Radon transform on a harmonic manifold

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Abstract

We extend to a large class of noncompact harmonic manifolds the inversion formulas for the Radon transform on horospheres in hyperbolic spaces or Damek-Ricci spaces. Horospheres are defined here as level hypersurfaces of Busemann functions. The proof uses harmonic analysis on the manifolds considered, developed in a recent paper by Biswas, Knieper and Peyerimhoff; we also give a concise proof of their Fourier inversion theorem for harmonic manifolds.

Keywords: harmonic manifold, Radon transform, harmonic analysis, Busemann function, horosphere.

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Introduction

Harmonic analysis and integral geometry have close links with each other. In various geometric frameworks the (generalized) Fourier transform of a function u is obtained by integrating it against eigenfunctions of the Laplace operator. Decomposing this integral according to the level sets of an eigenfunction, one can prove a so-called «projection slice theorem» which links the Radon transform Ru, given by the integrals of u over such sets, with the Fourier transform of u. A Fourier inversion formula for u may then lead to an inversion formula for the integral transform R. One of the simplest examples is given by the exponentials $x \mapsto e^{2i\pi\langle\xi,x\rangle}$, eigenfunctions of the Laplace operator in $\mathbb{R}^n$, whose level sets are the hyperplanes $\langle\xi, x\rangle = constant$. The classical Fourier inversion theorem yields a Radon inversion formula which reconstructs u from its integrals over hyperplanes.

Among many examples of this problem let us mention Helgason's Radon transform over horospheres of a symmetric space of the noncompact type [12]. As symmetric spaces of rank one, the hyperbolic spaces are a special case, which also extends in a different direction to a large class of spaces (non symmetric in general) known as Damek-Ricci spaces or harmonic NA groups [9]. The purpose of this note is to extend the latter cases one step further, to all simply connected harmonic manifolds with purely exponential volume growth. For this we shall use harmonic analysis on these manifolds as developed in a recent paper by Biswas, Knieper and Peyerimhoff [4]. There are no Lie groups here; horospheres are defined by means of Busemann functions.

In Section 1 we recall basic facts about harmonic manifolds, Busemann functions and we give explicit geometric expressions of these functions and the corresponding horospheres for hyperbolic spaces and for Damek-Ricci spaces. In Section 2 we summarize the main results of [4] in harmonic analysis, radial then non radial; for the reader's convenience we provide a concise version of the proof of their Fourier inversion theorem (Theorem 7). In Section 3 we introduce the horosphere Radon transform R on a harmonic manifold with purely exponential volume growth and its dual transform R^*. Our main result is Theorem 11, giving two versions of an inversion formula for R.

One of them is $u = R^*\Lambda R u$ (where Λ is a certain operator), similar to Helgason's Theorem 3.13 in [12] Chapter II. The other uses a «shifted dual Radon transform», a method introduced by Radon in his pioneering 1917 paper dealing with lines in $\mathbb{R}^2$.

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Notation. If X is a topological space we denote by $C(X)$, resp. $C_c(X)$, the space of functions $X \to \mathbb{C}$ which are continuous, resp. continuous with compact support.
If X is a (smooth real) manifold, we denote by T_oX its tangent space at $o \in X$ and by $\mathcal{D}(X)$ the space of C^∞ functions $X \to \mathbb{C}$ with compact support.
Throughout the paper X will be a *complete Riemannian manifold*, $\|.\|$ the norm of tangent vectors given by the Riemannian structure and $d(x, y)$ the geodesic distance between two points $x, y \in X$. Given $o \in X$, Exp_o is the exponential mapping at o: for any unit vector $v \in T_oX$, $\gamma(t) = \mathrm{Exp}_o(tv)$ is the geodesic defined by $\gamma(0) = o$ and $\gamma'(0) = v$, so that $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$. We shall also use the gradient ∇ and the Laplace-Beltrami operator $Lf = \mathrm{div}(\nabla f)$.

1 Harmonic manifolds and Busemann functions

1.1 General facts

We collect here some facts needed in the sequel, mainly taken from [4] Section 2, to which we refer for more details.

A Riemannian manifold X is *harmonic* if, for every origin $o \in X$, there exists a non constant harmonic function on a punctured neighborhood of o which is radial around o, i.e. only depends on the distance $d(o, x)$. For a few equivalent properties see e.g. [18] Théorème 4. A simply connected complete noncompact harmonic manifold has no conjugate points (Allamigeon's theorem) thus, by the Cartan-Hadamard theorem, the exponential mapping $\mathrm{Exp}_o : T_oX \to X$ at o is a global diffeomorphism onto, for every $o \in X$.

Let S_oX denote the unit sphere in T_oX and let $r > 0$. If X is harmonic the Jacobian of the map $v \mapsto \mathrm{Exp}_o(rv)$ from S_oX into X only depends on r and is called the density function $A(r)$ of X. The harmonic manifold X is said to be of *purely exponential volume growth* if there exist constants $C > 1$ and $\rho > 0$ such that, for all $r \geq 1$,

$$C^{-1}e^{2\rho r} \leq A(r) \leq Ce^{2\rho r}.$$

An equivalent property is that the volume of metric balls of radius r satisfies a similar inequality. The class of harmonic manifolds with purely exponential volume growth includes all known examples of non-flat noncompact harmonic manifolds.

Henceforth we assume X *is a simply connected harmonic manifold with purely exponential volume growth* and $o \in X$ is a given origin.

For $v \in S_oX$ let $\gamma_v(r) = \mathrm{Exp}_o(rv)$, $r \geq 0$, be the geodesic ray such that $\gamma_v(0) = o$ and $\gamma_v'(0) = v$. The corresponding *Busemann function* is defined by[1]

$$b_v(x) := \lim_{r\to+\infty} \left(d(o, \gamma_v(r)) - d(x, \gamma_v(r))\right) = \lim_{r\to+\infty} \left(r - d(x, \gamma_v(r))\right). \tag{1}$$

The limit exists since, by the triangle inequality, the function $r \mapsto r - d(x, \gamma_v(r))$ is increasing and bounded by $d(o, x)$. Also $|b_v(x) - b_v(y)| \leq d(x, y)$ and b_v is a Lipschitz function on X. A much stronger result can actually be proved: harmonic manifolds are Einstein manifolds, therefore analytic by the DeTurck-Kazdan theorem, and b_v is analytic on X (Ranjan and Shah [16], Theorem 3.1). It satisfies the partial differential equations

$$\|\nabla b_v(x)\| = 1 \ , \ Lb_v(x) = -2\rho \tag{2}$$

[1] This definition is the opposite of the classical one, so as to avoid minus signs at several places in the sequel, e.g. for the examples in Sections 1.2 and 1.3.

for $x \in X$, where L is the Laplace-Beltrami operator and ρ is the constant introduced above. The level sets of a Busemann function, called *horospheres*, may be viewed as spheres with center at infinity. They have constant mean curvature 2ρ.

A boundary structure for X can be introduced as follows. Let ∂X be the set of equivalence classes of geodesic rays in X, two rays γ_1 and γ_2 (with different origins) being equivalent if the distance $d(\gamma_1(r), \gamma_2(r))$ remains bounded for $r \to +\infty$. Let $\gamma(\infty) \in \partial X$ denote the class of a ray γ. Considering again $\gamma_v(r) = \mathrm{Exp}_o(rv)$, the map $v \mapsto \omega = \gamma_v(\infty)$ is a bijection of the unit sphere S_oX onto ∂X. The topology and the (normalized) measure on S_oX induced by the Riemannian norm can therefore be transferred to the boundary ∂X, providing it with a topology (which does not depend on o) and a (normalized) measure $d_o\omega$ (which depends on o), called the *visibility measure* from o.

We can then modify the definition of Busemann functions as follows. For $\omega \in \partial X$ let γ be any geodesic ray such that $\gamma(\infty) = \omega$. Then $\lim_{r\to+\infty} (d(o, \gamma(r)) - d(x, \gamma(r)))$ only depends on o, x and ω ([4], Lemma 2.1). We may thus define $B_o(x, \omega)$ by

$$B_o(x, \omega) := \lim_{r\to+\infty} (d(o, \gamma(r)) - d(x, \gamma(r))) = b_v(x), \tag{3}$$

where $v = \delta'(0) \in S_oX$ is the initial velocity of the ray δ such that $\delta(0) = o$ and $\delta(\infty) = \omega$ (thus $\delta(r) = \mathrm{Exp}_o(rv)$). The function $(x, \omega) \mapsto B_o(x, \omega)$ is continuous on $X \times \partial X$ and analytic with respect to x. Besides

$$\|\nabla_x B_o(x, \omega)\| = 1 \ , \ L_x B_o(x, \omega) = -2\rho \tag{4}$$

for $x \in X$, $\omega \in \partial X$.

An obvious consequence of definition (3) is that, when replacing o with a new origin a,

$$B_a(x, \omega) = B_o(x, \omega) - B_o(a, \omega). \tag{5}$$

Lemma 1 *Given $o \in X$, $t \in \mathbb{R}$ and $\omega \in \partial X$ let $S_t(\omega)$ denote the horosphere $\{x \in X | B_o(x, \omega) = t\}$. Then $d(o, S_t(\omega)) := \inf_{x\in S_t(\omega)} d(o, x) = |t|$.*

Proof. Since $|B_o(x, \omega)| \leq d(o, x)$ by the triangle inequality, $x \in S_t(\omega)$ implies $d(o, x) \geq |t|$. Let δ denote the geodesic such that $\delta(0) = o$ and $\delta(\infty) = \omega$. For any $r, s \in \mathbb{R}$ we have $d(\delta(s), \delta(r)) = |r - s|$ therefore

$$B_o(\delta(s), \omega) = \lim_{r\to+\infty} (r - |r - s|) = s.$$

In particular $\delta(t) \in S_t(\omega)$. Since $d(o, \delta(t)) = |t|$ this completes the proof. ■

We shall also need the following link between the visibility measures from o and a (see Theorem 1.4 in Knieper and Peyerimhoff [14] with $h = 2\rho$, proved for a wider class of harmonic manifolds):

$$d_a\omega = e^{2\rho B_o(a,\omega)} d_o\omega. \tag{6}$$

In the next two subsections we give geometric expressions of the Busemann functions for hyperbolic spaces, resp. Damek-Ricci spaces, showing the general harmonic analysis of [4] agrees with the classical theories developed in [12], resp. [2]. Hyperbolic spaces are actually a special case of Damek-Ricci spaces but, notation and tools being different, we shall consider both cases separately; for a detailed comparison see e.g. [18] Section 6.

1.2 Rank one symmetric spaces

Let $X = G/K$ be a symmetric space of the noncompact type, where G is a connected noncompact real semisimple Lie group with finite center and K is a maximal compact subgroup. A dot denotes the action of G on X: $g \cdot x = gg'K$ for $g, g' \in G$ and $x = g'K \in G/K$. We briefly recall the classical semisimple notation of Helgason's books [10], [11] and [12], to which we refer for details.

The Lie algebra of G decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the ± 1 eigenspaces of the Cartan involution, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the tangent space to G/K at the origin $o = K$.
Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. A nonzero linear form α on $\mathfrak{a}$ such that the joint eigenspace

$$\mathfrak{g}_\alpha = \{V \in \mathfrak{g} | \, [H, V] = \alpha(H)V \text{ for all } H \in \mathfrak{a}\}$$

is not $\{0\}$ is called a root of the pair $(\mathfrak{g}, \mathfrak{a})$. Here we assume X has rank one i.e. $\dim \mathfrak{a} = 1$, so that there are four roots at most: α, $-\alpha$ and (possibly) 2α, -2α. We write $\mathfrak{a} = \mathbb{R}H$ with H chosen so that $\alpha(H) = 1$. Then $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ is the Lie algebra of a nilpotent Lie subgroup N of G, leading to the Iwasawa decompositions $G = ANK = NAK$ where A is the (one-dimensional) subgroup with Lie algebra $\mathfrak{a}$.
A K-invariant scalar product on $\mathfrak{p}$, normalized so that H is a unit vector, can be constructed from the Killing form of $\mathfrak{g}$, whence a G-invariant Riemannian structure on X. The manifolds X thus obtained are all the hyperbolic spaces (real, complex, quaternionic and exceptional). They are harmonic with purely exponential volume growth.
Let M be the centralizer of A in K. The map $k \mapsto \mathrm{Ad}(k)H$ from K into $\mathfrak{p}$ given by the adjoint action induces a diffeomorphism $\omega = kM \mapsto v = \mathrm{Ad}(k)H$ of the homogeneous space K/M onto the unit sphere of $\mathfrak{p}$.
In the Iwasawa decomposition $G = ANK$ let $A(g) \in \mathbb{R}$ (identified with the line $\mathfrak{a} = \mathbb{R}H$) be defined by $g \in (\exp A(g))NK$. Thus $g = ka_t n$ with $a_t := \exp tH$ implies $t = A(k^{-1}g)$. From the properties of the subgroups A, N, K and M of G it follows easily that $A(k^{-1}g)$ only depends on the left cosets $x = gK$ and $\omega = kM$; following [12] Chapter III let us write it as $A(x, \omega) := A(k^{-1}g)$.

Proposition 2 *The Busemann function of a rank one symmetric space of the noncompact type G/K, relative to its origin $o = K$, is*

$$b_v(x) = A(x, \omega) = A(k^{-1}g)$$

for $x = gK \in G/K$, $\omega = kM \in K/M$ and $v = \mathrm{Ad}(k)H$.
Given $t \in \mathbb{R}$ and $\omega = kM \in K/M$, the equation $A(x, \omega) = t$ defines the horosphere $ka_tN \cdot o$.

Proof. A unit tangent vector at o may be written as $v = \mathrm{Ad}(k)H$ and defines the geodesic $\gamma_v(r) = \mathrm{Exp}_o(rv) = ka_r.o$ from the origin. Using the G-invariance of the distance d on G/K, we have $d(x, \gamma_v(r)) = d(k^{-1} \cdot x, a_r \cdot o)$. By the Iwasawa decomposition $G = ANK$ we may write $k^{-1} \cdot x = a_t n \cdot o$ with $t = A(x, \omega) \in \mathbb{R}$, $\omega = kM$ and some (unique) $n \in N$. Thus, with $s := r - t \to +\infty$,

$$\begin{aligned} d(x, \gamma_v(r)) &= d(a_t n \cdot o, a_r \cdot o) = d(o, n^{-1}a_s \cdot o) \\ &= d(a_{-s} \cdot o, a_{-s}n^{-1}a_s \cdot o). \end{aligned}$$

But, writing $n^{-1} = \exp(V + Z)$ with $V \in \mathfrak{g}_\alpha$ and $Z \in \mathfrak{g}_{2\alpha}$, we see that

$$a_{-s}n^{-1}a_s = \exp\left(e^{-s}V + e^{-2s}Z\right)$$

tends to the identity element as $s \to +\infty$. It follows that

$$d(x, \gamma_v(r)) = d(a_{-s} \cdot o, o) + \varepsilon(r) = |s| + \varepsilon(r) = r - t + \varepsilon(r),$$

with $\varepsilon(r) \to 0$ as $r \to +\infty$; indeed, by the triangle inequality,

$$|\varepsilon(r)| = |d(a_{-s} \cdot o, a_{-s}n^{-1}a_s \cdot o) - d(a_{-s} \cdot o, o)| \leq d(a_{-s}n^{-1}a_s \cdot o, o).$$

Therefore

$$b_v(x) = \lim_{r \to +\infty} (r - d(x, \gamma_v(r))) = t = A(x, \omega).$$

The last assertion of the proposition follows from the definitions. ■

Remark. The distance $d(x, \gamma_v(r)) = d(o, n^{-1}a_s \cdot o)$ might be computed exactly by the classical method of $SU(2, 1)$ reduction (see [10] Chapter IX §3). The above proof avoids this.

1.3 Damek-Ricci spaces

Also known as a «harmonic NA group», a Damek-Ricci space is a simply connected solvable Lie group S, semi-direct product of a two-step nilpotent group N by a one-dimensional group A isomorphic to the additive group of $\mathbb{R}$, with some additional properties summarized below. Its Lie algebra $\mathfrak{s}$ decomposes as $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}H$, a direct sum of vector subspaces with $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$, $[\mathfrak{v}, \mathfrak{z}] = 0$, $[\mathfrak{z}, \mathfrak{z}] = 0$ and $[H, V] = \frac{1}{2}V$, $[H, Z] = Z$ for all $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$. Here $\mathfrak{v} \oplus \mathfrak{z}$, resp. $\mathbb{R}H$, is the Lie algebra of N, resp. A.

Remark. In the special case of a rank one symmetric space considered in the previous section, one has $S = NA = G/K$ with $G = NAK$, $\mathfrak{v} = \mathfrak{g}_\alpha$ and $\mathfrak{z} = \mathfrak{g}_{2\alpha}$. Note the different choice of H in that case however, where we had $[H, V] = V$ and $[H, Z] = 2Z$ for $V \in \mathfrak{g}_\alpha$, $Z \in \mathfrak{g}_{2\alpha}$.

Given scalar products $\langle ., . \rangle$ on $\mathfrak{v}$ and $\mathfrak{z}$ we equip $\mathfrak{s}$ with the scalar product

$$\langle X, X' \rangle = \langle V, V' \rangle + \langle Z, Z' \rangle + tt' \text{ for } X = V + Z + tH \ , \ X' = V' + Z' + t'H,$$

and the norm defined by $\|X\|^2 = \|V\|^2 + \|Z\|^2 + t^2$.

For $Z \in \mathfrak{z}$, let $J_Z : \mathfrak{v} \to \mathfrak{v}$ denote the linear map defined by $\langle J_Z V, V' \rangle = \langle Z, [V, V'] \rangle$ for all $V, V' \in \mathfrak{v}$. We assume $J_Z^2 V = - \|Z\|^2 V$ for all $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$. It follows that $J_Z V$ is orthogonal to V and $\|J_Z V\| = \|Z\| \|V\|$.

It is convenient to realize the Damek-Ricci space as the vector space $S = \mathfrak{v} \times \mathfrak{z} \times \mathbb{R}$ with the group law

$$(V, Z, t)(V', Z', t') = \left(V + e^{t/2} V', Z + e^t Z' + \frac{e^{t/2}}{2} [V, V'], t + t' \right).$$

The subgroup N is defined by $t = 0$ and the subgroup A by $V = Z = 0$. Any $x \in S$ decomposes in a unique way as $x = na_t$ with $n = (V, Z, 0) \in N$ and $a_t = (0, 0, t) \in A$. We call $t = t(x)$ the A-component of x; clearly $t(xy) = t(x) + t(y)$ for $x, y \in S$.

The Lie group S is equipped with the Riemannian structure defined by the left invariant metric which coincides with $\langle ., . \rangle$ on the tangent space $\mathfrak{s}$ at the identity element $o = (0, 0, 0)$ of S. It is then a harmonic manifold with purely exponential volume growth. Let $d(., .)$ denote the corresponding left-invariant Riemannian distance on S.

Another useful realization of S is obtained from the *Cayley transform* $C : S \to B$, a diffeomorphism of S onto the open unit ball B in $\mathfrak{s}$ defined by

$$C(V, Z, t) = \left(\frac{(1 + u)V - J_Z V}{(1 + u)^2 + \|Z\|^2}, \frac{2Z}{(1 + u)^2 + \|Z\|^2}, \frac{-1 + u^2 + \|Z\|^2}{(1 + u)^2 + \|Z\|^2} \right) \in B \subset \mathfrak{v} \times \mathfrak{z} \times \mathbb{R}, \quad (7)$$
$$\text{with } u = e^t + \frac{1}{4} \|V\|^2$$

(see [8], or [18] Section 4.4). In the unit ball model the geodesics from the origin are simply the diameters of B:

$$C(\mathrm{Exp}_o(rX)) = \mathrm{th}\, \frac{r}{2} \cdot X, \quad (8)$$

where $X \in \mathfrak{s}$ is a unit tangent vector at o and $r \in \mathbb{R}$.

Replacing e^t by 0 in (7) we also obtain a diffeomorphism

$$\begin{aligned} C_\infty : (V, Z) &\mapsto C(V, Z, -\infty) \\ N &\to \partial B \backslash \{H\} \end{aligned} \quad (9)$$

of N onto $\partial B \backslash \{H\}$, the unit sphere of $\mathfrak{s}$ with the point H removed. Thus, adding to N a point ∞, $N \cup \{\infty\}$ may be seen as the set of points at infinity in S.

The *geodesic symmetry* σ with respect to the origin o of S is defined by $\sigma(\mathrm{Exp}_o(rX)) = \mathrm{Exp}_o(-rX)$. In other words, remembering (8),

$$C(\sigma(x)) = -C(x)$$

for any $x \in S$. From the definition (7) of C it follows that

$$\sigma(V,Z,t) = \left(\lambda^{-1}(-uV + J_Z V)\,, -\lambda^{-1}Z, t - \log\lambda\right)$$
$$\text{with } u = e^t + \frac{1}{4}\|V\|^2 \ , \ \lambda = u^2 + \|Z\|^2 .$$

It is then easily checked that, for any $s \in \mathbb{R}$, the map $x \mapsto \sigma(a_s x)$ is an involutive diffeomorphism of S, that is

$$\sigma(a_s x) = a_{-s}\sigma(x) = \sigma(a_s)\sigma(x) \text{ for } s \in \mathbb{R}, x \in S. \tag{10}$$

Introducing the A-component $\widetilde{t}(x) := t(\sigma(x))$ this implies

$$\widetilde{t}(a_s x) = \widetilde{t}(x) - s. \tag{11}$$

Let us also note that

$$d(o,x) = d(o,\sigma(x)) = d(o,x^{-1}) \tag{12}$$

in view of the definition of σ and the left-invariance of d.
We refer the reader to [7], [8] or to our expository notes [18], Chapter 4, for more details.

Proposition 3 *The Busemann function of a Damek-Ricci space $S = NA$, relative to its origin o, is*

$$b_X(x) = \widetilde{t}(n(X)^{-1}x) - \widetilde{t}(n(X)^{-1}),$$

where $x \in S$ and X is a unit tangent vector at the origin with $X \neq H = (0,0,1)$. Here $n(X) \in N$ is characterized by $C_\infty(n(X)) = X$ (see also (14) below) and, for $y \in S$, $\widetilde{t}(y) := t(\sigma(y))$ denotes the A-component of its symmetrical point $\sigma(y)$.
For $X = H$, we have

$$b_H(x) = t(x),$$

where $t(x)$ is the A-component of x.
The horospheres $b_X(x) =$ constant are the sets $n(X)\sigma(Na_t)$ if $X \neq H$, resp. Na_t if $X = H$, for some fixed $t \in \mathbb{R}$.

This result is given (with different proofs) in [3], Lemma 3.1 and [13], Theorem 4. The following proof avoids using the explicit expression of distances in a Damek-Ricci space.

Proof. Let $X = V_0 + Z_0 + t_0 H \in \mathfrak{s}$ be a given unit vector tangent to S at o and $\gamma(r) = \mathrm{Exp}_o(rX)$, $r \in \mathbb{R}$, be the geodesic defined by $\gamma(0) = o$ and $\gamma'(0) = X$. By (8) the Cayley transform C maps γ onto a diameter of B:

$$C(\gamma(r)) = RX = RV_0 + RZ_0 + Rt_0 H \ , \text{ with } R := \mathrm{th}\,\frac{r}{2}, r \in \mathbb{R},$$

and by [7] p. 14 (or [18] Théorème 9) $\gamma(r) \in S = NA$ decomposes as $\gamma(r) = n(r)a_{-\tau(r)}$ with

$$\begin{aligned} n(r) &= \frac{2R}{\chi(RX)}\left((1 - Rt_0)V_0 + RJ_{Z_0}V_0, Z_0, 0\right) \in N, \\ a_{-\tau(r)} &= (0,0,-\tau(r)) \in A \ , \ \tau(r) = \log\left(\chi(RX)/(1-R^2)\right), \\ \chi(RX) &= (1 - Rt_0)^2 + R^2\|Z_0\|^2 > 0. \end{aligned} \tag{13}$$

As r tends to $+\infty$ we have $\chi(RX) \to \chi(X) = (1-t_0)^2 + \|Z_0\|^2$. Since $\|V_0\|^2 + \|Z_0\|^2 + t_0^2 = 1$, we see that $\chi(X) > 0$ if and only if $X \neq H$.
Assume first $X \neq H$. Then $\tau(r) \to +\infty$ as $r \to +\infty$ by (13) and $n(r) \to n(X)$ with

$$n(X) := \frac{2}{\chi(X)}\left((1-t_0)V_0 + J_{Z_0}V_0, Z_0, 0\right). \tag{14}$$

Since $C(\gamma(r)) = C(n(r)a_{-\tau(r)}) = \operatorname{th}\frac{r}{2}\cdot X$ we infer $C_\infty(n(X)) = X$ for $r \to +\infty$, with C_∞ defined by (9). The map (14) $X \mapsto n(X)$ is thus the diffeomorphism C_∞^{-1} of the unit sphere of $\mathfrak{s}$ (with the point H removed) onto the subgroup N of S.
To compute the distance $d(x, \gamma(r))$ for $x \in S$ we use the left-invariance of d together with $d(o, y) = d(o, \sigma(y)^{-1})$ (see (12)). Writing $n = n(r)$ and $\tau = \tau(r)$ for short we obtain, in view of (10),

$$d(x, \gamma(r)) = d(o, a_\tau n^{-1} x) = d(o, \left(\sigma(n^{-1}x)\right)^{-1} a_\tau).$$

Let us decompose $\left(\sigma(n^{-1}x)\right)^{-1} = n' a_t$. Here n' and t depend on r and have finite limits as $r \to +\infty$ because of (14), with $t = -\widetilde{t}(n^{-1}x) \to -\widetilde{t}(n(X)^{-1}x)$. Writing $s := t + \tau$ we obtain

$$d(x, \gamma(r)) = d(o, n' a_s) = d(a_{-s}, a_{-s} n' a_s).$$

As in the proof of Proposition 2 we have $a_{-s} n' a_s \to o$ as $s \to +\infty$ therefore

$$d(x, \gamma(r)) = d(a_{-s}, o) + \varepsilon_1(r) = s + \varepsilon_1(r)$$

with $\lim_{r\to+\infty} \varepsilon_1(r) = 0$; indeed the triangle inequality implies

$$|\varepsilon_1(r)| = |d(a_{-s}, a_{-s} n' a_s) - d(a_{-s}, o)| \le d(a_{-s} n' a_s, o) \to 0.$$

Finally

$$d(x, \gamma(r)) = \tau(r) - \widetilde{t}(n(X)^{-1}x) + \varepsilon_2(r), \tag{15}$$

with $\lim_{r\to+\infty} \varepsilon_2(r) = 0$ and $\tau(r)$ given by (13). Subtracting the same equality for $x = o$ we obtain the first result of the proposition.
The case $X = H$ is easier. Here $\gamma(r) = (0, 0, r) = a_r$ and, writing $x = n a_t$ with $t = t(x)$,

$$d(x, \gamma(r)) = d(n a_t, a_r) = d(a_t, n^{-1} a_r) = d(a_{t-r}, a_{-r} n^{-1} a_r).$$

Since $a_{-r} n^{-1} a_r \to o$ for $r \to +\infty$ we infer as above

$$d(x, \gamma(r)) = d(o, a_{t-r}) + \varepsilon(r) = r - t + \varepsilon(r)$$

and the result follows.
Horospheres. For $X \neq H$, $b_X(x) = \mathit{constant}$ is equivalent to $\widetilde{t}(n(X)^{-1}x) = t$ for some t, that is $\sigma(n(X)^{-1}x) \in N a_t$. For $X = H$, $b_H(x) = t$ is equivalent to $x \in N a_t$. Note that $N a_t = a_t N$. ■

Remark. Since $d(o, \gamma(r)) = r$, (15) implies $\tau(r) = r + \widetilde{t}(n(X)^{-1}) + \varepsilon(r)$. Comparing with (13), it follows that $\log(\chi(X)/4) = \widetilde{t}(n(X)^{-1}) = \widetilde{t}(n(X))$.

2 Harmonic analysis on a harmonic manifold

From now on X will denote a simply connected harmonic manifold with purely exponential volume growth, dx its Riemannian measure and o an (arbitrary) origin in X. In this section we summarize the main results of [4] about harmonic analysis on X. In the same way as for harmonic analysis on symmetric spaces, one begins with the special case of radial functions.

2.1 Radial harmonic analysis

Let $d_o(x) := d(o, x)$ denote the distance function from the origin. A function u on X is *radial around* o if it is of the form $u = f \circ d_o$ for some function f on $(0, \infty)$. Then u belongs to $L^1(X, dx)$ if and only if f belongs to $L^1((0, \infty), A(r)dr)$, with the classical integral formula in geodesic polar coordinates

$$\int_X u(x)dx = \int_0^\infty f(r)A(r)dr$$

where $A(r)$ is the density function. Let L be the Laplace-Beltrami operator of X. For $f \in C^\infty(0,\infty)$ we have the radial part formula

$$L(f \circ d_o) = (L_r f) \circ d_o \text{ with } L_r := \frac{d^2 f}{dr^2} + \frac{A'(r)}{A(r)} \frac{df}{dr}. \tag{16}$$

Similarly, replacing the distance by a Busemann function, for $g \in C^\infty(\mathbb{R})$ and $\omega \in \partial X$ the function $x \mapsto v(x) = g(B_o(x,\omega))$ is constant on horospheres of X and we have

$$L\left(g \circ B_o(.,\omega)\right) = (L_h g) \circ B_o(.,\omega) \text{ with } L_h := \frac{d^2 g}{dt^2} - 2\rho \frac{dg}{dt}. \tag{17}$$

Formulas (16) and (17) easily follow from $\|\nabla d_o\| = 1$, $L d_o = (A'/A) \circ d_o$ and (4).

For $\lambda \in \mathbb{C}$ let φ_λ denote the unique function on $[0,\infty)$ such that $L_r \varphi_\lambda = -\left(\lambda^2 + \rho^2\right) \varphi_\lambda$ and $\varphi_\lambda(0) = 1$. It extends to a smooth even function on $\mathbb{R}$; furthermore $\varphi_\lambda = \varphi_{-\lambda}$ by uniqueness. As shown in Section 4 of [4], the theory of Chebli-Trimèche hypergroups applies to the present situation, with further information from Bloom and Xu [5]. The Fourier transform defined by

$$\widetilde{f}(\lambda) = \int_0^\infty f(r)\varphi_\lambda(r)A(r)dr$$

is thus inverted (under suitable assumptions on $f : [0,\infty) \to \mathbb{C}$) by

$$f(r) = \int_0^\infty \widetilde{f}(\lambda)\varphi_\lambda(r)d\beta(\lambda),$$

with the Plancherel measure $d\beta(\lambda) = C|c(\lambda)|^{-2}d\lambda$ where $C > 0$ is a constant and the generalized Harish-Chandra function $c(\lambda)$ is a complex function on $\mathbb{C}\backslash 0$.

Going back to our harmonic manifold X, let us introduce the *spherical function* (with respect to o)

$$\varphi_{\lambda,o}(x) := \varphi_\lambda(d(o,x)). \tag{18}$$

In view of (16) $\varphi_{\lambda,o}$ is characterized by the following properties: it is an eigenfunction of L with eigenvalue $-\left(\lambda^2 + \rho^2\right)$, which is radial around o and satisfies $\varphi_{\lambda,o}(o) = 1$. Thus $\varphi_{\lambda,o} = \varphi_{-\lambda,o}$. The *spherical transform* of a function $u \in \mathcal{D}(X)$, radial around o, is then defined by

$$\widetilde{u}^o(\lambda) := \int_X u(x)\varphi_{\lambda,o}(x)dx \ , \ \lambda \in \mathbb{C}, \tag{19}$$

and we have the inversion formula ([4], Theorem 4.6)

$$u(x) = \int_0^\infty \widetilde{u}^o(\lambda)\varphi_{\lambda,o}(x)d\beta(\lambda) \ , \ x \in X. \tag{20}$$

2.2 General harmonic analysis

Because the horospherical part L_h of L in (17) is a differential operator with constant coefficients, the function $v(x) = e^{(-i\lambda+\rho)B_o(x,\omega)}$ is, for any $\lambda \in \mathbb{C}$ and $\omega \in \partial X$, an eigenfunction of L with eigenvalue $-\left(\lambda^2 + \rho^2\right)$, satisfying $v(o) = 1$. The Fourier transform of a (not necessarily radial) function $u \in C_c(X)$ is defined by means of these eigenfunctions:

$$\widetilde{u}^o(\lambda,\omega) := \int_X u(x)e^{(-i\lambda+\rho)B_o(x,\omega)}dx. \tag{21}$$

This definition depends on the choice of an origin: when replacing o with $a \in X$ the identity (5) gives

$$\widetilde{u}^a(\lambda,\omega) = e^{(i\lambda-\rho)B_o(a,\omega)}\widetilde{u}^o(\lambda,\omega). \tag{22}$$

The aim of this section is to state the Fourier inversion theorem of [4] for non radial functions and to provide a concise version of the proof given in [4]. The *mean value operator* M_o is a natural tool to reduce the analysis of non radial functions to the radial case. For $u \in C(X)$ and $x \in X$ it is defined by the spherical mean

$$M_o u(x) := \int_{S_o X} u(\mathrm{Exp}_o\, rv) d_o\sigma(v) \tag{23}$$

of u on the geodesic sphere with center o and radius $r = d(o, x)$. Here $S_o X$ is the unit sphere in the tangent space $T_o X$ and $d_o\sigma$ is its normalized measure induced by the Riemannian norm on $T_o X$. We thus obtain a function $M_o u$, radial around o, with $(M_o u)(o) = u(o)$.

The definition of $M_o u$ may also be written as an integral over the orthogonal group K of $T_o X$, with its normalized Haar measure dk:

$$M_o u(\mathrm{Exp}_o\, \xi) = \int_K (u \circ \mathrm{Exp}_o)(k\xi) dk \ , \ \xi \in T_o X.$$

This shows the operator M_o maps the spaces $C(X)$, $C_c(X)$ and $\mathcal{D}(X)$ into themselves. Abusing notation, it will sometimes be convenient to write $M_o u(x)$ as $M_o u(r)$ with $r = d(o, x)$. In spherical coordinates around o we thus have

$$\int_X u(x) dx = \int_0^\infty M_o u(r) A(r) dr \tag{24}$$

for $u \in C_c(X)$, where A is the density function of X. This implies

$$\int_X u(x) M_o v(x) dx = \int_X M_o u(x) v(x) dx \tag{25}$$

for $u \in C_c(X)$, $v \in C(X)$, both sides being equal to $\int_0^\infty M_o u(r) M_o v(r) A(r) dr$.

Let us take up again the spherical function $\varphi_{\lambda,o}$ in (18). By the general properties of harmonic manifolds M_o commutes with the Laplace operator L. Thus

$$\varphi_{\lambda,o} = M_o \left(e^{(-i\lambda+\rho) B_o(.,\omega)} \right)$$

for every $\omega \in \partial X$, in view of the characterization of $\varphi_{\lambda,o}$ in Section 2.1. It then follows from (25) that definitions (19) and (21) agree if $u \in C_c(X)$ is radial around o, i.e. $M_o u = u$:

$$\widetilde{u}^o(\lambda, \omega) = \widetilde{u}^o(\lambda) \tag{26}$$

for $\lambda \in \mathbb{C}$, $\omega \in \partial X$. Our next goal is to prove Corollary 6, expressing $\varphi_{\lambda,o}$ as an integral over ∂X.

Lemma 4 *For $o, x \in X$, $\lambda \in \mathbb{C}$ and $r \geq 0$*

$$M_x \varphi_{\lambda,o}(r) = \varphi_{\lambda,o}(x) \varphi_\lambda(r).$$

This formula, noted by Szabó [19] p. 9, is a harmonic manifold analog of the functional equation of spherical functions for symmetric spaces ([11] Chapter IV §2).

Proof. Since M_x commutes with L, the function $u(y) := M_x \varphi_{\lambda,o}(y)$ is an eigenfunction of L with eigenvalue $-(\lambda^2 + \rho^2)$, radial around x. As a function of $r = d(x, y)$ it is therefore a constant multiple of $\varphi_\lambda(r)$, with the coefficient $u(x) = \varphi_{\lambda,o}(x)$. ■

Proposition 5 *Let $o, x \in X$, a simply connected harmonic manifold with purely exponential volume growth*[2] *and let $b_v(x) = \lim_{r\to\infty}(r - d(x, \mathrm{Exp}_o\, rv))$, $v \in S_o X$, denote the Busemann function (with respect to o). Let $f \in C(\mathbb{R})$. Then the function*

$$x \longmapsto \int_{S_o X} f(b_v(x)) d_o\sigma(v)$$

is radial around o.

[2] Using a result of Szabó [19] the proof given in [4] shows this proposition actually holds for all noncompact simply connected harmonic manifolds.

Proof. *(i)* It suffices to prove that, for any $f \in C(\mathbb{R})$ and $r \geq 0$, the integral

$$g(x,r) := \int_{S_o X} f(d(x, \operatorname{Exp}_o rv)) d_o\sigma(v)$$

is a radial function of x around o.
Indeed, replacing $f(t)$ by $f(r-t)$ the proposition will follow since, as $r \to \infty$,

$$f(r - d(x, \operatorname{Exp}_o rv)) \to f(b_v(x))$$

for each v and the estimate $|r - d(x, \operatorname{Exp}_o rv)| \leq d(o,x)$ shows we can apply the dominated convergence theorem.
(ii) Given $f \in C(\mathbb{R})$ the function $y \mapsto u(y) := f(d(x,y))$ is continuous on X and radial around x. Since $g(x,r) = M_o u(r)$, we must prove that $M_o u(r)$ only depends on $d(o,x)$ and r. We claim it will suffice to do it for any $u \in \mathcal{D}(X)$ which is radial around x.
Indeed, for any $u \in C(X)$ there exists a sequence (u_k) in $\mathcal{D}(X)$ such that $u_k \to u$ as $k \to \infty$, uniformly on every compact subset of X. For a Euclidean space this follows from the classical use of cutoff and regularization functions, and the result transfers to X by means of (e.g.) the global diffeomorphism $\operatorname{Exp}_x : T_x X \to X$. Besides, if u is radial around x, we may assume each u_k is radial around x too, replacing it if necessary by $M_x u_k \in \mathcal{D}(X)$ which converges to $M_x u = u$ uniformly on every compact subset of X. This being done, we have $M_o u_k \to M_o u$, which implies our claim.
(iii) Thus let $u \in \mathcal{D}(X)$ be radial around x and $v(y) := M_o u(y)$. Then $v \in \mathcal{D}(X)$ is radial around o and its Fourier transform (19) is

$$\widetilde{v}^o(\lambda) = \int_X M_o u(y) \varphi_{\lambda,o}(y) dy = \int_X u(y) \varphi_{\lambda,o}(y) dy$$

by (25). Writing $u(y) = f(d(x,y))$ and using spherical coordinates *around* x the latter integral becomes, by (24) and Lemma 4,

$$\widetilde{v}^o(\lambda) = \int_0^\infty f(r) M_x \varphi_{\lambda,o}(r) A(r) dr = \widetilde{f}(\lambda) \varphi_{\lambda,o}(x) \text{ with } \widetilde{f}(\lambda) = \int_0^\infty f(r) \varphi_\lambda(r) A(r) dr.$$

Note that the density function $A(r)$ is the same for the origins o and x. The radial Fourier inversion formula (20) now gives

$$M_o u(y) = v(y) = \int_0^\infty \widetilde{f}(\lambda) \varphi_{\lambda,o}(x) \varphi_{\lambda,o}(y) d\beta(\lambda).$$

Remembering (18), the right-hand side only depends on $d(o,x)$ and $d(o,y)$. The proof is complete. ■

Corollary 6 *Let X be a simply connected harmonic manifold with purely exponential volume growth. Its spherical functions with respect to $o \in X$ are given by*

$$\varphi_{\lambda,o}(x) = \int_{S_o X} e^{(-i\lambda+\rho) b_v(x)} d_o\sigma(v) = \int_{\partial X} e^{(-i\lambda+\rho) B_o(x,\omega)} d_o\omega$$

for $x \in X$, $\lambda \in \mathbb{C}$.

Proof. We know from (2) the first integral defines an eigenfunction of L with eigenvalue $-\left(\lambda^2 + \rho^2\right)$, which equals 1 for $x = o$. By Proposition 5 it is a radial function of x around o. It must therefore coincide with $\varphi_{\lambda,o}(x)$. The definitions of B_o and the visibility measure $d_o\omega$ in Section 1.1 show the second integral agrees with the first. ■

We can now prove the Fourier inversion formula, which is one of the main results of [4], extending a similar theorem for Damek-Ricci spaces proved in 1997 by Astengo, Camporesi and Di Blasio [2].

Theorem 7 *[4] Let X be a simply connected harmonic manifold with purely exponential volume growth. Given an origin $o \in X$ we have, for $u \in \mathcal{D}(X)$ and $x \in X$,*

$$u(x) = \int_0^\infty \int_{\partial X} \widetilde{u}^o(\lambda, \omega) e^{(i\lambda+\rho)B_o(x,\omega)} d_o\omega d\beta(\lambda).$$

The Plancherel measure $d\beta(\lambda)$ was introduced in Section 2.1.

Proof. Let us apply radial Fourier analysis to $v = M_o u$, radial around o. By (25) and Corollary 6,

$$\begin{aligned}\widetilde{v}^o(\lambda) &= \int_X M_o u(x) \varphi_{\lambda,o}(x) dx = \int_X u(x) \varphi_{\lambda,o}(x) dx \\ &= \int_X u(x) dx \int_{\partial X} e^{(-i\lambda+\rho)B_o(x,\omega)} d_o\omega.\end{aligned}$$

Thus, remembering the definition (21),

$$\widetilde{M_o u}^o(\lambda) = \int_{\partial X} \widetilde{u}^o(\lambda, \omega) d_o\omega \tag{27}$$

for any $u \in \mathcal{D}(X)$. The radial inversion formula (20) at o for the function v now gives

$$u(o) = M_o u(o) = \int_0^\infty \int_{\partial X} \widetilde{u}^o(\lambda, \omega) d_o\omega d\beta(\lambda).$$

Replacing the origin o with x by means of (6) and (22) we obtain the result. ■

3 The horosphere Radon transform

We shall need the following version of the classical coarea formula (see e.g. Chavel [6] p. 160).

Proposition 8 *Let X be a connected Riemannian manifold. Given a C^1 function $\varphi : X \to \mathbb{R}$ such that the gradient $\nabla\varphi$ never vanishes on X, let S_t denote the hypersurface defined by $S_t = \{x \in X | \varphi(x) = t\}$, $t \in \mathbb{R}$. Then, for any $f \in C_c(X)$,*

$$\int_X f(x) d\mu(x) = \int_{\mathbb{R}} dt \int_{S_t} \frac{f(x)}{\|\nabla\varphi(x)\|} d\mu_t(x),$$

where $d\mu$ is the Riemannian measure on X and $d\mu_t$ is the induced Riemannian measure on S_t.

Proof. Using a partition of unity it suffices to prove the result for supp f contained in a coordinate neighborhood. Since $\nabla\varphi \neq 0$, in the neighborhood of every point of X we can take a local coordinate system $\alpha : x \mapsto u = (u_1, ..., u_n)$ such that $\varphi(x) = \left(\varphi \circ \alpha^{-1}\right)(u) = u_1$. Let $ds^2 = \sum_{i,j=1}^n g_{ij}(u) du_i du_j$ denote the corresponding coordinate expression of the Riemannian metric of M and $g(u) := \det(g_{ij}(u))_{1 \le i,j \le n}$. Then

$$\begin{aligned}\int_X f(x) d\mu(x) &= \int_{\mathbb{R}^n} (f \circ \alpha^{-1})(u) \sqrt{g(u)} du \\ &= \int_{\mathbb{R}} dt \int_{\mathbb{R}^{n-1}} (f \circ \alpha^{-1})(t, u') \sqrt{g(t, u')} du'\end{aligned}$$

with $du = du_1 \cdots du_n$, $u' = (u_2, ..., u_n)$ and $du' = du_2 \cdots du_n$.
In our coordinate system S_t is defined by $u_1 = t$, with the induced Riemannian metric $ds^2 = \sum_{i,j=2}^n g_{ij}(t, u') du_i du_j$; let $g_t(u') := \det(g_{ij}(t, u'))_{2 \le i,j \le n}$. Then, using the customary notation

$(g^{ij})_{1\le i,j\le n}$ for the inverse matrix of $(g_{ij})_{1\le i,j\le n}$, we have $g^{11} = g^{-1}g_t$ since g_t is the cofactor of g_{11} in g, that is $g(t,u') = g_t(u')/g^{11}(t,u')$. The components of the gradient $\nabla\varphi$ are $\nabla_i\varphi = \sum_{j=1}^n g^{ij}\partial_j\left(\varphi\circ\alpha^{-1}\right) = g^{i1}$, whence $\|\nabla\varphi\|^2 = \sum_{i,j=1}^n g_{ij}g^{i1}g^{j1} = g^{11}$. We obtain

$$\sqrt{g(t,u')} = \left\|((\nabla\varphi)\circ\alpha^{-1})(t,u'))\right\|^{-1}\sqrt{g_t(u')}.$$

But $\sqrt{g_t(u')}du'$ is the coordinate expression of $d\mu_t$ and the result follows. ∎

With $\varphi(x) = d(o,x)$ where o is an origin in X, we have $\|\nabla\varphi(x)\| = 1$ and Proposition 8 implies the classical integral formula in spherical coordinates. We shall now use it with Busemann functions.

3.1 Definition and properties of the Radon transform

Going back to a simply connected harmonic manifold X with purely exponential volume growth as in Section 2, its Riemannian measure dx and an (arbitrary) origin o, we consider the **horosphere Radon transform** of a compactly supported continuous function $u \in C_c(X)$, defined by

$$R_o u(t,\omega) := \int_{S_t(\omega)} u(x)d\mu_t(x)\ ,\ t\in\mathbb{R}, \omega\in\partial X,$$

integral of u over the horosphere

$$S_t(\omega) := \{x\in X | B_o(x,\omega) = t\}$$

with respect to the induced Riemannian measure $d\mu_t$. The subscript in R_o reminds that this definition depends on the chosen origin: replacing o by $a \in X$ we have

$$R_a u(t,\omega) = R_o u(t + B_o(a,\omega),\omega), \tag{28}$$

since the identity (5) shows the equations $B_a(x,\omega) = t$ and $B_o(x,\omega) = t + B_o(a,\omega)$ define the same horosphere.

For $u \in C_c(X)$, resp. $\mathcal{D}(X)$, and $\omega \in \partial X$, the function $t \mapsto R_o u(t,\omega)$ belongs to $C_c(\mathbb{R})$, resp. $\mathcal{D}(\mathbb{R})$. Indeed $R_o u(t,\omega)$ vanishes for t outside the compact set $\{B_o(x,\omega), x\in \operatorname{supp} u\}$ and the continuity, resp. smoothness, with respect to t is clear when using a partition of unity and the same coordinates as in the proof of Proposition 8.

Since $\|\nabla_x B_o(x,\omega)\| = 1$ by (4), Proposition 8 yields

$$\int_X u(x)dx = \int_{\mathbb{R}} R_o u(t,\omega)dt$$

for any $\omega \in \partial X$. More generally, replacing $u(x)$ by $u(x)v(B_o(x,\omega),\omega)$ where v is a continuous function on $\mathbb{R}\times\partial X$,

$$\int_X u(x)v(B_o(x,\omega),\omega)dx = \int_{\mathbb{R}} R_o u(t,\omega)v(t,\omega)dt \tag{29}$$

for $u \in C_c(X)$ and $\omega \in \partial X$.

Let us introduce the **dual Radon transform** R_o^* of a function v on $\mathbb{R}\times\partial X$, defined by

$$R_o^* v(x) := \int_{\partial X} v(B_o(x,\omega),\omega)d_o\omega \tag{30}$$

with the visibility measure $d_o\omega$ introduced in Section 1.1. It is the integral of v over the set of horospheres containing a given $x \in X$.

Proposition 9 *On a simply connected harmonic manifold X with purely exponential volume growth,*
(i) the transforms R_o and R_o^* are dual to each other:

$$\int_X u(x) R_o^* v(x) dx = \int_{\mathbb{R}\times\partial X} R_o u(t,\omega) v(t,\omega) dt d_o\omega$$

for $u \in C_c(X)$ and $v \in C(\mathbb{R}\times\partial X)$
(ii) the horosphere Radon transform is related to the Fourier transform on X by

$$\widetilde{u}^o(\lambda,\omega) = \left(e^{\rho t} R_o u(t,\omega)\right)^\frown(\lambda,\omega) \tag{31}$$

where $u \in C_c(X)$, $\lambda \in \mathbb{C}$, $\omega \in \partial X$ and $\frown$ denotes the classical one-dimensional Fourier transform on the t variable: $\widehat{v}(\lambda) := \int_{\mathbb{R}} e^{-i\lambda t} v(t) dt$.

The second result is a «projection slice theorem» for harmonic manifolds.

Proof. *(i)* Integrate (29) over $\omega \in \partial X$ with respect to the measure $d_o\omega$.
(ii) Apply (29) with $v(t) = e^{(-i\lambda+\rho)t}$. ∎

Corollary 10 *(i) If $u \in C_c(X)$ is radial around o, its Radon transform $R_o u(t,\omega)$ does not depend on ω. We shall write it as $R_o u(t)$.*
(ii) For an arbitrary $u \in C_c(X)$,

$$R_o M_o u(t) = \int_{\partial X} R_o u(t,\omega) d_o\omega \ , \ t \in \mathbb{R}.$$

(iii) For $u, v \in C_c(X)$ with v radial around o,

$$R_o(u * v) = (R_o u) * (R_o v).$$

On the right-hand side of *(iii)* is a convolution product on the real variable t and, on the left-hand side, is the convolution product on X defined by

$$(u * v)(x) := \int_X u(y) f(d(x,y)) dy$$

if $v(x) = f(d(o,x))$.

Proof. *(i)* and *(ii)* follow from (31), together with (26) for *(i)* and (27) for *(ii)*.
(iii) From the equality

$$(u * v)^{\sim o}(\lambda,\omega) = \widetilde{u}^o(\lambda,\omega)\widetilde{v}^o(\lambda)$$

(see [4] Section 7) we infer

$$\left(e^{\rho t} R_o(u * v)\right)^\frown(\lambda,\omega) = \left(e^{\rho t} R_o u\right)^\frown(\lambda,\omega) \left(e^{\rho t} R_o v\right)^\frown(\lambda)$$

by (31). Therefore

$$\begin{aligned} e^{\rho t} R_o(u * v)(t,\omega) &= \int_{\mathbb{R}} e^{\rho s} R_o u(s,\omega) e^{\rho(t-s)} R_o v(t-s) ds \\ &= e^{\rho t} (R_o u * R_o v)(t,\omega). \end{aligned}$$

∎

Remark. Taking again $u \in C_c(X)$ radial around o, we have

$$\left(e^{\rho t} R_o u\right)^\frown(\lambda) = \widetilde{u}^o(\lambda) = \int_X u(x) \varphi_{\lambda,o}(x) dx$$

by (31) and (19). This is an even function of λ since $\varphi_{\lambda,o} = \varphi_{-\lambda,o}$, therefore

$$\mathcal{A}u(t) := e^{\rho t} R_o u(t) \tag{32}$$

is an even function of $t \in \mathbb{R}$, called the **Abel transform** of the radial function u. This transform was introduced and studied by Peyerimhoff and Samiou [15].

3.2 Main results

Let $S \in \mathcal{S}'(\mathbb{R})$ be the distribution defined by

$$\langle S(t), f(t) \rangle := \int_0^\infty \widehat{f}(\lambda) d\beta(\lambda) \ , \ f \in \mathcal{S}(\mathbb{R}), \tag{33}$$

where $\widehat{}$ is the classical Fourier transform and $d\beta(\lambda) = C|c(\lambda)|^{-2}d\lambda$ is the Plancherel measure introduced in Section 2.1. The estimates for $\lambda \in \mathbb{R}$

$$|c(\lambda)|^{-1} = O(|\lambda|) \text{ for } |\lambda| \le K$$
$$|c(\lambda)|^{-1} = O(|\lambda|^{(n-1)/2}) \text{ for } |\lambda| \ge K,$$

where K is a constant ([4] Section 4.1), show that S is actually a tempered distribution on $\mathbb{R}$.

We shall now prove two inversion formulas for the horosphere Radon transform R_o. One of them (35) is similar to the classical results for symmetric spaces ([12] Chapter II, Theorem 3.13) and Damek-Ricci spaces ([1], or [18] Corollaire 26). In those cases the function $c(\lambda)$ and the measure $d\beta(\lambda)$ defining S are explicitly given by quotients of gamma functions.

The other inversion formula (34) uses the **shifted dual Radon transform** operator $R^*_{o,t}$ defined by (see (6))

$$R^*_{o,t}v(x) := \int_{\partial X} v(t + B_o(x,\omega), \omega) d_x\omega = \int_{\partial X} v(t + B_o(x,\omega), \omega) e^{2\rho B_o(x,\omega)} d_o\omega$$

with $t \in \mathbb{R}$, $x \in X$ and $v \in C(\mathbb{R} \times \partial X)$, a method initiated by Johann Radon for lines in $\mathbb{R}^2$. The equation $B_o(y,\omega) = t + B_o(x,\omega)$ for y, equivalent to $B_x(y,\omega) = t$ by (5), defines a horosphere at distance $|t|$ from x (see Lemma 1). We are thus integrating over the set of horospheres at a given distance from x, instead of the set of horospheres containing x as in the dual transform R^*_o in (30). The interest of this method is that, having proved a Radon inversion formula at the origin for the special case of radial functions (such as (36) in the proof below), a general inversion formula follows easily by means of the shifted dual transform; cf. [17] Section 6.2 within the framework of homogeneous spaces of Lie groups.

Theorem 11 *Let X be a simply connected harmonic manifold with purely exponential volume growth and $o \in X$ a given origin. For $u \in \mathcal{D}(X)$ and $x \in X$ we have*

$$u(x) = \left\langle S(t), e^{\rho t} R^*_{o,t} R_o u(x) \right\rangle . \tag{34}$$

Variant: let Λ denote the operator defined by $\Lambda v(t,\omega) := e^{\rho t}(\check{S} * e^{\rho t} v)(t,\omega)$ (convolution on the real variable t), with $\check{S}(t) = S(-t)$. Then

$$u(x) = R^*_o \Lambda R_o u(x). \tag{35}$$

Proof. Assuming first u is radial around o, the radial Fourier inversion formula (20) at the origin gives

$$\begin{aligned} u(o) &= \int_0^\infty \widetilde{u}^o(\lambda) d\beta(\lambda) = \int_0^\infty \left(e^{\rho t} R_o u\right)^\frown(\lambda) d\beta(\lambda) \\ &= \left\langle S(t), e^{\rho t} R_o u(t) \right\rangle \end{aligned} \tag{36}$$

in view of (31) and the definition of S.
For an arbitrary $u \in \mathcal{D}(X)$ we infer, applying this to the radial function $M_o u$,

$$\begin{aligned} u(o) = M_o u(o) &= \left\langle S(t), e^{\rho t} R_o M_o u(t) \right\rangle \\ &= \left\langle S(t), e^{\rho t} \int_{\partial X} R_o u(t,\omega) d_o\omega \right\rangle \end{aligned}$$

by Corollary 10 *(ii)*. We may then replace o with any $x \in X$:

$$u(x) = \left\langle S(t), e^{\rho t} \int_{\partial X} R_x u(t,\omega) d_x\omega \right\rangle.$$

Going back to the origin o we have $R_x u(t,\omega) = R_o u(t+B_o(x,\omega),\omega)$ by (28) and $d_x\omega = e^{2\rho B_o(x,\omega)} d_o\omega$ by (6). This proves (34).
The inversion formula (34) may also be written as

$$u(x) = \int_{\partial X} \left\langle S(t), e^{\rho t} R_o u(t + B_o(x,\omega),\omega) \right\rangle e^{2\rho B_o(x,\omega)} d_o\omega. \tag{37}$$

But, writing $B = B_o(x,\omega)$ for short and changing t to $-t$, we have

$$\begin{aligned} \left\langle S(t), e^{\rho(t+2B)} R_o u(t+B,\omega) \right\rangle &= e^{\rho B} \left\langle \check{S}(t), e^{\rho(B-t)} R_o u(B-t,\omega) \right\rangle \\ &= e^{\rho B} (\check{S} * e^{\rho t} R_o u)(B,\omega), \end{aligned}$$

therefore

$$u(x) = \int_{\partial X} e^{\rho B_o(x,\omega)} (\check{S} * e^{\rho t} R_o u)(B_o(x,\omega),\omega) d_o\omega = R_o^* \Lambda R_o u(x).$$

■

Corollary 12 *Let $u \in \mathcal{D}(X)$ be radial around o. The Abel transform (32) is inverted by*

$$u(x) = \left\langle S(t), \int_{\partial X} \mathcal{A}u(t + B_o(x,\omega)) e^{\rho B_o(x,\omega)} d_o\omega \right\rangle.$$

Proof. This follows immediately from (37). ■

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