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X-RAY TRANSFORM ON DAMEK-RICCI SPACES

To Jan Boman on his seventy-fifth birthday.

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Abstract. Damek-Ricci spaces, also called harmonic NA groups, make up a large class of harmonic Riemannian manifolds including all hyperbolic spaces. We prove here an inversion formula and a support theorem for the X-ray transform, i.e. integration along geodesics, on those spaces.

Using suitably chosen totally geodesic submanifolds we reduce the problems to similar questions on low-dimensional hyperbolic spaces.

1. Introduction

Trying to reconstruct a function on a manifold knowing its integrals over a certain family of submanifolds is one of the main problems of integral geometry. In the framework of Riemannian manifolds a natural choice is the family of all geodesics: the simple example of lines in Euclidean space has suggested naming *X-ray transform* the corresponding integral operator, associating to a function f its integrals $Rf(\xi)$ along all geodesics ξ of the manifold.

But few explicit formulas are known to recover f from Rf, only valid (to the best of my knowledge) for Riemannian symmetric spaces: see Helgason [6] for Euclidean spaces, hyperbolic spaces or spheres, Helgason [7] or the author's paper [10] for general Riemannian symmetric spaces of the noncompact type.

We prove here an inversion formula and a support theorem for the X-ray transform on Damek-Ricci spaces (also called harmonic NA groups). Introduced in 1992 [3] as a negative answer to a 1944 question by Lichnérowicz: "*is a harmonic Riemannian manifold necessarily a symmetric space?*", those spaces were actively studied in the 90's. They make up a large class of harmonic Riemannian manifolds including all hyperbolic spaces.

Though the main results in analysis on Damek-Ricci spaces (see [4], [8]) look quite similar to their hyperbolic space analogs, their proofs are more delicate however for lack of the familiar compact isotropy group of the origin in the hyperbolic case. This difficulty arises here too; we circumvent it by reducing our problem to some totally geodesic submanifold, isometric to the complex hyperbolic space $H^2(\mathbb{C})$.

In Sections 2 to 4 we develop the necessary tools. In Section 2 we recall, with slight change, the inversion formula obtained in [10] for Riemannian symmetric spaces (Theorem 2.1). Section 3 contains a brief summary about Damek-Ricci

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spaces and a close look at the special case of hyperbolic spaces. Section 4 is devoted to the construction of totally geodesic subgroups of a Damek-Ricci space. Our main results (Theorems 5.1 and 6.1) are proved in Sections 5 and 6.

For another inversion formula in integral geometry over Damek-Ricci spaces (the "horocycle transform"), see [8] Corollaire 26.

2. X-ray transform on symmetric spaces

The *X-ray transform* $f \mapsto Rf$ on a complete Riemannian manifold M is defined by

$$Rf(\xi) = \int_{x \in \xi} f(x) \; dm_\xi(x) \; ,$$

where f is a function on M, ξ is an arbitrary geodesic and dm_ξ is the arc length measure on ξ. The integral converges if f is, for instance, continuous and rapidly decreasing with respect to the Riemannian distance on M.

An inversion formula, reconstructing f from Rf, was proved for $M = G/K$, a Riemannian symmetric space of the noncompact type, by Helgason [7] and independently in [10]. Theorem 2.1 below is a restatement of the main result of [10], in a slightly modified form better suited to our needs.

Let us recall first a few classical semisimple notations; see [5] for more details. In this section G denotes a connected noncompact real semisimple Lie group with finite center, K a maximal compact subgroup of G, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition of the Lie algebra (eigenspaces of the Cartan involution θ), $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, and α a root of the pair $(\mathfrak{g}, \mathfrak{a})$ i.e. a nonzero linear form on $\mathfrak{a}$ such that the joint eigenspace

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | \; [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$$

is not $\{0\}$.

The Killing form $B(X,Y) = \mathrm{tr}\,(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ of $\mathfrak{g}$ gives the invariant scalar product $< X, Y >= -B(X, \theta Y)$ of $X, Y \in \mathfrak{g}$ and the norm $|X| = \sqrt{-B(X, \theta X)}$. Let $H_\alpha \in \mathfrak{a}$ be the dual vector to α defined by $B(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{a}$; a different normalization of H_α was chosen in [10] but this is insignificant. We set $|\alpha| = |H_\alpha|$.

The manifold G/K is equipped with the G-invariant Riemannian metric defined by this scalar product on $\mathfrak{p}$ (identified to the tangent space to G/K at the origin o). As usual $\exp : \mathfrak{g} \to G$ will denote the exponential mapping of the group G and Exp the exponential mapping of G/K at the origin, a global diffeomorphism of $\mathfrak{p}$ onto G/K.

Let α be a fixed root. Taking $\xi_0 = \mathrm{Exp}\, \mathbb{R} H_\alpha$ as the origin in the space of geodesics the X-ray transform of a function f on G/K satisfies, for $g \in G$,

$$Rf(g \cdot \xi_0) = \int_{\xi_0} f(g \cdot x) \; dm_{\xi_0}(x) = |\alpha| \int_{\mathbb{R}} f(g \cdot \mathrm{Exp}\, tH_\alpha) \; dt$$

since $|H_\alpha| = |\alpha|$, where dots denote the natural action of G on G/K. We also need the *shifted dual transform* of a function φ on the space of geodesics, defined by

$$R_\gamma^* \varphi(x) = \int_K \varphi(gk\gamma \cdot \xi_0) \; dk \text{ if } x = g \cdot o \; ,$$

where the shift γ is a given element of G and dk is the normalized Haar measure on K. Roughly speaking R_γ^* integrates φ over a set of geodesics at a given distance from the point $x \in G/K$. When γ is the identity element R_γ^* is the classical dual transform.

Note that the group G does not act transitively on the space of all geodesics unless G/K has rank one. In the next theorem, having chosen $\xi_0 = \operatorname{Exp} \mathbb{R} H_\alpha$ as the origin, we use a shift arising from the Lie subalgebra $\mathfrak{g}_\alpha$.

Theorem 2.1. *Let G/K be a Riemannian symmetric space of the noncompact type and let α be any root of the pair $(\mathfrak{g}, \mathfrak{a})$. Taking $\xi_0 = \operatorname{Exp} \mathbb{R} H_\alpha$ as the origin in the space of geodesics, let R denote the X-ray transform obtained by integrating over geodesics in a family containing all $g \cdot \xi_0$, $g \in G$.*
For any nonzero $X \in \mathfrak{g}_\alpha$ this transform is inverted by

$$f(x) = -\frac{\sqrt{2}}{\pi |X|} \int_0^\infty \frac{\partial}{\partial t} \left(R^*_{\exp tX} Rf(x) \right) \frac{dt}{t} \ ,$$

for $f \in C_c^\infty(G/K)$, $x \in G/K$.

Proof. This was proved in [10] if the vector $Y = \frac{1}{2}(X - \theta X)$ has norm $|Y| = |\alpha|^{-1}$; the factor in front of the integral was $-|\alpha|/\pi$ then. But considering $Z = \frac{1}{2}(X + \theta X)$ we easily have $X = Y + Z$, $|Y| = |Z|$ and $|X| = \sqrt{2}|Y|$, thus $-|\alpha|/\pi = -\sqrt{2}/\pi|X|$. The formula now extends to an arbitrary $X \in \mathfrak{g}_\alpha$, since changing t into λt with $\lambda > 0$ in the integral amounts to replacing X with λX. □

3. Damek-Ricci spaces

A *Damek-Ricci space*, or *harmonic NA group*, is a simply connected Lie group S with Lie algebra $\mathfrak{s}$ satisfying the following assumptions 1 to 4:

1. *Direct sum decomposition.* $\mathfrak{s}$ is a direct sum of vector subspaces $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$, with $\mathfrak{a}$ one-dimensional. Let H be some fixed nonzero vector in $\mathfrak{a}$ (see 2 below); elements of $\mathfrak{s}$ will be written as $V + Z + tH$ with $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$ and $t \in \mathbb{R}$.
2. *Lie bracket.* We assume $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$, $[\mathfrak{v}, \mathfrak{z}] = 0$, $[\mathfrak{z}, \mathfrak{z}] = 0$ and $[H, V] = \frac{1}{2} V$, $[H, Z] = Z$ for all $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$. The general Lie bracket in $\mathfrak{s}$ is thus given by

$$[V + Z + tH, V' + Z' + t'H] = \frac{1}{2}(tV' - t'V) + (tZ' - t'Z + [V, V']) \ . \tag{3.1}$$

3. *Scalar product.* $\mathfrak{v}$ and $\mathfrak{z}$ are equipped with scalar products $< \ , \ >$, extended to a scalar product on $\mathfrak{s}$ as

$$< V + Z + tH, V' + Z' + t'H > = < V, V' > + < Z, Z' > + tt' \ . \tag{3.2}$$

   The group S is equipped with the left-invariant Riemannian metric defined by this scalar product on $\mathfrak{s}$. Let $\|.\|$ denote the corresponding norm on $\mathfrak{s}$; thus $\|H\| = 1$.

4. *The maps J_Z.* For $Z \in \mathfrak{z}$ let $J_Z : \mathfrak{v} \to \mathfrak{v}$ denote the linear map defined by

$$< J_Z V, V' > = < Z, [V, V'] > \tag{3.3}$$

   for $V, V' \in \mathfrak{v}$. We finally assume that, for all $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$,

$$J_Z^2 V = -\|Z\|^2 V \ . \tag{3.4}$$

From (3.3) and (3.4) it is easily shown that

$$\|J_Z V\| = \|Z\| \cdot \|V\| \ , \ [V, J_Z V] = \|V\|^2 Z \ . \tag{3.5}$$

From assumptions 1 to 3 it follows that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ is a nilpotent Lie algebra, $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{n}$, and $\mathfrak{s}$ is solvable with orthogonal decomposition $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ into eigenspaces of $\operatorname{ad} H$ with respective eigenvalues 1/2, 1, 0. At the group level we have $S = NA$, a semi-direct product, where N and A are the Lie subgroups of S with Lie algebras $\mathfrak{n}$ and $\mathfrak{a}$.

Damek-Ricci spaces generalize rank one symmetric spaces of the noncompact type (hyperbolic spaces). The latter are, in the notation of Section 2, the spaces G/K with G of rank one i.e. $\dim \mathfrak{a} = 1$. The classical Iwasawa decomposition $G = NAK$, where N, resp. A, is a nilpotent, resp. one-dimensional abelian, Lie subgroup of G, gives a diffeomorphism $\varphi : na \mapsto naK$ of the solvable group NA onto G/K, intertwining the left translation by $x \in NA$ with the natural action on G/K of the same element $x \in G$. A G-invariant Riemannian metric on G/K therefore corresponds via φ with a left invariant metric on NA. Forgetting G and K the Riemannian manifold G/K can thus be studied as the solvable group NA with this left invariant metric.

The next proposition links both points of view precisely. Let $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ where α and 2α denote the positive roots of the pair $(\mathfrak{g}, \mathfrak{a})$ with respective eigenspaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{2\alpha}$; let $p = \dim \mathfrak{g}_\alpha$, $q = \dim \mathfrak{g}_{2\alpha}$ (with $q = 0$ if 2α is not a root). The above-mentioned Iwasawa decomposition of G is given by its Lie subgroups N, A with Lie algebras $\mathfrak{n}$, $\mathfrak{a}$. On $\mathfrak{g}$ we consider the scalar product

$$< X', X'' >_{\mathfrak{g}} = -\frac{2}{p+4q} B(X', \theta X'') \ ; \tag{3.6}$$

the Killing form B is here normalized so that $H \in \mathfrak{a}$ defined by $\alpha(H) = 1/2$ is a unit vector. In this notation we have the following

Proposition 3.1. *(Hyperbolic spaces as Damek-Ricci spaces [9]) Equipped with the Lie bracket induced by $\mathfrak{g}$ and the scalar product*

$$< Y' + H', Y'' + H'' >_{\mathfrak{s}} = \frac{1}{2} < Y', Y'' >_{\mathfrak{g}} + < H', H'' >_{\mathfrak{g}} \quad ,$$

(with $Y', Y'' \in \mathfrak{n}$, $H', H'' \in \mathfrak{a}$), the Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ satisfies assumptions 1 to 4 with $\mathfrak{v} = \mathfrak{g}_\alpha$, $\mathfrak{z} = \mathfrak{g}_{2\alpha}$ and $J_Z V = [Z, \theta V]$ for $V \in \mathfrak{g}_\alpha$, $Z \in \mathfrak{g}_{2\alpha}$.
The group $S = NA$ is a Damek-Ricci space, isometric to the hyperbolic space G/K equipped with the G-invariant metric given by the scalar product induced on $\mathfrak{p}$ by $< , >_{\mathfrak{g}}$.

Proof. Easy by means of the linear isomorphism $\frac{1}{2}(I - \theta)$ of $\mathfrak{s}$ onto $\mathfrak{p}$ (the projection parallel to $\mathfrak{k}$ in $\mathfrak{g}$). See [9] §6.1 for details. □

General Damek-Ricci spaces are also solvable groups NA with one-dimensional A, but not necessarily arising from Iwasawa decompositions of semisimple Lie groups; excepting the hyperbolic spaces they are not symmetric spaces. For more details we refer to Damek and Ricci [3][4], Cowling *et al.* [1][2] or to our expository notes [8][9].

4. Totally geodesic subgroups

A (connected) submanifold M' of a (connected) Riemannian manifold M is said to be *totally geodesic in M* if each geodesic of M' (with respect to the metric induced by M) is a geodesic of M . Let us recall the following general

Proposition 4.1. *(Cowling* et al. *[1]) Let S be a Lie group with the left invariant Riemannian metric defined by a scalar product $< , >$ on its Lie algebra $\mathfrak{s}$. Let S_0 be a Lie subgroup of S with Lie algebra $\mathfrak{s}_0$.*
Then S_0 is totally geodesic in S if and only if

$$< X, [X, Y] >= 0$$

for all $X \in \mathfrak{s}_0$ and all $Y \in \mathfrak{s}_0^\perp$ (the orthogonal complement of $\mathfrak{s}_0$ in $\mathfrak{s}$).

Proof. See the Appendix to [1] or, for a shorter proof, Proposition 2.1 of [2]. □

Going back to Damek-Ricci spaces Proposition 4.1 implies a more specific result.

Corollary 4.2. *Let S_0 be a Lie subgroup containing A in a Damek-Ricci space $S = NA$ and let $\mathfrak{s}_0$ be its Lie algebra. Then $\mathfrak{s}_0 = \mathfrak{v}_0 \oplus \mathfrak{z}_0 \oplus \mathfrak{a}$ with $\mathfrak{v}_0 = \mathfrak{s}_0 \cap \mathfrak{v}$, $\mathfrak{z}_0 = \mathfrak{s}_0 \cap \mathfrak{z}$, and S_0 is totally geodesic in S if and only if*

$$J_Z \mathfrak{v}_0 \subset \mathfrak{v}_0 \text{ for all } Z \in \mathfrak{z}_0 \ .$$

When this condition holds S_0 is a Damek-Ricci space with the scalar product induced by S.

Proof. The decomposition of $\mathfrak{s}_0$ is given by the eigenspaces of its endomorphism $\operatorname{ad} H$.

Let $\mathfrak{v}_1$, resp. $\mathfrak{z}_1$, denote the orthogonal complement of $\mathfrak{v}_0$ in $\mathfrak{v}$, resp. $\mathfrak{z}_0$ in $\mathfrak{z}$. Then $\mathfrak{s}_0^\perp = \mathfrak{v}_1 \oplus \mathfrak{z}_1$ and, writing $X = V_0 + Z_0 + tH$ an element of $\mathfrak{s}_0$ and $Y = V_1 + Z_1$ an element of $\mathfrak{s}_0^\perp$, with $V_i \in \mathfrak{v}_i$, $Z_i \in \mathfrak{z}_i$, we have by (3.1)

$$[X, Y] = [V_0 + Z_0 + tH, V_1 + Z_1] = \frac{t}{2} V_1 + tZ_1 + [V_0, V_1]$$

therefore

$$\begin{aligned} < X, [X, Y] > &= < V_0 + Z_0 + tH, \frac{t}{2} V_1 + tZ_1 + [V_0, V_1] > \\ &= \frac{t}{2} < V_0, V_1 > + t < Z_0, Z_1 > + < Z_0, [V_0, V_1] > \\ &= < Z_0, [V_0, V_1] > = < J_{Z_0} V_0, V_1 > \end{aligned}$$

by (3.3). The condition of Proposition 4.1 is thus equivalent to $J_{Z_0} \mathfrak{v}_0 \subset \mathfrak{v}_0$ for all $Z_0 \in \mathfrak{z}_0$.

When this holds $\mathfrak{s}_0$ satisfies assumptions 1 to 4 with the scalar product induced by $\mathfrak{s}$: for $Z \in \mathfrak{z}_0$ the map J_Z for $\mathfrak{s}_0$ is the restriction to $\mathfrak{v}_0$ of the map J_Z for $\mathfrak{s}$. □

Proposition 4.3. *Let S be a Damek-Ricci space with Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ and let $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$ be nonzero vectors. Then*

$$\mathfrak{s}_0 = \mathbb{R}V \oplus \mathbb{R}J_Z V \oplus \mathbb{R}Z \oplus \mathbb{R}H$$

is a solvable Lie subalgebra of $\mathfrak{s}$.
The corresponding Lie subgroup S_0 is totally geodesic in S and isometric to the complex hyperbolic space $H^2(\mathbb{C}) = G_*/K_*$ with $G_* = SU(1,2)$, $K_* = S(U(1) \times U(2))$ and the metric arising from (3.6).

Proof. This is a Damek-Ricci analog of the classical $SU(1,2)$ reduction for rank one semisimple Lie groups.

We may assume $\|V\| = \|Z\| = 1$; by (3.5) $W = J_Z V$ is also a unit vector in $\mathfrak{v}$. By (3.1) and (3.5) again we have

$$\begin{aligned} &[H, V] = V/2 \ , \ [H, W] = W/2 \ , \ [H, Z] = Z \ , \\ &[V, W] = Z \ , \quad [Z, V] = 0 \ , \quad [Z, W] = 0 \ , \end{aligned} \tag{4.1}$$

and $\mathfrak{s}_0$ is a solvable Lie subalgebra. The space $\mathfrak{v}_0 = \mathbb{R}V \oplus \mathbb{R}W$ is stable under J_Z since $J_Z W = -V$ by (3.4), and the Lie subgroup S_0 of S with Lie algebra $\mathfrak{s}_0$ is totally geodesic by Corollary 4.2 with $\mathfrak{z}_0 = \mathbb{R}Z$.

A realization of S_0 can be obtained by considering the classical group $G_* = SU(1,2)$ and its maximal compact subgroup $K_* = S(U(1) \times U(2))$. All semisimple notations related to G_* will bear the subscript $_*$. Let $\mathfrak{g}_* = \mathfrak{n}_* \oplus \mathfrak{a}_* \oplus \mathfrak{k}_*$ be the Iwasawa decomposition arising from $\mathfrak{a}_* = \mathbb{R}H_*$ with

$$H_* = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

The bracket relations (4.1) mean that the mapping

$$vV + wW + zZ + tH \longmapsto \frac{1}{2}\begin{pmatrix} iz & v-iw & t-iz \\ v+iw & 0 & -v-iw \\ t+iz & v-iw & -iz \end{pmatrix} , \tag{4.2}$$

with $v, w, z, t \in \mathbb{R}$, is a Lie algebra isomorphism of $\mathfrak{s}_0$ onto the solvable subalgebra $\mathfrak{s}_* = \mathfrak{n}_* \oplus \mathfrak{a}_*$ of $\mathfrak{g}_*$.

This gives a Lie group isomorphism of S_0 onto the subgroup $S_* = N_*A_*$ of G_* hence, composing with $na \mapsto naK_*$, a diffeomorphism of S_0 onto the homogeneous space $G_*/K_* = H^2(\mathbb{C})$. In order to apply Proposition 3.1 (with $p = 2$, $q = 1$) we endow $\mathfrak{g}_*$ with the metric (3.6)

$$< X', X'' >_{\mathfrak{g}_*} = -\frac{2}{6}B_*(X', \theta_* X'') = 2\,\mathrm{tr}\,\left(X'\ {}^t\overline{X''}\right)$$

where ^{t} means transpose. Indeed $B_*(X', X'') = 6\,\mathrm{tr}\,(X'X'')$ and $\theta_* X'' = -\ {}^t\overline{X''}$ for the Lie algebra $\mathfrak{g}_* = \mathfrak{su}(1,2)$. The corresponding scalar product on $\mathfrak{s}_*$ is, by Proposition 3.1,

$$< Y' + H', Y'' + H'' >_{\mathfrak{s}_*} = \mathrm{tr}\,\left(Y'\ {}^t\overline{Y''}\right) + 2\,\mathrm{tr}\,\left(H'\ {}^t\overline{H''}\right)$$

with $Y', Y'' \in \mathfrak{n}_*$, $H', H'' \in \mathfrak{a}_*$. Applying this to matrices given by (4.2) (with $t' = t'' = 0$ for Y', Y'', and $H' = t'H_*$, $H'' = t''H_*$) yields $v'v'' + w'w'' + z'z'' + t't''$, in agreement with the scalar product in $\mathfrak{s}_0$ with respect to its orthonormal basis (V, W, Z, H). The diffeomorphism of S_0 onto $H^2(\mathbb{C})$ is therefore an isometry, as claimed. □

5. Inversion formula for Damek-Ricci spaces

For lack of a compact group K inversion formulas for the X-ray transform on a Damek-Ricci space cannot be directly obtained by the method of Section 2. The difficulty can be circumvented however by means of totally geodesic submanifolds, drawing inspiration from [7] and [10].

On a Damek-Ricci space $S = NA$ let exp denote the exponential mapping of the group S and Exp the exponential mapping of the Riemannian manifold S at the origin o (the identity element). They are (distinct) global diffeomorphisms of $\mathfrak{s}$ onto S ([9] p. 18 and 24). Let $\xi_0 = \mathrm{Exp}\,(\mathbb{R}H) = A$ be the geodesic tangent to H at o.

Theorem 5.1. *Let S be a Damek-Ricci space and V be any nonzero vector in $\mathfrak{v}$. The X-ray transform R on S is inverted by*

$$f(x) = -\frac{1}{\pi\sqrt{3}\,\|V\|}\int_0^\infty \frac{\partial}{\partial t}\left(R^*_{\exp tV} Rf(x)\right)\frac{dt}{t} ,$$

for $f \in C_c^\infty(S)$, $x \in S$, where R^* is defined by (5.2) below.

Proof. Let S_0 be the totally geodesic subgroup of S given by Proposition 4.3 from the given V and any nonzero $Z \in \mathfrak{z}$. We work at the origin first, restricting f to S_0.

By Proposition 4.3 and its proof we have a Lie group isomorphism $S_0 \to S_* = N_*A_*$ onto a solvable sugbroup of $G_* = SU(1,2)$ and an isometry of S_* onto

$G_*/K_* = H^2(\mathbb{C})$. Using $\widetilde{}$ for objects tranferred from S_0 to G_*/K_* by the composed diffeomorphism, or from $\mathfrak{s}_0$ to $\mathfrak{s}_* \subset \mathfrak{g}_*$, we have by Theorem 2.1 at the origin of G_*/K_*

$$\widetilde{f}(o) = -\frac{\sqrt{2}}{\pi|\widetilde{V}|}\int_0^\infty \frac{\partial}{\partial t}\left(\int_{K_*} R\widetilde{f}(k\cdot \exp t\widetilde{V}\cdot \widetilde{\xi}_0)\ dk\right)\frac{dt}{t}\ ,$$

where dots mean here the natural action of G_* on G_*/K_*. Going back to S_0 we obtain

$$f(o) = -\frac{\sqrt{2}}{\pi|\widetilde{V}|}\int_0^\infty \frac{\partial}{\partial t}\left(\int_{K_*} Rf(k\cdot \exp tV\cdot \xi_0)\ dk\right)\frac{dt}{t}\ . \tag{5.1}$$

Indeed the mapping exp for S_0 (restriction to $\mathfrak{s}_0$ of exp for S) corresponds with exp for S_* (restriction to S_* of exp for G_*) in view of the Lie isomorphism (4.2), and the left translation by $\exp tV$ in S_0 with the action of $\exp t\widetilde{V}$ on G_*/K_*. Abusing notations we have still denoted by $k\cdot(\cdots)$ the isometry of S_0 corresponding with the action of $k \in K_*$ on the hyperbolic space.

The norm used in (5.1) is given by $|\widetilde{V}|^2 = -B_*(\widetilde{V}, \theta_*\widetilde{V})$ therefore, by (3.6) with $p = 2$, $q = 1$ and Proposition 3.1 for $\mathfrak{g}_*$,

$$|\widetilde{V}|^2 = -B_*(\widetilde{V}, \theta_*\widetilde{V}) = 3\left\|\widetilde{V}\right\|^2_{\mathfrak{g}_*} = 6\left\|\widetilde{V}\right\|^2_{\mathfrak{s}_*}\ .$$

Now Proposition 4.3 implies $\left\|\widetilde{V}\right\|_{\mathfrak{s}_*} = \|V\|_{\mathfrak{s}_0} = \|V\|$ (the norm in $\mathfrak{s}$) so that $|\widetilde{V}| = \sqrt{6}\,\|V\|$ and the constant factor in (5.1) is $-1/\pi\sqrt{3}\,\|V\|$.

We finally introduce the following analog of the *shifted dual transform* of a function φ on the set of all geodesics of S:

$$R^*_\gamma\varphi(x) = \int_{K_*}\varphi(x\cdot k\cdot\gamma\cdot\xi_0)\ dk\ ,\ x\in S\ ,\ \gamma\in S_0\ . \tag{5.2}$$

Let us explain the notation: ξ_0 is a geodesic of S and S_0, $\gamma\cdot\xi_0$ is the geodesic of S_0 (and S) obtained by left action of γ in S_0. With $k\cdot$ as above, $k\cdot\gamma\cdot\xi_0$ is a geodesic of S_0, therefore of S. Finally $x\cdot k\cdot\gamma\cdot\xi_0$ is the geodesic of S obtained by left action of $x\in S$. The integral, taken with respect to the normalized Haar measure dk on the compact group K_*, converges if $\varphi(x\cdot k\cdot\gamma\cdot\xi_0)$ is a continuous function of k.

Thus the integral over K_* in (5.1) is $R^*_{\exp tV}Rf(o)$, and the theorem follows by left action of x. □

Remark. When written down explicitly our inversion formula only involves the geodesics $\xi = x\cdot k\cdot\exp tV\cdot\xi_0$ with $k\in K_*$, $t\in\mathbb{R}$, lying at distance $\frac{1}{\sqrt{2}}\|V\|\arg\operatorname{sh} t$ from x. Indeed the distance used here on S_0 is $1/\sqrt{6}$ times the distance in [10], which was $|Y|\arg\operatorname{sh} t = \frac{1}{\sqrt{2}}|\widetilde{V}|\arg\operatorname{sh} t$ (see [10], final Remark), and $|\widetilde{V}| = \sqrt{6}\,\|V\|$.

6. Support theorem on Damek-Ricci spaces

On a Riemannian manifold M with origin o and distance d we shall say (following [6] p. 120) that a function f is *exponentially decreasing* if $e^{kd(o,x)}f(x)$ is bounded on M for any $k\geq 0$. Because of the triangle inequality this definition does not depend on the choice of o.

Theorem 6.1. *Let S be a Damek-Ricci space and let $R>0$. For an exponentially decreasing continuous function f on S the following are equivalent:*
(i) $f(x) = 0$ *for any point* $x\in S$ *such that* $d(o,x)\geq R$

(ii) $Rf(\xi) = 0$ *for any geodesic* ξ *of* S *such that* $d(o, \xi) \geq R$.

Proof. (i) implies (ii): obvious.
(ii) implies (i). Let $x \in S$ with $d(o, x) \geq R$. Since Exp is a global diffeomorphism there exists $X \in \mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$ such that $x = \operatorname{Exp} X$ (and $\|X\| = d(o, x)$). This vector decomposes as $X = vV + zZ + tH$ where $V \in \mathfrak{v}$, $Z \in \mathfrak{z}$ are unit vectors and $v, z, t \in \mathbb{R}$. Let $\mathfrak{s}_0 = \mathbb{R}V \oplus \mathbb{R}J_Z V \oplus \mathbb{R}Z \oplus \mathbb{R}H$.

By Proposition 4.3 the corresponding S_0 is totally geodesic in S, contains x and is isometric to the hyperbolic space $H^2(\mathbb{C})$. Considering the restriction of f to S_0 our assumption implies $Rf(\xi) = 0$ for any geodesic ξ of S_0 such that $d(o, \xi) \geq R$. Helgason's Corollary 4.1 of [6] p. 120 then applies to S_0, hence $f(x) = 0$. □

Remark. Helgason's result is also proved by restriction to a totally geodesic submanifold, this time isometric to the two-dimensional hyperbolic space $H^2(\mathbb{R})$.

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