INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

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Abstract. In the framework of homogeneous spaces of Lie groups, we propose a synthetic survey and several generalizations of various inversion formulas from the literature on Radon transforms, obtained by group-theoretic tools such as invariant differential operators and harmonic analysis.

We introduce a general concept of shifted Radon transform, which also leads to simple inversion formulas and solves wave equations.

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Contents

1. Introduction

The Radon transform on a manifold X associates to a function u on this manifold its integrals $Ru(y)$ over a given family Y of submanifolds y (equipped with suitable measures). One of the main problems of integral geometry is to recover u from Ru by means of an explicit inversion formula. The dual Radon transform R^* then enters the picture in a natural way: it maps functions on Y into functions on X, by integrating (with respect to a suitable measure) over all submanifolds $y \in Y$ which contain a given point $x \in X$.

Here we assume that a Lie group G acts transitively on both X and Y, so that they are homogeneous space $X = G/K$, $Y = G/H$ where K, H are Lie subgroups of G ; besides K will be compact throughout the paper. Our main examples for X will be Riemannian symmetric spaces of the noncompact type, often assumed to have rank one (hyperbolic spaces). For Y they will be a family of totally geodesic submanifolds of X, or the family of horocycles.

We first look for a left inverse of R of the following form

$$u(x) = DR^*Ru(x) \ , \tag{*}$$

where D is some operator acting on functions on X. In all known examples D is an integro-differential operator, sometimes even differential. The purpose of the present paper is to emphasize three simple ideas leading to such results (or related to them), sometimes hidden under long calculations dealing with some specific example. As a benefit we can unify several proofs from the literature, and obtain some generalizations.

a. The first idea stems from Proposition 3 (section 3.1) : R^*R is always a *convolution operator* on X, by a K-invariant measure S. Besides S can be easily written down explicitly on rank one examples (Propositions 4 and 5). The problem is thus to find a convolution inverse D to S. We study it in section 4 for noncompact isotropic spaces (i.e. all Euclidean or hyperbolic spaces), looking for D as a polynomial of the Laplace-Beltrami operator of

X with given fundamental solution S. This can be done for the Radon transform on even-dimensional totally geodesic submanifolds (with an additional assumption, see Theorem 8), or on horocycles of odd-dimensional hyperbolic spaces (Theorem 9).

Another natural approach is to seek the convolution inverse D by means of K-invariant harmonic analysis on X. We discuss this in section 5 for the totally geodesic Radon transform on hyperbolic spaces. Unfortunately it seems difficult to find D explicitly by this method, except under the assumptions of Theorem 8 (proved by simpler tools) or for the case of $X = H^n(\mathbb{R})$ (already solved by Berenstein and Tarabusi [1]).

b. The second idea goes back to Johann Radon himself, and will be developed here in full generality. If we replace R^* by the *shifted dual transform* R_t^*, obtained by integrating over submanifolds y at distance t (in some sense) from a point x, we may prove new inversion formulas for R. More precisely for $X = G/K$, $Y = G/H$ we consider (section 6.1)

$$Ru(gH) = \int_H u(ghK)\,dh \ ,\ R_t^* v(gK) = \int_K v(gktH)\,dk \ ,$$

where g, t are elements of G, u is a function on X and v on Y. Of course $R_t^* = R^*$ when t is the identity. It is then quite elementary to observe (section 6.2) that an inversion formula of R at the origin x_o for K-invariant u, say $u(x_o) = < T_{(y)}, Ru(y) >$, implies the following new result

$$u(x) = < T_{(t)}, R_t^* Ru(x) > \qquad (**)$$

for arbitrary u and x. The notation $T_{(t)}$ means that the operator T now acts on the shift variable t, instead of x as in (*). Applying this method to the horocycle transform on Riemannian symmetric spaces of the noncompact type, we obtain a new proof of Helgason's inversion formulas (Theorem 13 and Corollary 20). In Theorem 14 the same method.is applied to the totally geodesic transform, thus extending to all classical hyperbolic spaces known results for the real ones.

c. It is now an intriguing question to compare the results (*) and (**) of methods a and b. For the 2-dimensional totally geodesic transform on $X = H^3(\mathbb{R})$, Helgason ([10]) obtained a curious "*amusing formula*" by equating the right-hand sides of (*) and (**). In sections 6.4 and 6.5 we give direct proofs of such formulas, for the Laplace operator first (Proposition 16), then for general invariant differential operators (Theorem 17).

The content of these results is easily understood on the example of the Radon transform on all hyperplanes of $X = \mathbb{R}^{2k+1}$ (see section 6.4 for more details). Here the inversion formulas (*), resp. (**), are

$$Cu(x) = L_x^k R^* Ru(x) \ , \text{resp. } Cu(x) = \partial_t^{2k} R_t^* Ru(x)\big|_{t=0} \ ,$$

where C is a constant factor and L is the Euclidean Laplacian. Passing from one to the other is thus an immediate consequence of the *wave equation*

$$L_x R_t^* v(x) = \partial_t^2 R_t^* v(x)$$

for $v = Ru$ and all x and t. In Proposition 16 and Theorem 17 we construct solutions of some generalized wave equations, some of them only valid when t is the identity (but this suffices for our purpose). Such results may have independent interest, providing explicit solutions of certain multitemporal wave equations by means of shifted dual Radon transforms, which appear as integrals of elementary "plane" waves (Proposition 19, for horocycles).

One last remark : explicit inversion formulas for the totally geodesic Radon transform seem rather difficult to obtain, and most of them in the literature are only given for spaces of constant curvature. We obtain here some results for $X = H^n(\mathbb{F})$, with $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, provided that the tangent spaces to the geodesic submanifolds under consideration are

$\mathbb{F}$-vector spaces (section 4.3.c, Theorem 14, Proposition 16). This seems to be the simplest case after $\mathbb{R}^n$ and $H^n(\mathbb{R})$.

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Notations.

a. General notations. As usual $\mathbb{R}, \mathbb{C}, \mathbb{H}$ respectively denote the fields of real numbers, complex numbers and quaternions. When considering vector spaces on $\mathbb{H}$, the scalars will act on the right.

If X is a (real C^∞) manifold, $C(X)$ is the space of complex-valued continuous functions on X, $C_c(X)$ the subspace of compactly supported functions and $\mathcal{D}(X)$ the subspace of compactly supported C^∞ functions; $\mathcal{D}'(X)$ is the space of distributions and $\mathcal{E}'(X)$ the subspace of compactly supported distributions.

If T is an operator (e.g. differential) on a space of functions on X, a notation like $T_{(x)}f(x, y)$ means that T acts on the variable x, not y.

If G is a (real) Lie group, let e, $\mathfrak{g}$, exp, Ad, ad respectively denote its origin, Lie algebra, exponential mapping, adjoint representations of G and $\mathfrak{g}$. When G acts on X, we shall write $g \cdot x$, or sometimes $\tau(g)x$ or even $\tau_X(g)x$, for the point obtained when $g \in G$ acts on the point $x \in X$. In particular, for $V \in \mathfrak{g}$, it is convenient to write $g \cdot V$ for $\mathrm{Ad}(g)V$. In this context, $\mathbb{D}(X)$ is the algebra of linear differential operators on X which commute to the action of G, and $\mathbb{D}(G)$ refers to the special case when G acts onto itself by left translations.

If X is a Riemannian manifold, $\Sigma(x, r)$ will denote the sphere with center $x \in X$ and radius $r \geq 0$. Also $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in the Euclidean space $\mathbb{R}^n$.

b. Riemannian homogeneous spaces. Let G be a Lie group, K a *compact* subgroup and $\mathfrak{g}$, $\mathfrak{k}$ their Lie algebras. The homogeneous manifold $X = G/K$ can be provided with a G-invariant Riemannian structure. Indeed a scalar product can be taken on $\mathfrak{g}$, invariant under the compact group $\mathrm{Ad}_G(K)$; then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$, the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, is a K-invariant (i.e. stable under $\mathrm{Ad}_G(K)$) vector subspace which can be identified with the tangent space to X at the origin $x_o = K$. Carrying by the action of G the K-invariant scalar product on $\mathfrak{p}$ we thus obtain a Riemannian structure on X, and elements of G are isometries.

We shall also consider $Y = G/H$, where H is another Lie subgroup of G.

c. Riemannian symmetric spaces (see [8] chap.IV or [15] chap. XI for their basic properties). A special case of the previous one, they are the homogeneous spaces $X = G/K$, where G is a connected Lie group provided with an involutive automorphism θ and K is a compact subgroup which lies between the group of all fixed points of θ in G and its identity component. The differential of θ at e induces a Lie algebra automorphism of $\mathfrak{g}$ and the eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (same notations as before).

The exponential mapping of the symmetric space is $\mathrm{Exp} : \mathfrak{p} \to X$, related to $\exp : \mathfrak{g} \to G$ by $\mathrm{Exp}\, V = (\exp V)K$ for $V \in \mathfrak{p}$. The curve $\mathrm{Exp}\, \mathbb{R}V$ is the geodesic of X which is tangent to the vector V at the origin $x_o = K$.

d. Riemannian symmetric spaces of the noncompact type. Assuming further that G is a connected non compact real semisimple Lie group with finite center and K a maximal compact subgroup, we obtain the subclass of Riemannian symmetric spaces of the noncompact type, particularly interesting because of their rich (and well-known) structure arising from the theory of root systems. The map $\mathrm{Exp} : \mathfrak{p} \to X$ is then a global diffeomorphism onto.

We briefly recall some classical semisimple notations, as used for instance in Helgason's books. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Related to the restricted root system of the pair $(\mathfrak{g}, \mathfrak{a})$ are the eigenspaces $\mathfrak{g}_\alpha$, the *Iwasawa decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of the Lie algebra and $G = KAN$ for the group (unique decomposition of each element of G into a product of factors in the respective subgroups); the subgroups A, resp. N, of G are abelian, resp. nilpotent. The half sum of positive roots (counted with multiplicities) is a linear form ρ on $\mathfrak{a}$; we write $a^\rho = e^{\rho(\log a)}$ for $a \in A$. Let M, resp. M', denote the centralizer, resp. normalizer, of A in K. Then $W = M'/M$ is a finite group called the Weyl group.

Let y_o denote the orbit $N \cdot x_o \subset X$. The *horocycles* of X are the submanifolds $g \cdot y_o$, for $g \in G$. Since $g \cdot y_o = y_o$ (globally) if and only if $g \in MN$, the space of all horocycles is $Y = G/MN$.

e. Isotropic Riemannian symmetric spaces. A Riemannian manifold X is called *isotropic* if, for every $x \in X$ and every pair of unit tangent vectors V, W to X at x, there exists an isometry of X leaving x fixed and mapping V to W. The connected isotropic Riemannian manifolds are the Euclidean spaces $\mathbb{R}^n$, the *hyperbolic spaces* i.e. the Riemannian symmetric spaces of the noncompact type and of rank one ($\dim \mathfrak{a} = 1$), and their compact analogues, spheres and projective spaces. The compact spaces will not be considered in this paper, so that most of our examples will be taken from the list

$$\mathbb{R}^n, H^n(\mathbb{R}), H^n(\mathbb{C}), H^n(\mathbb{H}), H^{16}(\mathbb{O}) \ .$$

Among them we shall often restrict ourselves to the *classical hyperbolic spaces* $H^n(\mathbb{F})$, with $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

2. Geometric setting

2.1. Double fibrations of homogeneous spaces. The general group-theoretic setting for Radon transforms, introduced by Helgason in the sixties, is motivated by the well-known example of points and hyperplanes in the Euclidean space $\mathbb{R}^n$. The set of points and the set of hyperplanes are both homogeneous spaces of the isometry group of $\mathbb{R}^n$, and it turns out that the fundamental "incidence" relation (a point x belongs to a hyperplane y), as well as the defining integral of the Radon transform, have simple expressions in terms of Lie groups and invariant measures. This observation suggests considering the following general situation.

Let X and Y be two manifolds, *with given origins* $x_o \in X$ and $y_o \in Y$, and assume a real Lie group G *acts transitively on both manifolds* X and Y. Two elements $x \in X$ and $y \in Y$ are said to be *incident* if there exists some $g \in G$ such that $x = g \cdot x_o$ and $y = g \cdot y_o$. Roughly speaking, if we think of g as a motion, this means that x and y have the same relative position as the origins x_o and y_o.

A more convenient formulation is obtained in terms of the isotropy subgroups K, resp. H, of x_o, resp. y_o, in G. They are closed Lie subgroups of G, and the manifolds X, Y can be identified with the homogeneous spaces of left cosets G/K, G/H respectively; in particular we may write $x_o = K$, $y_o = H$, $g \cdot x_o = gK$, etc. The points $x = g'K \in X$ and $y = g''H \in Y$ are then incident if and only if there exists $g \in G$ such that $g'K = g \cdot x_o = gK$ and $g''H = g \cdot y_o = gH$, in other words if *the left cosets $g'K$ and $g''H$, as subsets of G, are not disjoint* (they meet at g).

Given $y = g''H$, we see that x is incident to y if and only if $x = g''hK$ for some $h \in H$. Given $x = g'K$, the point y is incident to x if and only if $y = g'kH$ for some $k \in K$.

In the above example X, resp. Y, is the set of points, resp. hyperplanes, of $\mathbb{R}^n$ and G is the group of all isometries. But hyperplanes can also be viewed as subsets of

$X = \mathbb{R}^n$, and the incidence relation boils down to the familiar "the point x belongs to the hyperplane y" if and only if the chosen origin x_o belongs to the chosen origin y_o. Lemma 1 below extends this fact to Riemannian manifolds. More general incidence relations can be considered, however, and will be helpful in section 6.

Clearly, the group G acts transitively on the subset Z of $X \times Y$ consisting of all incident couples $(x, y) = (g \cdot x_o, g \cdot y_o)$, with $K \cap H$ as the isotropy subgroup of the origin $(x_o, y_o) \in Z$. Thus $Z = G/(K \cap H)$ can be endowed with a structure of manifold, and the present setting can be summarized by the following *double fibration of homogeneous spaces*

$$\begin{array}{ccc} Z = G/(K \cap H) \subset X \times Y & & \\ \downarrow \qquad\qquad \searrow & & \\ X = G/K \qquad Y = G/H \ , & & \end{array}$$

where the arrows denote the natural projections.

Radon transforms can be studied with more general double fibrations of manifolds X, Y, Z (without groups), as introduced by Gelfand et al. [4]. We refer to Guillemin and Sternberg [6] p.340, 370 for their basic properties; this theory has been developed in several papers by Boman, Quinto, and others.

2.2. Group-theoretic Radon transforms. Let G be a real Lie group and K a (closed) Lie subgroup, equipped with left-invariant Haar measures dg, dk respectively. If the homogeneous space G/K admits a G-invariant measure $d(gK)$, the measures can then be normalized so that

$$\int_G f(g)\, dg = \int_{G/K} d(gK) \int_K f(gk)\, dk \ ,$$

for any $f \in C_c(G)$. This applies in particular if K is compact (on invariant measures, see [9] chap. I §1).

Throughout the paper G will be a Lie group, K a *compact* subgroup, and H a (closed) Lie subgroup of G with left-invariant measure dh. The Haar measure dk of K will be normalized by $\int_K dk = 1$.

Let u be a (complex-valued) function on $X = G/K$. Its *Radon transform* is the function Ru on $Y = G/H$ defined by

$$Ru(gH) = \int_H u(ghK)\, dh \ ,$$

for $g \in G$, whenever this makes sense (e.g. if $u \in C_c(X)$). The left invariance of dh implies that the integral only depends on the left coset gH of g. Given $y = gH$ in $Y = G/H$, the value $Ru(y)$ is an integral of u over all x incident to y. A more precise statement can be given in the following important example.

Example. Let X be a connected Riemannian manifold, G a transitive Lie group of *isometries* of X and K the isotropy subgroup of some origin $x_o \in X$; then K is compact ([8] p.204) and $X = G/K$. Let y_o be a given *closed submanifold of* X, *containing* x_o, and let Y be the set of all submanifolds $y = g \cdot y_o$ of X, with $g \in G$.

The set H of all $h \in G$ such that $h \cdot y_o = y_o$ (i.e. the submanifold y_o is globally invariant under h) is a closed Lie subgroup of G. Indeed if $h_n \in H$ converges to h in G, for any $x \in y_o$ the point $\lim h_n \cdot x = h \cdot x$ belongs to y_o; similarly $h^{-1} \cdot x \in y_o$, so that $h \cdot y_o = y_o$. Thus $Y = G/H$ can be endowed with a structure of manifold and we obtain a double fibration of homogeneous spaces.

The following lemma allows computing the Radon transform without knowing H explicitly.

Lemma 1. *Keeping the notation of this example, assume furthermore that $y_o = G' \cdot x_o$ is a closed orbit of the origin $x_o = K$ under some Lie subgroup G' of G.*
Then $G' \subset H \subset G'K$ and $y_o = H \cdot x_o$. The incidence relation between $X = G/K$ and $Y = G/H$ is simply $x \in y$. Besides, the left-invariant Haar measures dh, dg' of the groups H, G' can be normalized so that

$$\begin{aligned} Ru(y) &= \int_H u(gh \cdot x_o)\, dh = \int_{G'} u(gg' \cdot x_o)\, dg' \\ &= \int_y u(x)\, dm_y(x) \ , \end{aligned}$$

where dm_y is the Riemannian measure induced by X on its submanifold $y = g \cdot y_o$.

The subgroup H can of course be strictly bigger than G'. This occurs for instance if y_o is a line in $X = \mathbb{R}^n$ and G' is the group of translations along this line, or a horocycle in a Riemannian symmetric space X of the noncompact type (for which $G' = N$ and $H = MN = NM$ in the usual semisimple notations).
Proof. If $y_o = G' \cdot x_o$, then H obviously contains G' and it follows that

$$y_o = G' \cdot x_o \subset H \cdot x_o \subset y_o \ ,$$

whence $H \cdot x_o = G' \cdot x_o$ and $H \subset G'K$.

A point $x \in X$ is incident to $y = g \cdot y_o \in Y$ if and only if there exists $h \in H$ such that $x = gh \cdot x_o$, i.e. $x \in gH \cdot x_o = g \cdot y_o = y$.

An isometry g transforms the Riemannian measure of y_o into the Riemannian measure of $y = g \cdot y_o$, and it suffices to prove the integral formula for $g = e$. Now $y_o = H \cdot x_o$ can be identified to the homogeneous space $H/(H \cap K)$, and dm_{y_o} (which is invariant under all isometries of y_o) to an H-invariant measure on this space. The Haar measure dh can therefore be normalized so that the corresponding measure on $H/(H \cap K)$ satisfies

$$\begin{aligned} \int_{y_o} u(x)\, dm_{y_o}(x) &= \int_{H/(H\cap K)} u(h \cdot x_o)\, d(h(H \cap K)) \\ &= \int_H u(h \cdot x_o)\, dh = Ru(y_o) \ . \end{aligned}$$

The proof is similar for $\int_{G'}$, whence the lemma. ∎

Going back to general double fibrations, the *Radon dual transform* of a (continuous, say) function v on $Y = G/H$ is the function on $X = G/K$ defined by

$$R^* v(gK) = \int_K v(gkH)\, dk \ ,$$

for $g \in G$, an integral of v over all y incident to $x = gK$. The word “dual” is of course motivated by the classical projective duality between points and hyperplanes in the basic example, but it stems from the following proposition too.

Proposition 2. *Let $X = G/K$ with K compact, and assume that $Y = G/H$ has a G-invariant measure. Let $u \in C_c(X)$, $v \in C(Y)$. Then $Ru \in C_c(Y)$, $R^*v \in C(X)$ and*

$$\int_X u(x)\, R^* v(x)\, dx = \int_Y Ru(y)\, v(y)\, dy = \int_Z u(x)\, v(y)\, dz$$

where dx, dy, dz are the respective G-invariant measures on X, Y and $Z = G/(K \cap H)$.

In the latter integral $u(x)v(y)$ is considered as a function of $z = (x, y)$ on Z (section 2.1). We omit the proof, a classical exercise on invariant integrals (cf. [9] p.144 and [11] p.41); all groups are assumed unimodular there, but the proof only uses the invariant measures on the homogeneous spaces, thus extends to the present situation.

Proposition 2 allows a natural extension of the transforms R and R^* to distributions. Given $u \in \mathcal{E}'(X)$, the distribution $Ru \in \mathcal{E}'(Y)$ is defined by

$$< Ru, v >=< u, R^* v >$$

for all test functions $v \in C^\infty(Y)$. Similarly, given $v \in \mathcal{D}'(Y)$, the distribution $R^*v \in \mathcal{D}'(X)$ is defined by

$$< R^*v, u >=< v, Ru >$$

for all $u \in \mathcal{D}(X)$. Again we refer to Helgason [11] p.42 for details, based on the compactness of K. These definitions do extend the Radon integrals for functions, as Proposition 2 shows, when identifying a function u with the distribution $u(x)\, dx$, and similarly for v.

3. Convolution on X and inversion of R

3.1. A convolution formula. Again G is a Lie group, K a *compact* subgroup, $X = G/K$ and $\tau(g)$ denotes the natural action of G on X, i.e. $\tau(g)x = g \cdot x$.

a. A general result. Let $S_1, S_2 \in \mathcal{D}'(X)$ be two distributions on X, with S_2 assumed K-invariant. By analogy with the group case (if K were the trivial subgroup), the convolution $S_1 * S_2 \in \mathcal{D}'(X)$ can be defined by

$$\begin{aligned} < \ & S_1 * S_2, \varphi >=< S_1(g_1K), < S_2(g_2K), \varphi(g_1g_2K) >> \\ = \ & < S_1(g_1K), < S_2, \varphi \circ \tau(g_1) >> \end{aligned} \tag{1}$$

for any $\varphi \in \mathcal{D}(X)$. Indeed, the K-invariance of S_2 implies that $< S_2, \varphi \circ \tau(g_1) >$ is a right K-invariant function of $g_1 \in G$, hence defines a function of $g_1K \in X$ to which S_1 can be applied (assuming that S_1 or S_2 has compact support). A more classical definition ([9] p.290) of $S_1 * S_2$ arises from the convolution on the group G itself, by means of the projection $G \to G/K$; it is easily checked that both definitions agree, but (1) will be more convenient here (and could be used even if K were not compact).

Proposition 3. *Let $X = G/K$ with K compact, and assume that $Y = G/H$ has a G-invariant measure. For any $u \in C_c(X)$ we have*

$$R^* Ru = u * S\ ,$$

a convolution on X. Here, denoting by δ the Dirac measure at the origin $x_o = K$ of X, the distribution $S = R^*R\delta$ is the K-invariant measure on X given by

$$< S, u >= R^* Ru(x_o) = \int_{K\times H} u(kh \cdot x_o)\, dk\, dh = Ru_K(y_o)\ ,$$

with $u_K(x) = \int_K u(k \cdot x)\, dk$ and $y_o = H$.

Proof. The definition of the Radon transforms R and R^* clearly show they intertwine the actions of G on X and Y (here denoted by $\tau_X(g)$, resp. $\tau_Y(g)$, for $g \in G$) :

$$R(u \circ \tau_X(g)) = (Ru) \circ \tau_Y(g)\ ,\ R^*(v \circ \tau_Y(g)) = (R^*v) \circ \tau_X(g)\ .$$

Therefore R^*R commutes with $\tau_X(g)$, hence is a right convolution operator. Indeed, let $\varphi \in \mathcal{D}(X)$ be a test function. The distribution S defined by $< S, \varphi >= R^*R\varphi(x_o)$ extends to a K-invariant positive linear form on $C_c(X)$, i.e. a measure, and

$$\begin{aligned} < \quad & u * S, \varphi > = < u(g \cdot x_o), < S, \varphi \circ \tau_X(g) >> \text{ by (1)} \\ = \quad & < u(g \cdot x_o), R^*R(\varphi \circ \tau_X(g))(x_o) > \\ = \quad & < u(g \cdot x_o), (R^*R\varphi)(g \cdot x_o) > \\ = \quad & < u, R^*R\varphi > = < R^*Ru, \varphi > \ . \end{aligned}$$

The last equality follows from the duality between R and R^* (Proposition 2). ∎

b. Totally geodesic transform on isotropic spaces. The following variant of Proposition 3 gives a more precise statement in a specific situation. Unifying and extending several results from the literature on totally geodesic Radon transforms on two-point homogeneous spaces (Helgason [9] p.104, 124 and 160, Berenstein and Casadio Tarabusi [1] p.618), it will lead to inversion formulas. Let $X = G/K$ be an *isotropic* connected non compact Riemannian manifold with distance d, where G is a transitive Lie group of isometries of X and K is the isotropy subgroup of some origin $x_o \in X$. Let y_o be a *totally geodesic* submanifold of X, containing x_o, and let Y be the set of all submanifolds $y = g \cdot y_o$ of X, with $g \in G$. We denote by $A(r)$, resp. $A_o(r)$, the Riemannian measure (area) of a sphere of radius r in X, resp. in y_o.

As explained in section 4.1.a below, Lemma 1 applies to this situation and the Radon transform can be written as

$$Ru(y) = \int_y u(x)\, dm_y(x) \ , \ u \in C_c(X) \ , \ y \in Y \ ,$$

where dm_y is the Riemannian measure induced by X on its submanifold y, and

$$R^*v(g \cdot x_o) = \int_K v(gk \cdot y_o)\, dk \ , \ v \in C(Y) \ , \ g \in G \ .$$

Note that we will not need here the group H nor an invariant measure on G/H, as opposed to Proposition 3.

Proposition 4. *With the above notation we have, for any $u \in C_c(X)$,*

$$R^*Ru = u * S$$

(convolution on X), where S is the K-invariant function on X defined by

$$S(x) = A_o(r)/A(r) \ , \ r = d(x_o, x) \ .$$

An explicit formula (5) for S will be given in section 4.1, after we introduce the relevant notations.

Proof. Fix $z = g \cdot x_o \in X$. The measure dm_y on $y = gk \cdot y_o$ corresponds to the measure dm_o on y_o by the isometry $x \mapsto gk \cdot x$, whence

$$R^*Ru(z) = \int_{y_o} \left(\int_K u(gk \cdot x)\, dk \right) dm_o(x) \ .$$

Now, X being isotropic, K-orbits are spheres centered at x_o. Since $\int_K dk = 1$, the above integral over K is the mean value $(M_r u)(z)$ of u over the sphere $\Sigma(z, r)$ with center z and radius $r = d(x_o, x)$. Therefore

$$\int_K u(gk \cdot x)\, dk = (M_r u)(z) = \frac{1}{A(r)} \int_{\Sigma(z,r)} u\, d\sigma \ ,$$

where $d\sigma$ is the Riemannian measure on $\Sigma(z,r)$, and

$$R^*Ru(z) = \int_{y_o} (M_r u)(z)\, dm_o(x) \ .$$

But, y_o being totally geodesic, the distance $r = d(x_o, x)$ between two points of y_o is the same in X and in y_o, and the latter integral can thus be computed in geodesic polar coordinates on y_o (with center x_o), as

$$\begin{aligned} R^*Ru(z) &= \int_0^\infty (M_r u)(z)\, A_o(r)\, dr \\ &= \int_0^\infty (M_r u)(z)\, A(r) f(r)\, dr \end{aligned}$$

with $f(r) = A_o(r)/A(r)$. This in turn can be viewed as an integral over X computed in polar coordinates (with center z), namely

$$R^*Ru(z) = \int_0^\infty f(r)\, dr \int_{\Sigma(z,r)} u\, d\sigma = \int_X u(x)\, f(d(z,x))\, dx \ .$$

Setting $z = g \cdot x_o$, $x = g' \cdot x_o$ it follows that, for any test function $\varphi \in \mathcal{D}(X)$,

$$\int_X R^*Ru(z)\, \varphi(z)\, dz = \int_{G\times G} u(g' \cdot x_o)\, f(d(g \cdot x_o, g' \cdot x_o))\, \varphi(g \cdot x_o)\, dg' dg \ .$$

Changing the variable g into $g = g'g''$ (with fixed g') in $\int dg$, we obtain from the left invariance of dg

$$\begin{aligned} \int_X R^*Ru(z)\, \varphi(z)\, dz &= \int_{G\times G} u(g' \cdot x_o)\, f(d(g'' \cdot x_o, x_o))\, \varphi(g'g'' \cdot x_o)\, dg' dg'' \\ &= < u * S, \varphi > \ , \end{aligned}$$

according to (1) and the definition of S in the proposition. ■

c. Horocycle transform on rank one spaces. Let $X = G/K$ be a Riemannian symmetric space of the noncompact type, $G = KAN$ an Iwasawa decomposition (cf. Notations, d) and $Y = G/MN$ the space of all horocycles in X. The corresponding dual Radon transforms are

$$Ru(gMN) = \int_N u(gnK)\, dn \ , \ R^*v(gK) = \int_K v(gkN)\, dk$$

for $u \in C_c(X)$, $v \in C(Y)$; MN has been replaced by N in the right-hand sides because K contains M..

We now specialize to rank one spaces, with positive roots α and (possibly) 2α. Let H be the basis vector of $\mathfrak{a}$ such that $\alpha(H) = 1$. Multiplying the Killing form scalar product on $\mathfrak{g}$ by a suitable factor, it will be convenient to assume that the corresponding norm on $\mathfrak{p}$ satisfies $\|H\| = 1$.

The exponential mapping $\exp : \mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha} \to N$ is a diffeomorphism onto, with jacobian 1 ; the Haar measure dn on N can therefore be chosen so that

$$\int_N f(n)\, dn = \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} f(\exp(Z+T))\, dZ dT \ ,$$

where dZ, resp. dT, is the Lebesgue measure on $\mathfrak{g}_\alpha$, resp. $\mathfrak{g}_{2\alpha}$, corresponding to the norm $\|.\|$.

Let $p = \dim \mathfrak{g}_\alpha$, $q = \dim \mathfrak{g}_{2\alpha}$, $\rho = (p/2) + q$, $n = p + q + 1 = \dim X$, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$. With the above normalizations we now have the following analogue of Proposition 4.

Proposition 5. *For the horocycle Radon transform on X, a rank one Riemannian symmetric space of the noncompact type, and $u \in C_c(X)$ we have*

$$R^* Ru = u * S \ ,$$

(convolution on X). Here S is the radial function on X given by

$$S(r) = 2^{(n-1)/2} \frac{\omega_{n-1}}{\omega_n} (\sinh r)^{-1} \, {}_2F_1 \left(\frac{\rho - 1}{2}, \frac{\rho}{2}; \frac{n-1}{2}; -\sinh^2 r \right) \ ,$$

with $r > 0$. For $X = H^n(\mathbb{R})$, i.e. $q = 0$, this reduces to

$$S(r) = 2^{(n-1)/2} \frac{\omega_{n-1}}{\omega_n} (\sinh r)^{-1} \left(\cosh \frac{r}{2} \right)^{3-n} \ .$$

Proof. We first assume $q = 0$.

The groups G and MN being unimodular, the space $Y = G/MN$ has a G-invariant measure ([11] p.100). By Proposition 3 it follows that $R^* Ru = u * S$, with

$$< S, u >= \int_N u(n \cdot x_o) \, dn = \int_{\mathfrak{g}_\alpha} u(\exp Z \cdot x_o) \, dZ$$

for any K-invariant function u on X (this will suffice to find the K-invariant function S).

By classical rank one computations ([8] p.414), the radial component $\exp(rH)$ of $\exp Z$ is given by

$$\exp Z \cdot x_o = k \exp(rH) \cdot x_o \ ,$$

with $k \in K$, $r \geq 0$ and $\|Z\| = 2\sqrt{2} \sinh(r/2)$. Using spherical coordinates in $\mathfrak{g}_\alpha = \mathbb{R}^{n-1}$ it follows that, for K-invariant u,

$$\begin{aligned} \int_N u(n \cdot x_o) \, dn &= \int_0^\infty u(\operatorname{Exp} rH) \, f(r) \, dr \ , \\ \text{with } f(r) &= 2^{(3/2)(n-1)-1} \omega_{n-1} \left(\sinh \frac{r}{2} \right)^{n-2} \cosh \frac{r}{2} \ . \end{aligned}$$

On the other hand, using the diffeomorphism Exp and spherical coordinates on $\mathfrak{p}$ we have

$$\int_X u(x) \, dx = \int_0^\infty u(\operatorname{Exp} rH) \, A(r) \, dr \ , \text{ with } A(r) = \omega_n (\sinh r)^{n-1}$$

(cf. section 4.1.b for more details). If $S(r) = f(r)/A(r)$ we thus have, for K-invariant u,

$$\int_N u(n \cdot x_o) \, dn = \int_0^\infty u(\operatorname{Exp} rH) \, S(r) \, A(r) \, dr = \int_X u(x) \, S(x) \, dx \ ,$$

as claimed.

The case $q \geq 1$ will not be used in the sequel; we sketch its proof, similar to the case $q = 0$. First

$$< S, u >= \int_N u(n \cdot x_o) \, dn = \int_{\mathfrak{g}_\alpha \times \mathfrak{g}_{2\alpha}} u(\exp(Z + T) \cdot x_o) \, dZ \, dT \ .$$

Then, by rank one computations ([8] p.414),

$$\begin{aligned} \exp(Z + T) \cdot x_o &= k \exp(rH) \cdot x_o \ , \ k \in K \ , \\ \cosh^2 r &= \left(1 + \frac{1}{4} \|Z\|^2 \right)^2 + \frac{1}{2} \|T\|^2 \ , \ r \geq 0 \ . \end{aligned}$$

Let $x = \|Z\|^2/4$, $y = \|T\|^2/2$. Using spherical coordinates in $\mathfrak{g}_\alpha = \mathbb{R}^p$ and $\mathfrak{g}_{2\alpha} = \mathbb{R}^q$ we obtain

$$\begin{aligned}\int_N u(n\cdot x_o)\,dn &= 2^{p-2+(q/2)}\omega_p\omega_q \int_0^\infty\int_0^\infty u(\exp(rH)\cdot x_o)\,x^{(p/2)-1}y^{(q/2)-1}dx\,dy \\ &= \int_0^\infty u(\exp(rH)\cdot x_o)\,f(r)\,dr\ .\end{aligned}$$

The latter expression follows from the change of variables $(x,r)\mapsto(x,y)$, with jacobian $\sinh 2r$; here

$$f(r) = 2^{p-2+(q/2)}\omega_p\omega_q \sinh 2r \int_0^{\cosh r-1} x^{(p/2)-1}\left(\cosh^2 r-(1+x)^2\right)^{(q/2)-1}dx\ .$$

Setting $x = t(\cosh r - 1)$ we find

$$\begin{aligned}f(r) &= 2^{(3p+q)/2}\omega_{n-1}(\sinh r)^{q-1}\left(\sinh\frac{r}{2}\right)^p \cosh r\times \\ &\times\frac{\Gamma((p+q)/2)}{\Gamma(p/2)\Gamma(q/2)}\int_0^1 t^{(p/2)-1}(1-t)^{(q/2)-1}\left(1+t\tanh^2\frac{r}{2}\right)^{(q/2)-1}dt \\ &= 2^{(3p+q)/2}\omega_{n-1}(\sinh r)^{q-1}\left(\sinh\frac{r}{2}\right)^p\cosh r\cdot {}_2F_1\left(\frac{p}{2},1-\frac{q}{2};\frac{p+q}{2};-\tanh^2\frac{r}{2}\right)\ ,\end{aligned}$$

by Euler's integral formula for the hypergeometric function. From a quadratic transformation formula for ${}_2F_1$ ([3] p.113, formula (35)) we finally obtain

$$f(r) = 2^{(n-1)/2}\omega_{n-1}(\sinh r)^{n-2}(\cosh r)^q\,{}_2F_1\left(\frac{\rho-1}{2},\frac{\rho}{2};\frac{n-1}{2};-\sinh^2 r\right)\ .$$

Thus, for K-invariant u,

$$\int_N u(n\cdot x_o)\,dn = \int_0^\infty u\left(\exp(rH)\cdot x_o\right)\,S(r)\,A(r)\,dr = \int_X u(x)\,S(x)\,dx\ ,$$

where $A(r) = \omega_n(\sinh r)^{n-1}(\cosh r)^q$ and $S(r) = f(r)/A(r)$. ∎

3.2. Radon inversion by convolution. Radon inversion formulas will follow from section 3.1 if we can solve for u the convolution equation $u * S = R^*Ru$, in the form

$$u = DR^*Ru\ . \tag{2}$$

To recover $u(x)$ from Ru the recipe will be to integrate $Ru(y)$ over all y incident to x, and to apply the operator D on the x variable.

As noted in the proof of Proposition 3, R^*R commutes with the action of G on X, and it is natural to look for a D with the same property, i.e. a convolution operator : if T is a distribution on X such that $S * T = \delta$, then

$$u = (R^*Ru) * T\ .$$

Though the question can be tackled by harmonic analysis on X (cf. section 5), a G-invariant linear differential operator D can sometimes be found directly, such that $DS = \delta$. Then (2) follows from the equality $u = u * DS = D(u * S)$. Indeed, for any test function φ,

$$\begin{aligned}< \ & D(u*S),\varphi> = < u*S, {}^tD\varphi> \\ = \ & < u(g\cdot x_o), < S, ({}^tD\varphi)\circ\tau(g)>> \text{ by (1)} \\ = \ & < u(g\cdot x_o), < S, {}^tD(\varphi\circ\tau(g))>>\ ,\end{aligned}$$

since the transpose operator ${}^tD$ is G-invariant too, as follows from the existence of a G-invariant measure on X. Finally,

$$\begin{aligned} < \quad & D(u * S), \varphi > = < u(g \cdot x_o), < DS, \varphi \circ \tau(g) >> \\ = \quad & < u * DS, \varphi > \quad , \end{aligned}$$

as claimed; assuming G unimodular (as in [9] p.291) is thus unnecessary here.

The method applies whenever we can *find a G-invariant differential operator D on X with given fundamental solution S.* We shall now investigate this question on the basis of Propositions 4 and 5.

4. Radon transforms on isotropic spaces

Throughout this section X will be an *isotropic* connected *noncompact* Riemannian manifold, that is a Euclidean space or a Riemannian globally symmetric space of rank one:

$$X = \mathbb{R}^n \text{ or } H^m(\mathbb{R}), H^{2m}(\mathbb{C}), H^{4m}(\mathbb{H}), H^{16}(\mathbb{O}) \ ,$$

where all superscripts denote the real dimension of these real, complex, quaternionic or Cayley hyperbolic spaces (cf. Wolf [18] §8.12). We first try to invert the d-geodesic Radon transform on X, defined by integrating over a family of d-dimensional totally geodesic submanifolds of X. At the end of this section we shall see that the same tools provide an inversion formula for the horocycle Radon transform on $H^{2k+1}(\mathbb{R})$.

4.1. Totally geodesic submanifolds. Our first goal is to describe these submanifolds and the corresponding functions S in Proposition 4.

a. Let $X = G/K$ be a *Riemannian symmetric space of the noncompact type (of arbitrary rank)*, where G is a connected semisimple Lie group and K a maximal compact subgroup (see Notations, c and d).

At the Lie algebra level, a totally geodesic submanifold of X is defined by a *Lie triple system*, i.e. a vector subspace $\mathfrak{s}$ of $\mathfrak{p}$ such that $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$. Then $\operatorname{Exp} \mathfrak{s}$ is totally geodesic in X and contains the origin x_o. Besides $\mathfrak{k}' = [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}$ and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{s}$ are Lie subalgebras of $\mathfrak{g}$. Let G' be the (closed) Lie subgroup with Lie algebra $\mathfrak{g}'$, and K' (with Lie algebra $\mathfrak{k}'$) be the isotropy subgroup of x_o in G'. Then

$$\operatorname{Exp} \mathfrak{s} = G'/K' = G' \cdot x_o \ ,$$

a closed symmetric subspace of X ([8] p.224-226, or [15] p. 234 sq.).

Now let Y be the set of all d-dimensional totally geodesic submanifolds $y = g \cdot y_o$ of X, with $g \in G$ and $y_o = \operatorname{Exp} \mathfrak{s} = G' \cdot x_o$. *Lemma 1 applies* : if H is the subgroup of all $h \in G$ such that $h \cdot y_o = y_o$, then $y_o = H \cdot x_o$, $Y = G/H$ and the incidence relation is $x \in y$.

It will be useful to note that the Lie algebra $\mathfrak{h}$ of H satisfies

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{s} \ , \ \mathfrak{h} \cap \mathfrak{k} \supset [\mathfrak{s}, \mathfrak{s}] \ , \ \mathfrak{h} \cap \mathfrak{p} = \mathfrak{s} \ . \tag{3}$$

Indeed the definition of H shows its invariance under the Cartan involution of G, whence the direct sum decomposition of $\mathfrak{h}$. Besides $\mathfrak{h}$ contains $\mathfrak{g}' = [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{s}$ by Lemma 1 and, for $V \in \mathfrak{h} \cap \mathfrak{p}$, the point $\exp V \cdot x_o = \operatorname{Exp} V$ belongs to $H \cdot x_o = \operatorname{Exp} \mathfrak{s}$, thus $V \in \mathfrak{s}$ by the injectivity of Exp on $\mathfrak{p}$.

By Lemma 1 the Radon transform of $u \in C_c(X)$ is given by

$$Ru(y) = \int_y u(x)\, dm_y(x) = \int_{\operatorname{Exp} \mathfrak{s}} u(g \cdot x)\, dm_{y_o}(x) \ ,$$

where dm_{y_o} is the Riemannian measure induced by X on its submanifold $y_o = \mathrm{Exp}\,\mathfrak{s}$.

b. Rank one case. We now restrict to the rank one case (hyperbolic spaces). Let $H \in \mathfrak{s}$ be a fixed non zero vector. The line $\mathfrak{a} = \mathbb{R}H$ is a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{s}$, and $\mathrm{Exp}\,\mathfrak{s}$ is again a symmetric space of rank one. The classical decomposition

$$\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$$

into eigenspaces of $(\mathrm{ad}\,H)^2$, with respective eigenvalues $0, (\alpha(H))^2, (2\alpha(H))^2$ (where α and 2α are the positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$), implies a similar decomposition of the invariant subspace $\mathfrak{s}$:

$$\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{s}_\alpha \oplus \mathfrak{s}_{2\alpha} \ ,$$

with $\mathfrak{s}_\alpha = \mathfrak{s} \cap \mathfrak{p}_\alpha$ and $\mathfrak{s}_{2\alpha} = \mathfrak{s} \cap \mathfrak{p}_{2\alpha}$. We set

$$\begin{aligned} p &= \dim \mathfrak{p}_\alpha \ , \ q = \dim \mathfrak{p}_{2\alpha} \ , \ n = \dim X = p + q + 1 \\ p' &= \dim \mathfrak{s}_\alpha \ , \ q' = \dim \mathfrak{s}_{2\alpha} \ , \ d = \dim \mathfrak{s} = p' + q' + 1 \ , \end{aligned}$$

with $q = q' = 0$ when 2α is not a root (case of real hyperbolic spaces).

Let us normalize the vector H by the condition $\alpha(H) = 1$. Multiplying if necessary the Riemannian metric of X by a constant factor, we may assume that the corresponding Euclidean norm on $\mathfrak{p}$ satisfies $\|H\| = 1$. Since Exp is a diffeomorphism of $\mathfrak{p}$ onto X, the integral of a function $u \in C_c(X)$ can be computed as

$$\int_X u(x)\,dx = \int_{\mathfrak{p}} u(\mathrm{Exp}\,Z) J(Z)\,dZ \ ,$$

where $J(Z) = \det_{\mathfrak{p}}(\sinh \mathrm{ad}\,Z / \mathrm{ad}\,Z)$ is the jacobian of Exp, a K-invariant function on $\mathfrak{p}$. If u is K-invariant on X, we simply write $u(r)$ for $u(\mathrm{Exp}\,Z) = u(\mathrm{Exp}\,rH)$ with $r = \|Z\|$ whence, computing with spherical coordinates on $\mathfrak{p}$,

$$\int_X u(x)\,dx = \int_0^\infty u(r) A(r)\,dr \ ,$$

where $A(r) = \omega_n r^{n-1} \det_{\mathfrak{p}}(\sinh \mathrm{ad}\,rH / \mathrm{ad}\,rH)$ is the area of the sphere with center x_o and radius r in X, and $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$. Taking account of the eigenvalues of $(\mathrm{ad}\,H)^2$ we obtain, with a parameter ε explained in the next remark,

$$A(r) = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon} \right)^p \left(\frac{\sinh 2\varepsilon r}{2\varepsilon} \right)^q = \omega_n \left(\frac{\sinh \varepsilon r}{\varepsilon} \right)^{n-1} (\cosh \varepsilon r)^q \ . \tag{4}$$

A similar expression gives $A_o(r)$ for the submanifold y_o (with d, p', q' instead of n, p, q). The distribution S in Proposition 4 is thus defined by the radial function

$$S(r) = A_o(r)/A(r) = (\omega_d/\omega_n) \left(\frac{\sinh \varepsilon r}{\varepsilon} \right)^{d-n} (\cosh \varepsilon r)^{q'-q} \ . \tag{5}$$

Remark. Here $\varepsilon = 1$ for spaces of the noncompact type, but (4) and (5) remain valid in the Euclidean case too, setting $\varepsilon = 0$ and $(\sinh \varepsilon r)/\varepsilon = r$: when $X = \mathbb{R}^n$ the geodesic submanifolds are the affine d-planes, $1 \le d \le n-1$, and

$$S(r) = (\omega_d/\omega_n)\, r^{d-n} \ .$$

The compact cases (projective spaces) might be dealt with similarly. One should then normalize H by $\alpha(H) = i$ and replace ε by i. Integrals with respect to r should run from 0 to the diameter ℓ of X, i.e. the first number $\ell > 0$ such that $A(\ell) = 0$.

4.2. An inversion formula. The G-invariant differential operators on an isotropic space X are the polynomials of its Laplace-Beltrami operator L ([9] p.288). In order to invert the d-geodesic Radon transform on X, section 3.2 suggests looking for a polynomial P such that the above distribution S is a fundamental solution of $P(L)$.

Motivated by (4) and (5), we introduce the family of radial functions $f_{a,b}$ on X defined by

$$f_{a,b}(r) = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^a (\cosh \varepsilon r)^b = \left(\frac{\sinh \varepsilon r}{\varepsilon}\right)^{a-b} \left(\frac{\sinh 2\varepsilon r}{2\varepsilon}\right)^b ,$$

where a and b are real constants and r is the distance from the origin x_o; in particular $f_{a,b}(r) = r^a$ for $\varepsilon = 0$. Thus

$$A(r) = \omega_n \, f_{n-1,q} \ , \ S(r) = (\omega_d/\omega_n) \ f_{d-n,q'-q}$$

with q, q', n and d as defined above.

Proposition 6. *Assume $\varepsilon = 0$ (Euclidean case), or $\varepsilon = 1$ and $b = 0$, or else $\varepsilon = 1$ and $b = 1 - q$ (hyperbolic cases). Then, for any integer $k \geq 1$, the function $f_{2k-n,b}$ defines a K-invariant distribution $F_{2k-n,b}$ on X such that*

$$P_k(L)F_{2k-n,b} = \omega_n 2^{k-1}(k-1)!(2-n)(4-n)\cdots(2k-n)\,\delta \ ,$$

where δ is the Dirac distribution at the origin x_o and P_k is the polynomial

$$P_k(x) = \prod_{j=1}^{k} \left(x + \varepsilon^2(n-2j-b)(2j+b+q-1)\right) \ .$$

The case $b = 0$, $n = 2k + 2$ was given by Schimming and Schlichtkrull [17] Theorem 6.1, as an example in their beautiful study of Hadamard's method and Helmholtz operators on harmonic manifolds.

Proof. By [9] p.313 the radial part of L is

$$\begin{aligned} \Delta &= \partial_r^2 + \frac{A'(r)}{A(r)}\partial_r = A(r)^{-1} \circ \partial_r \circ A(r) \circ \partial_r \\ &= \partial_r^2 + ((n-1)\varepsilon \coth \varepsilon r + q\varepsilon \tanh \varepsilon r)\,\partial_r \\ &= \partial_r^2 + (p\varepsilon \coth \varepsilon r + 2q\varepsilon \coth 2\varepsilon r)\,\partial_r \ . \end{aligned}$$

The proof of the proposition breaks up into a few easy facts. First we have for any $a, b \in \mathbb{R}$ the following equality of functions of $r > 0$

$$\begin{aligned} &\left(\Delta - \varepsilon^2(a+b)(a+n+b+q-1)\right) f_{a,b} \\ &\qquad = a(a+n-2)f_{a-2,b} - \varepsilon^2 b(b+q-1)f_{a,b-2} \ , \end{aligned} \tag{6}$$

which is immediate from $\Delta f = A^{-1}(Af')'$ and the identities

$$f'_{a,b} = af_{a-1,b+1} + \varepsilon^2 bf_{a+1,b-1} \ , \ f_{a,b} = f_{a,b-2} + \varepsilon^2 f_{a+2,b-2} \ .$$

Lemma 7. *For $a + n \geq 2$, $\varepsilon = 0$ or 1, the locally integrable function $f_{a,b}$ defines a K-invariant distribution $F_{a,b}$ on X such that*

$$\begin{aligned} &\left(L - \varepsilon^2(a+b)(a+n+b+q-1)\right) F_{a,b} \\ &= \begin{cases} a(a+n-2)F_{a-2,b} - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n > 2 \\ \omega_n a\,\delta - \varepsilon^2 b(b+q-1)F_{a,b-2} & \text{if } a+n = 2 \end{cases} \end{aligned}$$

(equality of distributions on X).

For example, taking $b=0$, resp. $b=1-q$, the lemma provides the following fundamental solutions (which coincide for $q=1$)

$$\begin{aligned}\left(L+\varepsilon^2(n-2)(q+1)\right)\left(\frac{\sinh\varepsilon r}{\varepsilon}\right)^{2-n} &= \omega_n(2-n)\delta\\ \left(L+2\varepsilon^2(n+q-3)\right)\left(\frac{\sinh\varepsilon r}{\varepsilon}\right)^{2-n}(\cosh\varepsilon r)^{1-q} &= \omega_n(2-n)\delta\ .\end{aligned}$$

In the flat case $\varepsilon=0$ they both reduce to $Lr^{2-n}=\omega_n(2-n)\delta$, a classical result for $\mathbb{R}^n$.

Proof. Due to the K-invariance of $f_{a,b}$ and L we need only consider K-invariant test functions $u\in\mathcal{D}(X)$. The integral

$$\int_X f_{a,b}\cdot u\,dx=\int_0^\infty f_{a,b}(r)u(r)A(r)\,dr=\omega_n\int_0^\infty f_{a+n-1,b+q}(r)u(r)\,dr\ ,$$

absolutely convergent if $a+n>0$, defines a distribution $F_{a,b}$ on X. In view of the symmetry and K-invariance of the Laplace operator we have

$$\begin{aligned}< & LF_{a,b},u>=<F_{a,b},Lu>\\ = & \int_0^\infty f_{a,b}(r)\Delta u(r)A(r)\,dr=\int_0^\infty f_{a,b}(Au')'\,dr\\ = & \left(Af'_{a,b}u\right)(0)-\left(Af_{a,b}u'\right)(0)+\int_0^\infty (Af'_{a,b})'u\,dr\ .\end{aligned}$$

If $a+n>2$ the function $Af_{a,b}$ vanishes at order $a+n-1$ at the origin, and $Af'_{a,b}$ at order $a+n-2$. Since $u(r)$ is smooth (this notation stands for $u(\operatorname{Exp} rH)$ with $\|H\|=1$), it follows that

$$<LF_{a,b},u>=\int_0^\infty \Delta f_{a,b}(r)u(r)A(r)\,dr\ ,$$

whence the result by (6).

The case $a+n=2$ is similar, in view of $(Af'_{a,b})(0)=\omega_n a$. ■

Proposition 6 now follows easily : letting

$$L_a=L-\varepsilon^2(a+b)(a+n+b+q-1)$$

we have, by Lemma 7,

$$L_aF_{a,b}=\begin{cases} a(a+n-2)\,F_{a-2,b} & \text{if } a+n>2\\ \omega_n a\,\delta & \text{if } a+n=2\end{cases}$$

whenever $\varepsilon^2 b(b+q-1)=0$. Since

$$P_k(L)=L_{2-n}L_{4-n}\cdots L_{2k-n}\ ,$$

the proposition follows by induction on k. ■

Theorem 8. *The d-geodesic Radon transform on a n-dimensional noncompact Riemannian isotropic space X can be inverted by means of a polynomial of its Laplace-Beltrami operator L, under the following assumptions (i) and (ii) :*

(i) d is even : $d=2k$ with $k\geq 1$

(ii) $X=\mathbb{R}^n$, or $\dim\mathfrak{s}_{2\alpha}=\dim\mathfrak{p}_{2\alpha}$, or else $\dim\mathfrak{s}_{2\alpha}=1$.

Then

$$Cu = P_k(L)R^*Ru$$

for any $u \in \mathcal{D}(X)$*, where* P_k *is the polynomial from Proposition 6 (with* $\varepsilon = 1$*,* $q = \dim \mathfrak{p}_{2\alpha}$ *and* $b + q = \dim \mathfrak{s}_{2\alpha}$ *if* X *is hyperbolic, or* $\varepsilon = 0$ *if* $X = \mathbb{R}^n$*) and*

$$C = \omega_d(-1)^k 2^{k-1}(k-1)!(n-2)(n-4)\cdots(n-2k)\ .$$

Proof. By (5) one has $S = (\omega_d/\omega_n)\, f_{a,b}$, with $a = d-n$ and $b = \dim \mathfrak{s}_{2\alpha} - \dim \mathfrak{p}_{2\alpha} = q' - q$ (section 4.1.b). The theorem follows from Proposition 6 and section 3.2. ■

Theorem 8 encompasses Helgason's Theorems 4.5 and 4.17 in [9] chapter I (with different normalizations from ours), as well as some generalizations (next section). See also Grinberg [5] for the case of projective spaces, where the polynomial P_k is related to representation theory.

4.3. Examples. According to assumption (*ii*), three types of totally geodesic Radon transforms can be inverted by Theorem 8. Putting aside the case of even-dimensional planes in the Euclidean space $X = \mathbb{R}^n$, we now describe some examples of the latter two.

The space $X = G/K$ is then one of the hyperbolic spaces, and the dual space Y consists of all geodesic submanifolds $g \cdot \operatorname{Exp} \mathfrak{s}$, $g \in G$, where $\mathfrak{s} \subset \mathfrak{p}$ is an even-dimensional Lie triple system. Let $\mathfrak{a} = \mathbb{R}H$ be any line in $\mathfrak{p}$, and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ be the corresponding root space decomposition.

a. A simple example is $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{p}_{2\alpha}$, assuming $\mathfrak{p}_{2\alpha} \neq 0$. Classical bracket relations (e.g. [8] p.335) imply that $\mathfrak{s}$ is a Lie triple system and, reading $\dim \mathfrak{p}_{2\alpha}$ from the classification of rank one spaces, $\dim \mathfrak{s}$ is 2, 4 or 8 ; here $\mathfrak{s}_\alpha = 0$ and $\mathfrak{s}_{2\alpha} = \mathfrak{p}_{2\alpha}$.

b. Another example is $\mathfrak{s} = \mathfrak{p}_\alpha$, assuming this space is even-dimensional. Bracket relations show $\mathfrak{s}$ is a Lie triple system. To obtain compatible root space decompositions of $\mathfrak{s}$ and $\mathfrak{p}$ we replace H by an $H' \in \mathfrak{s}$, whence the new root space decompositions with respect to $\mathfrak{a}' = \mathbb{R}H'$

$$\mathfrak{p} = \mathfrak{a}' \oplus \mathfrak{p}'_\alpha \oplus \mathfrak{p}'_{2\alpha}\ ,\ \mathfrak{s} = \mathfrak{a}' \oplus \mathfrak{s}'_\alpha \oplus \mathfrak{s}'_{2\alpha}\ .$$

It follows again from the classification that $\mathfrak{p}'_{2\alpha}$ and $\mathfrak{s}'_{2\alpha}$ have the same dimension in all cases, therefore coincide (Helgason [7] p.171, or [9] p.168). This example is motivated by the Radon transform on antipodal manifolds of compact symmetric spaces of rank one (loc. cit.).

c. Totally geodesic transform on classical hyperbolic spaces. Let $X = H^m(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, be one of the classical hyperbolic spaces. Then $X = G/K$ with $G = U(1,m;\mathbb{F})$, $K = U(1;\mathbb{F}) \times U(m;\mathbb{F})$, and the Cartan decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$, the space of all matrices

$$V = \begin{pmatrix} 0 & \overline{V_1} & \cdots & \overline{V_m} \\ V_1 & & & \\ \vdots & & (0) & \\ V_m & & & \end{pmatrix}\ ,\ V_i \in \mathbb{F}\ ,$$

can be identified with $\mathbb{F}^m$.

Let $\overline{V} \cdot W = \sum_{i=1}^m \overline{V_i} W_i$. For $U, V, W \in \mathfrak{p} = \mathbb{F}^m$, easy computations lead to

$$[U,[V,W]] = U\left(\overline{V} \cdot W - \overline{W} \cdot V\right) - V(\overline{W} \cdot U) + W(\overline{V} \cdot U) \tag{7}$$

($\mathbb{F}^m$ being considered as a $\mathbb{F}$-vector space, with scalars acting on the right). It follows that *any* $\mathbb{F}$*-subspace* $\mathfrak{s}$ *of* $\mathfrak{p}$ *is a Lie triple system.* Similarly, the natural inclusions $\mathbb{R}^m \subset$

$\mathbb{C}^m \subset \mathbb{H}^m$ show that any $\mathbb{R}$-subspace of $\mathfrak{p} \cap \mathbb{R}^m$, or any $\mathbb{C}$-subspace of $\mathfrak{p} \cap \mathbb{C}^m$, is a Lie triple system.

Let $H \neq 0$ be an element of $\mathfrak{p}$. The eigenspaces of $(\operatorname{ad} H)^2$ can be obtained from (7), whence the decomposition

$$\begin{aligned} \mathfrak{p} &= \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha} \ , \ \mathfrak{a} = \mathbb{R}H \ , \\ \mathfrak{p}_\alpha &= \{V \in \mathfrak{p} \mid \overline{H} \cdot V = 0\} \ , \ \mathfrak{p}_{2\alpha} = \{H\lambda \mid \lambda \in \mathbb{F}, \lambda + \overline{\lambda} = 0\} \ , \end{aligned}$$

with respective eigenvalues 0, $\overline{H} \cdot H$ and $4\left(\overline{H} \cdot H\right)$. A similar decomposition holds for $\mathfrak{s}$, if H is chosen in $\mathfrak{s}$. The spaces $\mathfrak{a} \oplus \mathfrak{p}_{2\alpha} = H\mathbb{F}$ and $\mathfrak{p}_\alpha$ are $\mathbb{F}$-subspaces of $\mathfrak{p}$, therefore Lie triple systems (as mentioned in a and b above). More generally, Theorem 8 applies to the following four families of totally geodesic submanifolds $\operatorname{Exp} \mathfrak{s}$; all superscripts in the table are real dimensions, with k, l, m strictly positive integers.

X	$\dim \mathfrak{p}_\alpha$	$\dim \mathfrak{p}_{2\alpha}$	$\mathfrak{s}$	$\dim \mathfrak{s}_\alpha$	$\dim \mathfrak{s}_{2\alpha}$	$y_o = \operatorname{Exp} \mathfrak{s}$
$H^m(\mathbb{R})$	$m-1$	0	(1)	$2k-1$	0	$H^{2k}(\mathbb{R})$
$H^{2m}(\mathbb{C})$	$2m-2$	1	(2)	$2k-2$	1	$H^{2k}(\mathbb{C})$
$H^{4m}(\mathbb{H})$	$4m-4$	3	(3)	$2k-2$	1	$H^{2k}(\mathbb{C})$
$H^{4m}(\mathbb{H})$	$4m-4$	3	(4)	$4l-4$	3	$H^{4l}(\mathbb{H})$

Case (1) : $\mathfrak{s}$ is any $\mathbb{R}$-subspace of $\mathfrak{p} = \mathbb{R}^m$, with $\dim_{\mathbb{R}} \mathfrak{s} = 2k \leq m$.
Case (2) : $\mathfrak{s}$ is any $\mathbb{C}$-subspace of $\mathfrak{p} = \mathbb{C}^m$, with $\dim_{\mathbb{C}} \mathfrak{s} = k \leq m$.
Case (3) : $\mathfrak{s}$ is any $\mathbb{C}$-subspace of $\mathbb{C}^m \subset \mathfrak{p} = \mathbb{H}^m$, with $\dim_{\mathbb{C}} \mathfrak{s} = k \leq m$.
Case (4) : $\mathfrak{s}$ is any $\mathbb{H}$-subspace of $\mathfrak{p} = \mathbb{H}^m$, with $\dim_{\mathbb{H}} \mathfrak{s} = l \leq m$.

d. Horocycle transform on real hyperbolic spaces. Proposition 6 also applies to this case, because of the similarity between the functions S obtained in Propositions 4 and 5.

Indeed, following the same steps as for geodesic submanifolds, one can find a polynomial of the Laplacian with fundamental solution S (case $q = 0$ in Proposition 5). Indeed $S(r)$ is now, up to a constant factor, $f_{-1,2-n}(r/2)$ in the notation of section 4.2 with $\varepsilon = 1$. Let

$$\Delta_{p,q} = \partial_r^2 + (p \coth r + 2q \coth 2r)\, \partial_r$$

be the radial part of the Laplacian and $g(r) = f(r/2)$. Then

$$4\left(\Delta_{p,0}\, g\right)(r) = \left(\Delta_{0,p}\, f\right)(r/2) \ ;$$

note that the roles of p and q have been interchanged. The next theorem now follows from Propositions 5 and 6, with $n = 2k+1$, $\varepsilon = 1$ and $b = 1 - p = 2 - n$.

Theorem 9. *(Helgason) The horocycle Radon transform on the odd-dimensional hyperbolic space $X = H^{2k+1}(\mathbb{R})$, $k \geq 1$, is inverted by*

$$Cu = Q_k(L) R^* R u \ ,$$

where $u \in \mathcal{D}(X)$, L is the Laplace-Beltrami operator of X,

$$C = \left(-\frac{\pi}{2}\right)^k \frac{(2k-1)!}{(k-1)!} \ , \ Q_k(x) = \prod_{j=1}^{k} \left(x + j(2k-j)\right) \ .$$

In [11] p.210, the normalization of the Riemannian metric on X differs from ours.

The result extends to the horocycle transform on a Riemannian symmetric space $X = G/K$ of the noncompact type, provided that the Lie algebra $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras (see Corollary 20 below). The spaces $H^{2k+1}(\mathbb{R})$ in Theorem 9 are the rank one spaces among those.

5. Harmonic analysis on X and inversion of R

As noted in section 3.2, an inversion formula for the Radon transform on $X = G/K$ follows from a convolution inverse T of the distribution S in Propositions 3 or 4: if $S * T = \delta$, then $u = (R^* Ru) * T$. Since S is K-invariant it is natural to search for a K-invariant T by means of harmonic analysis of radial functions on X, i.e. solving the equation

$$\widetilde{S}(\lambda)\,\widetilde{T}(\lambda) = 1$$

where $\widetilde{}$ denotes the spherical transform. We keep to the notations of section 4.1.b, dealing with the *d-geodesic transform on a n-dimensional hyperbolic space X.*

The spherical function φ_λ on X is the radial eigenfunction of the laplacian L defined by

$$L\varphi_\lambda = -\left(\lambda^2 + \rho^2\right)\varphi_\lambda \ , \ \varphi_\lambda(x_o) = 1 \ ,$$

where $\rho = (p/2) + q$ and λ is a real parameter. Writing down the radial part of L it follows that ([9] p.484)

$$\varphi_\lambda(r) = {}_2F_1\left(\frac{\rho + i\lambda}{2}, \frac{\rho - i\lambda}{2}; \frac{n}{2}; -\sinh^2 r\right) \tag{8}$$

where ${}_2F_1$ is the classical hypergeometric function. The spherical transform of a radial function S is then

$$\widetilde{S}(\lambda) = \int_X \varphi_\lambda(x) S(x)\, dx = \int_0^\infty \varphi_\lambda(r) S(r) A(r)\, dr \tag{9}$$

and, in view of the relevant expressions (4) and (5) of A and S (section 4.1 with $\varepsilon = 1$), we shall need the following lemma.

Lemma 10. *Let $a, b, \alpha, \beta, \gamma$ be complex numbers, with $0 < \operatorname{Re} a < \operatorname{Re}\gamma$, $\operatorname{Re}(a+b) < \operatorname{Re}\alpha$ and $\operatorname{Re}(a+b) < \operatorname{Re}\beta$. Then*

$$\int_0^\infty {}_2F_1(\alpha, \beta; \gamma; -s)\, s^{a-1}(1+s)^b ds$$
$$= \frac{\Gamma(\gamma)\Gamma(a)}{\Gamma(\gamma - a)} \frac{\Gamma(\alpha - a - b)\Gamma(\beta - a - b)}{\Gamma(\alpha - b)\Gamma(\beta - b)}\, {}_3F_2(a, -b, \alpha + \beta - \gamma - b; \alpha - b, \beta - b; 1) \ .$$

Here ${}_3F_2$ denotes the generalized hypergeometric series

$${}_3F_2(a, b, c; d, e; z) = \sum_{n \geq 0} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n} \frac{z^n}{n!} \ ,$$

with $(a)_o = 1$, $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$. The lemma follows from the change $s = t/(1-t)$, some classical identities for ${}_2F_1$ and term by term integration under $\int_0^1 (\cdots)\, dt$ of the power series expansion of ${}_2F_1$. We skip the details of the proof. Among various equivalent expressions which could be obtained similarly, the above one was chosen because of its obvious symmetry with respect to α and β.

Let $\mu = (1 - q' + \rho + i\lambda)/2$. Changing s into $-\sinh^2 r$ in Lemma 10 we obtain, in view of (4), (5), (8) and (9),

$$\widetilde{S}(\lambda) = \frac{\pi^{d/2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-d}{2}\right)} \left|\frac{\Gamma\left(\mu - \frac{d}{2}\right)}{\Gamma(\mu)}\right|^2 \cdot {}_3F_2\left(\frac{d}{2}, \frac{1-q'}{2}, \frac{q-q'}{2}; \mu, \overline{\mu}; 1\right) \ , \lambda \in \mathbb{R} \ , \tag{10}$$

assuming $\lambda \in \mathbb{R}$, $0 < d < n$ and $d < 1 - q' + \rho$; recall that $q = \dim \mathfrak{p}_{2\alpha}$, $q' = \dim \mathfrak{s}_{2\alpha}$.

Finding T such that $\widetilde{S}(\lambda)\widetilde{T}(\lambda) = 1$ seems intractable however, unless the ${}_3F_2$ factor is trivial. We are thus led to assume from now on (as in section 4.2)

$$q' = q \text{ or } q' = 1 \ , \tag{11}$$

so that ${}_3F_2(\cdots) = 1$. The conditions on d then reduce to $0 < d < 1 - q' + \rho$, ensuring the convergence of the integral $\widetilde{S}(\lambda)$.

If $d = 2k$ is an *even* integer, then $\widetilde{S}(\lambda)$ is the reciprocal of the polynomial

$$\frac{1}{\widetilde{S}(\lambda)} = C \prod_{j=1}^{k} \left(-\lambda^2 - \rho^2 + (n - 2j - q' + q)(2j + q' - 1)\right) \ ,$$

where C is a constant factor, in agreement with Theorem 8 (with $-\lambda^2 - \rho^2$ corresponding to L under the spherical transform).

If $d = p' + q' + 1$ is *odd*, then $p' = \dim \mathfrak{s}_\alpha$ and $q' = \dim \mathfrak{s}_{2\alpha}$ must have the same parity which, according to the classification of rank one symmetric spaces, can only occur for $q' = 0$ and p' even. Condition (11) now requires $q = 0$, and X must be a real hyperbolic space $H^n(\mathbb{R})$. This is the case studied by Berenstein and Tarabusi [1]; see also [14] p.101 for $X = H^2(\mathbb{R})$ and $d = 1$. To find the convolution inverse T the strategy is to consider

$$f_{a,b}(r) = (\sinh r)^a \, (\cosh r)^b \ ,$$

where a, b are chosen so that $\widetilde{f_{a,b}}(\lambda)$ has (by Lemma 10 again) an expression similar to (10) with trivial hypergeometric factor, and so that cross simplifications occur between the $|\Gamma(\cdots)|^2$ factors in the product $\widetilde{S}(\lambda)\,\widetilde{f_{a,b}}(\lambda)$. This product is then the reciprocal of a polynomial in λ^2 (as in the case d even), and the corresponding inversion formula is

$$u = P(L)\left((R^* R u) * f_{a,b}\right) \ ,$$

where P is a polynomial. We refer to [1] for details.

Unfortunately the method of spherical transforms sketched above seems to provide explicit inversion formulas for the d-geodesic Radon transform on X only when $q' = q$ or $q' = 1$ on the one hand (to get rid of ${}_3F_2$) *and* d even or $X = H^n(\mathbb{R})$ on the other hand. The only reachable results so far are thus the formulas already obtained in [1] for $H^n(\mathbb{R})$ and a new proof of the above Theorem 8. The method might however yield some new results in the wider class of Damek-Ricci spaces (or harmonic NA groups), where the dimension q can be an arbitrary integer.

6. Shifted Radon transforms, waves, and the amusing formula

On page 146 of [10], S. Helgason notes the "amusing formula"

$$LR^* Ru(x) = -\left.\frac{\partial}{\partial \tau} R^*_{t(\tau)} Ru(x)\right|_{\tau=1} \tag{12}$$

for the 2-geodesic Radon transform R on the hyperbolic space $X = H^3(\mathbb{R})$, where L is the Laplace-Beltrami operator of X and $x \in X$. In this formula, R^*_t is the generalized dual transform obtained by integrating over all 2-dimensional totally geodesic submanifolds *at distance* t from a point x, and $t = t(\tau)$ denotes the positive solution of the equation $\cosh t = 1/\tau$. In [10], or [11] p. 55, equation (12) is indirectly obtained by bringing together two different inversion formulas for R.

In this section we study general shifted transforms, a concept going back to Radon himself [16] for the line transform in $\mathbb{R}^2$, and we use them to derive inversion formulas.

They also provide solutions of wave-type equations; formula (12) can actually be seen as a wave equation at time $t = 0$. We shall give a direct proof of some generalized "amusing formulas", thus solving wave equations (called multitemporal when the time variable is multidimensional), and we use them to relate two different types of Radon inversion formulas (with or without shifts).

6.1. Shifts. As before, let $X = G/K$ and $Y = G/H$ be two homogeneous spaces, with K compact, and

$$Ru(g \cdot y_o) = \int_H u(gh \cdot x_o)\, dh$$

be the corresponding Radon transform of $u \in C_c(X)$.

Let $t \in G$ be a "*shift*", fixed at first. Replacing the origin $y_o = H$ in Y by the shifted origin $y_t = t \cdot y_o$, with stabilizer subgroup $H_t = tHt^{-1} \subset G$, we obtain the new identification $Y = G/H_t$, and a new incidence relation between X and Y. A point $x = g \cdot x_o \in X$ is now incident to $y \in Y$ if and only if there exists $\gamma \in G$ such that

$$x = \gamma \cdot x_o \text{ and } y = \gamma \cdot y_t = \gamma t \cdot y_o \ ,$$

i.e.

$$y = gkt \cdot y_o$$

for some $k \in K$. The corresponding *shifted dual transform* of $v \in C(Y)$ is

$$R_t^* v(g \cdot x_o) = \int_K v(gkt \cdot y_o)\, dk \ .$$

Remark. We now have two double fibrations

$$\begin{array}{ccccccc} & Z = G/(K \cap H) & & & & Z_t = G/(K \cap H_t) & \\ & \downarrow & \searrow & & & \downarrow & \searrow \\ & X = G/K & & Y = G/H \ , & & X = G/K & & Y = G/H_t \ , \end{array}$$

and we are dealing with the Radon transform R given by the first and the dual transform R_t^* given by the second. The transform R_t associated with the second diagram is

$$R_t u(g \cdot y_o) = \int_H u(ght^{-1} \cdot x_o)\, dh \ ;$$

but, excepting the proof of Proposition 12, it will not be used in the sequel.

Lemma 11. *Let $u \in C_c(X)$ and $g, t \in G$. Then*

$$(R_t^* Ru)(g \cdot x_o) = (Ru_g)(t \cdot y_o) \ ,$$

where u_g is the K-invariant function on X defined by

$$u_g(x) = \int_K u(gk \cdot x)\, dk \ .$$

Proof. Immediate, since

$$(R_t^* Ru)(g \cdot x_o) = \int_{K \times H} u(gkth \cdot x_o)\, dk\, dh = \int_H u_g(th \cdot x_o)\, dh \ .$$

■

Before proceeding we mention the following extension of Proposition 3 to shifted transforms. This result will not be used in the sequel.

Proposition 12. *Let G and H be unimodular, K compact, $X = G/K$ and $Y = G/H$. For any $u \in C_c(X)$ and $t \in G$ we have*

$$R_t^* Ru = u * S_t$$

(convolution on X). Here S_t is the K-invariant distribution on X defined by $S = R_t^* R\delta$, and δ is the Dirac distribution at the origin $x_o = K$ of X, i.e.

$$< S_t, u >= R^* R_t u(x_o) = \int_{K\times H} u(kht^{-1} \cdot x_o)\, dk\, dh\ .$$

Proof. The proof of Proposition 3 can be repeated here, with $R^* R_t$ as the dual of $R_t^* R$. The claim can also be checked directly, writing, for $\varphi \in \mathcal{D}(X)$,

$$< R_t^* Ru, \varphi >= \int_{G\times H} u(gth \cdot x_o)\varphi(g \cdot x_o)\, dg\, dh\ ,$$

and changing variables into $h' = h^{-1}$, $g' = gth$; the result follows easily, G and H being unimodular groups. Details are left to the reader. ∎

6.2. Radon inversion by shifts. The elementary Lemma 11 can be used in the following way. Assume the transform R can be inverted at the origin for K-invariant functions on X, say

$$u(x_o) =< T_{(y)}, Ru(y) > \ , \tag{13}$$

where T is some linear form on a space of functions on Y. Then, replacing u by the K-invariant function u_g in the lemma, we obtain

$$u(g \cdot x_o) = u_g(x_o) =< T, Ru_g > \ .$$

The roles of g and t can now be interchanged by Lemma 11, whence

$$u(x) =< T_{(t)}, R_t^* Ru(x) > \tag{14}$$

for arbitrary $u \in \mathcal{D}(X)$ and $x \in X$. The notation $T_{(t)}$ means that T now acts on the shift variable t, or $t \cdot y_o$ to be precise. Since $R_{kth}^* Ru(x) = R_t^* Ru(x)$ for $k \in K$ and $h \in H$, this variable may actually be taken in $K\backslash G/H$.

The general inversion formula (14) for R thus follows from the special case (13) of K-invariant functions at the origin, thanks to the shifted dual transform.

If X is an isotropic space, the above trick (replace u by u_g) simply means replacing $u(x)$ by its mean value over the sphere with center $g \cdot x_o$ and radius $d(x_o, x)$.

6.3. Examples. a. Horocycle transform. We first consider the horocycle Radon transform on $X = G/K$, a Riemannian symmetric space of the noncompact type. Using the classical semisimple notations related to an Iwasawa decomposition $G = KAN$ (see Notations, d), we take the point $x_o = K$, resp. the horocycle $y_o = N \cdot x_o$, as the origin in X, resp. in $Y = G/MN$. Then

$$Ru(g \cdot y_o) = \int_N u(gn \cdot x_o)\, dn$$

(integrating over M is unnecessary here) and the dual transform shifted by $a \in A$ is

$$R_a^* v(g \cdot x_o) = \int_K v(gka \cdot y_o)\, dk\ .$$

For K-invariant u the decomposition $g = kan$ gives

$$Ru(g \cdot y_o) = Ru(a \cdot y_o) = \int_N u(an \cdot x_o)\, dn = a^{-\rho} \mathcal{A}u(a) \ ;$$

the *Abel transform* $\mathcal{A}$ is defined by this equality.

For K-invariant $u \in \mathcal{D}(X)$ we have $\mathcal{A}u \in \mathcal{D}(A)$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. It is known from spherical harmonic analysis on X that the classical Fourier transform

$$\widehat{\mathcal{A}u}(\lambda) = \int_A a^{-i\lambda} \mathcal{A}u(a)\, da \ , \ \lambda \in \mathfrak{a}^* \ ,$$

coincides with the spherical transform of u, with the inversion formula ([9] p.454)

$$u(x_o) = C \int_{\mathfrak{a}^*} \widehat{\mathcal{A}u}(\lambda)\, |c(\lambda)|^{-2}\, d\lambda \ , \tag{15}$$

where C is a positive constant and $c(\lambda)$ is Harish-Chandra's function. Since $C \cdot |c(\lambda)|^{-2}$ has polynomial growth on $\mathfrak{a}^*$ its Fourier transform is a tempered distribution T on $A = \exp \mathfrak{a}$ such that

$$u(x_o) = < T, \mathcal{A}u > = < T_{(a)}, a^\rho Ru(a \cdot y_o) > \ .$$

Thus T inverts the Abel transform at the origin. By (14) we obtain the next theorem.

Theorem 13. *Let X be a Riemannian symmetric space of the noncompact type. Its horocycle Radon transform R can be inverted by*

$$u(x) = < T_{(a)}, a^\rho R_a^* Ru(x) > \ , \ x \in X \ ,$$

for $u \in \mathcal{D}(X)$. The distribution $T_{(a)}$ (acting on the variable $a \in A$) is, up to a constant factor, the Fourier transform of $|c(\lambda)|^{-2}$.

Remarks. (i)This extends a result by Berenstein and Tarabusi [2] for $X = H^n(\mathbb{R})$, obtained by direct calculations.
(ii) Helgason's original inversion formula ([11] p.116)

$$u(x) = R^* \Lambda \overline{\Lambda} Ru(x)$$

follows easily from Theorem 13. Indeed Helgason's operator $\Lambda\overline{\Lambda}$ is defined as follows ([11] p.111). Given $v \in \mathcal{D}(Y)$ and $g = kan \in G$, multiply $v(g \cdot y_o) = v(ka \cdot y_o)$ by a^ρ, take the Fourier transform with respect to $a \in A$, multiply it by $C \cdot |c(\lambda)|^{-2}$ (an even function of λ), take the inverse Fourier transform, and multiply by $a^{-\rho}$; the result is $\Lambda\overline{\Lambda} v(g \cdot y_o)$. In other words

$$\Lambda\overline{\Lambda} v(g \cdot y_o) = \Lambda\overline{\Lambda} v(ka \cdot y_o) = a^{-\rho} \left(T * (a^\rho v)\right)(ka \cdot y_o) \ ,$$

where $*$ is the convolution on A with respect to a. Let b denote a variable in A; since T is even we have

$$\begin{aligned} \Lambda\overline{\Lambda} v(g \cdot y_o) &= a^{-\rho} < T_{(b)}, (ab)^\rho v(kab \cdot y_o) > \\ &= < T_{(b)}, b^\rho v(kab \cdot y_o) > = < T_{(b)}, b^\rho v(gb \cdot y_o) > \ . \end{aligned}$$

Replacing v by Ru, g by gk and integrating with respect to $k \in K$ we obtain

$$\begin{aligned} R^* \Lambda\overline{\Lambda} Ru(g \cdot x_o) &= \int_K < T_{(b)}, b^\rho Ru(gkb \cdot y_o) > dk \\ &= < T_{(b)}, b^\rho \int_K Ru(gkb \cdot y_o)\, dk > = < T_{(b)}, b^\rho R_b^* Ru(g \cdot x_o) > \ . \end{aligned}$$

By Theorem 13 this is $u(g \cdot x_o)$, as claimed.
(iii) Note that T is supported at the origin if and only if $|c(\lambda)|^{-2}$ is a polynomial, i.e. if the Lie algebra $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras (see Corollary 20 below).

b. Totally geodesic transform on classical hyperbolic spaces. We retain the notation of section 4.3.c.

Theorem 14. *Let $X = H^m(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, be one of the classical hyperbolic spaces, let $\mathfrak{s}$ be any $\mathbb{F}$-vector subspace of $\mathfrak{p} = \mathbb{F}^m$, and T any unit vector orthogonal to $\mathfrak{s}$ in $\mathfrak{p}$.*

For the Radon transform defined by the totally geodesic submanifolds $y = g \cdot \operatorname{Exp} \mathfrak{s}$, of (real) dimension d, we have the following inversion formulas by means of shifted dual transforms, for $u \in \mathcal{D}(X)$ and $x \in X$,
(i) *If $d = 2k+1$ is odd, $k \geq 0$,*

$$2^k \pi^{k+1} u(x) = \left(\sigma^{-1}\partial_\sigma\right)^{k+1} \int_0^\sigma \left(R^*_{\exp t(\tau)T} Ru(x)\right) (\sigma^2 - \tau^2)^{-1/2} d\tau \bigg|_{\sigma=1} ,$$

where $t(\tau)$ denotes the positive solution of the equation $\cosh t = 1/\tau$.
(ii) *If $d = 2k$ is even, $k \geq 1$, there exists a polynomial of degree k*

$$Q_k(\lambda) = \frac{2^k k!}{(2k)!}\lambda^k + \cdots + (q'+1)(q'+3)\cdots(q'+2k-1) ,$$

with rational coefficients (depending on k and $q' = \dim \mathfrak{s}_{2\alpha}$), such that

$$(-2\pi)^k u(x) = Q_k(\partial_t^2)\left(R^*_{\exp tT} Ru(x)\right)_{t=0} .$$

Remarks. This extends a result proved by Helgason ([10] p.144, or [14] p.97) for $\mathbb{F} = \mathbb{R}$. In case (i), a look at the proof below shows that an arbitrary positive integer ℓ may be added to the exponents of $\sigma^{-1}\partial_\sigma$ and $\sigma^2 - \tau^2$; Helgason's result is obtained for $\ell = k$. From the proof of case (ii) we obtain for $k = 1, 2$

$$\begin{aligned} Q_1(\partial_t^2) &= \partial_t^2 + q' + 1 \\ Q_2(\partial_t^2) &= \frac{1}{3}\partial_t^4 + \left(2q' + \frac{14}{3}\right)\partial_t^2 + (q'+1)(q'+3) . \end{aligned}$$

Our d is of course even whenever $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. A comparison with section 4.3.c shows that (except for $\mathbb{F} = \mathbb{R}$) the present assumption on $\mathfrak{s}$ is stronger than in Theorem 8.
Proof of Theorem14. In order to use spherical coordinates on totally geodesic submanifolds of X, we need a lemma. As in section 4.3.c, the matrices in $\mathfrak{p}$ can be identified to vectors $V = (V_1, \ldots, V_m) \in \mathbb{F}^m$, and the scalar product of $T, V \in \mathfrak{p}$ is

$$(T, V) = \operatorname{Re}\left(\overline{T} \cdot V\right) \text{ , with } \overline{T} \cdot V = \sum_{i=1}^m \overline{T_i} V_i .$$

Let $\| \ \|$ be the corresponding norm.

Lemma 15. *Let $X = H^m(\mathbb{F})$ be a classical hyperbolic space.*
(i) *Let $T, V \in \mathfrak{p}$. In the geodesic triangle with vertices x_o (the origin of X), $\operatorname{Exp} T$ and $\exp T \cdot \operatorname{Exp} V$, the Riemannian lengths of the sides are $t = \|T\|$, $r = \|V\|$ and w given by*

$$\cosh^2 w = \left(\cosh t \cosh r + \frac{\sinh t}{t}\frac{\sinh r}{r}(T, V)\right)^2 + \left(\frac{\sinh t}{t}\frac{\sinh r}{r}|\overline{T} \cdot V - (T, V)|\right)^2 .$$

(ii) *Let $\mathfrak{s} \subset \mathfrak{p}$ be a Lie triple system. If $T \in \mathfrak{p}$ is orthogonal to $\mathfrak{s}$, the totally geodesic submanifold $\exp T \cdot \operatorname{Exp} \mathfrak{s}$ is at distance $t = \|T\|$ from the origin.*

Proof. (i) The Riemannian distance from x_o to $\operatorname{Exp} T$ is $\|T\| = t$. Transforming x_o and $\operatorname{Exp} V$ by the isometry $\exp T \in G$ shows that the second side of the triangle has length r. The third side is $w = \|W\|$, where W is the unique $W \in \mathfrak{p}$ such that $\operatorname{Exp} W = \exp T \cdot \operatorname{Exp} V$, in other words

$$\exp W = (\exp T \exp V)\, k$$

for some $k \in K$. The map $g \mapsto g\theta(g)^{-1}$, where θ is the Cartan involution of G, transforms this equality into

$$\exp 2W = \exp T \exp 2V \exp T\ .$$

By elementary matrix computations $T^3 = t^2 T$, and the exponential is

$$\exp T = I + \frac{\sinh t}{t} T + \frac{\cosh t - 1}{t^2} T^2\ ,$$

where I is the unit matrix. Now $\operatorname{tr} T = 0$ and $\operatorname{tr} T^2 = 2t^2$ is real, so that taking the traces we obtain

$$\operatorname{tr}(\exp 2W) = \operatorname{Re}\operatorname{tr}(\exp 2W) = \operatorname{Re}\operatorname{tr}(\exp 2T \exp 2V)\ ;$$

indeed $\operatorname{Re}\operatorname{tr}(gg') = \operatorname{Re}\operatorname{tr}(g'g)$ for $g, g' \in G$, even when $\mathbb{F} = \mathbb{H}$. Taking account of

$$\begin{aligned} \operatorname{Re}\operatorname{tr} TV &= 2(T,V)\ ,\ \operatorname{tr} T^2V = \operatorname{tr} TV^2 = 0\ , \\ \operatorname{Re}\operatorname{tr} T^2V^2 &= t^2r^2 + |\overline{T}\cdot V|^2\ , \end{aligned}$$

the expression of $\cosh w$ follows after some elementary calculations.
(ii) Let $y = \exp T \cdot \operatorname{Exp}\mathfrak{s}$. By (i) with $V \in \mathfrak{s}$ and $(T,V) = 0$, the distance w of the origin to the point $\operatorname{Exp} W = \exp T \cdot \operatorname{Exp} V$ of y is given by

$$\cosh^2 w = (\cosh t \cosh r)^2 + \left(\frac{\sinh t}{t}\frac{\sinh r}{r}|\overline{T}\cdot V|\right)^2\ .$$

Therefore $w \geq t$, with equality if and only if $V = 0$, and $\operatorname{Exp} T$ is the unique point of y closest to x_o (geodesic orthogonal projection of the origin on y). ■

Going back to Theorem 14, let $g \in G$ and let $y = g\cdot\operatorname{Exp}\mathfrak{s}$ be an arbitrary given totally geodesic submanifold, element of Y. The minimum distance between y and the origin x_o is obtained at a point $\operatorname{Exp} T \in y$, with $T \in \mathfrak{p}$. In particular there exists $V \in \mathfrak{s}$ such that $\operatorname{Exp} T = g\cdot\operatorname{Exp} V$, i.e. $(\exp T)k = g \exp V$ for some $k \in K$. But $\operatorname{Exp}\mathfrak{s}$ is globally invariant under the action of $\exp V$, so that $y = (\exp T)k \cdot \operatorname{Exp}\mathfrak{s} = \exp T \cdot \operatorname{Exp}(k\cdot\mathfrak{s})$. Changing notation, we may write $\mathfrak{s}$ for $k\cdot\mathfrak{s}$ and $y = \exp T \cdot \operatorname{Exp}\mathfrak{s}$.

Let $V \in \mathfrak{s}$. On the geodesic $\exp T \cdot \operatorname{Exp} sV$, $s \in \mathbb{R}$, contained in y, the minimum distance to x_o is obtained for $s = 0$. By Lemma 15 (i) with sV instead of V, this implies $(T,V) = 0$ so that T is orthogonal to $\mathfrak{s}$ and Lemma 15 (ii) applies.

Besides, if we assume $\mathfrak{s}$ is a $\mathbb{F}$-vector subspace of $\mathfrak{p}$ therefore a Lie triple system (section 4.3.c), the vector T must be orthogonal to all $V\lambda$, $V \in \mathfrak{s}$, $\lambda \in \mathbb{F}$, whence $\overline{T}\cdot V = 0$. By Lemma 15 the distance $w = w(t,r)$ between x_o and an arbitrary point $x = \exp T \cdot \operatorname{Exp} V$ of y is simply given by

$$\cosh w(t,r) = \cosh t \cosh r\ ,\ t = \|T\|\ ,\ r = \|V\|\ , \tag{16}$$

the same expression as for real hyperbolic spaces.

According to (13) and (14) we only need to invert R at the origin for a K-invariant function u. As shown in section 4.1.a, Lemma 1 applies and $Ru(y) = \int_y u(x)\, dm_y(x)$.

When u is radial the integral can be obtained in spherical coordinates on y with origin $\operatorname{Exp} T$, as

$$Ru(y) = \int_0^\infty u(w(t,r))A_o(r)\,dr \tag{17}$$

where $A_o(r) = \omega_d(\sinh r)^{d-1}(\cosh r)^{q'}$ is the area of spheres of radius r in y. By (16) and (17) Ru may be viewed as a smooth even function $Ru(t)$ of $t \in \mathbb{R}$.

The end of the proof is now similar to the case of $H^n(\mathbb{R})$, as given in [11] p.53 or [14] p.97. Let $\tau = (\cosh t)^{-1}$, and let $t = t(\tau) \geq 0$ denote the inverse function. Introducing the functions

$$\varphi(\tau) = \tau^{-d-q'}u(t(\tau))\ ,\ \psi(\tau) = \tau^{-1-q'}Ru(t(\tau))\ ,$$

which are C^∞ on $]0,1]$, (17) becomes

$$\psi(\tau) = \omega_d \int_0^\tau \varphi(\rho)\left(\tau^2-\rho^2\right)^{(d/2)-1}d\rho\ .$$

Proof of (i). The Abel type integral equation (18) can be inverted as usual : it implies that, for any $a > 0$, $\sigma > 0$,

$$\Gamma\left(\frac{d}{2}+a\right)\int_0^\sigma \psi(\tau)(\sigma^2-\tau^2)^{a-1}\tau d\tau = \pi^{d/2}\Gamma(a)\int_0^\sigma \varphi(\rho)\left(\sigma^2-\rho^2\right)^{(d/2)+a-1}d\rho$$

and, choosing $a > 0$ such that $N = (d/2)+a$ is a strictly positive integer, it follows easily that

$$2^{N-1}\pi^{d/2}\Gamma(a)\varphi(\sigma) = \sigma\left(\sigma^{-1}\partial_\sigma\right)^N\left(\int_0^\sigma \psi(\tau)(\sigma^2-\tau^2)^{a-1}\tau d\tau\right)\ .$$

If $d = 2k+1$ is odd, $k \geq 0$, the smallest such a is $1/2$ so that $N = k+1$ and

$$2^k\pi^{k+1}\varphi(\sigma) = \sigma\left(\sigma^{-1}\partial_\sigma\right)^{k+1}\left(\int_0^\sigma \psi(\tau)(\sigma^2-\tau^2)^{-1/2}\tau d\tau\right)\ ,\ \sigma > 0\ ;$$

the derivatives cannot be taken here under the integral. Besides d can only be odd for $\mathbb{F} = \mathbb{R}$ according to the assumption on $\mathfrak{s}$, and $q' = 0$ in that case. Going back to u and Ru we thus obtain for $\sigma = 1$

$$2^k\pi^{k+1}u(x_o) = \left(\sigma^{-1}\partial_\sigma\right)^{k+1}\int_0^\sigma Ru(t(\tau))(\sigma^2-\tau^2)^{-1/2}d\tau\Big|_{\sigma=1}\ ,$$

for any K-invariant $u \in \mathcal{D}(X)$. The claim follows by section 6.2.

Proof of (ii). If $d = 2k$ is even, $k \geq 1$, the integral equation (18) can be directly solved as

$$(2\pi)^k\varphi(\tau) = \tau\left(\tau^{-1}\partial_\tau\right)^k\psi(\tau)\ ,\ \tau > 0\ .$$

In particular, at the origin,

$$\begin{aligned}(2\pi)^k u(x_o) &= \left(\tau^{-1}\partial_\tau\right)^k\left(\tau^{-1-q'}Ru(t(\tau))\right)_{\tau=1}\\ &= \left(\partial_\tau^k+\cdots+(-1)^k(q'+1)(q'+3)\cdots(q'+2k-1)\right)Ru(t(\tau))\big|_{\tau=1}\ .\end{aligned}$$

To switch over to derivatives with respect to t we note that, if $g(\tau) = f(t)$ with $\tau = (\cosh t)^{-1} = 1-\frac{t^2}{2}+\cdots$, identification of Taylor expansions at $\tau = 1$, resp. $t = 0$, leads to

$$\left(-\frac{1}{2}\right)^k\frac{g^{(k)}(1)}{k!} = \frac{f^{(2k)}(0)}{(2k)!}+\cdots+a_k f''(0)\ ,$$

where dots are a sum of even derivatives of f multiplied by some rational coefficients (like a_k). Therefore

$$(-2\pi)^k u(x_o) = \left(\frac{2^k k!}{(2k)!}\partial_t^{2k} + \cdots + (q'+1)(q'+3)\cdots(q'+2k-1)\right) Ru(t)\Big|_{t=0} ,$$

for any K-invariant $u \in \mathcal{D}(X)$, whence the claim by section 6.2. ■

6.4. The amusing formula generalized. To motivate the forthcoming generalizations of the amusing formula (12) and their applications to Radon inversion, we briefly recall the classical example of points and hyperplanes in the Euclidean space $X = \mathbb{R}^n$. Let (ω, p) be parameters for the hyperplane defined by the equation $\omega \cdot x = p$, where ω is a unit vector, p is a real number and $\cdot$ is the scalar product. Given $t \in \mathbb{R}$ and a point $x \in \mathbb{R}^n$, the parameters $(\omega, p) = (\omega, t + \omega \cdot x)$ define a hyperplane at distance $|t|$ from x, and

$$R_t^* v(x) = \int_{S^{n-1}} v(\omega, t + \omega \cdot x)\, d\omega$$

is the corresponding shifted dual Radon transform, where $v(\omega, p) = v(-\omega, -p)$ is an arbitrary smooth even function on $S^{n-1} \times \mathbb{R}$. Changing ω into $-\omega$ in the integral shows that $R_t^* v(x)$ is an even function of t.

Since $\sum \omega_i^2 = 1$ it is easily checked that

$$\left(\partial_t^2 - \Delta_x\right) v(\omega, t + \omega \cdot x) = 0 ,$$

where Δ_x is the Euclidean Laplace operator acting on x. Thus $R_t^* v(x)$, as a function of (x, t) in $\mathbb{R}^n \times \mathbb{R}$, is a solution of the wave equation, being an integral of the *elementary plane waves* $v(\omega, t + \omega \cdot x)$. More generally, for any positive integer k,

$$\left(\partial_t^{2k} - \Delta_x^k\right) R_t^* v(x) = 0 . \tag{19}$$

For odd n we have, by Theorem 8 with $n = 2k+1$, $d = 2k$ and $\varepsilon = 0$, the following inversion formula for the Radon transform on hyperplanes

$$Cu(x) = \Delta_x^k R^* Ru(x) . \tag{20}$$

Putting $v = Ru$ in (19) and observing that $R^* = R_0^*$, we thus obtain a new inversion formula by means of the shifted dual transform

$$Cu(x) = \partial_t^{n-1} R_t^* Ru(x)\big|_{t=0} . \tag{21}$$

Formula (21) might also be proved directly by the method of section 6.2.

To extend formula (12) we first deal with the Laplace operator; general invariant operators will be considered in the next section.

Let G be a Lie group, K a compact subgroup and let L be the Laplace operator of the Riemannian manifold $X = G/K$ (cf. Notations, b). The operator L can be expressed by means of any orthonormal basis X_1,..., X_n of $\mathfrak{p}$ as

$$Lf(gK) = \sum_{j=1}^{n} \partial_s^2 f\left(g \exp\left(sX_j\right) K\right)\big|_{s=0} ,$$

with $f \in C^2(G/K)$, $g \in G$; indeed both sides are G-invariant operators on X which coincide at $g = e$.

Now let $Y = G/H$ where H is a Lie subgroup of G and, as before,

$$R^*v(gK) = \int_K v(gkH)\, dk \ , \ R_t^*v(gK) = \int_K v(gktH)\, dk$$

for $v \in C^2(Y)$ and $g, t \in G$. Then

$$LR^*v(gK) = \int_K \left(\sum_j \partial_s^2 v\left(g \exp\left(sX_j\right) kH\right)\Big|_{s=0} \right) dk \ . \tag{22}$$

But $\sum X_j^2$ is a K-invariant element in the symmetric algebra of $\mathfrak{p}$ and it follows that, for any $\varphi \in C^2(\mathfrak{p})$, $k \in K$,

$$\sum_j \partial_s^2 \varphi\left(sX_j\right)\Big|_{s=0} = \sum_j \partial_s^2 \varphi\left(s\left(k \cdot X_j\right)\right)\Big|_{s=0} \ .$$

Therefore k can be moved to the left of $\exp sX_j$ in (22) and we obtain

$$LR^*v(x) = \sum_j \partial_s^2 R^*_{\exp sX_j} v(x)\Big|_{s=0} \tag{23}$$

for $v \in C^2(Y)$, $x \in X$. If $\mathfrak{h} \cap \mathfrak{p}$ is a nontrivial subspace of $\mathfrak{p}$ and the basis (X_j) contains a basis of this subspace, the sum in (23) only runs over an orthonormal basis of the orthogonal subspace $(\mathfrak{h} \cap \mathfrak{p})^\perp$, due to the right H-invariance of v.

We now give a more specific result for the *geodesic Radon transform*, in the notation of section 4.1. If $\mathfrak{s}$ is a d-dimensional Lie triple system contained in $\mathfrak{p}$ and $y_o = \operatorname{Exp} \mathfrak{s}$ the corresponding totally geodesic submanifold of X, we take as Y the set of all $g \cdot y_o$ for $g \in G$. Then $Y = G/H$, where H is the subgroup of all $h \in G$ globally preserving y_o.

Proposition 16. *Let X be one of the classical hyperbolic spaces $H^n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Assume $\mathfrak{s}$ is a $\mathbb{F}$-vector subspace of $\mathfrak{p}$ and let $T \in \mathfrak{p}$ be any unit vector orthogonal to $\mathfrak{s}$. For $v \in C^2(Y)$, the shifted dual geodesic transform $R^*_{\exp tT} v$ is then an even function of $t \in \mathbb{R}$ and, for $x \in X$,*

$$LR^*v(x) = (n-d)\, \partial_t^2 R^*_{\exp tT} v(x)\Big|_{t=0}$$

where n and d denote the real *dimensions of X and $\mathfrak{s}$ respectively.*

In other words, the function $(x,t) \mapsto R^*_{\exp tT} v(x)$ is a solution at time $t = 0$ of the wave operator $L - (n-d)\partial_t^2$ on $X \times \mathbb{R}$.
Applying the proposition to $H^3(\mathbb{R})$ with $d = 2$ we obtain formula (12). Indeed, if $\varphi(t)$ is an even function of t, let ψ be defined by $\psi(\tau) = \varphi(t)$ with $\cosh t = 1/\tau$; then $-\psi'(1) = \varphi''(0)$.
Example. By Theorem 8 the 2-geodesic transform on $X = H^n(\mathbb{R})$ can be inverted by means of a second order differential operator :

$$-2\pi(n-2)u = (L + n - 2)R^*Ru \ ,$$

and Proposition 16 now yields the inversion formula

$$-2\pi u = \left(\partial_t^2 + 1\right) R^*_{\exp tT} Ru\Big|_{t=0} \ , \tag{24}$$

where $u \in \mathcal{D}(X)$ and $T \in \mathfrak{p}$ is any unit vector orthogonal to $\mathfrak{s}$. Formula (24) also follows from Theorem 14 (ii) with $k = 1$, $q' = 0$.

Proof of Proposition 16. The point is to show that the group $K\cap H$ acts transitively on the unit sphere of $\mathfrak{s}^\perp$, the orthogonal of $\mathfrak{s}$ in $\mathfrak{p}$.

For the scalar product $(T,V)=\operatorname{Re}\sum \overline{T_i}V_i$ on $\mathfrak{p}$ we have $(T,V\lambda)=(T\overline{\lambda},V)$, $\lambda\in\mathbb{F}$, therefore $\mathfrak{s}^\perp$ is a $\mathbb{F}$-subspace of $\mathfrak{p}$.

An element k of $K\cap H$ is characterized by $k\in K$ and $k\cdot \operatorname{Exp}\mathfrak{s}=\operatorname{Exp}\mathfrak{s}$, i.e. $k\cdot\mathfrak{s}=\mathfrak{s}$ (adjoint action). Let n', d' be the respective dimensions of $\mathfrak{p}$ and $\mathfrak{s}$ as $\mathbb{F}$-vector spaces. Taking a $\mathbb{F}$-basis of $\mathfrak{p}$ according to the decomposition $\mathfrak{p}=\mathfrak{s}\oplus\mathfrak{s}^\perp$, it follows that

$$K=U(1;\mathbb{F})\times U(n';\mathbb{F})\ ,\ K\cap H=U(1;\mathbb{F})\times U(d';\mathbb{F})\times U(n'-d';\mathbb{F})\ .$$

But $U(n'-d';\mathbb{F})$ acts transitively on the unit sphere of $\mathbb{F}^{n'-d'}$, which implies our claim.

If $T,T'\in\mathfrak{s}^\perp$ are two unit vectors, there exists $k_o\in K\cap H$ such that $k_o\cdot T=T'$. Thus

$$\begin{aligned}R^*_{\exp tT'}v(gK) &= \int_K v\left(gkk_o\exp(tT)k_o^{-1}H\right)\,dk\\ &= \int_K v\left(gk\exp(tT)H\right)\,dk=R^*_{\exp tT}v(gK)\ .\end{aligned}$$

In particular $R^*_{\exp tT}v$ is an even function of t.

Going back to (23), we now take as (X_j) an orthonormal $\mathbb{R}$-basis of $\mathfrak{p}$ according to the decomposition $\mathfrak{p}=\mathfrak{s}\oplus\mathfrak{s}^\perp$. The $n-d$ basis vectors in $\mathfrak{s}^\perp$ give the same contribution to the right hand side, whereas the d vectors in $\mathfrak{s}$ generate one parameters subgroups of H and give no contribution; indeed $\exp tV\cdot\operatorname{Exp}\mathfrak{s}=\operatorname{Exp}\mathfrak{s}$ for $V\in\mathfrak{s}$, since $\mathfrak{s}$ is a Lie triple system by section 4.3.c. This completes the proof. ■

6.5. Multitemporal waves. We shall now deal with general invariant differential operators. As before G is a Lie group, H a closed subgroup, K a compact subgroup, and $X=G/K$, $Y=G/H$. Let $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ be the respective Lie algebras, and $\mathfrak{t}$ a vector subspace of $\mathfrak{g}$ such that

$$\mathfrak{g}=(\mathfrak{k}+\mathfrak{h})\oplus\mathfrak{t}\ .$$

Let $K_1,\dots,K_p$ be a basis of $\mathfrak{k}$, complemented by $H_1,\dots,H_q\in\mathfrak{h}$ so that the K_i's and H_j's are a basis of $\mathfrak{k}+\mathfrak{h}$, and let $T_1,\dots,T_r$ be a basis of $\mathfrak{t}$. We shall use the same notations for the corresponding left-invariant vector fields on G, e.g.

$$K_if(g)=\left.\partial_s f\left(g\exp sK_i\right)\right|_{s=0}$$

with $f\in C^\infty(G)$, $g\in G$, $s\in\mathbb{R}$. We denote by $\mathbb{D}(G)$ the algebra of all left invariant differential operators on G, by $\mathbb{D}(G)^K$ the subalgebra of right K-invariant operators and by $\mathbb{D}(X)$ the algebra of G-invariant differential operators on X. For $s=(s_1,\dots,s_r)\in\mathbb{R}^r$, let

$$t(s)=\exp s_1T_1\cdots\exp s_rT_r\ .$$

We recall that, for $g,t\in G$,

$$R^*_tv(gK)=\int_K v(gktH)\,dk\ .$$

Theorem 17. *Let G be a Lie group, H,K Lie subgroups, with K compact and $X=G/K$, $Y=G/H$.*
(i) For any $P\in\mathbb{D}(X)$ there exists $Q(\partial)$, a constant coefficients differential operator on $\mathbb{R}^r$, with order$(Q)\le$ order(P), such that for any $v\in C^\infty(Y)$, $x\in X$,

$$PR^*v(x)=Q(\partial_s)R^*_{t(s)}v(x)\Big|_{s=0}\ . \tag{25}$$

(ii) Assume furthermore that $\mathfrak{t}$ is a Lie subalgebra of $\mathfrak{g}$ with $[\mathfrak{t},\mathfrak{h}] \subset \mathfrak{h}$, and let T denote the connected Lie subgroup of G with Lie algebra $\mathfrak{t}$. Then for any $P \in \mathbb{D}(X)$ there exists a right-invariant differential operator Q on T, with $order(Q) \leq order(P)$, such that

$$P_{(x)} R_t^* v(x) = Q_{(t)} R_t^* v(x) \tag{26}$$

for $v \in C^\infty(Y)$; here $P_{(x)}$ acts on the variable $x \in X$ and $Q_{(t)}$ acts on $t \in T$.

Thus $R_t^* v(x)$, as a function of $(x,t) \in X \times T$, solves the generalized "*multitemporal*" wave equation (26) with time variable in a multidimensional space. Similarly (25) can be viewed as a wave equation in the variables $(x,s) \in X \times \mathbb{R}^r$ at the time $s = 0$.

Proof. In order to work on G rather than on its homogeneous spaces, we define $w(g) = v(gH)$ and, for $g, t \in G$,

$$F(g,t) = (R_t^* v)(gK) = \int_K w(gkt)\, dk \ , \tag{27}$$

so that $F(gk, k'th) = F(g,t)$ for any $k, k' \in K$, $h \in H$, and

$$F(g,e) = (R^* v)(gK) = \int_K w(gk)\, dk \ .$$

Let $P \in \mathbb{D}(X)$ be given. Since K is compact the coset space $X = G/K$ is reductive and there exists $D \in \mathbb{D}(G)^K$ such that ([9] p.285)

$$(Pf)(gK) = D_{(g)}\left(f(gK)\right) \tag{28}$$

for $f \in C^\infty(X)$, $g \in G$.

To transfer derivatives from g to t we observe that, by the invariance of D under left translation by gk and right translation by k,

$$D_{(g)} w(gkt) = D_{(x)} w(gkxt)\big|_{x=e} \ ,$$

where g, x, t are variables in G. Integrating over K it follows that

$$D_{(g)} F(g,t) = D_{(x)} F(g,xt)\big|_{x=e} \ , \tag{29}$$

By the Poincaré-Birkhoff-Witt theorem, the differential operators

$$K_1^{\beta_1} \cdots K_p^{\beta_p} T_1^{\alpha_1} \cdots T_r^{\alpha_r} H_1^{\gamma_1} \cdots H_q^{\gamma_q}$$

(where all exponents are positive integers) are a basis of $\mathbb{D}(G)$. Setting apart the terms with $\beta = \gamma = 0$, we can thus write, for some $E_i, F_j \in \mathbb{D}(G)$ and some constant coefficients a_α,

$$D = D' + \sum_{i=1}^{p} K_i E_i + \sum_{j=1}^{q} F_j H_j \ , \ D' = \sum_\alpha a_{\alpha_1 \ldots \alpha_r} T_1^{\alpha_1} \cdots T_r^{\alpha_r} \ . \tag{30}$$

If we replace $D_{(x)}$ by (30) in (29), the second term $(K_i E_i)_{(x)} F(g,xt)\big|_{x=e}$ vanishes because $K_i \in \mathfrak{k}$ and $F(g,kxt) = F(g,t)$. In the third term the left invariant vector field $H_j \in\in \mathfrak{h}$ acts by

$$(H_j)_{(x)} F(g,xt) = \partial_s F(g, x \exp(sH_j)\, t)\big|_{s=0} \ ,$$

and this vanishes too whenever t normalizes H, because $F(g,xth) = F(g,xt)$.

Since $t = e$ in case (i), or $t \in T$ with $Ht = tH$ in case (ii), we finally obtain for both cases (in multi-index notation)

$$D_{(g)}F(g,t) = D'_{(x)}F(g,xt)\Big|_{x=e} = \tag{31}$$

$$= \sum_\alpha a_\alpha \, \partial_s^\alpha F(g, (\exp s_1T_1 \cdots \exp s_rT_r)\, t)|_{s=0} = \left(\sum_\alpha a_\alpha \partial_s^\alpha\right) F(g, t(s)t)\Big|_{s=0} .$$

Let the operator Q be defined by

$$Qf(t) = \sum_\alpha a_\alpha \partial_s^\alpha f(t(s)t)\Big|_{s=0} ,$$

a right invariant differential operator on the group T in case (ii). The theorem now follows from (27), (28) and (31) in both cases (i) and (ii). ■

6.6. Examples. Keeping the notations of the previous section, we shall illustrate Theorem 17.

a. Totally geodesic transform. As in section 4.1.a, let $X = G/K$ be a Riemannian symmetric space of the noncompact type and $y_o = \operatorname{Exp}\mathfrak{s}$ the origin in the dual space $Y = G/H$. By (3) we have $\mathfrak{k} + \mathfrak{h} = \mathfrak{k} \oplus \mathfrak{s}$, therefore Theorem 17 (i) applies with $\mathfrak{t} = \mathfrak{s}^\perp$, the orthogonal of $\mathfrak{s}$ in $\mathfrak{p}$.

b. Horocycle transform. Again $X = G/K$ is a Riemannian symmetric space of the noncompact type (see Notations, d), but the dual space is now the space of horocycles $Y = G/MN$. We recall Harish-Chandra's isomorphism of algebras ([9] p.306)

$$\Gamma : \mathbb{D}(X) \longrightarrow \mathbb{D}(A)^W ,$$

where $\mathbb{D}(A)^W$ is the subalgebra of W-invariant differential operators in $\mathbb{D}(A)$. The definition of Γ will be recalled during the next proof.

Proposition 18. *Given $v \in C^\infty(Y)$, the function of $x = gK$ and $a \in A$ given by*

$$w(x,a) = a^\rho R_a^* v(x) = a^\rho \int_K v(gkaN)\, dk$$

is a solution of the system of multitemporal wave equations

$$P_{(x)}w(x,a) = \Gamma(P)_{(a)}w(x,a) \ , \ P \in \mathbb{D}(X), x \in X, a \in A .$$

Proof. Theorem 17 (ii) applies here with $T = A$, the abelian subgroup from the Iwasawa decomposition $G = KAN$; indeed $\mathfrak{k} + \mathfrak{h} = \mathfrak{k} + \mathfrak{m} + \mathfrak{n} = \mathfrak{k} \oplus \mathfrak{n}$, and $\mathfrak{g} = (\mathfrak{k} \oplus \mathfrak{n}) \oplus \mathfrak{a}$, $[\mathfrak{a}, \mathfrak{h}] \subset [\mathfrak{a}, \mathfrak{m}] + [\mathfrak{a}, \mathfrak{n}] \subset \mathfrak{n} \subset \mathfrak{h}$. By (31) we thus have

$$P_{(x)}R_a^* v(x) = D'_{(a)}R_a^* v(x) \ , \tag{32}$$

where $D \in \mathbb{D}(G)^K$ is related to P by (28) and $D' \in \mathbb{D}(A)$ was characterized by

$$D - D' \in \mathfrak{k}\mathbb{D}(G) + \mathbb{D}(G)\mathfrak{n} . \tag{33}$$

To compare D' and $\Gamma(P)$ we recall that $\Gamma(P) = a^{-\rho} D_\mathfrak{a} \circ a^\rho$, where $D_\mathfrak{a} \in \mathbb{D}(A)$ is characterized by

$$D - D_\mathfrak{a} \in \mathfrak{n}\mathbb{D}(G) + \mathbb{D}(G)\mathfrak{k} . \tag{34}$$

Moreover $(Df)(a) = D_{\mathfrak{a}}(f(a))$ for $a \in A$, if $f \in C^\infty(G)$ is such that $f(ngk) = f(g)$ for any $g \in G$, $k \in K$, $n \in N$ ([9] p.302 sq.).

Taking $u \in \mathcal{D}(G)$ we have, by a classical integral formula,

$$\begin{aligned}\int_G Df(g) \cdot u(g)\, dg &= \int_{N\times A\times K} Df(a) \cdot u(nak)\, a^{-2\rho} dndadk \\ &= \int_{N\times A\times K} D_{\mathfrak{a}} f(a) \cdot u(nak)\, a^{-2\rho} dndadk\ . \end{aligned} \tag{35}$$

On the other hand, this integral can be written with the transpose operator ${}^tD$ as

$$\begin{aligned}\int_G Df(g) \cdot u(g)\, dg &= \int_G f(g)\, {}^tDu(g)\, dg \\ &= \int_A f(a)\, a^{-2\rho} da \int_{N\times K} \left({}^tDu\right)(nak)\, dndk\ .\end{aligned}$$

But ${}^tD \in \mathbb{D}(G)^K$ therefore, for any $g \in G$,

$$\int_{N\times K} \left({}^tDu\right)(ngk)\, dndk = \left({}^tD\right)_{(g)} \left(\int_{N\times K} u(ngk)\, dndk\right)\ .$$

The latter integral, as a function of g, is left N-invariant and right K-invariant so that

$$\int_{N\times K} \left({}^tDu\right)(nak)\, dndk = \left({}^tD\right)_{\mathfrak{a}} \left(\int_{N\times K} u(nak)\, dndk\right)\ .$$

Since $\left({}^tD\right)_{\mathfrak{a}} = {}^t\left(D'\right)$ obviously by (33) and (34), we obtain

$$\begin{aligned}\int_G Df(g) \cdot u(g)\, dg &= \int_A D'(f(a)a^{-2\rho})\, da \int_{N\times K} u(nak)\, dndk \\ &= \int_{N\times A\times K} \left(a^{2\rho} D' \circ a^{-2\rho}\right) f(a) \cdot u(nak)\, a^{-2\rho} dndadk\ ,\end{aligned}$$

for any $f \in C^\infty(A)$ and any $u \in \mathcal{D}(G)$. Comparing with (35) it follows that

$$D_{\mathfrak{a}} = a^{2\rho} D' \circ a^{-2\rho}\ ,\ D' = a^{-\rho}\Gamma(P) \circ a^{\rho}\ ,$$

whence the result by (32). ■

A slightly different proof can be obtained by decomposing the wave $a^\rho R_a^* v(gK)$ into *elementary horocycle waves* as follows. For $g \in G$ we denote by $A(g) \in A$ the A-component of g in the Iwasawa decompositions $G = NAK = ANK$ (we recall that A normalizes N), and by $K(g) \in K$ its K-component in the decompositions $G = KAN = KNA$.

Proposition 19. (i) *Given $f \in C^\infty(A)$ and $k \in K$, the function*

$$w(gK, a) = a^{-\rho} f(A(k^{-1}g)a)$$

is a solution of the system of multitemporal wave equations

$$P_{(x)} w(x, a) = \Gamma(P)_{(a)} w(x, a)\ ,\ P \in \mathbb{D}(X), x \in X, a \in A\ .$$

(ii) *Given $v \in C^\infty(Y)$, the function of $x = gK$ and $a \in A$ given by*

$$a^\rho R_a^* v(gK) = \int_K a^\rho v(gkaN)\, dk$$

is a solution of the same equations.

Part (i) is Proposition 8.5 in [12] p.118. Note that, k being fixed, the "wave surfaces" $A(k^{-1}g) = constant$ are parallel horocycles with the same normal $kM \in K/M$ (cf. [11] p.81). Indeed the equality $A(k^{-1}g) = a_o \in A$ is equivalent to $k^{-1}g \in a_o NK$, i.e. $g \cdot x_o \in ka_o \cdot y_o$.
If λ is a linear form on $\mathfrak{a}$ and $f(a) = a^{i\lambda+\rho}$, the result (i) implies that $A(k^{-1}g)^{i\lambda+\rho}$ is, as a function of gK, an eigenfunction of all invariant operators $P \in \mathbb{D}(X)$; this is a fundamental result for harmonic analysis on X.
Part (ii) provides a simpler proof and a generalization of Proposition 8.6 in [12] p.118, where v was the Radon transform Ru of some $u \in \mathcal{D}(X)$. We refer to [12] or [13] for a detailed study of those multitemporal wave equations.
Proof of Proposition 19. (i) Both sides of the wave equation are invariant under the action of K on X; we can therefore assume $k = e$. Now $w(gK, a) = a^{-\rho} f(A(g)a)$ is left N-invariant and right K-invariant as a function of g, and it will suffice to prove the result for $g = a \in A$.

By the decomposition (34) of D we have, for any $b \in A$,

$$D_{(g)}\left(f(A(g)b)\right)\big|_{g=a} = (D_{\mathfrak{a}})_{(a)}\left(f(ab)\right) = a^\rho \Gamma(P)_{(a)}\left(a^{-\rho} f(ab)\right) \ .$$

But $\Gamma(P)$ is an invariant differential operator on A, isomorphic to the additive group of a vector space, and we obtain

$$\begin{aligned} D_{(g)}\left(b^{-\rho} f(A(g)b)\right)\big|_{g=a} &= a^\rho \Gamma(P)_{(a)}\left((ab)^{-\rho} f(ab)\right) \\ &= a^\rho \Gamma(P)_{(b)}\left((ab)^{-\rho} f(ab)\right) \\ &= \Gamma(P)_{(b)}\left(b^{-\rho} f(ab)\right) = \Gamma(P)_{(b)}\left(b^{-\rho} f(A(g)b)\right)\big|_{g=a} \ . \end{aligned}$$

Thus (i) is proved for $g = a$.
(ii) Let $g \in G$, $k \in K$ and $k' = K(gk)$. Then $gk = k'a'n'$ with $a' \in A$ and $n' \in N$. It follows that $k'^{-1}g = a'n'k^{-1}$, therefore $a' = A(k'^{-1}g)$ and

$$gkaN = k'A\left(k'^{-1}g\right)aN \ .$$

For fixed g the map $k \mapsto K(gk) = k'$ is a diffeomorphism of K onto itself and, by the integral formula ([9] p.197)

$$\int_K F(k')\,dk = \int_K A(k'^{-1}g)^{2\rho}\,F(k')\,dk' \ ,$$

we have

$$\begin{aligned} a^\rho R_a^* v(gK) &= a^\rho \int_K v(gkaN)\,dk = a^\rho \int_K v(k'A(k'^{-1}g)aN)\,dk \\ &= a^{-\rho} \int_K \left(A(k'^{-1}g)a\right)^{2\rho} v(k'A(k'^{-1}g)aN)\,dk' \ . \end{aligned}$$

By (i) applied to the functions $f(a) = a^{2\rho} v(k'aN)$, $k' \in K$, this is a solution of the wave equations. ■

Corollary 20. *(Helgason) If $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras, there exists a differential operator $P \in \mathbb{D}(X)$ such that the horocycle Radon transform of $X = G/K$ is inverted by*

$$u(x) = PR^*Ru(x)$$

for $u \in \mathcal{D}(X)$, $x \in X$.

We prove it here by means of shifted transforms and wave equations ; see [11] p.116 for Helgason's original proof.

Proof. The assumption on $\mathfrak{g}$ implies that, in the notation of (15), $C \cdot |c(\lambda)|^{-2}$ is a W-invariant polynomial on $\mathfrak{a}^*$. Let $P \in \mathbb{D}(X)$ be the corresponding operator under the isomorphism $\Gamma : \mathbb{D}(X) \rightarrow \mathbb{D}(A)^W$, so that $\Gamma(P)(i\lambda) = C \cdot |c(\lambda)|^{-2}$. By Theorem 13 and Proposition 19 (*ii*) (with $v = Ru$) we have

$$\begin{aligned} u(x) &= \; < T_{(a)}, a^\rho R_a^* Ru(x) > = \Gamma(D)_{(a)} \left(a^\rho R_a^* Ru(x)\right)\big|_{a=e} \\ &= P_{(x)} \left(a^\rho R_a^* Ru(x)\right)\big|_{a=e} = P_{(x)} R^* Ru(x) \; . \end{aligned}$$

■

7. References

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