GEODESIC RADON TRANSFORMS ON SYMMETRIC SPACES

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à la mémoire d'André Cerezo

Abstract. Inversion formulas are given for the X-ray transform on all Riemannian symmetric spaces of the non-compact type, by means of shifted dual Radon transforms. One of these formulas is extended to a large class of totally geodesic Radon transforms on these spaces.

1. Introduction

1.1. Inverting the X-ray transform on a Riemannian manifold means rebuilding a function u on this manifold from the family of its integrals $Ru(\xi)$ over all geodesics ξ. In the most basic example the ξ's are the lines in a two-dimensional Euclidean plane; a nice inversion formula for this case was given by J. Radon in his pioneering 1917 article [12]:

$$u(x) = -\frac{1}{\pi}\int_0^\infty \frac{dF_x(t)}{t} \ ,$$

where $F_x(t)$ is the average of $Ru(\xi)$ over all lines ξ at distance t from the point x. He also mentioned without proof the corresponding result for a two-dimensional hyperbolic plane, with $\operatorname{sh} t$ instead of t in the denominator. After a long silence the problem was taken up again by many authors: S. Helgason, and later C. Berenstein and E. Casadio Tarabusi, S. Gindikin, S. Ishikawa, A. Kurusa, B. Rubin, among others. A brief and partial comparative survey will be given in Section 5 below. To the best of my knowledge, however, no inversion formula of the X-ray transform has been published yet beyond the case of spaces of constant curvature (Euclidean, spherical or hyperbolic geometry).

It will be shown here that J. Radon's announcement for the hyperbolic plane extends to the X-ray transform on all *Riemannian symmetric spaces of the non-compact type*, i.e. all homogeneous spaces $X = G/K$ where G is a non-compact semisimple Lie group and K is a maximal compact subgroup of G. Our method draws inspiration from S. Helgason's proof of the support theorem for this transform ([3] p.179). Inversion formulas will be proved also for more general Radon transforms on these spaces, with ξ running through a family of totally geodesic submanifolds of X, under some assumptions explained below.

1.2. Our inversion formula for the X-ray transform on X is

$$(1) \qquad u(x) = -\frac{|\alpha|}{\pi}\int_0^\infty \frac{\partial}{\partial t}\left(R^*_{\exp tY} Ru(x)\right)\frac{dt}{\operatorname{sh} t}$$

(Section 3, Theorem 2). Here u is an arbitrary smooth compactly supported function on X, x is any point of X, α is a root and the Radon transform Ru is obtained by integrating u over the geodesics. To explain the operator $R^*_{\exp tY}$ let x_0 be the origin of X, let ξ_0 be a given geodesic through x_0 and let γ belong to the group G. The classical dual Radon transform $R^*v(x)$ (or backprojection operator), which takes the mean value of a function $v(\xi)$ over geodesics ξ containing the point x, is here replaced by the *shifted dual tranform*

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$R_\gamma^* v(x)$ averaging v over geodesics at a given distance from x, namely at the same distance than the geodesic $\gamma \cdot \xi_0$ from x_0.

In (1) the direction of ξ_0 is H_α, the root vector corresponding to α, and Y is any vector in the eigenspace $\mathfrak{p}_\alpha$ with norm $|Y| = |\alpha|^{-1}$ (see Section 2.1 for definitions); thus Y is orthogonal to ξ_0 at the origin. The shift $\gamma = \exp tY$ gives $R_\gamma^* v(x)$, the average of $v(\xi)$ over geodesics ξ at a distance $|\alpha|^{-1}t$ from x.

To prove (1) the key observation is that, due to simple bracket relations involving H_α and Y, the relevant geometry is the two-dimensional hyperbolic geometry. In particular the hyperbolic Pythagorean theorem $\operatorname{ch} c = \operatorname{ch} a \operatorname{ch} b$ holds for right-angled geodesic triangles with sides a, b and hypotenuse c, and the inversion problem for R is easily reduced to inverting an integral equation of Abel type.

A variant of (1) is also given inTheorem 2, with $\exp tY$ replaced by a shift belonging to the nilpotent part of an Iwasawa decomposition of G, and dt/t instead of $dt/\operatorname{sh} t$.

Recently S. Helgason ([7]) has obtained another inversion formula valid when X has rank $l > 1$ (but not for $l = 1$). Instead of hyperbolic planes embedded in X he uses the Euclidean inversion formula for the X-ray transform in the l-dimensional flat totally geodesic submanifold $\operatorname{Exp} \mathfrak{a}$ of X.

In the notation of (1) J. Radon's result (extended to the X-ray transform on $\mathbb{R}^n$) would be

$$u(x) = -\frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial t} \left(R_{tY}^* Ru(x) \right) \frac{dt}{t} \ ,$$

where Y is any unit vector orthogonal to the line ξ_0 taken as the origin and tY is a translation of length t in $\mathbb{R}^n$.

1.3. The method does not extend exactly as such to higher dimensional totally geodesic Radon transforms, a topic dealt with in Section 4. The transform $Ru(\xi)$ under study is now obtained by integrating a function u on X over the submanifolds $\xi = g \cdot \xi_0$, where ξ_0 is a given totally geodesic submanifold containing the origin of X and g is arbitrary in G. When the tangent space $\mathfrak{s}$ to ξ_0 at x_0 has dimension greater than one it is no longer possible to choose the vector Y as before, providing a suitable shift for all directions in $\mathfrak{s}$. The difficulty can be circumvented by exchanging the roles of H_α and Y. In Theorem 3 we assume (i) and (ii)

(i) $\mathfrak{s}$ is a Lie triple system, i.e. $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$

(ii) $\mathfrak{s}$ is contained in the eigenspace $\mathfrak{p}_\alpha$ and α is an indivisible root.

Owing to (i) the submanifold $\xi_0 = \operatorname{Exp} \mathfrak{s}$ is totally geodesic in X. Assumption (ii) implies that the root vector H_α is orthogonal to ξ_0 at the origin and the hyperbolic Pythagorean theorem can be used again, so as to reduce the problem to some Abel integral equation. Roughly speaking (ii) means that $\mathfrak{s}$ is not too big (see Lemma 4). When (i) and (ii) hold we obtain the following inversion formulas (Theorem 3), where $d = \dim \mathfrak{s} = \dim \xi_0$:

- if $d = 2k$ is even there exists a polynomial P of degree k such that

$$u(x) = P\left(\frac{\partial^2}{\partial t^2}\right) \left(R_{\exp tH_\alpha}^* Ru(x) \right)\Big|_{t=0} \tag{2}$$

- if $d = 2k - 1$ is odd

$$u(x) = \sum_{l=1}^{k} a_{k,l} \int_0^\infty (\operatorname{ch} t)^{l-1} \left(\frac{1}{\operatorname{sh} t} \frac{\partial}{\partial t} \right)^l \left(R_{\exp tH_\alpha}^* Ru(x) \right) dt \tag{3}$$

where the $a_{k,l}$'s are some inductively defined coefficients. Two variants, valid for any d (even or odd), are given by (17') and (17") in Section 4.5.

Here again the shifted dual transform $R_\gamma^* v(x)$ with $\gamma = \exp tH_\alpha$ is an average of the function $v(\xi)$ over totally geodesic submanifolds ξ at distance $|\alpha|^{-1}t$ from x. In the special case of the X-ray transform ($d = k = 1$) it is easily checked that formula (3) is equivalent to (1) (Section 4.2).

1.4. Section 2 is devoted to notations, definitions, and generalities about shifted dual transforms. In Section 3 we prove our main result about the X-ray transform (Theorem 2). More general totally geodesic submanifolds are considered in Section 4, with Theorem 3 as the main result; examples and variants are also discussed. Finally, in Section 5, our results are compared with a few others from the literature.

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2. Preliminaries

2.1. **Notations.** As usual $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ will respectively denote the fields of real numbers, complex numbers, quaternions, and $\mathcal{D} = C_c^\infty$ the space of compactly supported C^∞ functions on a manifold. We write ∂_x as well as $\partial/\partial x$ for partial derivatives.

If G is a (real) Lie group let $\mathfrak{g}$, ad, exp denote the Lie algebra, its adjoint representation and the exponential mapping respectively. When G acts on a set we write $g \cdot x$ for the result of the action of $g \in G$ on x.

Throughout the paper G will be a connected non-compact real semisimple Lie group with finite center and K a maximal compact subgroup. We briefly recall some classical notations (see [2] for details). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra, with Cartan involution θ. Equipped with a G-invariant metric, the homogeneous space $X = G/K$ is a *Riemannian symmetric space of the non-compact type.* We shall denote its origin by $x_0 = K$ and by $d(x, y)$ the distance of the points x, y. The space $\mathfrak{p}$ can be identified with the tangent space to X at x_0, and the exponential mapping $\mathrm{Exp} : \mathfrak{p} \to X$ is a global diffeomorphism onto, related to exp by $\mathrm{Exp}\, V = \exp V \cdot x_0$ for $V \in \mathfrak{p}$. The curve $\mathrm{Exp}\, \mathbb{R}V$ is the geodesic tangent to V at the origin x_0.

A vector subspace $\mathfrak{s}$ of $\mathfrak{p}$ is called a *Lie triple system* if $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$. Then $\xi_0 = \mathrm{Exp}\, \mathfrak{s}$ is a totally geodesic submanifold of X, with tangent space $\mathfrak{s}$ at x_0.

Let ξ_0 be a connected submanifold of X and Ξ the set of all submanifolds $g \cdot \xi_0$, $g \in G$. For $\xi \in \Xi$ let dm_ξ be the measure on ξ induced by the Riemannian measure of X; when $\xi = \xi_0$ we write dm for dm_{ξ_0}. The *Radon transform* of a function u on X is the function on Ξ defined by

$$Ru(\xi) = \int_\xi u(x)\ dm_\xi(x)$$

if the integral converges. For $\gamma \in G$ the *shifted dual Radon transform* of a function v on Ξ is the function $R_\gamma^* v$ on X defined by

$$R_\gamma^* v(g \cdot x_0) = \int_K v(gk\gamma \cdot \xi_0)\ dk\ ,\ g \in G\ ,$$

where dk is the Haar measure of K normalized by $\int_K dk = 1$; this definition depends on the choice of the origin ξ_0 in Ξ. When γ is the identity R_γ^* reduces to the classical dual Radon transform R^*; if the origins are chosen so that ξ_0 contains x_0, all submanifolds $gk \cdot \xi_0$ in the integral contain the point $g \cdot x_0$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and α a root of $\mathfrak{g}$ with respect to $\mathfrak{a}$. This means that the joint eigenspace[1]

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\mathrm{ad}\, H)X = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

is not $\{0\}$. We shall also use the eigenspaces

$$\mathfrak{p}_\alpha = \{Y \in \mathfrak{p} \mid (\mathrm{ad}\, H)^2 Y = \alpha(H)^2 Y \text{ for all } H \in \mathfrak{a}\}\ .$$

The map $X \mapsto Y = \frac{1}{2}(X - \theta X)$ is a linear isomorphism of $\mathfrak{g}_\alpha$ onto $\mathfrak{p}_\alpha$.

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[1] No confusion should arise here with $X = G/K$!

The Killing form $B(Y,Z) = \mathrm{tr}(\mathrm{ad}\, Y \,\mathrm{ad}\, Z)$ of $\mathfrak{g}$ gives rise to the invariant scalar product $< Y, Z >= -B(Y, \theta Z)$ of $Y, Z \in \mathfrak{g}$ and to the norm

$$|Y| = \sqrt{-B(Y, \theta Y)} \ .$$

The space X will be equipped with the Riemannian metric corresponding to this norm on $\mathfrak{p}$.

2.2. **The role of shifted transforms.** All our inversion formulas for Radon transforms will be proved by means of the following general observation from [14] p. 234. Let u be a compactly supported continuous function on $X = G/K$ and ξ_0 a given totally geodesic submanifold of X. For $g \in G$ the function

$$u_g(x) = \int_K u(gk \cdot x)\, dk \ , \ x \in X \ ,$$

is K-invariant and $u_g(x_0) = u(g \cdot x_0)$; when X has rank one K acts transitively on spheres with center x_0, and $u_g(x)$ is the mean value of u over the sphere with center $g \cdot x_0$ and radius $d(x_0, x)$. Its Radon transform is, with $\gamma \in G$,

$$\begin{aligned} Ru_g(\gamma \cdot \xi_0) &= \int_{\xi_0} u_g(\gamma \cdot x)\, dm(x) = \int_{\xi_0} \int_K u(gk\gamma \cdot x)\, dm(x) dk \\ &= \int_K Ru(gk\gamma \cdot \xi_0)\, dk = R_\gamma^* Ru(g \cdot x_0) \end{aligned} \tag{4}$$

by definition of the shifted dual transform R_γ^*.

Now assume an inversion formula for R is known at the origin x_0 for K-invariant functions, say

$$u(x_0) =< T(\gamma), Ru(\gamma \cdot \xi_0) > \ , \tag{5}$$

where T is some linear form on a space of functions of the variable γ belonging to G (or to some submanifold of G). When u is arbitrary (5) applies to the K-invariant function u_g, whence $u(g \cdot x_0) =< T(\gamma), Ru_g(\gamma \cdot \xi_0) >$ i.e.

$$u(x) =< T(\gamma), R_\gamma^* Ru(x) > \tag{6}$$

for any $x \in X$. In the sequel *it will therefore suffice to work with K-invariant functions and to invert R at the origin*; the general case will follow immediately thanks to the shifted dual tranform.

3. The X-ray transform

Let $X = G/K$ be a Riemannian symmetric space of the non-compact type. We keep to the notation of Section 2.1. Let α be a root of $\mathfrak{g}$ with respect to $\mathfrak{a}$ and let $A_\alpha \in \mathfrak{a}$ be defined by $B(H, A_\alpha) = \alpha(H)$ for all $H \in \mathfrak{a}$. The norm $|\alpha|$ is defined by

$$|\alpha|^2 = |A_\alpha|^2 = B(A_\alpha, A_\alpha) = \alpha(A_\alpha) \ .$$

The vector $H_\alpha = |\alpha|^{-2} A_\alpha$, which satisfies $\alpha(H_\alpha) = 1$, will be more convenient here than A_α. Besides, given a non-zero root vector $Y \in \mathfrak{p}_\alpha$, let X be the unique vector in $\mathfrak{g}_\alpha$ such that $Y = \frac{1}{2}(X - \theta X)$ and let $Z = \frac{1}{2}(X + \theta X) \in \mathfrak{k}$.

As explained in the introduction hyperbolic planes embedded in X as totally geodesic submanifolds are essential to our method. The following easy lemma provides a large supply.

Lemma 1. *With the above notations*

$$|H_\alpha| = |\alpha|^{-1} \ , \ |Y| = |Z| = 2^{-1/2}|X| \ .$$

The linear span of H_α, Y and Z is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $sl(2,\mathbb{R})$ and $\mathrm{Exp}(\mathbb{R}H_\alpha \oplus \mathbb{R}Y)$ *is a totally geodesic submanifold of X isomorphic to the hyperbolic plane $H^2(\mathbb{R})$.*

If $|Y| = |\alpha|^{-1}$ *the adjoint action of* $\exp \mathbb{R}Z$ *on* H_α *is*

$$\mathrm{Ad}(\exp tZ)H_\alpha = (\cos t)H_\alpha - (\sin t)Y \ , \ t \in \mathbb{R} \ . \tag{7}$$

Proof. In view of the invariance properties of the Killing form we have $B(X, X) = B(\theta X, \theta X) = 0$, therefore

$$|Y|^2 = -B(Y, \theta Y) = -\frac{1}{2}B(X, \theta X) = \frac{1}{2}|X|^2$$

and similarly for $|Z|^2$.

Besides $[X, \theta X]$ belongs to $\mathfrak{p}$ and commutes to $\mathfrak{a}$, therefore belongs to $\mathfrak{a}$, and

$$B(H, [X, \theta X]) = B(\theta X, [H, X]) = -\alpha(H)|X|^2 = -|\alpha|^2|X|^2 B(H, H_\alpha)$$

for all $H \in \mathfrak{a}$. It follows that

$$[H_\alpha, X] = X \ , \ [H_\alpha, \theta X] = -\theta X \ , [X, \theta X] = -|\alpha|^2|X|^2 \ H_\alpha$$

and H_α, X, θX generate a Lie subalgebra of $\mathfrak{g}$ isomorphic to $sl(2, \mathbb{R})$, these generators respectively corresponding to the matrices

$$\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} , \frac{|\alpha||X|}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \frac{|\alpha||X|}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} .$$

Finally

$$\begin{gathered} [H_\alpha, Y] = Z \ , \ [H_\alpha, Z] = Y \ , \ [Y, Z] = -|\alpha|^2|Y|^2 H_\alpha \ , \\ (\mathrm{ad}\, H_\alpha)^2 Y = Y \ , \ (\mathrm{ad}\, Y)^2 H_\alpha = |\alpha|^2|Y|^2 H_\alpha \ , \end{gathered} \tag{8}$$

showing that $\mathbb{R}H_\alpha \oplus \mathbb{R}Y$ is a two-dimensional non-abelian Lie triple system.

By (8) both sides of (7) solve the linear differential equation

$$X'(t) = [Z, X(t)] \ , \ X(0) = H_\alpha \ ,$$

and the lemma follows. □

Remark. When X has rank one Lemma 1 gives all its totally geodesic submanifolds containing the origin and isomorphic to $H^2(\mathbb{R})$. Indeed, let H, Y be an orthogonal basis of the Lie triple system of such a manifold. Then $(\mathrm{ad}\, H)^2 Y$ is a linear combination of H and Y, orthogonal to H with respect to the Killing form, thus Y is an eigenvector of $(\mathrm{ad}\, H)^2$. If $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_\alpha \oplus \mathfrak{p}_{2\alpha}$ is the eigenspace decomposition of $\mathfrak{p}$ given by its maximal abelian subspace $\mathfrak{a} = \mathbb{R}H$, the vector Y must belong to the root space $\mathfrak{p}_\alpha$ or $\mathfrak{p}_{2\alpha}$, and H is proportional to H_α.

We can now derive two versions of an inversion formula for the X-ray transform.

Theorem 2. *Let* $X = G/K$ *be a Riemannian symmetric space of the non-compact type. Pick a root* α *of the pair* $(\mathfrak{g}, \mathfrak{a})$, *any vector* $Y \in \mathfrak{p}_\alpha$ *with* $|Y| = |\alpha|^{-1}$, *and let*[2] $H_\alpha \in \mathfrak{a}$, $X \in \mathfrak{g}_\alpha$ *be defined as in Lemma 1. Taking* $\xi_0 = \mathrm{Exp}\, \mathbb{R}H_\alpha$ *as the origin in the space of geodesics, let* R *be the X-ray transform defined by integration over geodesics of* X *in a family containing all* $g \cdot \xi_0$, $g \in G$.
Then R *is inverted by the following formulas*

$$u(x) = -\frac{|\alpha|}{\pi} \int_0^\infty \frac{\partial}{\partial t} \left(R^*_{\exp tY} Ru(x) \right) \frac{dt}{\mathrm{sh}\, t} \tag{9}$$

and

$$u(x) = -\frac{|\alpha|}{\pi} \int_0^\infty \frac{\partial}{\partial s} \left(R^*_{\exp sX} Ru(x) \right) \frac{ds}{s} \ , \tag{9'}$$

for $u \in \mathcal{D}(X)$, $x \in X$.

[2] No confusion should arise here with $X = G/K$!

Proof. By Section 2.2 it is enough to prove that

$$u(x_0) = -\frac{|\alpha|}{\pi}\int_0^\infty \frac{\partial}{\partial t}\left(Ru(\exp tY\cdot\xi_0)\right)\frac{dt}{\operatorname{sh} t} \tag{10}$$

$$= -\frac{|\alpha|}{\pi}\int_0^\infty \frac{\partial}{\partial s}\left(Ru(\exp sX\cdot\xi_0)\right)\frac{ds}{s} \tag{10'}$$

for any K-invariant $u \in \mathcal{D}(X)$. Here, taking account of $|H_\alpha| = |\alpha|^{-1}$,

$$Ru(g\cdot\xi_0) = \int_{\xi_0} u(g\cdot x)\ dm(x) = \frac{1}{|\alpha|}\int_{\mathbb{R}} u(g\cdot \operatorname{Exp} rH_\alpha)\ dr$$

for $g \in G$.

As in Lemma 1 let $Y = \frac{1}{2}(X - \theta X)$ and $Z = \frac{1}{2}(X + \theta X)$. Since $|Y| = |X|/\sqrt{2} = |\alpha|^{-1}$ the generators H_α, Y, Z will respectively correspond to

$$\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix},\ \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}$$

in the Lie algebra isomorphism of their linear span $\mathfrak{g}^* = \mathbb{R}H_\alpha \oplus \mathbb{R}Y \oplus \mathbb{R}Z$ with $sl(2,\mathbb{R})$. Note that $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$, with $\mathfrak{k}^* = \mathbb{R}Z$, $\mathfrak{p}^* = \mathbb{R}H_\alpha \oplus \mathbb{R}Y$, is a Cartan decomposition. Elementary matrix computations in $SL(2,\mathbb{R})$ can then give identities in G as we now explain.

Let $\varphi : sl(2,\mathbb{R}) \to \mathfrak{g}^*$ denote the above isomorphism and $\Phi : \widetilde{SL(2,\mathbb{R})} \to G^*$ the corresponding morphism of Lie groups from the universal covering of $SL(2,\mathbb{R})$ onto G^*, the Lie subgroup of G with Lie algebra $\mathfrak{g}^*$. Since $SL(2,\mathbb{R}) = \widetilde{SL(2,\mathbb{R})}/\Gamma$, where Γ is a discrete central subgroup of $\widetilde{SL(2,\mathbb{R})}$, an equality in $SL(2,\mathbb{R})$ will imply an equality modulo Γ in $\widetilde{SL(2,\mathbb{R})}$, whence by Φ an equality modulo $\Phi(\Gamma)$ in G^*. But, Φ being onto, $\Phi(\Gamma)$ is contained in the center of G^*, itself contained in the subgroup $K^* = \exp\mathbb{R}Z$ of G^* (see [2] p. 252). For instance the equality in $SL(2,\mathbb{R})$

$$\exp A \exp B = \exp C \exp D \exp E$$

with $A,\dots,E \in sl(2,\mathbb{R})$, implies in G^*

$$\exp\varphi(A)\exp\varphi(B) = k\exp\varphi(C)\exp\varphi(D)\exp\varphi(E)$$

for some $k \in \exp\mathbb{R}Z$ commuting to G^*.

Applying this principle, the Cartan decomposition ($G = KAK$)

$$\exp tY \exp rH_\alpha = k_1 \exp(wH_\alpha)k_2\ , \tag{11}$$

where k_1, k_2 belong to $\exp\mathbb{R}Z \subset K$ and $w = w(r,t) \geq 0$ is defined by

$$\operatorname{ch} w = \operatorname{ch} r \operatorname{ch} t\ , \tag{12}$$

follows from easy computations in $SL(2,\mathbb{R})$. The latter formula is the hyperbolic Pythagorean theorem. Therefore, for K-invariant u,

$$Ru(\exp tY\cdot\xi_0) = \frac{1}{|\alpha|}\int_{\mathbb{R}} u(\operatorname{Exp} w(r,t)H_\alpha)\ dr\ .$$

By (7) $\exp(\pi Z) \in K$ transforms H_α into $-H_\alpha$. Thus $u(\operatorname{Exp} wH_\alpha)$ is a compactly supported smooth even function of $w \in \mathbb{R}$, and there exists a compactly supported smooth function $\underline{u}$ on $[1,\infty[$ such that

$$u(\operatorname{Exp} wH_\alpha) = \underline{u}(\operatorname{ch} w)\ ,\ w \in \mathbb{R}$$

(see [13] p. 270 for a detailed proof), whence

$$Ru(\exp tY\cdot\xi_0) = \frac{1}{|\alpha|}\int_{\mathbb{R}} \underline{u}(\operatorname{ch} r \operatorname{ch} t)\ dr\ .$$

The left-hand side is thus a smooth even compactly supported function of $t \in \mathbb{R}$, which may be written as $\underline{Ru}(\tau)$ with $\tau = \operatorname{ch} t$ and

$$\underline{Ru}(\tau) = \frac{2}{|\alpha|} \int_0^\infty \underline{u}(\tau \operatorname{ch} r)\, dr \ .$$

This integral equation of Abel type can be solved for $\underline{u}$ in a classical way. First it implies the equality of integrals

$$\int_0^\infty \underline{Ru}(\tau \operatorname{ch} s) \frac{ds}{\operatorname{ch} s} = \frac{\pi}{|\alpha|} \int_\tau^\infty \underline{u}(\rho) \frac{d\rho}{\rho} \ , \ \tau \geq 1 \ . \tag{13}$$

Indeed the left-hand side is $2|\alpha|^{-1} \int\int \underline{u}(\tau \operatorname{ch} r \operatorname{ch} s) dr ds / \operatorname{ch} s$ and the double integral converges since $\underline{u}$ is compactly supported. By the change of variables $(r, s) \mapsto (\rho, \theta)$ with $\rho \geq \tau$ and $0 \leq \theta \leq \pi/2$ defined by

$$\rho = \tau \operatorname{ch} r \operatorname{ch} s \ , \ \sin \theta = \frac{\operatorname{sh} r}{\sqrt{\operatorname{ch}^2 r \operatorname{ch}^2 s - 1}} \tag{14}$$

we have $dr ds / \operatorname{ch} s = d\rho d\theta / \rho$ and (13) follows. Then, taking derivatives of (13) with respect to τ at $\tau = 1$, we obtain

$$-\frac{\pi}{|\alpha|} \underline{u}(1) = \int_0^\infty \left(\underline{Ru}\right)' (\operatorname{ch} s)\, ds \ .$$

In view of $\underline{u}(1) = u(x_0)$ and $\underline{Ru}(\operatorname{ch} t) = Ru(\exp tY \cdot \xi_0)$ this is (9).

To deduce (9') from (9) we use the Iwasawa decomposition ($G = KNA$)

$$\exp tY = k \exp((\operatorname{sh} t) X) a \tag{15}$$

with $k \in \exp \mathbb{R} Z$ and $a \in \exp \mathbb{R} H_\alpha$. By the principle explained above (15) follows again from easy computations in $SL(2, \mathbb{R})$. Then, for K-invariant u,

$$Ru(\exp tY \cdot \xi_0) = Ru(\exp(\operatorname{sh} t) X \cdot \xi_0)$$

and (9') follows with $s = \operatorname{sh} t$. □

Remarks. (i) By (11) the point $\exp tY \cdot \operatorname{Exp} rH_\alpha = k_1 \cdot \operatorname{Exp} wH_\alpha$ is at distance $|\alpha|^{-1} w$ from the origin; as r varies this is minimum for $r = 0$ by (12). Therefore the point $\operatorname{Exp} tY$ is the orthogonal projection of x_0 on the geodesic $\exp tY \cdot \xi_0$, and the shifted dual transform $R^*_{\exp tY}$ integrates over a family of geodesics at distance $|\alpha|^{-1} t$ from the point considered. For $R^*_{\exp sX}$ the distance is $|\alpha|^{-1} t$ given by $\operatorname{sh} t = s$.
(ii) Different choices of α lead to different inversion formulas. But, α being chosen, the choice of $Y \in \mathfrak{p}_\alpha$ (with $|Y| = |\alpha|^{-1}$) is irrelevant: indeed two such vectors lie in the same K-orbit since both can be transformed into H_α by the action of K (see (7) with $t = -\pi/2$).

4. Totally geodesic Radon transforms

4.1. **Inversion formulas.** Radon transforms on a large class of d-dimensional geodesic submanifolds can be inverted by a method similar to the above one for $d = 1$. But finding a shift suitable for all directions in these submanifolds requires reversing the roles: in the next theorem the direction of ξ_0 is assumed to lie in some eigenspace $\mathfrak{p}_\alpha$ and the shift is defined by the corresponding root vector H_α. The (more natural) opposite choice was made for $d = 1$ in Theorem 2. Both results are equivalent in this case however, as explained in Section 4.2 below.

Our main result can be formulated in several ways; variants will be given in Section 4.5. We keep to the previous notation: $X = G/K$ is a Riemannian symmetric space of the non-compact type, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, α a root of the pair $(\mathfrak{g}, \mathfrak{a})$ and the root vector $H_\alpha \in \mathfrak{a}$ is defined by $B(H, H_\alpha) = |\alpha|^{-2} \alpha(H)$ for all $H \in \mathfrak{a}$. We recall that α is called *indivisible* if $\alpha/2$ is not a root.

Theorem 3. *Let $\mathfrak{s}$ be a d-dimensional Lie triple system contained in the eigenspace $\mathfrak{p}_\alpha$, where α is an indivisible root.*
Let Ξ be a family of d-dimensional totally geodesic submanifolds of X, containing $\xi_0 = \operatorname{Exp}\mathfrak{s}$ (taken as the origin) and all $g \cdot \xi_0$ for $g \in G$.
Then ξ_0 is a rank one Riemannian symmetric space of the non-compact type, or a single geodesic, and the Radon transform R defined by integration over the elements of Ξ is inverted by the following formulas.

- *If $d = 2k$ is even there exists a polynomial of degree k with rational coefficients*
$$P(t) = \frac{2^k k!}{(2k)!} t^k + \cdots$$
such that, for all $u \in \mathcal{D}(X)$, $x \in X$,
$$(-2\pi)^k |\alpha|^{-d} u(x) = P\left(\frac{\partial^2}{\partial t^2}\right) \left(R^*_{\exp tH_\alpha} Ru(x)\right)\Big|_{t=0} . \tag{16}$$
- *If $d = 2k-1$ is odd*
$$(-2\pi)^k |\alpha|^{-d} u(x) = 2\sum_{l=1}^{k} a_{k,l} \int_0^\infty (\operatorname{ch} t)^{l-1} \left(\frac{1}{\operatorname{sh} t}\frac{\partial}{\partial t}\right)^l \left(R^*_{\exp tH_\alpha} Ru(x)\right) dt \tag{17}$$
for all $u \in \mathcal{D}(X)$, $x \in X$. The coefficients $a_{k,l}$ are positive integers inductively defined by
$$p_1(t) = t \ , \ p_{k+1}(t) = (t + 2k)p_k(t) + tp'_k(t) \ , \ p_k(t) = \sum_{l=1}^{k} a_{k,l} t^l \ .$$
In particular $a_{k,1} = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)$ and $a_{k,k} = 1$.

Proof. (i) Structure of ξ_0. The exponential map of ξ_0, which is the restriction of Exp to $\mathfrak{s}$, is a diffeomorphism of $\mathfrak{s}$ onto ξ. In particular ξ is simply connected and it follows ([11] p. 147 or [2] p. 244) that
$$\xi_0 = \xi^- \times \xi^0 \times \xi^+ ,$$
a direct product decomposition of Riemannian symmetric spaces, with ξ^0 Euclidean, ξ^- of the compact type and ξ^+ of the non-compact type. This corresponds to the decomposition
$$\mathfrak{s} = \mathfrak{s}^- \oplus \mathfrak{s}^0 \oplus \mathfrak{s}^+$$
of the Lie triple system. Now Exp should induce a diffeomorphism of the vector space $\mathfrak{s}^-$ onto the compact ξ^-, whence $\mathfrak{s}^- = 0$ and $\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^+$.

Besides the eigenspace $\mathfrak{p}_\alpha = (I - \theta)\mathfrak{g}_\alpha$ is contained in $\mathfrak{p}$ and in the Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ generated by $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. The root α being indivisible, $\mathfrak{g}'$ is a semisimple Lie algebra of real rank one with root space decomposition ([2] p. 407)
$$\mathfrak{g}' = \mathfrak{g}_{2\alpha} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha} . \tag{18}$$
Thus $\mathfrak{p} \cap \mathfrak{g}'$ and $\mathfrak{p}_\alpha$ a fortiori contain no abelian subspace of dimension greater than one.

Now let $\mathfrak{b}$ be a maximal abelian subspace of $\mathfrak{s}^+$. Then $\mathfrak{s}^0 \oplus \mathfrak{b}$ is an abelian subspace of $\mathfrak{s} \subset \mathfrak{p}_\alpha$, therefore one-dimensional at most whence $\mathfrak{s}^0 = 0$ or $\mathfrak{b} = 0$. In the first case $\xi_0 = \xi^+$ and $\dim \mathfrak{b} = 1$, in the latter $\mathfrak{s}^+ = 0$ and ξ_0 is one-dimensional. This implies the first assertion of Theorem 3.

(ii) Integration over ξ_0. All vectors relevant to the proof lie in the above rank one subalgebra $\mathfrak{g}'$. Let $'$ denote notions relative to $\mathfrak{g}'$, e.g. $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ etc. As a maximal abelian subspace of $\mathfrak{p}'$ we now use $\mathbb{R}Y$ instead of $\mathbb{R}H_\alpha$, with $Y \in \mathfrak{s} \subset \mathfrak{p}_\alpha \subset \mathfrak{p}'$ and $|Y| = |H_\alpha| = |\alpha|^{-1}$. As in Section 3 let $X \in \mathfrak{g}_\alpha \subset \mathfrak{g}'$ be such that $Y = \frac{1}{2}(X - \theta X)$ and let $Z = \frac{1}{2}(X + \theta X)$. By (7)
$$Y = \operatorname{Ad} k(H_\alpha) \ , \text{ with } k = \exp\left(-\frac{\pi}{2}Z\right) \in K' = K \cap G' . \tag{19}$$

Therefore $\operatorname{ad} Y$, as an endomorphism of $\mathfrak{g}'$, has the same eigenvalues as $\operatorname{ad} H_\alpha$ i.e. 0, $\pm\alpha(H_\alpha) = \pm 1$ and $\pm 2\alpha(H_\alpha) = \pm 2$. Thus

$$\mathfrak{p}' = \mathbb{R}Y \oplus \mathfrak{p}'_1 \oplus \mathfrak{p}'_2 \ ,$$

a decomposition into eigenspaces of $(\operatorname{ad} Y)^2$ with respective eigenvalues 0, 1 and 4. Similarly the stable subspace $\mathfrak{s} \subset \mathfrak{p}'$ decomposes as

$$\mathfrak{s} = \mathbb{R}Y \oplus \mathfrak{s}_1 \oplus \mathfrak{s}_2$$

with $\mathfrak{s}_1 = \mathfrak{s} \cap \mathfrak{p}'_1$, $\mathfrak{s}_2 = \mathfrak{s} \cap \mathfrak{p}'_2$ and the same respective eigenvalues for $(\operatorname{ad} Y)^2$. Setting $p = \dim \mathfrak{s}_1$, $q = \dim \mathfrak{s}_2$ we have $d = \dim \mathfrak{s} = p + q + 1$.

The jacobian $J(T)$ of $\operatorname{Exp} : \mathfrak{s} \to \xi_0$ at $T \in \mathfrak{s}$ is a radial function of T. Since $|T| = |rY|$ with $r = |\alpha||T|$ we have

$$\begin{aligned} J(T) = J(rY) &= \det \left(\frac{\operatorname{sh}(r \operatorname{ad} Y)}{r \operatorname{ad} Y} \right)\Bigg|_{\mathsf{s}} \\ &= \left(\frac{\operatorname{sh} r}{r} \right)^p \left(\frac{\operatorname{sh} 2r}{2r} \right)^q = r^{1-d} (\operatorname{sh} r)^{d-1} (\operatorname{ch} r)^q \ . \end{aligned}$$

The integral over ξ_0 of a radial function f is therefore

$$\begin{aligned} \int_{\xi_0} f(y)\ dm(y) &= \int_{\mathsf{s}} f(\operatorname{Exp} T) J(T) dT \\ &= \frac{2\pi^{d/2}|\alpha|^{-d}}{\Gamma(d/2)} \int_0^\infty f(\operatorname{Exp} rY)(\operatorname{sh} r)^{d-1}(\operatorname{ch} r)^q dr \ , \end{aligned} \tag{20}$$

with $Y \in \mathfrak{s}$, $|Y| = |\alpha|^{-1}$.

(iii) The Radon integral. By Section 2.2 it will suffice to work with a K-invariant function $u \in \mathcal{D}(X)$ and to prove (16) and (17) at x_0, with $R^*_{\exp tH_\alpha} Ru(x)$ replaced by $Ru(\exp tH_\alpha . \xi_0)$. The latter can be computed by means of a Cartan decomposition, easily checked in $SL(2, \mathbb{R})$ as (11)(12) above: for any $T \in \mathfrak{s}$ there exist $k_1, k_2 \in K$ such that

$$\exp tH_\alpha \exp T = k_1 \exp(wH_\alpha) k_2 \ ,$$

with $w = w(r,t) \geq 0$ defined by

$$\operatorname{ch} w = \operatorname{ch} t \operatorname{ch} r \ , \ r = |\alpha||T| \ .$$

By (20) it follows that

$$\begin{aligned} Ru(\exp tH_\alpha . \xi_0) &= \int_{\xi_0} u(\exp tH_\alpha . y)\ dm(y) \\ &= \frac{2\pi^{d/2}|\alpha|^{-d}}{\Gamma(d/2)} \int_0^\infty u(\exp tH_\alpha . \operatorname{Exp} rY)(\operatorname{sh} r)^{d-1}(\operatorname{ch} r)^q dr \\ &= \frac{2\pi^{d/2}|\alpha|^{-d}}{\Gamma(d/2)} \int_0^\infty u(\operatorname{Exp} w(r,t) H_\alpha)(\operatorname{sh} r)^{d-1}(\operatorname{ch} r)^q dr \ . \end{aligned}$$

As in the proof of Theorem 2 we may now write

$$u(\operatorname{Exp} wH_\alpha) = \underline{u}(\operatorname{ch} w) \ , \ Ru(\exp tH_\alpha . \xi_0) = \underline{Ru}(\tau) \ , \ \tau = \operatorname{ch} t \ .$$

Then

$$\underline{Ru}(\tau) = \frac{2\pi^{d/2}|\alpha|^{-d}}{\Gamma(d/2)} \int_0^\infty \underline{u}(\tau \operatorname{ch} r)(\operatorname{sh} r)^{d-1}(\operatorname{ch} r)^q dr \ , \tag{21}$$

or else, with $\rho = \tau \operatorname{ch} r$,

$$\tau^{d-1+q} \underline{Ru}(\tau) = \frac{2\pi^{d/2}|\alpha|^{-d}}{\Gamma(d/2)} \int_\tau^\infty \underline{u}(\rho)(\rho^2 - \tau^2)^{\frac{d}{2}-1} \rho^q d\rho \ . \tag{22}$$

(iv) Assume d even, $d = 2k$. Then repeated applications of the derivation $\tau^{-1}\partial_\tau$ lead to the following inversion of (22):

$$(-2\pi)^k |\alpha|^{-d} \tau^{q-1} \underline{u}(\tau) = \left(\tau^{-1}\partial_\tau\right)^k \left(\tau^{2k-1+q} \underline{Ru}(\tau)\right)$$

and, for $\tau = 1$,

$$(-2\pi)^k |\alpha|^{-d} u(x_0) = \left(\partial_\tau^k + \cdots + (q+1)(q+3)\cdots(q+2k-1)\right) \underline{Ru}(\tau)\Big|_{\tau=1}$$

where the operator $\left(\partial_\tau^k + \cdots\right)$ is a polynomial in ∂_τ of degree k with integer coefficients depending on k and q. But $\underline{Ru}(\tau) = Ru(\exp tH_\alpha.\xi_0)$ is an even function of t and, identifying Taylor expansions at $\tau = 1$ resp $t = 0$, related by $\tau - 1 = \operatorname{ch} t - 1 = \frac{t^2}{2} + \cdots$, we obtain a triangular system of linear relations between derivatives which is solved as

$$\partial_\tau^l \underline{Ru}(1) = \left(\frac{2^l l!}{(2l)!}\partial_t^{2l} + \cdots + a_l \partial_t^2\right) Ru(\exp tH_\alpha \cdot \xi_0)\Big|_{t=0}$$

for $l \geq 1$, where the dots are a sum of even derivatives of decreasing order, mutliplied by some rational coefficients (like a_l). The result (16) follows for $d = 2k$.

(v) Assume d odd, $d = 2k - 1$. Since $d = p + q + 1$ the multiplicities p and q must have the same parity. By the classification of symmetric spaces of rank one (or by Araki's results on multiplicities, [2] p. 530), q must then vanish and the geodesic submanifold ξ_0 is isomorphic to $H^d(\mathbb{R})$. To invert (21) or (22), an integral equation of Abel type, we need the integral formula (extending (13) in Section 3)

$$\int_0^\infty \underline{Ru}(\tau \operatorname{ch} s) \frac{ds}{\operatorname{ch} s} = \frac{\pi^k |\alpha|^{-d}}{(k-1)!} \int_\tau^\infty \underline{u}(\rho) \left(\frac{\rho^2}{\tau^2} - 1\right)^{k-1} \frac{d\rho}{\rho}\ , \ \tau \geq 1\ , \tag{23}$$

which follows from (21) after some straightforward calculations by means of the change of variables (14). The right-hand side of (23) is similar to the one in (22), and (23) is now inverted by repeated applications of the derivation $\tau^3\partial_\tau$:

$$(-2\pi)^k |\alpha|^{-d} \tau^{2k} \underline{u}(\tau) = 2\left(\tau^3 \partial_\tau\right)^k \left(\int_0^\infty \underline{Ru}(\tau \operatorname{ch} s)\frac{ds}{\operatorname{ch} s}\right)\ . \tag{24}$$

It is easily checked by induction that, for any smooth function f,

$$\left(\tau^3\partial_\tau\right)^k \left(f(\tau\sigma)\right) = \tau^{2k} \sum_{l=1}^k a_{k,l} (\tau\sigma)^l f^{(l)}(\tau\sigma)$$

with $a_{k+1,l} = a_{k,l-1} + (2k + l)a_{k,l}$, which is equivalent to the claimed induction formula satisfied by the polynomials p_k. Taking $f = \underline{Ru}$, $\sigma = \operatorname{ch} s$ and $\tau = 1$, it follows that

$$(-2\pi)^k |\alpha|^{-d} u(x_0) = 2\sum_{l=1}^k a_{k,l} \int_0^\infty (\operatorname{ch} s)^{l-1} (\underline{Ru})^{(l)}(\operatorname{ch} s)\, ds\ .$$

Going back to $Ru(\exp sH_\alpha \cdot \xi_0) = \underline{Ru}(\operatorname{ch} s)$ we have

$$(\underline{Ru})^{(l)}(\operatorname{ch} s) = \left(\frac{1}{\operatorname{sh} s}\frac{\partial}{\partial s}\right)^l \left(Ru(\exp sH_\alpha \cdot \xi_0)\right)$$

and (17) is proved at x_0 for a K-invariant function u. □

Remark. As in Section 3, the point $\operatorname{Exp} tH_\alpha$ is the orthogonal projection of the origin x_0 on the geodesic submanifold $\exp tH_\alpha.\xi_0$, and the shifted dual transform $R^*_{\exp tH_\alpha}$ integrates over a family of submanifolds at distance $|\alpha|^{-1}t$ from the point considered.

4.2. **Example 1: the X-ray transform.** For $d = 1$ formula (17) reduces to

$$u(x) = -\frac{|\alpha|}{\pi}\int_0^\infty \frac{\partial}{\partial t}\left(R^*_{\exp tH_\alpha} Ru(x)\right)\frac{dt}{\operatorname{sh} t}\ , \tag{25}$$

with $\xi_0 = \operatorname{Exp}\mathbb{R}Y$, $Y \in \mathfrak{p}_\alpha$ and (for convenience) $|Y| = |\alpha|^{-1}$. This is (9) with H_α and Y interchanged.

Actually (25) *is equivalent to* (9). Indeed by (7) we have $\operatorname{Ad} k_0(H_\alpha) = -Y$ and $\operatorname{Ad} k_0(Y) = H_\alpha$ with $k_0 = \exp\left(\frac{\pi}{2}Z\right) \in K$, therefore the set $\Xi = \{g\cdot\xi_0, g \in G\}$ of geodesics remains the same if the origin $\xi_0 = \operatorname{Exp}\mathbb{R}Y$ is replaced by $\xi_0' = k_0\cdot\xi_0 = \operatorname{Exp}\mathbb{R}H_\alpha = k_0^{-1}\cdot\xi_0$. Besides

$$\exp tH_\alpha\cdot\xi_0 = k_0\exp tY\cdot\xi_0'\ .$$

Letting R^*, resp. R'^*, denote the dual Radon transform when the origin is ξ_0, resp. ξ_0', it then follows from (4) that

$$\begin{aligned} R^*_{\exp tH_\alpha} Ru(g\cdot x_0) &= \int_K Ru(gk\exp tH_\alpha\cdot\xi_0)\,dk = \int_K Ru(gk\exp tY\cdot\xi_0')\ dk \\ &= R'^*_{\exp tY} Ru(g\cdot x_0)\ . \end{aligned}$$

This implies our claim.

4.3. **Example 2: the classical hyperbolic spaces.** Let $X = H^n(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ be one of the classical hyperbolic spaces. Then $X = G/K$ with $G = U(1,n;\mathbb{F})$, $K = U(1;\mathbb{F})\times U(n;\mathbb{F})$, and the Cartan decomposition is $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ where $\mathfrak{p}$, the space of all matrices

$$V = \begin{pmatrix} 0 & \overline{V_1} & \cdots & \overline{V_n} \\ V_1 & & & \\ \vdots & & (0) & \\ V_n & & & \end{pmatrix}\ ,\ V_i \in \mathbb{F}\ ,$$

can be identified with $\mathbb{F}^n$.

Let $\overline{V}\cdot W = \sum_{i=1}^n \overline{V_i}W_i$. The scalar product of $V, W \in \mathfrak{p}$ (as a real vector space) is $\operatorname{Re}(\overline{V}\cdot W)$ up to a constant factor. For $U, V, W \in \mathfrak{p} = \mathbb{F}^n$, easy computations lead to

$$[U,[V,W]] = U\left(\overline{V}\cdot W - \overline{W}\cdot V\right) - V(\overline{W}\cdot U) + W(\overline{V}\cdot U)\ ; \tag{26}$$

here $\mathbb{F}^n$ is considered as a $\mathbb{F}$-vector space, with scalars acting on the right.

Having chosen $H \in \mathfrak{p}$, $H \neq 0$, the eigenspaces of $(\operatorname{ad} H)^2$ can be obtained from (26) whence the decomposition

$$\begin{gathered} \mathfrak{p} = \mathfrak{a}\oplus\mathfrak{p}_\alpha\oplus\mathfrak{p}_{2\alpha}\ \text{, with } \mathfrak{a} = \mathbb{R}H \text{ and} \\ \mathfrak{p}_\alpha = \{V \in \mathfrak{p}\mid \overline{H}\cdot V = 0\}\ ,\ \mathfrak{p}_{2\alpha} = \{H\lambda \mid \lambda\in\mathbb{F}, \lambda+\overline{\lambda} = 0\}\ . \end{gathered}$$

The respective eigenvalues are 0, $\overline{H}\cdot H$ and $4\left(\overline{H}\cdot H\right)$.

Lemma 4. *Any $\mathbb{F}$-subspace $\mathfrak{s}$ of $\mathfrak{p}$ is a Lie triple system.*
A real vector subspace $\mathfrak{s}$ of $\mathfrak{p}$ is contained in $\mathfrak{p}_\alpha$ (for some choice of $H \in \mathfrak{p}$) if and only if the $\mathbb{F}$-subspace of $\mathfrak{p}$ generated by $\mathfrak{s}$ is not $\mathfrak{p}$ itself.

Proof. The first assertion is immediate from (26) and the second from the above expression of $\mathfrak{p}_\alpha$. ∎

4.4. **Example 3: the eigenspaces.** Again let α be an indivisible root for $X = G/K$, an arbitrary Riemannian symmetric space of the non-compact type. The assumptions of Theorem 3 are satisfied by $\mathfrak{s} = \mathfrak{p}_\alpha$.

Indeed, for any linear form λ on $\mathfrak{a}$ let

$$\mathfrak{k}_\lambda = \{Z \in \mathfrak{k} \mid (\operatorname{ad} H)^2 Z = \lambda(H)^2 Z \text{ for all } H \in \mathfrak{a}\} .$$

Then (see [2] p.335)

$$\begin{aligned} &[\mathfrak{p}_\alpha, \mathfrak{p}_\alpha] \subset \mathfrak{k}_{2\alpha} + \mathfrak{k}_0 , \\ &[\mathfrak{p}_\alpha, [\mathfrak{p}_\alpha, \mathfrak{p}_\alpha]] \subset [\mathfrak{p}_\alpha, \mathfrak{k}_{2\alpha}] + [\mathfrak{p}_\alpha, \mathfrak{k}_0] \subset (\mathfrak{p}_{3\alpha} + \mathfrak{p}_\alpha) + \mathfrak{p}_\alpha = \mathfrak{p}_\alpha \end{aligned}$$

since 3α is not a root. Thus $\mathfrak{p}_\alpha$ is a Lie triple system.

4.5. **Variants.** The method of proof of Theorem 3 can provide other inversion formulas such as (17') or (17") below, both valid for any d, even or odd. Here n is any integer such that $n > d/2$, and $C_n = (-1)^n 2^{n-1} \pi^{d/2} \Gamma\left(n - \frac{d}{2}\right) |\alpha|^{-d}$.

$$C_n u(x) = \left(\tau^{-1}\partial_\tau\right)^n \left(\int_\tau^\infty R^*_{\exp sH_\alpha} Ru(x) \sigma^{d+q} \left(\sigma^2 - \tau^2\right)^{n-1-(d/2)} d\sigma \right)\Bigg|_{\tau=1} \tag{17'}$$

$$C_n u(x) = \left(\tau^{-1}\partial_\tau\right)^n \left(\tau^d \int_\tau^\infty R^*_{\exp sH_\alpha} Ru(x) \sigma^{d-2n+q} \left(\sigma^2 - \tau^2\right)^{n-1-(d/2)} d\sigma \right)\Bigg|_{\tau=1} \tag{17"}$$

with $\sigma = \operatorname{ch} s$, $s \geq 0$, under the integrals.

For $d = 2k$ the smallest n is $k+1$ and (17') gives back the results of (iv) in the proof of the theorem. For $d = 2k-1$ the smallest n is k and (17')(17") are variants of (17). Formula (17) was preferred in Theorem 3 because of its similarity with Theorem 2. Changing σ to $1/\sigma$ and τ to $1/\tau$ it may be checked also that (17") generalizes Theorem 14 (i) in [14] p. 237, itself a generalization of Helgason's result for $H^n(\mathbb{R})$ in [4] p. 144 or [6] p. 97.

Let us sketch brief proofs of (17') and (17"). From (22) it follows that, for any a and any $n > d/2$,

$$\int_\tau^\infty \underline{Ru}(\sigma)\sigma^a \left(\sigma^2 - \tau^2\right)^{n-1-(d/2)} d\sigma = \frac{2\pi^{d/2}|\alpha|^{-d}}{\Gamma(d/2)} \int_\tau^\infty \underline{u}(\rho)\rho^q A(\rho)\, d\rho , \tag{27}$$
$$\text{with } A(\rho) = \int_\tau^\rho \sigma^{a-d-q+1} \left(\rho^2 - \sigma^2\right)^{(d/2)-1} \left(\sigma^2 - \tau^2\right)^{n-1-(d/2)} d\sigma .$$

The latter integral is hypergeometric, but boils down to an elementary function when $a = d+q$, resp. $a = d - 2n + q$. It is easily computed by replacing the variable σ by $x \in]0,1[$ such that

$$\sigma^2 = x\rho^2 + (1-x)\tau^2 \text{ , resp. } \sigma^{-2} = x\rho^{-2} + (1-x)\tau^{-2} .$$

Up to a constant factor the right-hand side of (27) becomes

$$\int_\tau^\infty \underline{u}(\rho)\rho^q \left(\sigma^2 - \tau^2\right)^{n-1} d\rho \text{ , resp. } \tau^{-d} \int_\tau^\infty \underline{u}(\rho)\rho^{d-2n+q} \left(\sigma^2 - \tau^2\right)^{n-1} d\rho .$$

If n is an integer (17') resp. (17") follow by applying n times the operator $\tau^{-1}\partial_\tau$.

5. Notes

In this final section the above method and results will be compared with a few others from the literature, restricting ourselves to the X-ray transform. The following short list of related works is of course far from exhaustive.

5.1. X-ray transforms are inverted here by means of shifted dual transforms. This method, initiated by J. Radon (1917) [12] for $\mathbb{R}^2$ and $H^2(\mathbb{R})$, was later extended by S. Helgason to $H^n(\mathbb{R})$ (1958, published[3] in 1990 [4]). In our present notation Helgason's result is

$$u(x) = \frac{1}{\pi}\partial_\tau \left(\int_0^\tau R^*_{\exp sH_\alpha} Ru(x) \frac{d\sigma}{\sqrt{\tau^2 - \sigma^2}} \right)\Big|_{\tau=1}$$

with $\sigma = 1/\operatorname{ch} s$, $s \geq 0$, under the integral. Changing σ to $1/\sigma$ and τ to $1/\tau$, this is (17") above for $k = 1$.

5.2. A different method was used by C. Berenstein and E. Casadio Tarabusi (1991). For the X-ray transform on $H^n(\mathbb{R})$, $n \geq 4$, they obtain ([1] p. 628)

$$u(x) = -(L + n - 2)SR^*Ru(x) \tag{28}$$

where L is the Laplace-Beltrami operator, R^* is the classical dual Radon transform and S is the convolution operator by a suitable radial function $S(r)$ on $H^n(\mathbb{R})$. Observing that R^*R itself is a convolution operator (by a radial function proportional to $(\operatorname{sh} r)^{1-n}$), their idea was to choose S so that the composition SR^*R could be inverted by a differential operator. This was accomplished by means of radial harmonic analysis on the space, leading to

$$S(r) = C\,(\operatorname{sh} r)^{1-n} \operatorname{ch} r \ , \text{ with } C = \frac{\left(\Gamma\left(\frac{n-1}{2}\right)\right)^2}{4\pi^{(n/2)+1}\Gamma\left(n/2\right)} \ . \tag{29}$$

5.3. Yet another approach is B. Rubin's (2002). For the X-ray transform on $H^n(\mathbb{R})$, $n \geq 4$, he proves that ([15] p. 208)

$$u(x) = -(L + n - 2)R^{*1}Ru(x) \ , \tag{30}$$

where R^{*1} is the integral operator transforming a function φ on the space Ξ of all geodesics into the function

$$R^{*1}\varphi(x) = C' \int_\Xi \varphi(\xi)(\operatorname{sh} d(x,\xi))^{2-n} d\xi \ , \text{ with } C' = \frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{4\pi^{(n+1)/2}\Gamma(n/2)} \ ; \tag{31}$$

as before d denotes the distance. Note that Rubin's operator R^{*1} integrates over all geodesics ξ, whereas the dual transform R^* integrates only over geodesics passing through a given point x, and the shifted dual transform R^*_γ over geodesics at a given distance from x.

The similarity between (28) and (30) is explained by the next lemma.

Lemma 5. *Retain the above notation on $H^n(\mathbb{R})$. Then*

$$SR^*\varphi = R^{*1}\varphi$$

for any function φ on Ξ such that the right-hand side is an absolutely convergent integral.

Proof. The operators S, R^* and R^{*1} commute with the action of G on $X = H^n(\mathbb{R})$, it will therefore suffice to prove the result at the origin x_0. By the duality between R and R^* the integral

$$SR^*\varphi(x_0) = \int_X R^*\varphi(x)S(d(x_0,x))\ dx$$

may be written as

$$SR^*\varphi(x_0) = \int_\Xi \varphi(\xi)RS(\xi)\ d\xi \ , \text{ with } RS(\xi) = \int_\xi S(d(x_0,x))\ dm_\xi(x) \ .$$

[3] "The formula for d odd seemed unreasonably complicated compared to [the formula] for d even, and the case $d = 1$, [which] is the X-ray transform, had not acquired its later distinction through tomography", Helgason commented on this 32 years delay ([4] p.142).

Let x_1 be the orthogonal projection of x_0 on the geodesic ξ. Using the distance r to x_1 as a coordinate on ξ the latter integral becomes

$$RS(\xi) = 2\int_0^\infty S(w)\ dr$$

with $w = d(x_0, x)$, $t = d(x_0, x_1) = d(x_0, \xi)$, $r = d(x_1, x)$ and $\operatorname{ch} w = \operatorname{ch} t \operatorname{ch} r$, as in (12). By (29) $S(w) = C\,(\operatorname{sh} w)^{1-n} \operatorname{ch} w$ therefore, for $t > 0$,

$$RS(\xi) = 2C\int_0^\infty \frac{\operatorname{ch} t \operatorname{ch} r}{\left(\operatorname{ch}^2 t \operatorname{ch}^2 r - 1\right)^{(n-1)/2}} dr\ ,$$

and changing the variable r into $x = \left(\operatorname{ch}^2 t \operatorname{ch}^2 r - 1\right)^{-1} \operatorname{sh}^2 t$, $0 < x < 1$, it is easily checked that

$$RS(\xi) = C\frac{\pi^{1/2}\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}(\operatorname{sh} t)^{2-n} = C'\left(\operatorname{sh} d(x_0, \xi)\right)^{2-n}\ .$$

□

5.4. S. Ishikawa's method is completely different ([8][9][10]), involving harmonic analysis on the non-Riemannian symmetric space $\Xi = G/H$. Though not explicitly written in these articles, inversion formulas might be obtained from them; it would be interesting to interpret them geometrically.

5.5. The present Theorem 3 extends the similar Theorem 14 from our previous paper ([14] p. 237), valid for the classical hyperbolic spaces $H^n(\mathbb{F})$ only, and under the stronger assumption that $\mathfrak{s}$ is a (strict) $\mathbb{F}$-vector subspace of $\mathfrak{p}$ (cf. Lemma 4). Besides, the inversion formula for odd-dimensional submanifolds is now given a (hopefully) more manageable form.

References

[1] BERENSTEIN, C. and E. CASADIO TARABUSI, Inversion formulas for the k-dimensional Radon transform in real hyperbolic spaces, *Duke Math. J.* 62 (1991), 613-631.
[2] HELGASON, S., *Differential geometry, Lie groups and symmetric spaces*, Academic Press 1978.
[3] HELGASON, S., *Groups and geometric analysis*, Academic Press 1984.
[4] HELGASON, S., The totally geodesic Radon transform on constant curvature spaces, *Contemp. Math.* 113 (1990), 141-149.
[5] HELGASON, S., *Geometric analysis on symmetric spaces*, Math. Surveys and Monographs 39, Amer. Math. Soc. 1994.
[6] HELGASON, S., *The Radon transform*, second edition, Birkhäuser 1999.
[7] HELGASON, S., private communication 2004.
[8] ISHIKAWA, S., The range characterizations of the totally geodesic Radon transform on the real hyperbolic space, *Duke Math. J.* 90 (1997), 149-203.
[9] ISHIKAWA, S., Symmetric subvarieties in compactifications and the Radon transform on Riemannian symmetric spaces of the noncompact type, *J. Funct. Anal.* 204 (2003), 50-100.
[10] ISHIKAWA, S., The Radon transform for double fibrations of semisimple symmetric spaces, preprint 2004.
[11] LOOS, O., *Symmetric spaces I: general theory*, Benjamin 1969.
[12] RADON, J., Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten, *Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math. Nat. Kl.* 69 (1917), 262-277; English translation in DEANS, S., *The Radon transform and some of its applications*, Krieger 1993, 204-217.
[13] ROUVIÈRE, F., Sur la transformation d'Abel des groupes de Lie semi-simples de rang un, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 10 (1983), 263-290.
[14] ROUVIÈRE, F., Inverting Radon transforms: the group-theoretic approach, *Enseign. Math.* 47 (2001), 205-252.
[15] RUBIN, B., Radon, cosine and sine transforms on real hyperbolic space, *Adv. Math.* 170 (2002), 206-223.

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