Quasipatterns solutions of the Swift-Hohenberg PDE

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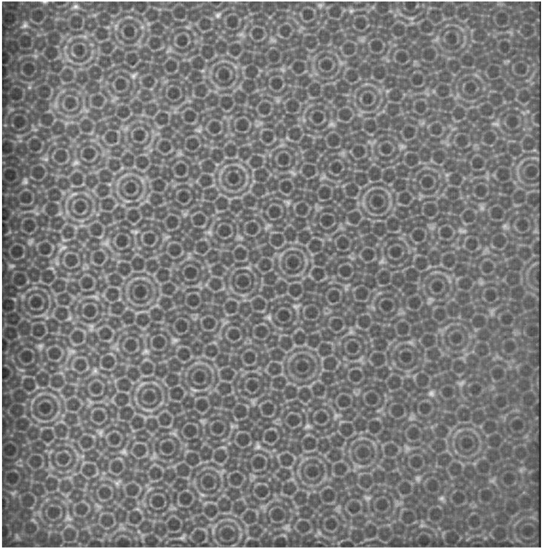
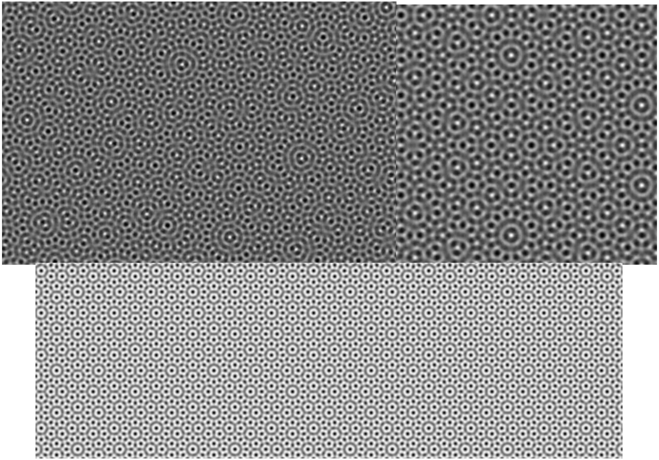
collaboration with B.Braaksma and L.Stolovitch

Quasipatterns experiments



Experiment of Faraday type. Kudrolli, Pier, Gollub 1998

Quasipatterns on Swift-Hohenberg PDE



Numerical computation. Rucklidge-Silber 2009
Swift-Hohenberg PDE

Steady Swift-Hohenberg equation in $\mathbb{R}^2$

$$(1+\Delta)^2 u = \mu u - u^3,\ \mathbf{x} \in \mathbb{R}^2 \to u(\mathbf{x}) \in \mathbb{R}$$

$$e^{i\mathbf{k}.\mathbf{x}} \in Ker\{(1+\Delta)^2 - \mu\}$$

iff Dispersion equation holds:

$$(1-|\mathbf{k}|^2)^2 = \mu,\ \mathbf{k} \in \mathbb{R}^2$$

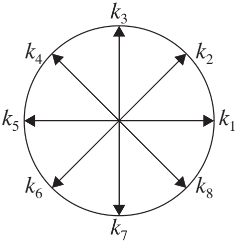
For $\mu = 0$ all wave vectors $\mathbf{k}$ with $|\mathbf{k}| = 1$ are **critical**
We choose to look for solutions, quasiperiodic in $\mathbb{R}^2$, invariant under rotations of angle π/q and with μ close to 0.

Quasilattices

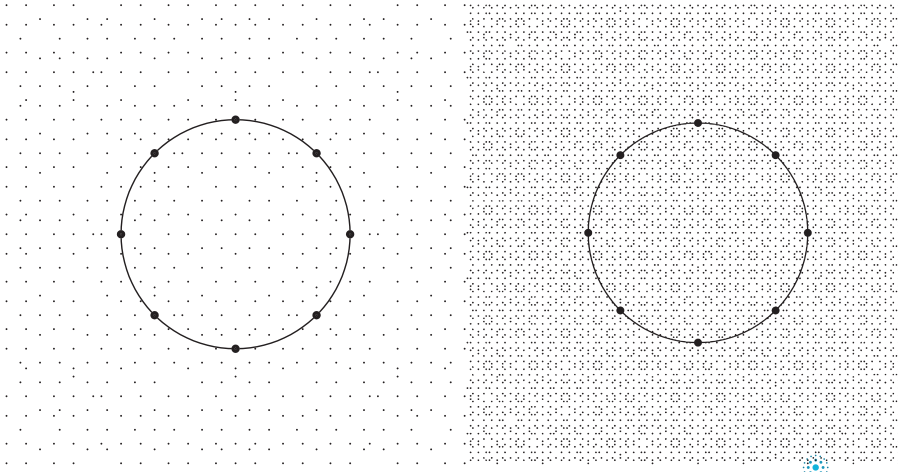
$$u = \Sigma_{\mathbf{k}\in\Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}},\ \ \mathbf{k}_j = e^{i(j-1)\pi/q},\ \ u^{(\mathbf{k})} = \overline{u}^{(-\mathbf{k})}$$

$$\Gamma = \{\mathbf{k} = \sum_{j=1,\dots 2q} m_j \mathbf{k}_j,\quad m \in \mathbb{N}^{2q},\ (\mathbf{k}_j, \mathbf{k}_{j+1}) = \pi/q\}$$

For $q = 1, 2, 3$ Γ is a lattice leading to a periodic pattern
For $q \geq 4$ Γ is a quasilattice leading to a quasipattern



Example $q = 4$, the 8 wavevectors which form the generators of the quasilattice



Example with $q = 4$, The truncated quasilattices $\Gamma_9$and Γ_{27}. The small dots mark the combinations of up to 9 or 27 of the 8 basis vectors.

Formal Lyapunov-Schmidt method

$$
\begin{aligned}
L_0 u &= \mu u - u^3, \quad L_0 = (1+\Delta)^2, \\
u &= \sum_{n\geq 0} \epsilon^{2n+1} u_{2n+1} \text{ invariant under rotations } \mathbf{R}_{\pi/q}, \\
\mu &= \sum_{n\geq 1} \epsilon^{2n} \mu_{2n}
\end{aligned}
$$

$$L_0 u_1 = 0, \; u_1 = \sum_{j=1}^{2q} e^{i\mathbf{k}_j.\mathbf{x}} \text{ unique eigenvector invariant under } \mathbf{R}_{\pi/q}$$

$$L_0 u_3 = \mu_2 u_1 - u_1^3, \; \mu_2 = 3(2q-1) \text{ (compatib. cond.: rhs } \perp \text{ to } u_1).$$

$$u_3 = \sum_{\mathbf{k}=\mathbf{k}_j+\mathbf{k}_l+\mathbf{k}_r} \alpha_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}, \text{ uniquely determ. } \perp \text{ to } u_1$$

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Formal series

Assume u_{2k+1}, μ_{2k} known for $k = 1, ..., n-1$, then u_{2n+1}, μ_{2n} are determined by

$$L_0 u_{2n+1} = \mu_{2n} u_1 - \sum_{l+r+s=n-1} u_{2l+1} u_{2r+1} u_{2s+1}, \; u_{2n+1} \in \{u_1\}^{\perp}$$

Compatibility condition gives μ_{2n}, then we need to invert L_0 in using

$$L_0^{-1} e^{i\mathbf{k}\cdot\mathbf{x}} = (1 - |\mathbf{k}|^2)^{-2} e^{i\mathbf{k}\cdot\mathbf{x}}, \; \mathbf{k} \neq \mathbf{k}_j, j = 1, ..., 2q$$

Problem: Estimate u_{2n+1}, μ_{2n}
$\Rightarrow$ Small divisor problem

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Diophantine estimate

$\mathbb{Q}$ vector space $span\{\mathbf{k}_j; j = 1, ..2q\}$ has dimension d where $d/2 = l_0 + 1 \leq q/2$ is the degree of the minimal Polynomial for the algebraic nb $\omega = 2\cos\pi/q$ (coef in $\mathbb{Z}$).

$$\text{for } q = 4, 5, 6, \ \omega = \sqrt{2}, \ \frac{1+\sqrt{5}}{2}, \ \sqrt{3}, \ l_0 + 1 = 2$$

$$\mathbf{k} = \Sigma_{j=1}^{2q} m_j \mathbf{k}_j = \frac{1}{\mathfrak{d}} \Sigma_{s=1}^{d} m_s^* \mathbf{k}_s^*, \ \mathbf{m}^* = (m_1^*, .., m_d^*) \in \mathbb{Z}^d$$

$$N_{\mathbf{k}} = \Sigma_{s=1}^{d} |m_s^*| \text{ notice that } \mathfrak{d} = 1 \text{ for } q = 4, 5, ...12$$

$$(|\mathbf{k}|^2 - 1)^2 \geq c(1 + N_{\mathbf{k}}^2)^{-2l_0}, \ \text{if } \mathbf{k} \neq \mathbf{k}_j, j = 1, ..2q$$

Algebra

The quasi-lattice Γ possesses the property that the only solutions of

$$|\mathbf{k}|^2 - 1 = 0, \ \mathbf{k} \in \Gamma$$

are $\mathbf{k}_j$, $j = 1, ...2q$.
This results from the Kronecker-Weber theorem, saying that every abelian extension of $\mathbb{Q}$ is cyclotomic.

$\Rightarrow$ the kernel of $\mathbf{L}_0$ is $2q$ - dimensional
kernel of $\mathbf{L}_0$, invariant under $R_{\pi/q}$ is 1-dimensional

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Spaces of quasi-periodic functions

Sobolev like spaces

$$\mathcal{H}_s = \left\{ u = \sum_{\mathbf{k}\in\Gamma} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}; ||u||_s^2 = \sum_{\mathbf{k}\in\Gamma} (1+N_{\mathbf{k}}^2)^s |u_{\mathbf{k}}|^2 < \infty \right\}$$

$$\langle w, v\rangle_s = \sum_{\mathbf{k}\in\Gamma} (1+N_{\mathbf{k}}^2)^s w_{\mathbf{k}} \overline{v_{\mathbf{k}}}$$

Lemma

Assume $q \geq 4$, then for $s > d/2$, for any $u \in \mathcal{H}_s$ and any $v \in \mathcal{H}_0$

$$||uv||_0 \leq c_s ||u||_s ||v||_0.$$

For $s \geq s' > d/2$ and $u, v \in \mathcal{H}_s$, then,

$$\|uv\|_s \leq C(s,s')(\|u\|_s \|v\|_{s'} + \|u\|_{s'} \|v\|_s).$$

for some positive constant $C(s,s')$ that depends only on s and s'. For $\ell \geq 0$ and $s > \ell + d/2$, $\mathcal{H}_s$ is continuously embedded into $\mathcal{C}^\ell$.

Approximated solution

Gevrey estimate:

$$||u_{2n+1}||_s + |\mu_{2n}| \leq \delta K^{2n+1}((2n+1)!)^{4l_0}$$

Theorem

(G.I., A.Rucklidge 2009) Let $q \geq 4$, $s > d/2, s \geq 4$ *then there exists* K *and* $C > 0$ *such that for* $\epsilon < \epsilon_0$, *there exists* $\bar{u}(\epsilon) \in \mathcal{H}_s$ *with the formal asymptotic expansion computed above and satisfying*

$$||(1+\Delta)^2\bar{u}(\epsilon) - \bar{\mu}(\epsilon)\bar{u}(\epsilon)) + [\bar{u}(\epsilon)]^3||_{s-4} \leq Ce^{-\frac{K}{\epsilon^{1/8l_0}}}$$

Existence of quasipatterns

Theorem

(Braaksma, Iooss, Stolovitch 2015) Let $q \geq 4$ *be an integer and let* $d = 2(l_0 + 1)$ *be the dimension of the* $\mathbb{Q}$*-vector space spanned by the wave vectors* $\mathbf{k}_j$, $j = 1, ..., 2q$. *Then, there exists* $s_0 > d/2$, $\varepsilon_0 > 0$, *and* $0 < \epsilon_2 < \varepsilon_0$ *such that, for any* $s \geq s_0$, $|\varepsilon| < \epsilon_2$, *there exists a set* $\mathcal{E}$ *contained in* $[-\epsilon_2, \epsilon_2]$, *of asymptotic full measure as* ϵ *tends to* 0, *such that for* $\varepsilon \in \mathcal{E}$, *there exists a quasipattern solution* (U, μ) *of the steady Swift-Hohenberg equation, invariant under the rotation* $\mathbf{R}_{\pi/q}$, *of the form*

$$\begin{aligned} U &= U_\varepsilon + \varepsilon^{2p} V(\varepsilon) \in \mathcal{H}_{s_0}, U_\varepsilon = \varepsilon u_1 + \varepsilon^3 u_3 + ...\varepsilon^{2p-1} u_{2p-1} \\ \mu &= \mu_\varepsilon + \varepsilon^{2p+2} h(\varepsilon),\ \mu_\varepsilon = \varepsilon^2 \mu_2 + ... + \varepsilon^{2p} \mu_{2p} \end{aligned}$$

where V *and* εh *are* C^1, $\mu_2 = 3(2q-1) > 0$ *and coefficients* μ_{2n}, u_1, u_{2n+1} *are the ones of the formal asymptotic expansion.*

Idea of Proof 1

$$
\begin{aligned}
u &= U_\varepsilon + \varepsilon^{2p}\widetilde{u},\ \widetilde{u} \in \{u_1\}^\perp, \\
U_\varepsilon &= \varepsilon u_1 + \varepsilon^3 \widetilde{U}_\varepsilon,\ \widetilde{U}_\varepsilon = u_3 + ...\varepsilon^{2p-4} u_{2p-1} \perp u_1 \\
\mu &= \mu_\varepsilon + \widetilde{\mu},\ \ \mu_\varepsilon = \varepsilon^2 \mu_2 + ...\varepsilon^{2p}\mu_{2p}.
\end{aligned}
$$

Range equation:

$$(\mathbf{L}_0 - \widetilde{\mu})\widetilde{u} + g(\varepsilon, \widetilde{\mu}) + \mathcal{B}_\varepsilon \widetilde{u} + \mathcal{C}_\varepsilon(\widetilde{u}) = 0,$$

where $g(\varepsilon, \widetilde{\mu}) = \widetilde{\mu}\varepsilon^{3-2p}\widetilde{U}_\varepsilon - \varepsilon \mathbf{Q}_0 f_\varepsilon$, $\mathcal{B}_\varepsilon$ is linear and $O(\varepsilon^2)$ in any $\mathcal{H}_s$, and $\mathcal{C}_\varepsilon$ is at least quadratic and $O(\varepsilon^{2p+1})$ in $\mathcal{H}_s, s > d/2$. We expect, for suitable $\widetilde{\mu} \in (-\varepsilon^{2p-2}, \varepsilon^{2p-2})$, to solve this range equation with respect to $\widetilde{u}$ which should be of order $O(\varepsilon)$, and put it into the

Bifurcation equation:

$$\langle u_1, u_1\rangle \widetilde{\mu} - 3\varepsilon^{2p+1}\langle u_1^2 \widetilde{u}, u_1\rangle = \mathcal{O}(\varepsilon^{2p+2}),$$

Then we solve with respect to $\widetilde{\mu}$, and find $\widetilde{\mu} = O(\varepsilon^{2p+2})$.

Idea of Proof (continued 1)

Solving the Range equation, we have a small divisor problem:

$$\widetilde{\mathbf{L}_0}^{-1} e^{i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{(|\mathbf{k}|^2 - 1)^2} e^{i\mathbf{k}\cdot\mathbf{x}}$$

with $(|\mathbf{k}|^2 - 1)^2 \geq cN_{\mathbf{k}}^{-4l_0}$
Nash-Moser method needs to invert the differential $\mathcal{L}_{\varepsilon,V} - \widetilde{\mu}\mathbb{I}$ at any V near 0, where $\mathcal{L}_{\varepsilon,V}$ acts in $\mathbf{Q}_0\mathcal{H}_t,\ \ t \geq 0$, $\mathbf{Q}_0$ is the orthogonal projection on $\{u_1\}^{\perp}$in $\mathcal{H}_t,\ t \geq 0$, and $\mathcal{L}_{\varepsilon,V}$ is defined by

$$\mathcal{L}_{\varepsilon,V} = \mathbf{L}_0 - \mu_\varepsilon\mathbb{I} + 3\mathbf{Q}_0(U_\varepsilon^2\cdot) - 6\varepsilon^{2p}\mathbf{Q}_0(U_\varepsilon V\cdot) - 3\varepsilon^{4p}\mathbf{Q}_0[(V)^2\cdot].$$

$\mathbf{L}_0$ is selfadjoint in all $\mathbf{Q}_0\mathcal{H}_s,\ s \geq 0$, and its spectrum is $\mathbb{R}^+$ (constant coefficients)

Inverse of the differential $\mathcal{L}_{\varepsilon,V} - \widetilde{\mu}\mathbb{I}$

Definition

Truncation of the space. Let $s \geq 0$ and $N > 1$ be an integer, we define $E_N := \Pi_N \mathbf{Q}_0 \mathcal{H}_s$, which consists in keeping in the Fourier expansion of $\widetilde{u} \in \mathbf{Q}_0 \mathcal{H}_s$ only those $\mathbf{k} \in \Gamma$ such that $N_{\mathbf{k}} \leq N$. By construction we obtain

$$||(\Pi_N \mathbf{L}_0 \Pi_N)^{-1}||_s \leq c_0 (1+N)^{4l_0}.$$

Inverse of $\mathcal{L}_{\varepsilon,V} - \widetilde{\mu}\mathbb{I}$ for $N < M_\varepsilon$, (elementary perturbation theory)

Lemma

Let $S > s_0 > d/2$ and $\varepsilon_0 > 0$ small enough and $\alpha \in (\mathcal{E}_1 \cap \mathcal{E}_0) \cup \mathcal{E}_{\mathbb{Q}}$. Then for $0 < \varepsilon \leq \varepsilon_0$ and $N \leq M_\varepsilon$ with $M_\varepsilon := \left[\frac{c_1}{\varepsilon^{1/2l_0}}\right]$ and $(\varepsilon, \widetilde{\mu}, V) \in [-\varepsilon_0, \varepsilon_0] \times [-\varepsilon^{2p-2}, \varepsilon^{2p-2}] \times E_N$, the following holds for $s \in [s_0, S]$ and V such that $||V||_s \leq 1$,
$||(\Pi_N(\mathcal{L}_{\varepsilon,V} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}||_s \leq 2c_0(1+N)^{4l_0}$

Inverse of $\mathcal{L}_{\varepsilon,V} - \widetilde{\mu}\mathbb{I}$ for large N

define $\Lambda := \{(\varepsilon, \tilde{\mu}); \varepsilon \in [-\varepsilon_0, \varepsilon_0], \tilde{\mu} \in [-\varepsilon^{2p-2}, \varepsilon^{2p-2}]\}$, and for $M > 0$, $s_0 > d/2$,

$$\begin{aligned} \mathcal{U}_M^{(N)} \quad : \quad &= \{V \in C^1(\Lambda, E_N); V(0, \widetilde{\mu}) = 0, \\ &||V||_{s_0} \leq 1, ||\partial_\varepsilon V||_{s_0} \leq M, ||\partial_{\widetilde{\mu}} V||_{s_0} \leq (M/\varepsilon^{2p-2})\}. \end{aligned}$$

For $V \in \mathcal{U}_M^{(N)}$, we consider the operator

$$\begin{aligned} \Pi_N(\mathcal{L}_{\varepsilon,V(\varepsilon,\widetilde{\mu})} - \widetilde{\mu}\mathbb{I})\Pi_N \quad = \quad & \Pi_N \mathbf{L}_0 \Pi_N - \widetilde{\mu}\mathbb{I}_N + \varepsilon^2 \mathcal{B}_1^{(N)}(\varepsilon) + \\ & + \varepsilon^{2p+1} \mathcal{B}_2^{(N)}(\varepsilon, V(\varepsilon, \widetilde{\mu})), \end{aligned}$$

$\Pi_N \mathbf{L}_0 \Pi_N$, $\mathcal{B}_1^{(N)}$, $\mathcal{B}_2^{(N)}$ selfadjoint in $\Pi_N \mathbf{Q}_0 \mathcal{H}_0$ and analytic in their arguments.

Eigenv. of $\Pi_N(\mathcal{L}_{\varepsilon,V(\varepsilon,\widetilde{\mu})} - \widetilde{\mu}\mathbb{I})\Pi_N$ are (see Kato, thm 6.1 and 6.10)

$$\sigma_j(\varepsilon, \widetilde{\mu}) = s_j(\varepsilon) + f_j(\varepsilon, \widetilde{\mu}) - \widetilde{\mu},$$

where s_j is analytic and f_j is Lipschitz in $(\varepsilon, \widetilde{\mu})$ (Lidskii theorem) and $|f_j(\varepsilon_2, \widetilde{\mu_2}) - f_j(\varepsilon_1, \widetilde{\mu_1})| \leq c[\varepsilon^{2p}|\varepsilon_2 - \varepsilon_1| + \varepsilon^3|\widetilde{\mu_2} - \widetilde{\mu_1}|]$

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Inverse of $\mathcal{L}_{\varepsilon,V} - \widetilde{\mu}\mathbb{I}$ for large N (continued 1)

Bad set of $\tilde{\mu}$

$$B^{(N)}_{\varepsilon,\gamma}(V) = \Big\{\widetilde{\mu} \in [-\varepsilon_0, \varepsilon_0]; (\varepsilon, V) \in [-\varepsilon_0, \varepsilon_0] \times \mathcal{U}^{(N)}_M, \exists j \in \{1, ...\mathcal{N}\}, |\sigma_j(\varepsilon, \widetilde{\mu})| < \frac{\gamma}{N^\tau}\Big\}$$

$$B^{(N)}_{\varepsilon,\gamma}(V) = \cup_{j=1}^{\mathcal{N}}(\widetilde{\mu}_j^-(\varepsilon), \widetilde{\mu}_j^+(\varepsilon)),$$

$$0 < \widetilde{\mu}_j^+(\varepsilon) - \widetilde{\mu}_j^-(\varepsilon) \leq \frac{4\gamma}{N^\tau}, \ \mathcal{N} \leq bN^d$$

$$\mathrm{meas}(B^{(N)}_{\varepsilon,\gamma}(V)) \leq \frac{4b\gamma}{N^{\tau-d}},$$

$\widetilde{\mu}_j^\pm(\varepsilon)$ are Lipschitz continuous with a small Lip constant.

Good set of $\tilde{\mu}$: $G^{(N)}_{\varepsilon,\gamma}(V) := [-\varepsilon_0, \varepsilon_0] \backslash B^{(N)}_{\varepsilon,\gamma}(V)$.

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Inverse of $\mathcal{L}_{\varepsilon,V} - \widetilde{\mu}\mathbb{I}$ for large N (continued 2)

Lemma

Assume $\gamma \leq \widetilde{\gamma} = 2^{2l_0+1}c_0)$ *and* $\tau > d + 3 + (4p+4)l_0$. *For* $V \in \mathcal{U}_M^{(N)}$ *and* $|\varepsilon| \leq \varepsilon_0$ *fixed, then if* $\widetilde{\mu} \in G_{\varepsilon,\gamma}^{(N)}(V) \cap [-\varepsilon^{2p-2}, \varepsilon^{2p-2}]$, $N > 1$

$$||(\Pi_N(\mathcal{L}_{\varepsilon,V(\varepsilon,\widetilde{\mu})} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}||_0 \leq \frac{N^\tau}{\gamma}.$$

Moreover, for $N > M_\varepsilon$, *the measure of the "bad set"* $B_{\varepsilon,\gamma}^{(N)}(V)$ *is bounded by* $4b\gamma/N^{\tau-d}$, *while it is* 0 *for* $N \leq M_\varepsilon$.

This estimate is in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_0)$. In fact, we need to obtain a tame estimate for $(\Pi_N(\mathcal{L}_{\varepsilon,V(\varepsilon,\widetilde{\mu})} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}$ in $\mathcal{L}(\mathbf{Q}_0\mathcal{H}_s)$ for $s > 0$, with an exponent on N not depending on s.

We use Bourgain 1995, Craig 2000, Berti-Bolle 2010 with a suitable adaptation.

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Inverse of the differential in $\mathcal{H}_s$ for large N (continued 3)

Singular set in $\mathbb{Z}^d$: $S(N) := \{\mathbf{z} \in \Gamma(N);\ (1 - |\mathbf{k}(\mathbf{z})|^2)^2 < \rho\}$ with

$$\begin{aligned} \mathbf{k}(\mathbf{z}) &= \mathfrak{d}^{-1} \sum_{s=1}^{d} z_s \mathbf{k}_s^*,\ \ \mathbf{z} = (z_1, ..., z_s) \in \mathbb{Z}^d \\ \Gamma(N) &:= \{\mathbf{z} \in \mathbb{Z}^d;\ 0 \leq |\mathbf{z}| \leq N,\ \mathbf{k}(\mathbf{z}) \in \Gamma \backslash \{\mathbf{k}_j, j = 1, ..., 2q\}\}. \end{aligned}$$

Useful lemma (uses Bourgain 1995, Craig 2000, Berti-Bolle 2010)
There exists $\rho_0 > 0$ independent of N such that if $\rho \in]0, \rho_0]$ then $S(N) = \bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha$ is a union of disjoint clusters Ω_α satisfying :

- $(H1)$, for all $\alpha \in \mathcal{A}$, $M_\alpha \leq 2m_\alpha$ where $M_\alpha = \max_{\mathbf{z} \in \Omega_\alpha} |\mathbf{z}|$ and $m_\alpha = \min_{\mathbf{z} \in \Omega_\alpha} |\mathbf{z}|$;
- $(H2)$, there exists $\delta = \delta(d) \in]0, 1[$ independent of N such that if $\alpha, \beta \in \mathcal{A}, \alpha \neq \beta$ then $\mathrm{dist}(\Omega_\alpha, \Omega_\beta) := \min_{\mathbf{z} \in \Omega_\alpha, \mathbf{z}' \in \Omega_\beta} |\mathbf{z} - \mathbf{z}'| \geq \frac{(M_\alpha + M_\beta)^\delta}{2}$

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Basic ingredient for the Lemma above

Define the positive definite matrix $\mathbf{A}$ in $\mathbb{Z}^d$:

$$\mathfrak{d}^2|\mathbf{k}(\mathbf{z})|^2 = \langle \mathbf{z}, \mathbf{A}\mathbf{z}\rangle, \ \mathbf{A} = \sum_{r=1}^{l_0} \mathbf{A}_r \omega^r$$

where $\omega = 2\cos\pi/q$, and matrices $\mathbf{A}_r$ have integer coefficients.
Then, for any $\mathbb{Q}$-linearly independent family $\{\mathbf{e}_j, j = 1, ..., d_0 \leq d\}$ in $\mathbb{Z}^d$, let consider the matrix $\mathbf{M}$ such that $M_{l,m} = \langle \mathbf{e}_l, \mathbf{A}\mathbf{e}_m\rangle$.
We have $det\mathbf{M} = \sum_{r=0}^{l_0} q_r \omega^r$.
Then, (see G.I.-A.R. 2010 Lemma 2.1) there exists $C > 0$ such that

$$|det\mathbf{M}| \geq \frac{C}{|\mathbf{q}|^{l_0}}$$

Inverse of $\mathcal{L}_{\varepsilon,V} - \widetilde{\mu}\mathbb{I}$ in $\mathcal{H}_s$ for large N (the end)

Define the set of "good' $\widetilde{\mu}$ for all $K \leq N$:

$$\mathcal{G}_{\varepsilon,\gamma}^{(N)}(V) := \cap_{M_\varepsilon < K \leq N} G_{\varepsilon,\gamma}^{(K)}(V) \cap [-\varepsilon^{2p-2}, \varepsilon^{2p-2}]$$

Lemma

Let $\tau > d + 3 + (4p+4)l_0$ and $s_0 \geq \frac{d}{2} + \frac{d+\tau}{\delta} + 1$, where δ is the number introduced in separation property (H2), and define $m := 2\tau + 3d/2$. Assume moreover that $0 < \gamma \leq \widetilde{\gamma} = 1/(2^{2l_0+1}c_0)$, and $(\varepsilon, \widetilde{\mu}, V) \in \Lambda \times \mathcal{U}_M^{(N)}$, with $|\varepsilon| \leq \epsilon_1, \widetilde{\mu} \in \mathcal{G}_{\varepsilon,\gamma}^{(N)}(V)$, ϵ_1 small enough. Let $\overline{s} > s_0$. Then for all $s \in [s_0, \overline{s}]$ there exists $K(s) > 0$ such that for any $h \in \Pi_N \mathbf{Q}_0 \mathcal{H}_s$, we have

$$||(\Pi_N(\mathcal{L}_{\varepsilon,V} - \widetilde{\mu}\mathbb{I})\Pi_N)^{-1}h||_s \leq K(s)\frac{N^m}{\gamma}(||h||_s + ||V||_s||h||_{s_0})$$

The proof follows Berti-Bolle 2010

Resolution of the Range equation

Define $\widetilde{\mu} = \varepsilon^{2p-2}\hat{\mu}$, then the range equation takes the form

$$\mathcal{F}(\varepsilon, \hat{\mu}, \widetilde{u}) := (\mathbf{L}_0 - \varepsilon^{2p-2}\hat{\mu})\widetilde{u} + \hat{g}(\varepsilon, \hat{\mu}) + \mathcal{B}_\varepsilon \widetilde{u} + \mathcal{C}_\varepsilon(\widetilde{u}) = 0$$

$(\varepsilon, \hat{\mu}, \widetilde{u}) \to \mathcal{F}(\varepsilon, \hat{\mu}, \widetilde{u})$ is analytic from $[-\varepsilon_0, \varepsilon_0] \times [-1, 1] \times \mathbf{Q}_0\mathcal{H}_{s+4}$ to $\mathbf{Q}_0\mathcal{H}_s$, and $\mathcal{F}(-\varepsilon, \hat{\mu}, -\widetilde{u}) = -\mathcal{F}(\varepsilon, \hat{\mu}, \widetilde{u})$.

$$\begin{aligned}
\hat{g}(0, \hat{\mu}) &= 0, \text{ and for } ||V||_{s_0} \leq 1, s \geq s_0 > d/2 \\
||D_{\widetilde{u}}\mathcal{F}(\varepsilon, \hat{\mu}, V)v||_s &\leq C(s)(||v||_{s+4} + \varepsilon^{2p+1}||v||_{s_0}||V||_s) \\
||D^2_{\widetilde{u}}\mathcal{F}(\varepsilon, \hat{\mu}, V)(v, h)||_s &\leq C(s)\varepsilon^{2p+1}(||h||_s||v||_{s_0} + ||h||_{s_0}||v||_s + \\
&\quad +||V||_s||h||_{s_0}||v||_{s_0}
\end{aligned}$$

$||\Pi_N u||_{s+r} \leq (1+N)^r||u||_s, \ ||(\mathbb{I} - \Pi_N)u||_s \leq (1+N)^{-r}||u||_{r+s}$
Π_N is a "smoothing operator", and for $V \in \mathcal{U}_M^{(N)}$, $\widetilde{\mu} \in \mathcal{G}_{\varepsilon,\gamma}^{(N)}(V)$,
$||\{\Pi_N D_{\widetilde{u}}\mathcal{F}(\varepsilon, \hat{\mu}, V)\Pi_N\}^{-1}v||_s \leq K(s)\frac{N^m}{\gamma}(||h||_s + ||V||_s||h||_{s_0})$

Resolution of the Range equation (continued)

For $\tilde{V}(\varepsilon, \widetilde{\mu}) \in \mathcal{U}_M^{(N)}$, then $V(\varepsilon, \hat{\mu}) := \tilde{V}(\varepsilon, \varepsilon^{2p-2}\hat{\mu})$ is C^1 with first derivatives bounded by M in $\mathcal{H}_{s_0}$.

Theorem

Let s_0 and $\tilde{\gamma}$ be as above. Then for all $0 < \gamma < \tilde{\gamma}$ there exist $\epsilon_2(\gamma) \in [0, \varepsilon_0]$ and a C^1-map $V : (-\epsilon_2, \epsilon_2) \times [-1, 1] \to \mathcal{H}_{s_0+4}$ such that $V(0, \hat{\mu}) = 0$ and if $|\varepsilon| \leq \epsilon_2$, $\hat{\mu} \in ([-1, 1] \setminus C_{\varepsilon,\gamma})$, the function $V(\varepsilon, \hat{\mu})$ is solution of $\mathcal{F}(\varepsilon, \hat{\mu}, V) = 0$. Here $C_{\varepsilon,\gamma}$ is a subset of $[-1, 1]$ which is a Lipschitz function of ε and has Lebesgue-measure less than $C\gamma|\varepsilon|^3$ for some constant $C > 0$ independent of ε and γ. Moreover, $V(-\varepsilon, \hat{\mu}) = -V(\varepsilon, \hat{\mu})$.

The proof uses Nash-Moser method, following Berti-Bolle-Procesi 2010.

Resolution of the Bifurcation equation

$$\widetilde{\mu} = \varepsilon^{2p-2}\hat{\mu}$$

Bifurcation equation:

$$\begin{aligned}\widetilde{\mu}u_1 &= 3\varepsilon^{2p-1}\mathbf{P}_0(U_\varepsilon^2 V(\varepsilon,\hat{\mu})) + \varepsilon^{2p+2}\mathbf{P}_0 f_\varepsilon^{(1)} \\ &\quad +3\varepsilon^{4p-1}\mathbf{P}_0(U_\varepsilon V^2) + \varepsilon^{6p-1}\mathbf{P}_0 V^3\end{aligned}$$

and using the implicit function theorem for $\varepsilon \in (-\epsilon_2, \epsilon_2)$,

(H) $\widetilde{\mu} = \varepsilon^{2p+2}h(\varepsilon),\ \varepsilon h(\varepsilon)$ odd function $\in C^1$

The only valid values for ε are the one giving "good" $\widetilde{\mu}$'s.

Structure of the "bad set" in the space $(\varepsilon, \widetilde{\mu})$

Define bad strips;

$$BS_N(V) = \{(\varepsilon, \widetilde{\mu}) \in \Lambda;\ \text{there exists } j \text{ with } \tilde{\mu} \in [\widetilde{\mu}_j^-(\varepsilon), \widetilde{\mu}_j^+(\varepsilon)]\}$$

$$\sum_j |\widetilde{\mu}_j^+(\varepsilon) - \widetilde{\mu}_j^-(\varepsilon)| \leq \frac{c\gamma}{N^{\tau-d}} \leq \frac{c\gamma\varepsilon^{2p+1}}{N^3}$$

$$\widetilde{\mu}_j^\pm(\varepsilon) = s_j^{(N)}(\varepsilon) + g_j^\pm(\varepsilon),\ \ s_j^{(N)}(\varepsilon) = s_j^{(N)}(0) + 3\varepsilon^2 + \mathcal{O}(\varepsilon^4)$$

$$|g_j^\pm(\varepsilon_2) - g_j^\pm(\varepsilon_1)| \leq c\varepsilon^4|\varepsilon_2 - \varepsilon_1|$$

$BS_N(V)$ is a union of thin Lipschitz strips in the plane $(\varepsilon, \widetilde{\mu})$
For the proof of the range theorem, we choose $\widetilde{\mu}$ outside of $\cup_{n\in\mathbb{N}} BS_{N_n}(V_{n-1})$ where $N_n = [N_0(\gamma)]^{2^n}$, and V_n are the successive points in the Newton iteration process.

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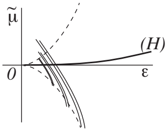
Transversality

Let $\tilde{\mu}$ be any one of the limiting curves of the bad strips given by $\cup_{n\in\mathbb{N}} BS_{N_n}(V_{n-1})$, then the form of $\widetilde{\mu}_j^{\pm}(\varepsilon)$ leads to

$$|\tilde{\mu}(\varepsilon + h) - \tilde{\mu}(\varepsilon)| \geq c|\varepsilon||h|,$$

which is OK for intersecting transversally the bifurcation curve (H) (slope of order ε^{2p+1}). From the resolution of the Range equation the Measure of "bad" $\widetilde{\mu}$'s $< C\gamma|\varepsilon|^{2p+1}$ hence measure of "bad" ε's

$$< \frac{C\gamma|\varepsilon|^{2p+1}}{min|\text{slope}|} < \frac{C\gamma|\varepsilon|^{2p+1}}{c|\varepsilon|} \leq C'\gamma\varepsilon^{2p}$$



The complementary subset in $(0, \varepsilon_2)$, is the good set of $|\varepsilon|$, which is of asymptotic full measure since $[|\varepsilon| - C'\gamma\varepsilon^{2p}]/|\varepsilon| \to 1$ as $\varepsilon \to 0$.

Theorem

Theorem

(B.I.S. 2015) Let $q \geq 4$ be an integer and let $d = 2(l_0 + 1)$ be the dimension of the $\mathbb{Q}$-vector space spanned by the wave vectors $\mathbf{k}_j$, $j = 1, ..., 2q$. Then, there exists $s_0 > d/2$, $\varepsilon_0 > 0$, and $0 < \epsilon_2 < \varepsilon_0$ such that, for any $s \geq s_0$, $|\varepsilon| < \epsilon_2$, there exists a set $\mathcal{E}$ contained in $[-\epsilon_2, \epsilon_2]$, of asymptotic full measure as ϵ tends to 0, such that for $\varepsilon \in \mathcal{E}$, there exists a quasipattern solution (U, μ) of the steady Swift-Hohenberg equation, invariant under the rotation $\mathbf{R}_{\pi/q}$, of the form

$$
\begin{aligned}
U &= U_\varepsilon + \varepsilon^{2p} V(\varepsilon) \in \mathcal{H}_{s_0}, U_\varepsilon = \varepsilon u_1 + \varepsilon^3 u_3 + ...\varepsilon^{2p-1} u_{2p-1} \\
\mu &= \mu_\varepsilon + \varepsilon^{2p+2} h(\varepsilon),\ \mu_\varepsilon = \varepsilon^2 \mu_2 + ... + \varepsilon^{2p} \mu_{2p}
\end{aligned}
$$

where V and εh are C^1, $\mu_2 = 3(2q-1) > 0$ and coefficients μ_{2n}, u_1, u_{2n+1} are the ones of the formal asymptotic expansion.

References on Quasipatterns

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