = 4$ and $\varepsilon = (-1, 1, -1, 1)$ yield $p_\varepsilon(x) = x_1 x_3$. The map $\varepsilon \mapsto p_\varepsilon$ is a bijection of E onto the set of divisors of $x_1 \cdots x_n$.

Let $\varepsilon, \eta \in E$. A minus sign occurs in $p_\varepsilon(\eta x) = p_\varepsilon(\eta_1 x_1, ..., \eta_n x_n)$ each time there is a factor x_i, that is $\varepsilon_i = -1$, and the corresponding η_i is -1. Therefore

$$p_\varepsilon(\eta x) = a_{\varepsilon,\eta} p_\varepsilon(x) \text{ with } a_{\varepsilon,\eta} := (-1)^{k(\varepsilon,\eta)}, \tag{27}$$

where $k(\varepsilon, \eta)$ denotes the number of indices i such that $\varepsilon_i = \eta_i = -1$.

Example. For $n = 2$ the matrix $(a_{\varepsilon,\eta})$ is given by the table:

	p_ε	1	x_1	x_2	x_1x_2
	ε	$++$	$-+$	$+-$	$--$
η					
$++$		1	1	1	1
$-+$		1	-1	1	-1
$+-$		1	1	-1	-1
$--$		1	-1	-1	1

Our inversion formula for R will be inferred from the following combinatorial lemma.

Lemma 7 *The set $E = \{1, -1\}^n$ being provided with some ordering, the $2^n \times 2^n$ matrix $A = (a_{\varepsilon,\eta})_{\varepsilon,\eta \in E}$ is symmetric and $A^2 = 2^n I$ (where I is the unit matrix).*

Proof. The symmetry is clear by the definition of $k(\varepsilon, \eta)$.
For $\varepsilon, \eta, \zeta \in E$ we have $k(\varepsilon, \eta\zeta) = k(\varepsilon, \eta) + k(\varepsilon, \zeta)$ since $\varepsilon_i = \eta_i \zeta_i = -1$ is equivalent to $\varepsilon_i = -1$ and $\eta_i = -1$, $\zeta_i = 1$ or (exclusive or) $\varepsilon_i = -1$ and $\eta_i = 1$, $\zeta_i = -1$. Therefore

$$a_{\varepsilon,\eta} a_{\varepsilon,\zeta} = a_{\varepsilon,\eta\zeta}. \tag{28}$$

Besides, for fixed $\eta \in E$,

$$\begin{aligned}\prod_{i=1}^n (1 + \eta_i x_i) &= 1 + \sum_i \eta_i x_i + \sum_{i<j} \eta_i \eta_j x_i x_j + \cdots + \eta_1 \cdots \eta_n x_1 \cdots x_n \\ &= \sum_{\varepsilon \in E} p_\varepsilon(\eta x) = \sum_{\varepsilon \in E} a_{\varepsilon,\eta} p_\varepsilon(x).\end{aligned}$$

Taking $x_1 = \cdots = x_n = 1$ this gives the sum of elements in each column (or row) of A:

$$\sum_{\varepsilon \in E} a_{\varepsilon,\eta} = \prod_{i=1}^n (1 + \eta_i) = \begin{cases} 2^n \text{ if } \eta = (1, ..., 1) \\ 0 \text{ otherwise.} \end{cases}$$

Now (28) implies

$$\sum_{\varepsilon\in E} a_{\varepsilon,\eta}a_{\varepsilon,\zeta} = \begin{cases} 2^n \text{ if } \eta\zeta = (1,...,1) \\ 0 \text{ otherwise.} \end{cases}$$

But $\eta\zeta = (1,...,1)$ is equivalent to $\eta_i = \zeta_i$ for all i, that is $\eta = \zeta$. Remembering the symmetry of A, we infer that $A^2 = 2^n I$. ■

Let us consider $Sf(x) := \sum_{\eta\in E} f(\eta x)$. Replacing f by $p_\varepsilon f$ we obtain, in view of (27),

$$S(p_\varepsilon f)(x) = \sum_{\eta\in E} (p_\varepsilon f)(\eta x) = p_\varepsilon(x) \sum_{\eta} a_{\varepsilon,\eta} f(\eta x),$$

which can be inverted by $A^{-1} = 2^{-n}A$ (Lemma 7) as

$$f(\eta x) = 2^{-n} \sum_{\varepsilon\in E} a_{\varepsilon,\eta} p_\varepsilon(x)^{-1} S(p_\varepsilon f)(x)$$

for each $\eta \in E$. By (26) applied to each $p_\varepsilon f$ we have

$$S(p_\varepsilon f)(x) = m^n (x_1 \cdots x_n)^{m-1} \psi^* (\mathcal{F}\widehat{Rp_\varepsilon f}(1,\cdot))(x)$$

on $\mathbb{R}^n_+$ and the latter equations show that f can be reconstructed in each quadrant of $\mathbb{R}^n$ from the 2^n nonlinear Radon transforms Rf, $R(x_i f)$, $R(x_i x_j f)$,..., $R(x_1 \cdots x_n f)$.

Summarizing we have proved the following theorem. Let us recall our notation: $\widehat{Rf} = \widehat{Rf}(1,\lambda)$ is given by (23) with $\lambda \in \mathbb{R}^n$, $\mathcal{F}$ is the inverse Fourier transform of tempered distributions on $\mathbb{R}^n$, ψ^* is the pullback of distributions by $\psi(x) = (x_1^m, ..., x_n^m)$, $E = \{1,-1\}^n$ and p_ε, $a_{\varepsilon,\eta}$ are defined before Lemma 7.

Theorem 8 *The nonlinear Radon transform (7) is inverted by the following formulas, where $f \in \mathcal{D}(\mathbb{R}^n)$.*
(i) if m is odd

$$f(x) = m^n (x_1 \cdots x_n)^{m-1} (\psi^* \mathcal{F}\widehat{Rf})(x)$$

(equality of distributions on the open set $x_1 \neq 0, ..., x_n \neq 0$);
(ii) if m is even: for $\eta \in E$,

$$f(\eta x) = \left(\frac{m}{2}\right)^n \sum_{\varepsilon\in E} a_{\varepsilon,\eta} p_\varepsilon(x)^{-1} (x_1 \cdots x_n)^{m-1} (\psi^* \mathcal{F}\widehat{Rp_\varepsilon f})(x)$$

(equality of distributions on the open set $x_1 > 0, ..., x_n > 0$).

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