ON RADON TRANSFORMS

AND THE KAPPA OPERATOR

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1. Introduction

In 1917 Johann Radon solved the following problem : find a function f on the Euclidean plane $\mathbb{R}^2$ knowing its integrals

$$Rf(\xi) = \int_\xi f$$

along all lines ξ in the plane. The operator R is now called the *Radon transform.* Apart from an important contribution by Fritz John (1938) the problem fell into oblivion for about four decades, until it was given a nice general differential geometric framework on the one hand and applications in medicine or physics on the other hand. In a radiograph of the human body the brightness of each point is determined by the absorption of X-ray light by bones and tissues, integrated along each ray.

More generally let X be a manifold and let Y be a family of submanifolds ξ of X equipped with measures dm_ξ (e.g. induced by a Riemannian measure on X). The Radon transform of a function f on X is the function Rf on Y defined by

$$Rf(\xi) = \int_{x\in\xi} f(x)\ dm_\xi(x)\ ,\ \xi \in Y\ ,$$

if the integral converges. The study of R is part of *integral geometry.*
Problem 1 (inversion formula) : Reconstruct f from Rf. A natural tool here is the *dual Radon transform* $\varphi \mapsto R^*\varphi$, mapping functions φ on Y into functions on X, with

$$R^*\varphi(x) = \int_{\xi\ni x} \varphi(\xi)\ dm_x(\xi)\ ,\ x \in X\ .$$

Thus φ is integrated over all all submanifolds ξ containing the point x, with respect to a suitably chosen measure dm_x. The corresponding differential geometric framework (Helgason, Gelfand, Guillemin,...) is a *double fibration*

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

where Z is the submanifold of $X \times Y$ consisting of all couples (x, ξ) such that $x \in \xi$.
Problem 2 (range theorem) : Characterize the image under R of various function spaces on X.

Problem 3 (support theorem) : Let K be a compact subset of X (e.g. a closed ball if X is a complete Riemannian manifold). Prove that $\operatorname{supp} f$ is contained in K if and only if $Rf(\xi) = 0$ for all submanifolds $\xi \in Y$ disjoint from K.

Only Problem 1 will be considered here. In section 2 we will obtain inversion formulas for homogeneous spaces X and Y by means of tools from Lie group theory (Helgason and others). Two methods are presented, using either convolutions or shifted dual transforms. A completely different approach is given in section 3 by means of a rather mysterious differential form (the kappa operator) introduced by Gelfand and his school.

2. Radon transforms on homogeneous spaces

a. Going back to J. Radon's original problem, it was a fundamental observation by S. Helgason (1966) that the set X of points and the set Y of lines in the Euclidean plane are both homogeneous spaces of a same group G, the isometry group of $\mathbb{R}^2$. This led him to study the following general situation ([7] chap. II, [9] chap. I).

Let X and Y be two manifolds with given origins $x_0 \in X$ and $\xi_0 \in Y$ and assume a Lie group G acts transitively on both manifolds X and Y. The elements $x \in X$ and $\xi \in Y$ are said to be *incident* if they have the same relative position as x_0 and ξ_0, i.e. if there exists some $g \in G$ such that $x = g \cdot x_0$ and $\xi = g \cdot \xi_0$. In most examples the ξ's are submanifolds of X, the origins are chosen so that $x_0 \in \xi_0$ and the incidence relation is simply $x \in \xi$.

Let K be the subgroup of G which stabilizes x_0. It is a closed Lie subgroup of G, a point $x = g \cdot x_0$ identifies with the left coset gK and the manifold X with the homogeneous space G/K. Likewise $Y = G/H$ where H is the stabilizer of ξ_0 in G.

The incidence relation translates into group-theoretic terms : $x = g_1 \cdot x_0$ and $\xi = g_2 \cdot \xi_0$ are incident if and only if there exists $g \in G$ such that $g_1 \cdot x_0 = g \cdot x_0$ and $g_2 \cdot \xi_0 = g \cdot \xi_0$, i.e. iff $g_1 k = g$ and $g_2 h = g$ for some $k \in K$, $h \in H$, i.e. iff the left cosets $g_1 K$ and $g_2 H$ are not disjoint in G. The set Z of all incident couples (x, ξ) identifies with the homogeneous space $G/K \cap H$.

Let us assume that *K is compact*. Then the *Radon transform* of a function f on X can be defined as

$$Rf(g \cdot \xi_0) = \int_H f(gh \cdot x_0)\, dh \ , \ g \in G$$

where dh is a left invariant measure on H. It is easily checked that f is here integrated over all x incident to $\xi = g \cdot \xi_0$. Likewise the *dual Radon transform* of a function φ on Y is

$$R^*\varphi(g \cdot x_0) = \int_K \varphi(gk \cdot \xi_0)\, dk \ ,$$

an integral of φ over all ξ incident to $x = g \cdot x_0$.

b. A detailed study of R has been be carried out by Helgason when X is a Riemannian symmetric space of the noncompact type and Y is a family of submanifolds of X. Let us start with the basic example of the two-dimensional hyperbolic space

$X = H^2(\mathbb{R})$. This Riemannian manifold can be realized as the open unit disc $|z| < 1$ in $\mathbb{C}$ equipped with the metric

$$ds^2 = 4\frac{|dz|^2}{(1-|z|^2)^2}\ ,\ |dz|^2 = dx^2 + dy^2\ .$$

Here $G = SU(1,1)$, the group of matrices $g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}$ with $a, b \in \mathbb{C}$ and $\det g = 1$, acting on $\mathbb{C}$ by

$$g \cdot z = \frac{az+b}{\overline{b}z+\overline{a}}\ .$$

The origin $x_0 = 0$ is stabilized by the subgroup K defined by $b = 0$ whence $X = G/K$. Now the Euclidean lines of $\mathbb{R}^2$ in Radon's original problem admit two analogues :

- *geodesics* of X, i.e. circles (or lines) meeting orthogonally the circle at infinity $|z| = 1$
- *horocycles* of X, i.e. curves orthogonal to a family of "parallel" geodesics (geodesics having a common point at infinity), i.e. circles contained in $|z| < 1$ and tangent to $|z| = 1$.

Which is best ? The former may seem more natural from a geometric point of view, but the latter turns out to be closely related to harmonic analysis on X. Indeed its Laplace operator admits a family of eigenfunctions which are constant on each horocycle. The horocycle Radon transform is thus linked to the analogue of the Fourier transform on X.

All this extends to the n-dimensional hyperbolic space $X = H^n(\mathbb{R})$, with $(n-1)$-dimensional horospheres instead of horocycles, and even to all Riemannian symmetric spaces of the noncompact type.

c. The following proposition ([12] p. 215) gives a first clue towards an inversion formula of R.

Proposition 1 *In the general setting of* ***a*** *above assume that $Y = G/H$ has a G-invariant measure. Then R^*R is a convolution operator on $X = G/K$.*

Convolution on X is deduced from convolution on the Lie group G by means of the natural projection $G \to G/K$.

Idea of proof. From the definitions of R and R^* it is clear that R^*R commutes with the action of G on X and the proposition easily follows.

To invert R it is therefore sufficient to find an inverse of this convolution and harmonic analysis on X provides natural tools for that. In the next theorem C denotes various nonzero constant factors, r is the (Riemannian) distance from the origin in X, Δ is the Laplace operator of X and f is an arbitrary function in $C_c^\infty(X)$.

Theorem 2 *(i) For the Radon transform on lines in $\mathbb{R}^n$ one has*

$$R^*Rf = C\ f * r^{1-n}$$
$$f = \Delta S R^* R f = C\ (-\Delta)^{1/2} R^* R f$$

where S is the convolution operator by $C\ r^{1-n}$.
(ii) For the Radon transform on geodesics in $H^n(\mathbb{R})$ one has

$$R^*Rf = C\ f * (\sinh r)^{1-n}$$
$$f = (\Delta + n - 2)SR^*Rf$$

where S is the convolution operator by $C\ \ (\sinh r)^{1-n} \cosh r$.
(iii) For the Radon transform on horospheres of $H^n(\mathbb{R})$ one has

$$R^*Rf = C\ f * (\sinh r)^{-1} \left(\cosh \frac{r}{2}\right)^{3-n}$$
$$f = P(\Delta)R^*Rf \ \text{ if } n = 2k+1\ ,$$

where P is a polynomial of degree k.

In $\mathbb{R}^n$ the symbol $*$ denotes the usual convolution whereas if $g(r)$ is a radial function on $H^n(\mathbb{R})$

$$(f * g)\,(x) = \int_{H^n(\mathbb{R})} f(y)g(d(x,y))\ dm(y)\ ,$$

where $d(.,.)$ is the Riemannian distance and dm is the Riemannian measure.
See [7] p. 29 for (i), [1] p. for (ii) and [12] p. 218 and 229 for (iii).

d. A different method, initiated by Radon himself in his 1917 paper, makes use of the *shifted dual Radon transform* defined (in the group-theoretic setting of **a**) by

$$R^*_\gamma\varphi(g \cdot x_0) = \int_K \varphi(gk\gamma \cdot \xi_0)\ dk\ ,$$

with $g, \gamma \in G$. The origin ξ_0 in Y is now replaced by the shifted origin $\gamma \cdot \xi_0$. Roughly speaking, instead of integrating φ over all ξ containing $x = g \cdot x_0$ as was done by R^* (in most examples at least), we now integrate over all ξ at a given distance from x. The shifted transform turns out to be a convenient tool to prove inversion formulas. In Theorems 3 and 4 below the origins are chosen so that $x_0 \in \xi_0$ and f is an arbitrary function in $C_c^\infty(X)$.

Theorem 3 *(i) For the Radon transform on lines in $\mathbb{R}^n$ one has*

$$f(x) = -\frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial t} \left(R^*_{\gamma(t)}Rf(x)\right) \frac{dt}{t}$$

where $\gamma(t)$ is a translation of length t orthogonally to the line ξ_0.
(ii) Let $X = G/K$ be a Riemannian symmetric space of the noncompact type. The Radon transform on geodesics of X is inverted by

$$f(x) = C \int_0^\infty \frac{\partial}{\partial t} \left(R^*_{\gamma(t)}Rf(x)\right) \frac{dt}{\sinh t}$$

with $\gamma(t) = \exp tV \in G$, V being a suitably chosen vector orthogonal to ξ_0 in the tangent space to X at x_0.

Formula (i) extends Radon's result to the n-dimensional Euclidean space, (ii) is proved in [13] and independently in [10].

Idea of proof. It suffices to work at $x = x_0$ and, taking averages over K, with K-invariant (radial) functions f. Then (i) is proved by solving an integral equation of Abel type. For (ii) Lie bracket computations show that V and the tangent to ξ_0 at x_0 generate a two-dimensional hyperbolic subspace of X. In this $H^2(\mathbb{R})$ the integrals can be written down explicitly and the problem boils down again to an Abel integral equation.

Theorem 4 *Let $X = G/K$ be a Riemannian symmetric space of the noncompact type. The Radon transform on horospheres of X is inverted by*

$$f(x) = < T(a), R_a^* Rf(x) >$$

where the variable a runs over A, the Abelian subgroup in an Iwasawa decomposition $G = KAN$, and $T(a)$ is a distribution on A.

See [12] p. 236 for a proof. Here shifted dual transforms provide a simpler proof of an inversion formula due to Helgason (1964, see [9] p.116). Up to a factor T is the Fourier transform of $|c(\lambda)|^{-2}$ where c is Harish-Chandra's famous function. If $X = H^{2k+1}(\mathbb{R})$ (more generally if all Cartan subalgebras are conjugate in the Lie algebra of G) $|c(\lambda)|^{-2}$ is a polynomial and T, supported at the origin of A, is a differential operator : the inversion formula from Theorem 2 (iii) can then be deduced from Theorem 4.

3. The Kappa operator

Introduced by Gelfand, Graev and Shapiro (1967) to study the Radon transform on k-dimensional planes in $\mathbb{C}^n$ the "kappa operator" has been developed in several papers by the Russian school (Gelfand, Gindikin, Graev - and Goncharov who related it to the language of $\mathcal{D}$-modules). But apart from Grinberg's paper [6] it has never been used by others (to the best of my knowledge), though this differential form seems to provide an efficient tool to invert Radon transforms in various situations. Here we introduce it in the context of the n-dimensional hyperbolic space $X = H^n(\mathbb{R})$, following [3] chapter 5 and several papers by Gindikin [4] [5].

a. Preliminaries. Let M be a n-dimensional manifold, ω a volume form on M and S the hypersurface defined by $\varphi(x) = 0$ where $\varphi : M \to \mathbb{R}$ is a C^∞ function with $d\varphi(x) \neq 0$ for $x \in S$. Let V be a vector field on M which is *transversal* to S, i.e. $V\varphi(x) \neq 0$ for any $x \in S$. Then the $(n-1)$-form

$$\mu = \frac{1}{V\varphi}\, i_V\omega \text{ (restricted to } S) \tag{1}$$

is a volume form on S such that $d\varphi \wedge \mu = \omega$. Here i_V denotes the *interior product* defined by

$$(i_V\omega)(V_1, ..., V_{n-1}) = \omega(V, V_1, ..., V_{n-1})$$

for any tangent vectors $V_1, ..., V_{n-1}$.

Integration over S with respect to μ defines a distribution $\delta(\varphi)$ supported in S :

$$< \delta(\varphi), f >= \int_S f \ \mu \ , \ f \in C_c^\infty(M) \ , \tag{2}$$

and it is easily checked that

$$\varphi\delta(\varphi) = 0 \ , \ \delta(\lambda\varphi) = \lambda^{-1}\delta(\varphi) \tag{3}$$

if λ is any strictly positive C^∞ function on M.

In the framework of microlocal analysis $\delta(\varphi)$ can also be defined as the *pullback* $\delta(\varphi) = \varphi^*\delta$ of the Dirac measure on $\mathbb{R}$, which is given by the oscillatory integral (see [11] Theorem 8.2.4)

$$< \delta(\varphi), f >= \int_{\mathbb{R}} dt \int_M f(x) e^{it\varphi(x)} \omega(x) \ . \tag{4}$$

b. Radon transform on a quadric. Let $\varepsilon_0 = 1, \varepsilon_i = -1$ for $i \geq 1$ and

$$Q(x) = \sum_{i=0}^n \varepsilon_i x_i^2 = x_0^2 - x_1^2 - \cdots - x_n^2 \ .$$

From now on we consider the upper sheet of the hyperboloid

$$X = \{x \in \mathbb{R}^{n+1} | Q(x) = 1 \text{ and } x_0 > 0\}$$

and the Radon transform obtained by integrating over sections of X by all hyperplanes

$$\xi \cdot x \equiv \xi_0 x_0 + \cdots + \xi_n x_n + \xi_{n+1} = 0$$

that is, for $f \in C_c^\infty(X)$, $\xi \in \mathbb{R}^{n+2}\backslash 0$,

$$Rf(\xi) = < \delta(\xi \cdot x), f(x) > \ . \tag{5}$$

In view of (1) above we may take the *Euler vector field* $V = \sum_{j=0}^n x_j(\partial/\partial x_j)$ as a transversal field to $X \subset \mathbb{R}^{n+1}$ and

$$\begin{aligned} \omega &= \frac{1}{VQ} i_V (dx_0 \wedge \cdots \wedge dx_n) \\ &= \frac{1}{2} \sum_{j=0}^n (-1)^j x_j dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n \end{aligned} \tag{6}$$

(with dx_j removed) as a volume form on X.

As such we get an *overdetermined problem* of integral geometry : reconstruct a function f of n variables (the dimension of X) from a function of $(n+1)$ (dimension of the space of ξ's, up to a factor). It is therefore to be expected that Rf satisfies additional conditions (Proposition 5 below) and that knowing its restriction to some n-dimensional submanifold should suffice to reconstruct f.

Let us point out two interesting such submanifolds. For $\xi_{n+1} = 0$ we obtain all planes through the origin and their intersections with X are the $(n-1)$-dimensional *totally geodesic hypersurfaces* of the hyperbolic space X. For $Q(\xi_0, ..., \xi_n) = 0$ we obtain the *horospheres* of X. This can be checked for instance by projecting X onto the plane $x_0 = 0$ in $\mathbb{R}^{n+1}$ from the point $(-1, 0, ..., 0)$. The image of X is the open unit ball of $\mathbb{R}^n$ (the Poincaré model of hyperbolic geometry), geodesics hypersurfaces become $(n-1)$-spheres orthogonal to the unit sphere and horospheres become $(n-1)$-spheres tangent to the unit sphere (cf. **2.b** above).

Proposition 5 *Let* $P(\partial_\xi) = \sum_{j=0}^{n+1} \varepsilon_j (\partial/\partial_{\xi_j})^2$. *For* $f \in C_c^\infty(X)$ *the function* Rf *on* $\mathbb{R}^{n+2} \backslash 0$ *satisfies*

$$Rf(\lambda\xi) = \lambda^{-1} Rf(\xi) \ , \ \lambda > 0 \ ,$$
$$P(\partial_\xi) Rf(\xi) = 0 \ .$$

Proof. The homogeneity follows from (3). The differential equation is best seen from (4) :

$$Rf(\xi) = \int_{\mathbb{R}} dt \int_X f(x) e^{it\xi\cdot x} \omega(x)$$

implies

$$P(\partial_\xi) Rf(\xi) = -\int_{\mathbb{R}} t^2 dt \int_X f(x)(Q(x) - 1) e^{it\xi\cdot x} \omega(x) = 0$$

since $Q(x) = 1$ on X. ■

c. The kappa operator at last. It will act on functions of $\xi \in \mathbb{R}^{n+2} \backslash 0$. Let

$$d\xi = d\xi_0 \wedge \cdots \wedge d\xi_{n+1}$$
$$E = \sum_{j=0}^{n+1} \xi_j \frac{\partial}{\partial \xi_j}$$

be the volume form and Euler vector field respectively. For $\varphi \in C^\infty(\mathbb{R}^{n+2} \backslash 0)$ we consider the differential n-form on $\mathbb{R}^{n+2} \backslash 0$

$$\kappa\varphi = i_E i_Z (d\xi) \tag{7}$$

where Z is a vector field on $\mathbb{R}^{n+2} \backslash 0$ to be chosen shortly and depending on φ. The goal is to obtain a *closed form* when $\varphi = Rf$ so that $\int_\gamma \kappa Rf$ will only depend on the homology class of the n-cycle γ.

First, using the Lie derivatives $L_E = d \circ i_E + i_E \circ d$, $L_Z = d \circ i_Z + i_Z \circ d$, and the divergence defined by $L_Z\,(d\xi) = (\operatorname{div} Z) d\xi$ we have

$$d\kappa\varphi = L_E(i_Z d\xi) - (\operatorname{div} Z) i_E d\xi \ . \tag{8}$$

Lemma 6 *For any linear differential operator* P *with* C^∞ *coefficients on* $\mathbb{R}^{n+2} \backslash 0$ *and any* $\varphi, u \in C^\infty(\mathbb{R}^{n+2} \backslash 0)$ *there exists a vector field* $Z = Z(P, \varphi, u)$ *on* $\mathbb{R}^{n+2} \backslash 0$ *such that*

$$\operatorname{div} Z = u \ P\varphi - \varphi \ {}^tPu \ ,$$

where ${}^tP$ *is the transpose of* P *with respect to* $d\xi$. *In other words* $d\,(i_Z d\xi) = \big(uP\varphi - \varphi {}^tPu\big)\, d\xi$.

Proof. (i) If P is a vector field we may take $Z(P,\varphi,u)=\varphi uP$. Indeed for any vector field V and any smooth function f

$$\mathrm{div}(fV)=Vf+f\,\mathrm{div}\,V\ .$$

This classical formula is easily checked from

$$\begin{aligned}\mathrm{div}(fV)d\xi &= L_{fV}d\xi = d(i_{fV}d\xi)=d(f\ i_Vd\xi)\\ &= df\wedge i_Vd\xi+f\ d(i_Vd\xi)=df\wedge i_Vd\xi+f\,(\mathrm{div}\,V)\,d\xi\end{aligned}$$

together with

$$\begin{aligned}0 &= i_V(df\wedge d\xi)=(i_Vdf)\wedge d\xi-df\wedge i_Vd\xi\\ &= (Vf)\,d\xi-df\wedge i_Vd\xi\ .\end{aligned}$$

Taking now $V=\varphi uP$ and for f an arbitrary *compactly supported* function it follows that

$$\int(\varphi uPf+f\,\mathrm{div}(\varphi uP))\,d\xi=\int\mathrm{div}(f\varphi uP)d\xi=0$$

by Stokes formula. But

$$\begin{aligned}\int\varphi uPf\ d\xi &= \int(uP(f\varphi)-fuP\varphi)\,d\xi\\ &= \int f\left(\varphi{}^tPu-uP\varphi\right)d\xi\ ,\end{aligned}$$

whence $\mathrm{div}(\varphi uP)=u\ P\varphi-\varphi\ {}^tPu$ as claimed.
(ii) If $P=P_1P_2$ is a product of two operators we may take

$$Z(P_1P_2,\varphi,u)=Z(P_1,P_2\varphi,u)+Z(P_2,\varphi,{}^tP_1u)\ .$$

The lemma follows by linearity and induction on the order of P. ∎

If Z is chosen according to the lemma with $i_Zd\xi$ *homogeneous of degree* 0 *and* ${}^tPu=0$, (8) becomes

$$d\kappa\varphi=-(uP\varphi)\ i_Ed\xi$$

so that $\kappa\varphi$ *is a closed form if* (and only if) $P\varphi=0$.

Going back to our problem with $P=P(\partial_\xi)=\sum_{j=0}^{n+1}\varepsilon_j(\partial/\partial_{\xi_j})^2={}^tP$, $\varepsilon_0=1,\varepsilon_j=-1$ for $j\geq 1$, we may take, according to Lemma 6,

$$Z(P,\varphi,u)=\sum_{j=0}^{n+1}\varepsilon_j\left(\frac{\partial\varphi}{\partial\xi_j}u-\varphi\frac{\partial u}{\partial\xi_j}\right)\frac{\partial}{\partial\xi_j}\tag{9}$$

and, for fixed $x\in X$,

$$u(\xi)=u_x(\xi)=(\xi\cdot x)^{1-n}\ .\tag{10}$$

Indeed ${}^tP(\partial_\xi)u_x=n(n-1)(Q(x)-1)(\xi\cdot x)^{-1-n}=0$ and $i_Zd\xi$ will have the required homogeneity if φ is homogeneous of degree -1. Summarizing we obtain

Proposition 7 *Let $x \in X$ and $\varphi \in C^\infty(\mathbb{R}_\xi^{n+2} \backslash 0)$, homogeneous of degree -1. The differential n-form $\kappa_x\varphi = i_E i_Z(d\xi)$ (where E is the Euler vector field and Z is given by (9) and (10)) is closed if and only if $P(\partial_\xi)\varphi = 0$.*
This holds in particular if $\varphi = Rf$.

A technical difficulty arises here : our u_x is singular when $\xi \cdot x = 0$, i.e. when the hyperplane defined by ξ contains the given point x. It can be circumvented by replacing $(\xi \cdot x)^{1-n}$ with the distribution

$$u_x(\xi) = (\xi \cdot x - i0)^{1-n} = \lim_{\varepsilon \to 0^+} (\xi \cdot x - i\varepsilon)^{1-n} \ ,$$

i.e. the pullback of

$$(t - i0)^{1-n} = t^{1-n} + \frac{i\pi(-1)^n}{(n-2)!}\delta^{(n-2)}(t) \tag{11}$$

under the map $\xi \mapsto \xi \cdot x$. Then $\kappa_x\varphi$ becomes a n-form with distribution valued coefficients and Proposition 7 remains valid. Explicitly

$$Z = \sum_{j=0}^{n+1} Z_j \frac{\partial}{\partial \xi_j} \ , \ Z_j = \varepsilon_j \left(\frac{\partial \varphi}{\partial \xi_j} u_x - \varphi \frac{\partial u_x}{\partial \xi_j} \right) \ , \ \kappa_x\varphi = \sum_{j=0}^{n+1} Z_j \omega_j \ , \tag{12}$$

where

$$\omega_0 = i_E \left(d\xi_1 \wedge \cdots \wedge d\xi_{n+1} \right) = \sum_{k=1}^{n+1} (-1)^{k-1} \xi_k d\xi_1 \wedge \cdots \wedge \widehat{d\xi_k} \wedge \cdots \wedge d\xi_{n+1} \tag{13}$$

and the subsequent ω_j are obtained from ω_0 by cyclic permutation of $\xi_0, ..., \xi_{n+1}$ and multiplication by $(-1)^{(n-1)j}$.
Remark. The above construction of $\kappa_x\varphi$ extends to all nondegenerate quadratic form P on $\mathbb{R}^{n+2}$ (see [4]).

d. Inversion formula.

Theorem 8 *(Gindikin) Let $f \in C_c^\infty(X)$ where X is the upper sheet of the n-dimensional hyperboloid. Let γ be any n-dimensional submanifold of $\mathbb{R}^{n+2} \backslash 0$ homologous to the cycle γ_0 defined in the proof below. The Radon transform on X is then inverted by*

$$\int_\gamma \kappa_x Rf = C \ f(x) \ , \ x \in X \ ,$$

where $C = -(2i\pi)^n/(n-2)!$.

Proof. Let us take

$$\gamma_0 = \left\{ \xi \in \mathbb{R}^{n+2} \ \middle| \ \xi_0 = \xi_1 \text{ and } \xi_1^2 + \cdots + \xi_n^2 = 1 \right\} \ .$$

Any homologous n-cycle in $\mathbb{R}^{n+2} \backslash 0$ will give the same integral since $\kappa_x Rf$ is a closed n-form by Proposition 7. For instance as a more natural but equivalent choice

γ might be defined by $\xi_0 = \xi_1$ and $\xi_0^2 + \xi_1^2 + \cdots + \xi_n^2 = 1$. The first condition means that we consider hyperplanes containing $(1, -1, 0, ..., 0) \in X$.

We now show the claim boils down to the classical inversion formula for the Radon transform on hyperplanes of $\mathbb{R}^n$.

(i) Restricting $Rf(\xi)$ to $\xi_0 = \xi_1$ we obtain the function

$$\begin{aligned}\psi(\xi_1, ..., \xi_{n+1}) &\equiv Rf(\xi_1, \xi_1, ..., \xi_{n+1}) \\ &= < \delta(\xi_1(x_0 + x_1) + \xi_2 x_2 + \cdots + \xi_n x_n + \xi_{n+1}), f(x) > \quad .\end{aligned}$$

The brackets are here computed by means of the volume form ω on X (see (6) above). The map $x = (x_0, ..., x_n) \mapsto y = (y_1, y_2, ..., y_n)$ with $y_1 = x_0 + x_1$, $y_2 = x_2$, ..., $y_n = x_n$ induces a diffeomorphism of X onto the half space $y_1 > 0$ in $\mathbb{R}^n$. In view of

$$\begin{gathered}Q(x) = x_0^2 - x_1^2 - \cdots - x_n^2 = 1 \\ x_0 dx_0 = x_1 dx_1 + \cdots + x_n dx_n \\ y_1 dx_1 \wedge \cdots \wedge dx_n = x_0 dy_1 \wedge \cdots \wedge dy_n\end{gathered}$$

it is readily checked that the volume form $\omega(x)$ on X becomes

$$\widetilde{\omega}(y) = \frac{1}{2y_1} dy_1 \wedge \cdots \wedge dy_n \ .$$

To f corresponds the function $\widetilde{f}(y) = f(x)$, compactly supported in $y_1 > 0$. Then

$$\psi(\xi_1, ..., \xi_{n+1}) = < \delta(\xi_1 y_1 + \cdots + \xi_n y_n + \xi_{n+1}), \widetilde{f}(y) > \quad ,$$

where the brackets are now computed by means of $\widetilde{\omega}$. In other words ψ is the Radon tranform of the function $\widetilde{f}(y_1, ..., y_n)/2y_1$ over hyperplanes of $\mathbb{R}^n$.

Let $c = -(2i\pi)^n/(n-1)!$. From a *classical inversion formula* (see e.g. [3] p. 11) it follows that

$$\begin{gathered}c\ f(x) = c\ \widetilde{f}(y) = \\ = 2y_1 \int_{S^{n-1}\times\mathbb{R}} \psi(\xi_1, ..., \xi_{n+1}) \left(\xi_1 y_1 + \cdots + \xi_n y_n + \xi_{n+1} - i0\right)^{-n} \sigma \wedge d\xi_{n+1} \qquad (14)\end{gathered}$$

an integral taken over all $\xi_{n+1} \in \mathbb{R}$ and all $(\xi_1, ..., \xi_n)$ in the unit sphere S^{n-1} with volume form

$$\sigma = \sum_{j=1}^{n} (-1)^{j-1} \xi_j d\xi_1 \wedge \cdots \wedge \widehat{d\xi_j} \wedge \cdots \wedge d\xi_n \ .$$

(ii) Let us now consider the restriction to γ of $\kappa_x \varphi$ with $\varphi \in C^\infty(\mathbb{R}^{n+2}\backslash 0)$, homogeneous of degree -1. From the definition of γ we have $d\xi_0 = d\xi_1$, $d\xi_1 \wedge \cdots \wedge d\xi_n = 0$, and $\omega_0 = -\omega_1 = \sigma \wedge d\xi_{n+1}$, $\omega_j = 0$ for $j \geq 2$ by (13). On γ (12) thus reduces to

$$\begin{gathered}\kappa_x \varphi = (Z_0 - Z_1)\omega_0 \\ = \left(\left(\frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_1} \right) \varphi . u_x - \varphi . \left(\frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_1} \right) u_x \right) \sigma \wedge d\xi_{n+1} \ .\end{gathered}$$

But, for $\varphi = Rf$ and ψ as above,

$$\left(\left(\frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_1}\right)\varphi\right)(\xi_1, \xi_1, ..., \xi_{n+1}) = \frac{\partial \psi}{\partial \xi_1}(\xi_1, ..., \xi_{n+1})$$

and, for $u_x = (\xi \cdot x - i0)^{1-n}$,

$$\left(\frac{\partial}{\partial \xi_0} + \frac{\partial}{\partial \xi_1}\right) u_x = (1-n)(x_0 + x_1)(\xi \cdot x - i0)^{-n} \ .$$

Therefore on γ

$$\kappa_x Rf = \left(\frac{\partial \psi}{\partial \xi_1}(\xi \cdot x - i0)^{1-n} + (n-1) y_1 \psi \ (\xi \cdot x - i0)^{-n}\right) \sigma \wedge d\xi_{n+1} \ .$$

Integrating the first term by parts we obtain

$$\int_\gamma \kappa_x Rf = 2(n-1) y_1 \int \psi \ (\xi \cdot x - i0)^{-n} \sigma \wedge d\xi_{n+1} \ ,$$

where the latter integral is taken over all $\xi_{n+1} \in \mathbb{R}$ and all $(\xi_1, ..., \xi_n) \in S^{n-1}$. Comparing with (14) we are done. ■

e. Examples.

Geodesic hypersurfaces of X. As noted above geodesics hypersurfaces of the hyperbolic space are the sections of X by hyperplanes through the origin of $\mathbb{R}^{n+1}$, i.e. $\xi \cdot x = 0$ with $\xi_{n+1} = 0$. Now Theorem 8 applies with the cycle γ replaced by

$$\gamma_1 = \left\{ \xi \in \mathbb{R}^{n+2} \,\middle|\, \xi_{n+1} = 0 \text{ and } \xi_1^2 + \cdots + \xi_n^2 = 1 \right\} \ .$$

Indeed one can find in the group $SL(n+2, \mathbb{R})$ a continuous curve g_t, $0 \leq t \leq 1$, from the identity to the matrix

$$g_1 = \begin{pmatrix} 0 & 0....0 & 1 \\ 0 & Id_n & 0 \\ -1 & 10...0 & 0 \end{pmatrix} \ .$$

Applying it to ξ we obtain a continuous deformation of γ to γ_1 and $\int_{\gamma_1} \kappa_x Rf = C \ f(x)$.
The result can be made explicit by means of (12) and (13). In ω_j, $0 \leq j \leq n$, each term contains ξ_{n+1} or $d\xi_{n+1}$ hence $\omega_j = 0$ on γ_1 and

$$\kappa_x \varphi = \left(\varphi \frac{\partial u_x}{\partial \xi_{n+1}} - \frac{\partial \varphi}{\partial \xi_{n+1}} u_x\right) \omega_{n+1} \ .$$

The corresponding inversion formula eventually simplifies to ([3] p. 154)

$$2 \int_{\gamma_1} Rf(\xi)(\xi \cdot x - i0)^{-n} \omega_{n+1}(\xi) = C \ f(x) \ .$$

Horospheres of X. We need here all hyperplanes $\xi \cdot x = 0$ such that

$$Q(\xi) = \xi_0^2 - \xi_1^2 - \cdots - \xi_n^2 = 0 \ .$$

Adding as before the condition $\xi_1^2 + \cdots + \xi_n^2 = 1$ we consider the cycle

$$\gamma_2 = \left\{ \xi \in \mathbb{R}^{n+2} \,\middle|\, \xi_0 = 1 \text{ and } \xi_1^2 + \cdots + \xi_n^2 = 1 \right\} \ .$$

By continuous deformation the condition $\xi_0 = 1$ can be changed into $\xi_0 = 0$, then to $\xi_0 = \xi_1$ by a continuous path in $SL(n+2, \mathbb{R})$ as above. Thus Theorem 8 applies with γ replaced by γ_2 and leads to the inversion formula ([3] p.156)

$$\int_{\gamma_2} Rf(\xi)(\xi \cdot x - i0)^{-n} \omega_0(\xi) = C \ f(x) \ .$$

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