RADON TRANSFORM ON GRASSMANNIANS

AND THE KAPPA OPERATOR

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Abstract. The Radon transform considered here is defined by integrating a function over p-dimensional affine subspaces in $\mathbb{R}^n$. Viewing those planes as graphs, a general inversion formula follows easily from a projection slice theorem. For even p it may also be written by means of a differential form given by the so-called kappa operator.

We also discuss the special case of Radon transform on Lagrangian p-planes in $\mathbb{R}^{2p}$, and give an overview of two range theorems.

The aim of this expository note is to provide an elementary approach to some methods and tools introduced and developed by the Russian school in the field of integral geometry on Grassmannians.

1. INTRODUCTION

By p-plane we mean a p-dimensional affine subspace of the affine space $\mathbb{R}^n$. Assuming $1 \leq p \leq n-1$ let $q = n-p$; points in $\mathbb{R}^n$ will be written as $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$. A generic p-plane can be defined as a graph :

$$\mathcal{P}(u, v) = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q | y = ux + v\} \ ,$$

where u is a linear map of $\mathbb{R}^p$ into $\mathbb{R}^q$ and v is a vector in $\mathbb{R}^q$. The map $(u, v) \mapsto \mathcal{P}(u, v)$ is a bijection of $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) \times \mathbb{R}^q$ onto the set of p-planes meeting $0 \times \mathbb{R}^q$ transversally. Throughout the paper we identify $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ with the space of $p \times q$ real matrices.

Our Radon transform is given by integrals of a function f over the p-planes $\mathcal{P}(u, v)$:

$$Rf(u, v) = \int_{\mathbb{R}^p} f(x, ux + v) \, dx \ , \qquad (1)$$

where f is an arbitrary function in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions (and all derivatives) and dx denotes Lebesgue measure.

An inversion formula of the transform R can be obtained by the following steps **a**, **b** and sometimes **c**. For brevity we only write it at the origin in this introduction ; the general case follows by translation, or can be worked out directly as will be done in the next sections.

a. Projection slice theorem. Let $(\xi, \eta) \in \mathbb{R}^p \times \mathbb{R}^q$ and let $< \ , \ >$ denote the canonical scalar products in $\mathbb{R}^p$ and $\mathbb{R}^q$. The function $(x, y) \mapsto < \xi, x > + < \nu, y >$ is constant on $\mathcal{P}(u, v)$ if and only if (ξ, η) is orthogonal to this plane i.e. $\xi = - \, {}^t u \eta$ (where ${}^t u$ is the transpose of u) ; the constant value is then $< \eta, v >$. As an immediate consequence one obtains the following "*projection slice theorem*"

$$\widehat{f}(\xi,\eta) = \widehat{Rf}(u,\eta) \text{ if } \xi = -\ {}^t u\eta\ , \tag{2}$$

where

$$\widehat{f}(\xi,\eta) = \int_{\mathbb{R}^p\times\mathbb{R}^q} f(x,y)e^{-2i\pi(<\xi,x>+<\eta,y>)}dxdy\ , \tag{3}$$

$$\widehat{Rf}(u,\eta) = \int_{\mathbb{R}^q} Rf(u,v)e^{-2i\pi<\eta,v>}dv \tag{4}$$

are the classical Fourier transforms in $\mathbb{R}^n$, resp. $\mathbb{R}^q$.

b. Choice of an admissible family. There exists a smooth map $t \mapsto u(t)$ of $\mathbb{R}^p$ into $\mathcal{L}(\mathbb{R}^p,\mathbb{R}^q)$ such that $\varphi : (t,\eta) \mapsto (-\ {}^t u(t)\eta, \eta)$ is a diffeomorphism of $\mathbb{R}^p \times \mathbb{R}^q$ onto $\mathbb{R}^p \times \mathbb{R}^q$ (up to sets of measure 0). Examples of such maps will be given in Sections 2 and 4 below. By (2) $\widehat{f}$ is then determined by $\widehat{Rf}$ whence, by Fourier inversion in $\mathbb{R}^n$,

$$f(0,0) = \int \widehat{f}(\xi,\eta)d\xi d\eta = \int \widehat{Rf}(u(t),\eta)|J(t,\eta)|dtd\eta \tag{5}$$

(Theorem 1). The Jacobian J of φ is a homogeneous polynomial function of degree p with respect to η. Thus one recovers f from Rf by means of an *operator with symbol* $|J(t,\cdot)|$ *acting on* $Rf(u(t),\cdot)$, followed by integration over t.

c. Kappa operator. If $J(t,\eta)$ has constant sign ε we get a classical differential operator and the change $(t,\eta) \mapsto (\xi,\eta)$ in (5) can be written in terms of differential forms. This can only happen for p even. To a smooth function F of $(u,v) \in \mathcal{L}(\mathbb{R}^p,\mathbb{R}^q)\times\mathbb{R}^q$ the *kappa operator* (at the origin) associates the differential p-form on $\mathcal{L}(\mathbb{R}^p,\mathbb{R}^q)$

$$\kappa F(u) = \sum \frac{\partial}{\partial v_{j_1}} \cdots \frac{\partial}{\partial v_{j_p}} F(u,0)\ du_{j_1 1} \wedge \cdots \wedge du_{j_p p}\ , \tag{6}$$

where the u_{jk}'s are the matrix elements of u and the sum runs over all $j_1, ..., j_p$ from 1 to q. This definition (due to the Russian school, cf. [3]) is motivated by inversion formula (5) which is then equivalent to

$$(2i\pi)^p \varepsilon f(0,0) = \int_{u(\mathbb{R}^p)} \kappa Rf \tag{7}$$

as one readily checks under the assumption on J (Theorem 4).
Besides κRf is a *closed differential form* (Proposition 5). If $p < n-1$ this remarkable fact follows from a system of differential equations satisfied by Rf, which actually characterize the image under R of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ (Theorem 10).

Details are given in Sections 2 and 3. The special case of Radon transforms on p-dimensional Lagrangian subspaces of $\mathbb{R}^{2p}$ has been studied by Debiard and Gaveau, Grinberg in [1], [9], [10]. In Section 4 we show their inversion formulas easily follow from Sections 2 and 3 by restriction. In the final Section 5 we give two range theorems, but the more technical proofs will be only sketched here.

2. INVERSION FORMULAS

We keep to the notations of the introduction : let $f \in \mathcal{S}(\mathbb{R}^n)$, $u \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$, $v \in \mathbb{R}^q$ and

$$Rf(u,v) = \int_{\mathbb{R}^p} f(x, ux+v)\ dx\ .$$

Remark. When working in the *Euclidean* space $\mathbb{R}^n$ a more natural definition would make use of the Euclidean measure induced on $\mathcal{P}(u,v)$, instead of dx, and the above $Rf(u,v)$ should then be replaced by

$$\sqrt{\det\left(I + \ {}^t uu\right)}\ Rf(u,v)\ ,$$

where I is the unit $p \times p$ matrix and ^{t} means transpose. However we shall keep here to definition (1).

The following properties **a** to **d** of the Radon transform (1) are easily checked.

a. Homogeneity. For any constant $\lambda > 0$ let $f_\lambda(x,y) = f(\lambda x, \lambda y)$. Then

$$Rf_\lambda(u,v) = \lambda^{-p} Rf(u, \lambda v)\ . \tag{8}$$

b. Translation. For $(a,b) \in \mathbb{R}^p \times \mathbb{R}^q$ let $f_{a,b}(x,y) = f(x+a, y+b)$. Then

$$Rf_{a,b}(u,v) = Rf(u, v+b-ua)\ . \tag{9}$$

c. Partial differential equations. The Radon transform of f satisfies the following system of $pq(q-1)/2$ differential equations

$$\left(\partial_{v_i}\partial_{u_{jk}} - \partial_{v_j}\partial_{u_{ik}}\right) Rf(u,v) = 0\ ,\ 1 \le i,j \le q\ ,\ 1 \le k \le p\ , \tag{10}$$

where ∂_{v_i} means $\partial/\partial v_i$ etc.
Indeed

$$\partial_{u_{jk}}(f(x, ux+v)) = x_k(\partial_{y_j} f)(x, ux+v)$$
$$\partial_{v_i}\partial_{u_{jk}}(f(x, ux+v)) = x_k(\partial_{y_i}\partial_{y_j} f)(x, ux+v)$$

and derivatives can be taken under the integral sign in (1).

d. Projection slice theorem. Given $u \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ the partial Fourier transform $\widehat{Rf}(u, .)$ (see (4)) is the restriction of the Fourier transform $\widehat{f}$ (see (3)) to the q-dimensional vector subspace $\xi + \ {}^t u\eta = 0$ in $\mathbb{R}^p \times \mathbb{R}^q$, i.e.

$$\widehat{Rf}(u,\eta) = \widehat{f}(-\ {}^t u\eta, \eta) \tag{11}$$

for $u \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$, $\eta \in \mathbb{R}^q$.

Proof. Let u be fixed. In order to decompose the integral (3) into *slices* parallel to the plane $\mathcal{P}(u,v)$ we change variables according to $(x,v) \mapsto (x,y) = (x, ux+v)$:

$$\widehat{f}(\xi,\eta) = \int_{\mathbb{R}^p \times \mathbb{R}^q} f(x, ux+v) e^{-2i\pi\left(<\xi + \ {}^t u\eta, x> + <\eta, v>\right)} dx dv\ .$$

This simplifies if $\xi + {}^t u\eta = 0$ i.e. if $< \xi, x > + < \eta, y >$ (the phase function of the Fourier transform) is constant on $\mathcal{P}(u, v)$, i.e. if (ξ, η) is an orthogonal vector to this plane. This constant value is $< \eta, v >$ and

$$\widehat{f}(- {}^t u\eta, \eta) = \int_{\mathbb{R}^q} e^{-2i\pi<\eta,v>} dv \int_{\mathbb{R}^p} f(x, ux + v)\, dx = \widehat{Rf}(u, \eta)\ .$$

Absolute convergence of the integrals justifies the calculations.
Remark. By (11) $\widehat{f}$ is almost entirely determined by Rf. Indeed the linear map $u \mapsto \xi = - {}^t u\eta$ maps $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ *onto* $\mathbb{R}^p$ for any given $\eta \neq 0$ in $\mathbb{R}^q$, as shown by taking a rank one matrix $u = (\alpha_j \xi_k)$ with $\alpha_1, ..., \alpha_q$ chosen such that $\sum_1^q \alpha_j \eta_j = -1$. The points $(- {}^t u\eta, \eta)$, $u \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$, $\eta \in \mathbb{R}^q \backslash 0$, thus fill all of $\mathbb{R}^p \times (\mathbb{R}^q \backslash 0)$. Equation (11) and the Fourier inversion formula now imply injectivity of the Radon transform $f \mapsto Rf$. They also lead to an inversion formula for R (Theorem 1 below).

e. Inversion formulas. The dimension of the set of all (u, v) being $pq + q$, greater than $n = p + q$, we shall restrict u to some p-dimensional submanifold of $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ in order to reconstruct $f(x, y)$ from $Rf(u, v)$.

There exist an open subset Ω of $\mathbb{R}^q$ with complement of measure 0 and a C^∞ map $t \mapsto u(t)$ of $\mathbb{R}^p$ into $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ such that

$$\begin{aligned} \varphi : (t, \eta) &\longmapsto (\xi, \eta) = (- {}^t u(t)\eta, \eta) \\ \mathbb{R}^p \times \Omega &\longrightarrow \mathbb{R}^p \times \Omega \end{aligned} \tag{12}$$

is a diffeomorphism onto. For example one can take any constant nonzero vector $\alpha \in \mathbb{R}^q$ and the rank one matrix

$$u(t) = \alpha \otimes t \text{ i.e. } u_{jk}(t) = \alpha_j t_k\ ,\ 1 \leq j \leq q\ ,\ 1 \leq k \leq p\ , \tag{13}$$

so that ${}^t u(t)\eta =< \alpha, \eta > t$, together with

$$\Omega = \{\eta \in \mathbb{R}^q | < \alpha, \eta > \neq 0\}\ .$$

More general examples can be obtained by composing this map $t \mapsto u(t)$ with a diffeomorphism $t' \mapsto t$ of $\mathbb{R}^p$, or replacing $u(t)$ by $u(t)a(t)$ with $a(t) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p)$ and ${}^t a(t)t = t$.

The planes $\mathcal{P}(u(t), v)$ then make up a n-dimensional "*admissible submanifold*" leading to an inversion formula.

Theorem 1 *(General inversion formula) Let $f \in \mathcal{S}(\mathbb{R}^n)$ and let $t \mapsto u(t)$ be any map such that (12) is a diffeomorphism. Then, for any $(x, y) \in \mathbb{R}^n$,*

$$f(x, y) = \int_{\mathbb{R}^p \times \Omega} \widehat{Rf}(u(t), \eta) e^{2i\pi<\eta, y - u(t)x>} |J(t, \eta)|\, dt d\eta \tag{14}$$

where J, the Jacobian of φ, is

$$J(t, \eta) = (-1)^p \det \partial_t({}^t u(t)\eta) = (-1)^p \sum_{j_1, ..., j_p = 1}^{q} \eta_{j_1} ... \eta_{j_p} \frac{\partial(u_{j_1 1}(t), ..., u_{j_p p}(t))}{\partial(t_1, ..., t_p)}\ , \tag{15}$$

a homogeneous polynomial of degree p with respect to η.

In all our specific examples u will be linear in t and the bilinear map $(t, \eta) \mapsto {}^t u(t)\eta$ may be written as

$$ {}^t u(t)\eta = A(\eta)t \tag{16} $$

with $A(\eta) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p)$, linearly depending on η, hence $J(t, \eta) = (-1)^p \det A(\eta)$.

Corollary 2 *(Special inversion formula) In particular, with $u(t)$ as in (16), we have*

$$ (2i\pi)^p f(x, y) = \int_{\mathbb{R}^p} (|\det A(\partial_v)| Rf)\,(u(t), y - u(t)x)\ dt \tag{17} $$

where $|\det A(\partial_v)|$ is the operator with symbol $|\det A(\eta)|$ acting on the variable v in $Rf(u, v)$ (see (18) below). The integral converges absolutely.
If $u(t) = \alpha \otimes t$ as in (13) then $u(t)x =< t, x > \alpha$ and $|\det A(\partial_v)| = | < \alpha, \partial_v > |^p$, a differential operator for p even.

Proof of Theorem 1. Since Ω fills $\mathbb{R}^q$ up to a set of measure zero, we may change variables by means of (12) in the Fourier inversion formula and obtain

$$ \begin{aligned} f(x, y) &= \int_{\mathbb{R}^p \times \Omega} \widehat{f}(\xi, \eta) e^{2i\pi(<\xi, x> + <\eta, y>)} d\xi d\eta \\ &= \int_{\mathbb{R}^p \times \Omega} \widehat{f}(-\ {}^t u(t)\eta, \eta) e^{2i\pi(<-\ {}^t u(t)\eta, x> + <\eta, y>)} |J(t, \eta)|\ dt d\eta\ . \end{aligned} $$

Both integrals converge absolutely for $f \in \mathcal{S}(\mathbb{R}^n)$. But $\left(\widehat{f} \circ \varphi\right)(t, \eta) = \widehat{Rf}(u(t), \eta)$ by (11) and (12) ; the result follows. Besides (12) gives $\xi_k = -\sum_j u_{jk}\eta_j$ and

$$ \begin{aligned} &d\xi_1 \wedge \cdots \wedge d\xi_p \wedge d\eta_1 \wedge \cdots \wedge d\eta_q = \\ &\qquad = (-1)^p \sum_{j_1, \ldots, j_p} \eta_{j_1} \ldots \eta_{j_p} du_{j_1 1} \wedge \cdots \wedge du_{j_p p} \wedge d\eta_1 \wedge \cdots \wedge d\eta_q\ , \end{aligned} $$

hence the expression of J. ∎

Proof of Corollary 2. Here $J(t, \eta) = (-1)^p \det A(\eta)$ and (14) can be written more explicitly. If $P(\eta)$ is a homogeneous polynomial of degree p with respect to η, let the operator $|P(\partial_v)|$ be defined by

$$ |P(\partial_v)| F(v) = (2i\pi)^p \int_{\mathbb{R}^q} \widehat{F}(\eta) e^{2i\pi<\eta, v>} |P(\eta)|\ d\eta\ . \tag{18} $$

Then (17) is an immediate consequence of (14). For p even $| < \alpha, \partial_v > |^p$ is the differential operator $\left(\sum_{j=1}^q \alpha_j \partial_{v_j}\right)^p$. ∎

f. Link with the dual transform. Corollary 2 can be written in terms of the *dual Radon transform* R^*. Indeed let $F(u, v)$ be a function on $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) \times \mathbb{R}^q$ and let

$$ R^* F(x, y) = \int_{\mathbb{R}^p} F(u(t), y - u(t)x)\ dt \tag{19} $$

be the integral of F over the family of all p-planes $\mathcal{P}(u(t), v)$ containing the point (x, y). This definition of R^* of course depends on the choice of the map $t \mapsto u(t)$. Though natural in the present context it differs from Helgason's definition in the Euclidean case ([11] chapter I, §6), given by an integral over the orthogonal group.

Assuming absolute convergence of the integrals we have, for all $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^p \times \mathbb{R}^q} R^*F(x,y)f(x,y)\, dxdy &= \int F(u(t), y - u(t)x)f(x,y)\, dxdydt \\ &= \int F(u(t), v)f(x, u(t)x + v)\, dxdvdt \\ &= \int_{\mathbb{R}^p \times \mathbb{R}^q} F(u(t), v)Rf(u(t), v)\, dvdt\ .\end{aligned}$$

Thus R^* is actually dual to R.

In this notation Corollary 2 becomes

$$(2i\pi)^p f = R^* |\det A(\partial_v)| Rf \tag{20}$$

where R^* is given by (19). Since

$$R^* \partial_{v_j} F = \partial_{y_j} R^* F\ ,\ 1 \leq j \leq q\ ,$$

the example (13) $u(t) = \alpha \otimes t$ may be rewritten, for p even, as

$$(2i\pi)^p f = (< \alpha, \partial_y >)^p R^* R f\ . \tag{21}$$

3. THE KAPPA OPERATOR

Interesting simplifications occur in the inversion formula if the diffeomorphism (12) is orientation preserving (or reversing), i.e. if the Jacobian J given by (15) does not change sign in $\mathbb{R}^p \times \Omega$: see Corollary 2 with p even for an example. Throughout this section we thus *assume that* $J(t, \eta)$ *has a constant sign* ε ($= 1$ or -1) *for all* $(t, \eta) \in \mathbb{R}^p \times \Omega$.

This can only hold for even p. Indeed assume p odd and fix $t \in \mathbb{R}^p$. Then there exists an open set V in $\mathbb{R}^q$ such that $J(t, \eta)$, as a non identically zero polynomial of odd degree with respect to η, is strictly positive on V and strictly negative on $-V$. Both V and $-V$ meet the dense open set Ω, and J cannot have a constant sign on $\mathbb{R}^p \times \Omega$.

No absolute value is necessary in (14) then and Theorem 1 can be rewritten with differential forms instead of densities. Going over its proof again we assume $\mathbb{R}^p \times \mathbb{R}^q$ is oriented by the volume form

$$d\xi \wedge d\eta = d\xi_1 \wedge \cdots \wedge d\xi_p \wedge d\eta_1 \wedge \cdots \wedge d\eta_q\ .$$

Then

$$\begin{aligned}f(x, y) &= \int_{\mathbb{R}^p \times \Omega} \widehat{f}(\xi, \eta) e^{2i\pi(<\xi, x> + <\eta, y>)} d\xi \wedge d\eta \\ &= \varepsilon \int_{\mathbb{R}^p \times \Omega} \widehat{Rf}(u(t), \eta) e^{2i\pi <\eta, y - u(t)x>} J(t, \eta)\, dt \wedge d\eta\ ,\end{aligned}$$

with (recalling that p is even)

$$J(t,\eta)dt \wedge d\eta = \sum_{j_1,\dots,j_p} \eta_{j_1}...\eta_{j_p}\ d\eta_1 \wedge \cdots \wedge d\eta_q \wedge du_{j_1 1} \wedge \cdots \wedge du_{j_p p}$$

and all u_{jk}'s expressed as functions of t. By Fourier inversion for the integral over Ω, i.e. over $\mathbb{R}^q$, we obtain

$$\varepsilon(2i\pi)^p f(x,y) = \int_{t\in\mathbb{R}^p} \sum_{j_1,\dots,j_p} \left(\partial_{v_{j_1}} \cdots \partial_{v_{j_p}} Rf\right)(u(t), y-u(t)x)\ du_{j_1 1} \wedge \cdots \wedge du_{j_p p}$$

in which $du_{jk} = \sum_h \partial_{t_h} u_{jk}(t)\ dt_h$. This motivates the following

Definition 3 *Let $(x,y) \in \mathbb{R}^p \times \mathbb{R}^q$. To any smooth function F of $(u,v) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) \times \mathbb{R}^q$ the* kappa operator *$\kappa_{x,y}$ associates the differential p-form on $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ given by*

$$(\kappa_{x,y} F)(u) = \sum_{j_1,\dots,j_p=1}^{q} \left(\partial_{v_{j_1}} \cdots \partial_{v_{j_p}} F\right)(u, y-ux)\ du_{j_1 1} \wedge \cdots \wedge du_{j_p p}\ . \tag{22}$$

Summarizing we have proved

Theorem 4 *(Inversion formula with kappa operator) Let $f \in \mathcal{S}(\mathbb{R}^n)$ and let $t \mapsto u(t)$ be any map such that (12) is a diffeomorphism and its Jacobian (15) has constant sign ε. Then p is even and, for any $(x,y) \in \mathbb{R}^n$,*

$$\varepsilon(2i\pi)^p f(x,y) = \int_\gamma \kappa_{x,y} Rf \tag{23}$$

with $\gamma = u(\mathbb{R}^p)$.

In (23) $\int_\gamma \kappa_{x,y} Rf$ means the integral over $\mathbb{R}^p$ of the pullback $u^*(\kappa_{x,y} Rf)$. This is a *local* inversion formula, i.e. f can be reconstructed at (x,y) by means of integrals over p-planes close to this point. As noted above the assumption is satisfied with $\varepsilon = 1$ if $u(t) = \alpha \otimes t$ and p is even.
In [3] p. 61 the right-hand side of (23) is shown to vanish for odd p, so that κ does not yield an inversion formula.

Proposition 5 *(Properties of the kappa operator). For $(a,b), (x,y) \in \mathbb{R}^p \times \mathbb{R}^q$, $f \in \mathcal{S}(\mathbb{R}^n)$,*
(i) $\kappa_{a,b} Rf_\lambda = \kappa_{\lambda a, \lambda b} Rf$ with $f_\lambda(x,y) = f(\lambda x, \lambda y)$, $\lambda > 0$
(ii) $\kappa_{a,b} Rf = \kappa_{0,0} Rf_{a,b}$ with $f_{a,b}(x,y) = f(x+a, y+b)$
(iii) Assume F satisfies the $pq(q-1)/2$ differential equations, for $(u,v) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) \times \mathbb{R}^q$, $1 \le i,j \le q$, $1 \le k \le p$,

$$\left(\partial_{v_i} \partial_{u_{jk}} - \partial_{v_j} \partial_{u_{ik}}\right) F(u,v) = 0\ . \tag{24}$$

Then $\kappa_{x,y} F$ is a closed differential form. *In particular $\kappa_{x,y} Rf$* is closed.

Proof. *(i)* and *(ii)* are immediate from (8), (9) and (22).
(iii) If $q = 1$ i.e. $p = n-1$ the differential equations are trivial but the result is clear, since $u = (u_1, ..., u_p)$ in this case and κF has the form

$$\kappa_{x,y}F(u) = g(u_1, ..., u_p)\ du_1 \wedge \cdots \wedge du_p\ ,$$

obviously closed. We may thus assume $q > 1$. Then $d(\kappa_{x,y}F) = \sum_{k=1}^{p} (\omega_k - x_k\psi_k)$ with

$$\omega_k = \sum_{j,j_1,...,j_p=1}^{q} \partial_{u_{jk}}\partial_{v_{j_1}} \cdots \partial_{v_{j_k}} \cdots \partial_{v_{j_p}} F\ du_{jk} \wedge du_{j_1 1} \wedge \cdots \wedge du_{j_k k} \wedge \cdots \wedge du_{j_p p}\ ,$$

$$\psi_k = \sum_{j,j_1,...,j_p=1}^{q} \partial_{v_j}\partial_{v_{j_1}} \cdots \partial_{v_{j_k}} \cdots \partial_{v_{j_p}} F\ du_{jk} \wedge du_{j_1 1} \wedge \cdots \wedge du_{j_k k} \wedge \cdots \wedge du_{j_p p}\ .$$

Both forms vanish by symmetry of their coefficients with respect to j and j_k.
In view of (10) this holds for $F = Rf$. ■

4. RADON TRANSFORM ON LAGRANGIAN PLANES

The above results actually deal with the Radon transform restricted to the n-dimensional "admissible" family of planes $\mathcal{P}(u(t), v)$, $t \in \mathbb{R}^p$, $v \in \mathbb{R}^q$, where $u(t)$ is chosen so that (12) is a diffeomorphism. The rank one choice (13) leads to nice-looking inversion formulas : see Corollary 2 and (21). Other interesting choices come out of the case of Lagrangian planes.

Throughout this section we *assume* $p = q$ *i.e.* $n = 2p$. A p-dimensional vector subspace of $\mathbb{R}^{2p}$ is *Lagrangian* if and only if the symplectic form

$$\sigma((x, y), (x', y')) =< x, y' > - < x', y >$$

(with $x, x', y, y' \in \mathbb{R}^p$) vanishes on it identically. For a graph $y = ux$, $u \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p)$, this is equivalent to the symmetry $u = {}^t u$. Let S_p denote the space of *symmetric* $p \times p$ *real matrices*. We now consider the Radon transform (1) restricted to the manifold of *affine Lagrangian* p-planes $\mathcal{P}(u, v)$, $u \in S_p$, $v \in \mathbb{R}^p$.

Restricting (11) to $u \in S_p$ gives the projection slice theorem :

$$\widehat{Rf}(u, \eta) = \widehat{f}(-u\eta, \eta)\ ,\ u \in S_p\ ,\ \eta \in \mathbb{R}^p\ .$$

An inversion formula will follow as above for any choice of $t \mapsto u(t) \in S_p$ such that (12) is a diffeomorphism onto.

a. A Lagrangian inversion formula. As a first example one can take the diagonal matrix

$$u(t) = diag\,(t_1, ..., t_p)\ \ ,\ \Omega = \{\eta \in \mathbb{R}^p | \eta_1 \cdots \eta_p \neq 0\}\ ,$$

hence, by (20) with $A(\eta) = diag(\eta_1, ..., \eta_p)$, the inversion formula for Lagrangian planes (cf. [1])

$$(2i\pi)^p f(x, y) = R^*|\partial_{v_1} \cdots \partial_{v_p}|Rf \tag{25}$$

where $f \in \mathcal{S}(\mathbb{R}^{2p})$, $(x, y) \in \mathbb{R}^p \times \mathbb{R}^p$, R^* is defined by (19) (with the above $u(t)$) and $|\partial_{v_1} \cdots \partial_{v_p}|$ denotes the operator with symbol $|\eta_1 \cdots \eta_p|$ acting on v. But here the Jacobian

$J(t,\eta) = (-1)^p \eta_1 \cdots \eta_p$ changes sign in Ω and we have no analog of (23) with a kappa operator.

b. One more Lagrangian inversion formula. Another example is given, *for even* p, by the block diagonal matrix (cf. [10] for $p=2$)

$$u(t) = diag\left(\begin{pmatrix} t_1 & t_2 \\ t_2 & -t_1 \end{pmatrix}, ..., \begin{pmatrix} t_{p-1} & t_p \\ t_p & -t_{p-1} \end{pmatrix}\right) \ ,$$
$$\Omega = \{\eta \in \mathbb{R}^p | (\eta_1, \eta_2) \neq 0, ..., (\eta_{p-1}, \eta_p) \neq 0\} \ .$$

Then, in the notation of (16) above, $u(t)\eta = A(\eta)t$ with

$$A(\eta) = diag\left(\begin{pmatrix} \eta_1 & \eta_2 \\ -\eta_2 & \eta_1 \end{pmatrix}, ..., \begin{pmatrix} \eta_{p-1} & \eta_p \\ -\eta_p & \eta_{p-1} \end{pmatrix}\right) \ ,$$
$$\det A(\eta) = (\eta_1^2 + \eta_2^2) \cdots (\eta_{p-1}^2 + \eta_p^2) \ ,$$

hence by (20) the inversion formula for $f \in \mathcal{S}(\mathbb{R}^{2p})$, p even, $(x,y) \in \mathbb{R}^p \times \mathbb{R}^p$,

$$\begin{aligned} (2i\pi)^p f(x,y) &= R^* \Delta_{12}^v \Delta_{34}^v \cdots \Delta_{p-1,p}^v Rf \\ &= \Delta_{12}^y \Delta_{34}^y \cdots \Delta_{p-1,p}^y R^* Rf \end{aligned} \tag{26}$$

with $\Delta_{jk}^v = \partial_{v_j}^2 + \partial_{v_k}^2$, $\Delta_{jk}^y = \partial_{y_j}^2 + \partial_{y_k}^2$.

c. Lagrangians and the kappa operator.

Definition 6 *Let* $(x,y) \in \mathbb{R}^p \times \mathbb{R}^p$*. To any smooth function* F *of* $(u,v) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p) \times \mathbb{R}^p$ *the* Lagrangian kappa operator $\kappa_{x,y}^L$ *associates the restriction to* S_p*, the space of symmetric* $p \times p$ *matrices, of the differential* p*-form* $\kappa_{x,y}F$ *(Definition 3), i.e.*

$$\kappa_{x,y}^L F = \iota^* \left(\kappa_{x,y} F\right) \ , \tag{27}$$

where $\iota : S_p \hookrightarrow \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p)$ *is the canonical injection.*

Thus $\kappa_{x,y}^L F$ is a differential p-form on S_p which, in view of (22), only depends on the restriction of F to $S_p \times \mathbb{R}^p$. For $p=2$

$$\kappa_{x,y}^L F = \partial_{v_1}^2 F \ du_{11} \wedge du_{12} + \partial_{v_1}\partial_{v_2} F \ du_{11} \wedge du_{22} + \partial_{v_2}^2 F \ du_{12} \wedge du_{22} \ ,$$

with all derivatives of F computed at $(u,v) = (u, y - ux)$.

Theorem 7 *(Lagrangian inversion formula with kappa operator) Let* $f \in \mathcal{S}(\mathbb{R}^{2p})$ *and let* $t \mapsto u(t)$ *be any map from* $\mathbb{R}^p$ *into* S_p *such that* $(t,\eta) \mapsto (-u(t)\eta, \eta)$ *is a diffeomorphism of* $\mathbb{R}^p \times \Omega$ *onto itself (see (12)) with Jacobian of constant sign* ε*. Then* p *is even and, for any* $(x,y) \in \mathbb{R}^p \times \mathbb{R}^p$*,*

$$\varepsilon(2i\pi)^p f(x,y) = \int_\gamma \kappa_{x,y}^L Rf \tag{28}$$

with $\gamma = u(\mathbb{R}^p) \subset S_p$*.*

The right-hand side of (28) is the integral over $\mathbb{R}^p$ of the pullback $u^*\left(\kappa^L_{x,y}Rf\right)$.

The assumption of Theorem 7 is satisfied in example **b** above. Taking $p=2$ and $u(t)$ as in **b** i.e. $u_{21}=u_{12}$, $u_{22}=-u_{11}$, (28) reads

$$-4\pi^2 f(x,y)=\int_{\mathbb{R}^2}(\partial^2_{v_1}+\partial^2_{v_2})Rf(u,y-ux)\;du_{11}\wedge du_{12}\ ,$$

in agreement (up to a factor) with the result of [10].

Proof of Theorem 7. Theorem 4 applies with u replaced by $\iota\circ u:\mathbb{R}^p\to\mathcal{L}(\mathbb{R}^p,\mathbb{R}^p)$, hence

$$\varepsilon(2i\pi)^p f(x,y)=\int_{\mathbb{R}^p}(u^*\circ\iota^*)\left(\kappa_{x,y}Rf\right)=\int_{\mathbb{R}^p}u^*(\kappa^L_{x,y}Rf)\ .\ \blacksquare$$

d. Partial differential equations.

Proposition 8 *(i) Let $f\in\mathcal{S}(\mathbb{R}^{2p})$ and $(x,y)\in\mathbb{R}^p\times\mathbb{R}^p$. Then $\kappa^L_{x,y}Rf$ is a closed differential form on S_p.*
(ii) Let F be a smooth function on $\mathcal{L}(\mathbb{R}^p,\mathbb{R}^p)\times\mathbb{R}^p$ satisfying the differential equations (24) with $p>1$ (e.g. $F=Rf$). Let G be the restriction of F to $S_p\times\mathbb{R}^p$. Then for $(u,v)\in S_p\times\mathbb{R}^p$, $1\le j<k\le p$,

$$\left(\partial^2_{v_j}\partial_{u_{kk}}+\partial^2_{v_k}\partial_{u_{jj}}-\partial_{v_j}\partial_{v_k}\partial_{u_{jk}}\right)G(u,v)=0\ . \tag{29}$$

Proof. *(i)* By Definition 6 and Proposition 5*(iii)*

$$d\left(\kappa^L_{x,y}Rf\right)=d\left(\iota^*\kappa_{x,y}Rf\right)=\iota^*\left(d\kappa_{x,y}Rf\right)=0\ .$$

(ii) We take $(u_{jk})_{j\le k}$ as coordinates on S_p. If $I:S_p\times\mathbb{R}^p\hookrightarrow\mathcal{L}(\mathbb{R}^p,\mathbb{R}^p)\times\mathbb{R}^p$ denotes the canonical injection we have $G=F\circ I$ and

$$\begin{aligned}\partial_{u_{jj}}G&=\left(\partial_{u_{jj}}F\right)\circ I\ ,\ \partial_{v_j}G=\left(\partial_{v_j}F\right)\circ I\ ,\\ \partial_{u_{jk}}G&=\left((\partial_{u_{jk}}+\partial_{u_{kj}})F\right)\circ I\quad\text{ for } j<k\ .\end{aligned}$$

By (24) $\partial_{v_k}\partial_{u_{jk}}F=\partial_{v_j}\partial_{u_{kk}}F$ hence

$$\begin{aligned}\partial_{v_j}\partial_{v_k}\partial_{u_{jk}}G&=\left(\left(\partial_{v_j}\partial_{v_k}\partial_{u_{jk}}+\partial_{v_j}\partial_{v_k}\partial_{u_{kj}}\right)F\right)\circ I\\ &=\left(\left(\partial^2_{v_j}\partial_{u_{kk}}+\partial^2_{v_k}\partial_{u_{jj}}\right)F\right)\circ I\\ &=\left(\partial^2_{v_j}\partial_{u_{kk}}+\partial^2_{v_k}\partial_{u_{jj}}\right)G\ .\ \blacksquare\end{aligned}$$

e. Remark. As noted in [9] the situation would be different with *skew-symmetric* matrices instead of symmetric, despite the analogy with the Lagrangian Radon transform. Indeed restriction of (11) would give the projection slice theorem

$$\widehat{Rf}(u,\eta)=\widehat{f}(u\eta,\eta)\ ,$$

but $u\eta$ is now orthogonal to η and $\widehat{f}$ is not fully determined by this equation. The transform R is not injective in this case; see [9] p. 125 for details.

5. RANGE THEOREMS

The above tools also lead to range theorems for the Radon transform, i.e. characterizations of the image under R of the space $\mathcal{S}(\mathbb{R}^n)$. We shall only sketch the proofs of Theorems 10 and 12 below (due to Grinberg [9], Debiard and Gaveau [1]) with emphasis on the formal significance of the partial differential equations (Lemmas 11 and 13).

a. General case. In order to define the relevant notion of rapidly decreasing functions on the Grassmannian we need a notation : let $u = (u_{jk}) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$, $\eta \in \mathbb{R}^q$ and, for $1 \leq j \leq q$, $1 \leq k \leq p$,

$$L_{jk} = \sum_{i=1}^{q} u_{ik} \left(\sum_{l=1}^{p} u_{jl} \partial_{u_{il}} - \eta_i \partial_{\eta_j} \right) .$$

As before $\widehat{F}(u, \eta)$ denotes the partial Fourier transform on $\mathbb{R}^q$ of a function $F(u, v)$.

Definition 9 *A complex valued function F of $(u, v) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) \times \mathbb{R}^q$ is said to be* rapidly decreasing *if it is C^∞ and*
(i) for each u the function $v \mapsto F(u, v)$ belongs to $\mathcal{S}(\mathbb{R}^q)$
(ii) for any polynomial P in the differential operators

$$\partial_{u_{jk}} \ , \ \left\| I + \ {}^t u u \right\|_2^{-1/2} \partial_{\eta_j} \ , \ \sum_{i=1}^{q} u_{ik} \partial_{u_{il}} \ , \ L_{jk}$$

with $1 \leq j \leq q$, $1 \leq k, l \leq p$, there exist a positive integer N and a constant $C_N > 0$ such that, for all u, η,

$$P\widehat{F}(u, \eta) \leq C_N \left(1 + \left\|{}^t u \eta\right\| + \|\eta\|\right)^{-N} .$$

A more pleasant definition was given by Gonzalez [7][8] with a different parametrization of the manifold of p-planes. Here, writing them as graphs, we had interesting simplifications in the previous sections but at the expense of introducing artificial singularities at planes not transversal to $0 \times \mathbb{R}^q$. The role of the operators $\sum u_{ik} \partial_{u_{il}}$, L_{jk} is to deal with these singularities.

Theorem 10 *(Range theorem for p-planes) Let F be a function of $(u, v) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) \times \mathbb{R}^q$ with $n = p + q$. For $1 \leq p < n - 1$ the following are equivalent :*
(i) There exists $f \in \mathcal{S}(\mathbb{R}^n)$ such that $F = Rf$.
(ii) F is rapidly decreasing and $\left(\partial_{v_i} \partial_{u_{jk}} - \partial_{v_j} \partial_{u_{ik}}\right) F(u, v) = 0$ for $(u, v) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) \times \mathbb{R}^q$, $1 \leq i, j \leq q$, $1 \leq k \leq p$.
In (i) f is unique and given by (14), or (23) if J has constant sign.

As noted in [9] p. 120 no additional "moment conditions" on F are required for $p < n - 1$; see Richter [12] for a proof. Viewing the Grassmannian as a homogeneous space of the Euclidean motion group of $\mathbb{R}^n$ Gonzalez proves a more satisfactory result, with the differential operators in *(i)* replaced by a family of second order invariant differential operators under this group [7] or even by a single fourth order invariant differential operator [8].

Sketch of proof. *(i)* implies *(ii)* by (10) and estimates of Rf and its derivatives ([9] p. 116).

(ii) implies *(i)*. The key lemma is

Lemma 11 *Let F be a rapidly decreasing function of $(u,v) \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q) \times \mathbb{R}^q$. The following are equivalent :*
(ii) $\left(\partial_{v_i}\partial_{u_{jk}} - \partial_{v_j}\partial_{u_{ik}}\right) F = 0 \ , \ 1 \le i,j \le q \ , \ 1 \le k \le p \ .$
(ii') For all $u, u' \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ and all $\eta \neq 0$ in $\mathbb{R}^q$, the equality ${}^t u\eta = {}^t u'\eta$ implies $\widehat{F}(u,\eta) = \widehat{F}(u',\eta)$.

Proof of Lemma 11. Let $E_{jk} \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$ denote the matrix with 1 as entry (j,k) and 0 elsewhere. Considering the Fourier transform $\widehat{F}(u,\eta) = \int F(u,v)e^{-2i\pi<\eta,v>}dv$ we have

$$\left(\left(\partial_{v_h}\partial_{u_{jk}} - \partial_{v_j}\partial_{u_{hk}}\right) F\right)\widehat{}\,(u,\eta) = 2i\pi\ \partial_{s=0}\widehat{F}(u + s(\eta_h E_{jk} - \eta_j E_{hk}), \eta) \ . \qquad (30)$$

For $\eta \neq 0$ the matrices $A_{hjk} = \eta_h E_{jk} - \eta_j E_{hk}$, $1 \le h, j \le q$, $1 \le k \le p$, generate the kernel of the map $u \mapsto {}^t u\eta$. Indeed this map is onto by the Remark in Section 2.d, therefore its kernel has dimension $pq - p$ and, assuming $\eta_h \neq 0$ for some h, the A_{hjk} with $1 \le j \le q$, $j \neq h$, $1 \le k \le p$, are $p(q-1)$ independent elements of this kernel.
Replacing u by $u + sA_{hjk}$ in (30) we see that *(ii)* is equivalent to $\widehat{F}(u+sA_{hjk}, \eta) = \widehat{F}(u,\eta)$ for all s, h, j, k, i.e. to $\widehat{F}(u+u', \eta) = \widehat{F}(u,\eta)$ whenever ${}^t u'\eta = 0$. The lemma is proved. ■

By *(ii)* and Lemma 11 there exists a unique function ψ on $\mathbb{R}^p \times (\mathbb{R}^q \backslash 0)$ such that

$$\psi(-\,{}^t u\eta, \eta) = \widehat{F}(u,\eta)$$

for all $u \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^q)$, $\eta \in \mathbb{R}^q \backslash 0$. The technical point is to show that, F being rapidly decreasing, ψ extends to a function in $\mathcal{S}(\mathbb{R}^p \times \mathbb{R}^q)$. We admit it here ; see [9] Appendix II.

If there exists $f \in \mathcal{S}(\mathbb{R}^n)$ such that $Rf = F$ then the projection slice theorem (11) implies $\widehat{f} = \psi$. Conversely, if f denotes the inverse Fourier transform of ψ, we have $f \in \mathcal{S}(\mathbb{R}^n)$ and

$$\widehat{Rf}(u,\eta) = \widehat{f}(-\,{}^t u\eta, \eta) = \psi(-\,{}^t u\eta, \eta) = \widehat{F}(u,\eta) \ ,$$

hence $Rf = F$ and the theorem. ■

b. Lagrangian case. Let us now restrict to S_p, the space of symmetric matrices (with $p = q$) as in Section 4. Following [9] again we say that a function G is *rapidly decreasing on $S_p \times \mathbb{R}^p$* if it extends to a rapidly decreasing function on $\mathcal{L}(\mathbb{R}^p, \mathbb{R}^p) \times \mathbb{R}^p$.

Theorem 12 *(Range theorem for Lagrangian p-planes) Let G be a function of $(u,v) \in S_p \times \mathbb{R}^p$. For $p > 1$ the following are equivalent :*
(i) There exists $f \in \mathcal{S}(\mathbb{R}^{2p})$ such that G is the restriction of Rf to $S_p \times \mathbb{R}^p$.
(ii) G is rapidly decreasing and $\left(\partial^2_{v_j}\partial_{u_{kk}} + \partial^2_{v_k}\partial_{u_{jj}} - \partial_{v_j}\partial_{v_k}\partial_{u_{jk}}\right) G(u,v) = 0$ for $(u,v) \in S_p \times \mathbb{R}^p$, $1 \le j < k \le p$.
In (i) f is unique and given by one of the inversion formulas in Section 4.

Sketch of proof. *(i)* implies *(ii)* by Proposition 8 and estimates of Rf and its derivatives.
(ii) implies *(i)*. The key lemma is

Lemma 13 *Let G be a rapidly decreasing function of $(u,v) \in S_p \times \mathbb{R}^p$. The following are equivalent :*
(ii) $\left(\partial^2_{v_j}\partial_{u_{kk}} + \partial^2_{v_k}\partial_{u_{jj}} - \partial_{v_j}\partial_{v_k}\partial_{u_{jk}}\right) G(u,v) = 0$ *for* $(u,v) \in S_p \times \mathbb{R}^p$, $1 \le j < k \le p$
(ii') For all $u, u' \in S_p$ and all $\eta \in \mathbb{R}^p$ such that $\eta_1 \cdots \eta_p \neq 0$ the equality $u\eta = u'\eta$ implies $\widehat{G}(u,\eta) = \widehat{G}(u',\eta)$.

Proof of Lemma 13. Let $S_{jk} \in S_p$ denote the matrix with 1 as entries (j,k) and (k,j) and 0 elsewhere. Then

$$\left(\left(\partial^2_{v_j}\partial_{u_{kk}} + \partial^2_{v_k}\partial_{u_{jj}} - \partial_{v_j}\partial_{v_k}\partial_{u_{jk}}\right) G\right)^{\wedge}(u,\eta) =$$
$$= -4\pi^2\partial_{s=0}\widehat{G}\left(u + s\left(\eta_j^2 S_{kk} + \eta_k^2 S_{jj} - \eta_j\eta_k S_{jk}\right), \eta\right) . \qquad (31)$$

We now claim that for $\eta_1 \cdots \eta_p \neq 0$ the matrices

$$B_{jk} = \eta_j^2 S_{kk} + \eta_k^2 S_{jj} - \eta_j\eta_k S_{jk} \ , \ 1 \leq j < k \leq p \ ,$$

make up a basis of the kernel of the map $L_\eta : u \mapsto u\eta$.
Indeed $L_\eta(S_p) = \mathbb{R}^p$: for any $\xi \in \mathbb{R}^p$ the equation $u\eta = \xi$ is solved by

$$u = diag(\xi_1/\eta_1, ..., \xi_p/\eta_p) \in S_p \ .$$

Thus $\dim\ker L_\eta = p(p+1)/2 - p = p(p-1)/2$ and the $p(p-1)/2$ matrices B_{jk}, $j<k$, are linearly independent elements of this kernel since $\sum_{j<k}\lambda_{jk}B_{jk} = 0$ implies $\lambda_{jk}\eta_j\eta_k = 0$ hence $\lambda_{jk} = 0$. This proves the claim.
Replacing u by $u + sB_{jk}$ in (31) we see that *(ii)* is equivalent to $\widehat{G}(u+sB_{jk},\eta) = \widehat{G}(u,\eta)$ for all s, j, k and the lemma follows. ■

Let Ω be the set of all $\eta \in \mathbb{R}^p$ such that $\eta_1 \cdots \eta_p \neq 0$. By *(ii)* and Lemma 13 there exists a unique function ψ on $\mathbb{R}^p \times \Omega$ such that

$$\psi(-u\eta,\eta) = \widehat{G}(u,\eta)$$

for all $u \in S_p$, $\eta \in \Omega$. Then ψ extends to a function in $\mathcal{S}(\mathbb{R}^{2p})$ (admitted).

The proof ends as for Theorem 10. If there exists $f \in \mathcal{S}(\mathbb{R}^{2p})$ such that G equals Rf restricted to $S_p \times \mathbb{R}^p$ we must have $\widehat{f}(-u\eta,\eta) = \widehat{Rf}(u,\eta) = \widehat{G}(u,\eta) = \psi(-u\eta,\eta)$ for $u \in S_p$, $\eta \in \Omega$, hence $\widehat{f} = \psi$. Conversely, if f denotes the inverse Fourier transform of ψ, we have $f \in \mathcal{S}(\mathbb{R}^{2p})$ and

$$\widehat{Rf}(u,\eta) = \widehat{f}(-u\eta,\eta) = \psi(-u\eta,\eta) = \widehat{G}(u,\eta)$$

for $u \in S_p$, $\eta \in \Omega$, therefore G is the restriction of Rf to $S_p \times \mathbb{R}^p$. ■

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The present notes were mainly inspired by reading Gelfand, Gindikin and Graev [3] (who give a much deeper study of the kappa operator), Grinberg [9], Debiard and Gaveau [1], Gelfand and Gindikin [2]. Theorem 10 is taken from [9], Theorem 12 from [1] and [9]. For the case of Lagrangian planes see [1], [9] and [10].

For other occurrences of the kappa operator, in the context of integral geometry over quadrics, see [4] chapter 5, [5], [6] or even the notes [13].

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