Existence of quasipatterns, solutions of the Bénard - Rayleigh convection problem

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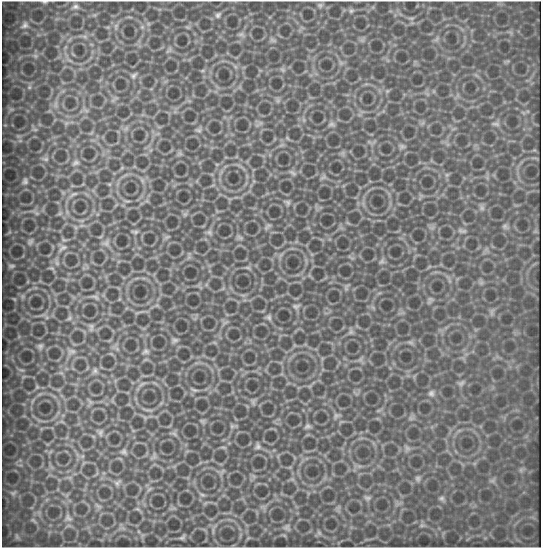
Collaboration with B.Braaksma (Groningen)

Quasipatterns experiments



Experiment of Faraday type. Kudrolli, Pier, Gollub 1998

Steady Bénard - Rayleigh system between two horizontal planes

$$\begin{aligned} V\cdot\nabla V+\nabla p &= \mathcal{P}(\theta e_z+\mathcal{R}^{-1/2}\Delta V), \\ V\cdot\nabla\theta &= \mathcal{R}^{-1/2}\Delta\theta+V\cdot e_z, \\ \nabla\cdot V &= 0. \end{aligned}$$

Boundary Conditions: $v^{(z)}=\theta=0$ in $z=0,1$.
either "rigid - rigid": $V^{(H)}=0$ in $z=0,1$,
or "rigid - free": $V^{(H)}=0$ in $z=0$, $\frac{\partial V^{(H)}}{\partial z}=0$ in $z=1$,
or "free - rigid": $\frac{\partial V^{(H)}}{\partial z}=0$ in $z=0$, $V^{(H)}=0$ in $z=1$.
We do not consider the "free-free" case: $\frac{\partial V^{(H)}}{\partial z}=0$ in $z=0,1$.
We choose to look for bifurcating solutions, quasiperiodic in $\mathbf{x}\in\mathbb{R}^2$, invariant under rotations of angle π/q.

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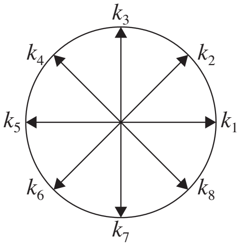
Quasilattices

$$u = \sum_{\mathbf{k}\in\Gamma} u^{(\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}}, \;\; u^{(-\mathbf{k})} = \overline{u}^{(\mathbf{k})}$$

$$\Gamma = \{\mathbf{k} = \sum_{j=1,\dots 2q} m_j \mathbf{k}_j, \;\; m \in \mathbb{N}^{2q}, \, |\mathbf{k}_j| = k_c, (\mathbf{k}_j, \mathbf{k}_{j+1}) = \pi/q\}$$

For $q = 1, 2, 3$ Γ is a lattice leading to a periodic pattern
See V.Yudovich et al (1963-67), W.Velte (1964-69),
K.Kirchgässner et al (1967-73), P.Rabinowitz (1968)

For $q \geq 4$ Γ is a quasilattice leading to a quasipattern



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Example $q = 4$, the 8 wavevectors which form the basis of the quasilattice

Diophantine estimate

$\mathbb{Q}$ vector space $span\{\mathbf{k}_j; j = 1, ..2q\}$ has dimension d,
$d/2 = l_0 + 1 \leq q/2$ is the degree of the minimal Polynomial for the algebraic integer $\omega = 2\cos\pi/q$ (coef in $\mathbb{Z}$ and first coef $= 1$).

$$\mathbf{k} = \Sigma_{j=1}^{2q} m_j \mathbf{k}_j = \frac{1}{\mathfrak{d}} \Sigma_{s=1}^{d} m_s^* \mathbf{k}_s^*, \ \mathbf{m}^* = (m_1^*, .., m_d^*) \in \mathbb{Z}^d$$

$$N_{\mathbf{k}} = \Sigma_{s=1}^{d} |m_s^*|$$

$$\begin{aligned} \text{for } q &= 4, 5, 6, \quad \omega = \sqrt{2}, \ \frac{1+\sqrt{5}}{2}, \ \sqrt{3}, \ l_0 = 1, \ d = 4 \\ \text{for } q &= 7, 9, \quad l_0 = 2, \ d = 6, \\ \text{for } q &= 8, 10, 12, \ l_0 = 3, \ d = 8, \ \text{for } q = 11, \ l_0 = 4, \ d = 10.... \end{aligned}$$

For $q = 4, 5, ..., 12$ then $\mathfrak{d} = 1$ and $\mathbf{k}_s^* = \mathbf{k}_s, s = 1, ..., d$.
In all cases, there exists $c > 0$ such that

$$(|\mathbf{k}|^2 - k_c^2)^2 \geq c(1 + N_{\mathbf{k}}^2)^{-2l_0}, \ \text{if } \mathbf{k} \neq \mathbf{k}_j, \ j = 1, ..2q$$

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Suitable formulation

Hilbert space for the 4-components vector field $u = (V, \theta)$:

$$\mathcal{K}_s = \{u = (V,\theta)(\mathbf{x},z) = \sum_{\mathbf{k}\in\Gamma} u_{\mathbf{k}}(z) e^{i\mathbf{k}\cdot\mathbf{x}}; \nabla \cdot V = 0, v_z|_{z=0,1} = 0,$$
$$\sum_{\mathbf{k}\in\Gamma} \left((1+N_{\mathbf{k}}^2)^s ||u_{\mathbf{k}}||^2_{L^2(0,1)} \right) < \infty\}$$

For $s > d/2, \lambda := \mathcal{R}^{-1/2} > 0$, finding a solution $u \in \mathcal{K}_s$ of

$$\lambda u - \mathcal{A}u + \mathcal{B}(u,u) = 0,$$

is equivalent to find a classical solution of Bénard-Rayleigh system. In $\mathcal{K}_s, \mathcal{A}$ is linear bounded, selfadjoint , $\mathcal{B}$ is quadratic, bounded. Operators $\mathcal{A}$ and $\mathcal{B}$ commute with $\mathbf{R}_\phi$ defined by

$$\mathbf{R}_\phi u = (R_\phi V(R_{-\phi}\mathbf{x}, z), \theta(R_{-\phi}\mathbf{x}, z))$$

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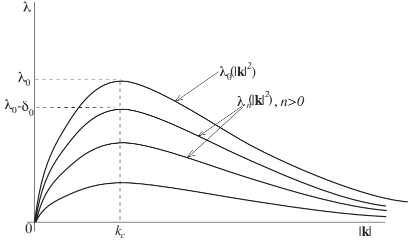
We are interested by quasipatterns solutions of the problem which are invariant under $\mathbf{R}_{\pi/q}$

Study of the linear equation :

$$(\mathcal{A} - \lambda)u = G \in \mathcal{K}_s$$

comes back to the study made by V.Yudovich (1966) in the periodic case, for any wave number $|\mathbf{k}|$.



positive eigenvalues of $\mathcal{A}$

Spectrum of $\mathcal{A}$: real interval $[-\lambda_0, \lambda_0]$
$\lambda = \lambda_0$ largest e.v. $Ker(\mathcal{A} - \lambda_0)$ spanned by
$\xi_j = \mathbf{R}_{\frac{\pi(j-1)}{q}} \left(\widehat{U}_{\mathbf{k}_1}(z) e^{i\mathbf{k}_1 \cdot \mathbf{x}} \right), \; j = 1, 2, ..., 2q,$ with $|\mathbf{k}_1| = k_c$

$$(\mathcal{A}-\lambda_0)u = G \in \mathcal{K}_s$$

Assume $\frac{d^2\lambda_0}{d|\mathbf{k}|^2}|_{k_c} \neq 0.$

For G satisfying compatibility conditions $\langle G, \xi_j \rangle = 0, j = 1, ..., 2q$

$$||u_{\mathbf{k}}||_0 \leq \frac{c(|\mathbf{k}|^2)}{(|\mathbf{k}|^2 - k_c^2)^2} ||G_{\mathbf{k}}||_0, \ \mathbf{k} \in \Gamma \text{ except } \mathbf{k}_j, j = 1, 2, ..., 2q$$

$c(|\mathbf{k}|^2)$ is analytic and $\mathcal{O}(|\mathbf{k}|^4)$ as $|\mathbf{k}|^2$ tends towards ∞

diophantine estimate: $\frac{1}{(|\mathbf{k}|^2 - k_c^2)^2} \leq C(1 + N_{\mathbf{k}}^2)^{2l_0}$ for $\mathbf{k} \neq \mathbf{k}_j$

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Formal series - Approximate solution

$$(\mathcal{A}-\lambda_0)u = -\mu u + \mathcal{B}(u,u), \;\; \lambda = \lambda_0 - \mu$$

$$u = \sum_{n\geq 1} \varepsilon^n u_n, , \;\; \mu = \sum_{n\geq 1} \varepsilon^n \mu_n, \text{ with } u_n \in \mathcal{K}_s, \;\; \langle u_n, u_1\rangle_0 = 0, n \geq 2$$

$$\begin{aligned}
(\mathcal{A}-\lambda_0)u_1 &= 0, \\
(\mathcal{A}-\lambda_0)u_2 &= -\mu_1 u_1 + \mathcal{B}(u_1,u_1) \\
(\mathcal{A}-\lambda_0)u_3 &= -\mu_1 u_2 - \mu_2 u_1 + 2\mathcal{B}(u_1,u_2).
\end{aligned}$$

$$u_1 = \sum_{1\leq j\leq 2q} \boldsymbol{\xi}_j, \text{ invariant under rotation } \mathbf{R}_{\pi/q}, \text{ spans } ker(\mathcal{A}-\lambda_0)$$

$$\langle \mathcal{B}(u_1,u_1), u_1\rangle_0 = 0, \text{ implies } \mu_1 = 0.$$

$$\mu_2\langle u_1,u_1\rangle_0 = \langle 2\mathcal{B}(u_1,u_2), u_1\rangle_0 = -\langle(\mathcal{A}-\lambda_0)u_2, u_2\rangle_0 > 0, \ldots$$

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Each step involves the pseudo-inverse $\widetilde{(\mathcal{A}-\lambda_0)}^{-1}$, implying only Gevrey series for $\sum_{n\geq 1}\varepsilon^n u_n$ and $\sum_{n\geq 1}\varepsilon^n \mu_n$.

New formulation

Idea: We decompose the system, as usual in bifurcation problems. The range equation contains the small divisor problem, we hope to use a parameter $\widetilde{\mu}$ able to move the whole spectrum of the linearized operator as this is used by Berti, Bolle, Procesi (2010).

$$
\begin{aligned}
u &= u_\varepsilon + h(\varepsilon, \mu') + \varepsilon^4 \widetilde{v}, \ \ \mu = \mu_\varepsilon + \varepsilon^3 \mu', \\
u_\varepsilon &= \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4, \ \ \widetilde{v} \in \{u_1\}^\perp \cap \mathcal{K}_s \\
\mu_\varepsilon &= \varepsilon^2 \mu_2 + \varepsilon^3 \mu_3, \ \ \widetilde{\mu} = \varepsilon^3 \mu'
\end{aligned}
$$

Range equ. $\mathfrak{L}_{\varepsilon,\widetilde{\mu}} \tilde{v} + g(\varepsilon, \widetilde{\mu}) - \varepsilon^4 \mathbf{Q}_0 \mathcal{B}(\tilde{v}, \tilde{v}) = 0,$

Bifurc. equ. $\widetilde{\mu} - \varepsilon^4 \mu_4 + \mathcal{O}[\varepsilon^3 (\varepsilon + ||\tilde{v}||)^2] = 0$

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu}} := \mathbf{Q}_0 (\mathcal{A} - \lambda_0) + \widetilde{\mu} \mathbb{I} + \mathcal{R}_{\varepsilon,\widetilde{\mu}}$$

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$\mathfrak{L}_{\varepsilon,\widetilde{\mu}}$ is analytic while $g(\varepsilon, \widetilde{\mu})$ is only C^2 in $(\varepsilon, \widetilde{\mu})$.
Main difficulty: Solve the Range equation with respect to $\tilde{v}$

Useful estimates

For $|\varepsilon| \leq \varepsilon_0, |\widetilde{\mu}| \leq \varepsilon_0, v \in \mathcal{K}_s, s \geq 0$

$$\begin{aligned} ||\mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_s &\leq c_s \varepsilon ||v||_s, \\ ||\partial_{\widetilde{\mu}} \mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_s + ||\partial^2_{\widetilde{\mu}^2} \mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_s + ||\partial^2_{\varepsilon\widetilde{\mu}} \mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_s &\leq c_s \varepsilon^2 ||v||_s, \\ ||\partial_{\varepsilon} \mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_s + ||\partial^2_{\varepsilon^2} \mathcal{R}_{\varepsilon,\widetilde{\mu}} v||_s &\leq c_s ||v||_s \end{aligned}$$

For $s_0 > d/2$, $||2\varepsilon^4 \mathbf{Q}_0 \mathcal{B}(V, v)||_s \leq c_s \varepsilon^4 (||V||_{s_0} ||v||_s + ||V||_s ||v||_{s_0})$.

For $|\widetilde{\mu}| \leq |\varepsilon|$

$$\begin{aligned} ||g(\varepsilon, \widetilde{\mu})||_s &\leq c_s \varepsilon^2, ||\partial_{\varepsilon,\widetilde{\mu}} g(\varepsilon, \widetilde{\mu})||_s \leq c_s \varepsilon^2, \\ ||\partial^2_{\widetilde{\mu}^2} g(\varepsilon, \widetilde{\mu})||_s &\leq c_s, ||\partial^2_{\varepsilon^2} g(\varepsilon, \widetilde{\mu})||_s \leq c_s \varepsilon^2, ||\partial^2_{\varepsilon\widetilde{\mu}} g(\varepsilon, \widetilde{\mu})||_s \leq c_s \varepsilon^2. \end{aligned}$$

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Splitting by π_0

We need to invert

$$\mathfrak{L}_{\varepsilon,\tilde{\mu},V} = \mathfrak{L}_{\varepsilon,\tilde{\mu}} - 2\varepsilon^4 \mathbf{Q}_0 \mathcal{B}(V,\cdot)$$

$$\lambda_0 - \lambda_0(|\mathbf{k}|^2) \geq 0,\ \ \lambda_0 - \lambda_j(|\mathbf{k}|^2) > \delta_0 > 0, j = 1, 2, ...$$

For $||\mathbf{k}| - k_c| > \delta_1$, $|\lambda_0 - \lambda_0(|\mathbf{k}|^2)| > \delta_0/2$.
Projection π_0: suppresses $\mathbf{k} \in \Gamma$ such that $||\mathbf{k}| - k_c| > \delta_1$
Inverting $\mathfrak{L}_{\varepsilon,\tilde{\mu},V}$ equiv to invert $\mathfrak{L}'_{\varepsilon,\tilde{\mu},V}$:

$$\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V} = \pi_0 \mathbf{Q}_0 (\mathcal{A} - \lambda_0) \mathbf{Q}_0 \pi_0 + \widetilde{\mu} + \mathfrak{B}_\varepsilon + \varepsilon^2 \widetilde{\mu} \mathfrak{C}_{\varepsilon,\widetilde{\mu}} + \mathfrak{R}_{\varepsilon,\widetilde{\mu},V}$$

$\mathfrak{B}_\varepsilon$, $\mathfrak{C}_{\varepsilon,\widetilde{\mu}}$ and $\mathfrak{R}_{\varepsilon,\widetilde{\mu},V}$ depend analytically on their arguments.
For $s \geq s_0 > d/2$

$$\begin{aligned}
||\mathfrak{B}_\varepsilon v||_s &\leq c\varepsilon ||v||_s, ||\mathfrak{C}_{\varepsilon,\widetilde{\mu}} v||_s + ||\partial_{\widetilde{\mu}} \mathfrak{C}_{\varepsilon,\widetilde{\mu}} v||_s \leq c||v||_s \\
||\mathfrak{R}_{\varepsilon,\widetilde{\mu},V} v||_s &\leq c\varepsilon^4 \{||V||_{s_0} ||v||_s + ||V||_s ||v||_{s_0}\}, \\
||\partial_{\widetilde{\mu}} \mathfrak{R}_{\varepsilon,\widetilde{\mu},V} v||_s &\leq c\varepsilon^4 \{||V||_{s_0} ||v||_s + ||V||_s ||v||_{s_0}\}
\end{aligned}$$

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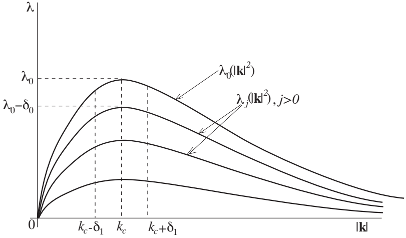
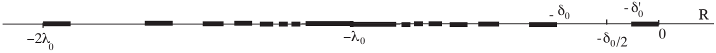
Splitting by π_0 - continued



positive eigenvalues of $\mathcal{A}$



Spectrum of $\pi_0\mathbf{Q}_0(\mathcal{A}-\lambda_0)\mathbf{Q}_0\pi_0$

Splitting by Π_N

Now, we truncate with the projection Π_N which
cuts the $\mathbf{k}$ such that $N_{\mathbf{k}} > N$.
This defines the space $E_N = \Pi_N \pi_0 \mathbf{Q}_0 \mathcal{K}_s$
still ∞ - dim, but with an isolated finite group of "small eigenvalues" perturbing $\lambda_0(|\mathbf{k}|^2) - \lambda_0, N_{\mathbf{k}} \leq N$.

$$\lambda_0 - \lambda_0(|\mathbf{k}|^2) \sim c((|\mathbf{k}|^2 - k_c^2)^2 \geq \frac{c}{(1+N^2)^{2l_0}}$$

Estimate of $(\Pi_N \mathfrak{L}'_{\varepsilon,\tilde{\mu},V}\Pi_N)^{-1}$ in $\Pi_N \pi_0 \mathbf{Q}_0 \mathcal{K}_s$, $N \leq M_\varepsilon$

Lemma

Let V satisfy $||V||_{s_0} \leq 1$, and assume $(\varepsilon, \tilde{\mu}) \in [0, \varepsilon_0] \times [-\varepsilon, \varepsilon]$. Then for $N \leq M_\varepsilon$, where $M_\varepsilon = \left[\frac{c_2}{\varepsilon^{1/4l_0}}\right]$, we have the following estimate for $s_0 > d/2$

$$\begin{aligned} ||(\Pi_N \mathfrak{L}'_{\varepsilon,\tilde{\mu},V}\Pi_N)^{-1} v||_s &\leq 2c(1+N^2)^{2l_0}\{||v||_s + ||V||_s||v||_{s_0})\}, \\ ||(\Pi_N \mathfrak{L}_{\varepsilon,\tilde{\mu},V}\Pi_N)^{-1} v||_s &\leq 2cc'(1+N^2)^{2l_0}\{||v||_s + ||V||_s||v||_{s_0})\}. \end{aligned}$$

Hint: Classical perturbation theory, based on the smallness of $\varepsilon(1+N^2)^{2l_0}$ for $N \leq M_\varepsilon$

For $M > 0$, $s_0 > d/2$ define

$$\mathcal{U}_M^{(N)} := \{u \in C^2([0,\varepsilon_1] \times [-\varepsilon,\varepsilon], E_N); u(0,\tilde{\mu}) = 0, \\ ||u||_{s_0} \leq 1, ||\partial^j_{\varepsilon,\tilde{\mu}} u||_{s_0} \leq M, j = 1,2\}$$

For $V \in \mathcal{U}_M^{(N)}$ let us denote $\mathfrak{L}^{(N,V)}_{\varepsilon,\tilde{\mu}} =: \Pi_N \mathfrak{L}'_{\varepsilon,\tilde{\mu},V(\varepsilon,\tilde{\mu})} \Pi_N$,

$$\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}} = (\widetilde{\mathcal{A} - \lambda_0})_N + \widetilde{\mu}\mathbb{I}d + \mathfrak{B}'^{(N)}_{\varepsilon} + \varepsilon^2 \mathfrak{C}'^{(N)}_{\varepsilon,\widetilde{\mu}},$$

$$(\widetilde{\mathcal{A} - \lambda_0})_N =: \Pi_N \pi_0 \mathbf{Q}_0 (\mathcal{A} - \lambda_0) \mathbf{Q}_0 \pi_0 \Pi_N$$

Now define the selfadjoint operator

$$\mathfrak{L}^{(N,V)}_{\varepsilon,\widetilde{\mu}} \mathfrak{L}^{(N,V)*}_{\varepsilon,\widetilde{\mu}} = \widetilde{\mu}^2 \mathbb{I}d + \widetilde{\mathfrak{B}}^{(N)}_{\varepsilon} + \widetilde{\mathfrak{C}}^{(N)}_{\varepsilon,\widetilde{\mu}}$$

$$\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)}\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*} = \widetilde{\mu}^2\mathbb{I}d + \widetilde{\mathfrak{B}}_{\varepsilon}^{(N)} + \widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)}$$

$$\begin{aligned}
\widetilde{\mathfrak{B}}_{\varepsilon}^{(N)} &= (\widetilde{\mathcal{A}-\lambda_0})_N^2 + \mathfrak{B}_{\varepsilon}^{\prime(N)}(\widetilde{\mathcal{A}-\lambda_0})_N + (\widetilde{\mathcal{A}-\lambda_0})_N\mathfrak{B}_{\varepsilon}^{\prime(N)*} + \\
&\quad +\mathfrak{B}_{\varepsilon}^{\prime(N)}\mathfrak{B}_{\varepsilon}^{\prime(N)*}, \\
\widetilde{\mathfrak{C}}_{\varepsilon,\widetilde{\mu}}^{(N)} &= \widetilde{\mu}[2(\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}_{\varepsilon}^{\prime(N)} + \mathfrak{B}_{\varepsilon}^{\prime(N)*}] + \\
&\quad +\varepsilon^2[(\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}_{\varepsilon}^{\prime(N)} + \widetilde{\mu}]\mathfrak{C}_{\varepsilon,\widetilde{\mu}}^{\prime(N)*} + \\
&\quad +\varepsilon^2\mathfrak{C}_{\varepsilon,\widetilde{\mu}}^{\prime(N)}[(\widetilde{\mathcal{A}-\lambda_0})_N + \mathfrak{B}_{\varepsilon}^{\prime(N)*} + \widetilde{\mu}] + \varepsilon^4\mathfrak{C}_{\varepsilon,\widetilde{\mu}}^{\prime(N)}\mathfrak{C}_{\varepsilon,\widetilde{\mu}}^{\prime(N)*},
\end{aligned}$$

Good set of $\tilde{\mu}$ (continued)

For $V \in \mathcal{U}_M^{(N)}$, then define the "good" set of $\tilde{\mu}$:

$$G_{\varepsilon,\gamma}^{(N)}(V) \quad =: \{\tilde{\mu} \in [-\varepsilon, \varepsilon];\ \text{for all } v \in E_N,$$
$$||\Pi_N \mathfrak{L}'_{\varepsilon,\tilde{\mu},V} \Pi_N)^{-1} v||_{s_0} \leq \frac{N^\tau}{\gamma} ||v||_{s_0}\}.$$

Consequence: if $\tilde{\mu} \in G_{\varepsilon,\gamma}^{(N)}(V)$, then $\mathfrak{L}_{\varepsilon,\tilde{\mu}}^{(N,V)} \mathfrak{L}_{\varepsilon,\tilde{\mu}}^{(N,V)*}$ has all its eigenvalues $\geq (\frac{\gamma}{N^\tau})^2$ in E_N.
Notice that $dim(E_N) = \mathcal{N} \leq bN^d$.

Bad set of $\tilde{\mu}$

For $V \in \mathcal{U}_M^{(N)}$ define the "bad" set of $\tilde{\mu}$

$$B_{\varepsilon,\gamma}^{(N)}(V) \quad =: \{\tilde{\mu} \in [-\varepsilon, \varepsilon];\ \text{there exists at least one eigenvalue } \sigma_j \text{ of } \mathfrak{L}_{\varepsilon,\tilde{\mu}}^{(N,V)} \mathfrak{L}_{\varepsilon,\tilde{\mu}}^{(N,V)*},\ \text{such that } 0 \leq \sigma_j \leq (\frac{\gamma}{N^\tau})^2\}.$$

the eigenvalues of $\mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)} \mathfrak{L}_{\varepsilon,\widetilde{\mu}}^{(N,V)*}$ take the form

$$\sigma_j(\varepsilon, \widetilde{\mu}) = \widetilde{\mu}^2 + f_j(\varepsilon, \widetilde{\mu}),\ f_j \text{ is } C^2 \text{ in } \widetilde{\mu}, \text{ and}$$

$$|f_j(\varepsilon_2, \widetilde{\mu}_2) - f_j(\varepsilon_1, \widetilde{\mu}_1)| \leq c(\delta_0' + \varepsilon)(|\varepsilon_2 - \varepsilon_1| + |\widetilde{\mu}_2 - \widetilde{\mu}_1|.$$

Assumption For $\widetilde{\mu} \in [-\varepsilon, \varepsilon]$, there exists $0 < k < 2$ with

$$|\partial_{\widetilde{\mu}} f_j(\varepsilon, \widetilde{\mu}_2) - \partial_{\widetilde{\mu}} f_j(\varepsilon, \widetilde{\mu}_1)| \leq k|\widetilde{\mu}_2 - \widetilde{\mu}_1|.$$

Lemma

Assume that $N > M_\varepsilon$, $d/2 < s_0$, $(\varepsilon, \tilde{\mu}) \in (0, \varepsilon_1] \times [-\varepsilon, \varepsilon]$, and $V \in \mathcal{U}_M^{(N)}$. Then there exists $C > 0$, such that the measure of $B_{\varepsilon,\gamma}^{(N)}(V)$ is bounded by $C\gamma/N^{\tau-d}$

Bad set of $\tilde{\mu}$ (proof of the bound)

Notice that in the Assumption above, we accept to loose some regularity for the second derivative of σ_j with respect to $\widetilde{\mu}$. Up to now we have no mean to control this loss.

For a "bad" $\widetilde{\mu}$, there exists j such that

$$0 \leq \widetilde{\mu}^2 + f_j(\varepsilon, \widetilde{\mu}) < \eta^2, \eta = \gamma/(N^\tau)$$

$$\text{define } \phi_\varepsilon(\widetilde{\mu}) =: \widetilde{\mu}^2 + f_j(\varepsilon, \widetilde{\mu})$$

$$\partial_{\widetilde{\mu}} \phi_\varepsilon(\widetilde{\mu}) = 2\widetilde{\mu} + \partial_{\widetilde{\mu}} f_j(\varepsilon, \widetilde{\mu})$$

increasing function of $\widetilde{\mu}$, cancelling at a unique $\widetilde{\mu} = \widetilde{\mu}_m$.

Since $\phi_\varepsilon(\widetilde{\mu}^\pm) = \eta^2$, then the convexity of ϕ_ε implies

$$\widetilde{\mu}^+ - \widetilde{\mu}^- \leq \frac{2\eta}{\sqrt{(1 - k/2)}}$$

Summing up for all eigenvalues, the measure of the set of bad $\widetilde{\mu}$ is bounded by

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$$\frac{2b\gamma}{\sqrt{(1 - k/2)}N^{\tau - d}}$$

Estimate of $(\Pi_N \mathfrak{L}'_{\varepsilon,\tilde{\mu},V}\Pi_N)^{-1}$ in $\Pi_N\pi_0\mathbf{Q}_0\mathcal{K}_{s_0}$

Lemma

Let $d = 2(l_0+1)$ be the dimension of the $\mathbb{Q}$- vector space spanned by the wave vectors $k_j, j = 1, ..., 2q$, and $\tau > d + 2 + 24l_0$. Let N be ≥ 1. Assume moreover that $0 < \gamma \leq \widetilde{\gamma} = \frac{c'}{c2^{2l_0+1}}$, $(\epsilon, \widetilde{\mu}, V) \in [0, \epsilon_1] \times [-\varepsilon, \varepsilon] \times \mathcal{U}_M^{(N)}$ with $\widetilde{\mu} \in G_{\varepsilon,\gamma}^{(N)}(V), \epsilon_1$ small enough. For $s_0 > \frac{d}{2}$, there exists $c' > 0$ independent of N and γ, such that for any $v \in E_N$, we have

$$||(\Pi_N \mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}v||_{s_0} \leq c'\frac{N^\tau}{\gamma}||v||_{s_0}$$

and the same estimate holds for $(\Pi_N \mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$.

We need an estimate in all $\mathcal{K}_s$, with an exponent on N independent of s. We may proceed as for the Swift-Hohenberg PDE, in adapting Bourgain 1995, Craig 2000, bert-Bolle 2010.

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Singular set in $\mathbb{Z}^d$:
$S(N) := \{\mathbf{z} \in \Gamma(N); (\lambda_0 - \lambda_0(|\mathbf{k}(\mathbf{z})|^2) < \rho, \mathbf{k}(\mathbf{z}) \in \Gamma(N)\}$ with

$$\mathbf{k}(\mathbf{z}) = \mathfrak{d}^{-1} \sum_{s=1}^{d} z_s \mathbf{k}_s^*, \ \mathbf{z} = (z_1, ..., z_s) \in \mathbb{Z}^d$$
$$\Gamma(N) := \{\mathbf{z} \in \mathbb{Z}^d; \ 0 \leq |\mathbf{z}| \leq N, \ \mathbf{k}(\mathbf{z}) \in \Gamma \backslash \{\mathbf{k}_j, j = 1, ..., 2q\}\}.$$

Useful lemma (uses Bourgain 1995, Craig 2000, Berti-Bolle 2010)
There exists $\rho_0 > 0$ independent of N such that if $\rho \in]0, \rho_0]$ then $S(N) = \bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha$ is a union of disjoint clusters Ω_α satisfying :

- $(H1)$, for all $\alpha \in \mathcal{A}$, $M_\alpha \leq 2m_\alpha$ where $M_\alpha = \max_{\mathbf{z} \in \Omega_\alpha} |\mathbf{z}|$ and $m_\alpha = \min_{\mathbf{z} \in \Omega_\alpha} |\mathbf{z}|$;
- $(H2)$, there exists $\delta = \delta(d) \in]0, 1[$ independent of N such that if $\alpha, \beta \in \mathcal{A}, \alpha \neq \beta$ then $\mathrm{dist}(\Omega_\alpha, \Omega_\beta) := \min_{\mathbf{z} \in \Omega_\alpha, \mathbf{z}' \in \Omega_\beta} |\mathbf{z} - \mathbf{z}'| \geq \frac{(M_\alpha + M_\beta)^\delta}{2}$

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Estimate of $(\Pi_N \mathfrak{L}'_{\varepsilon,\tilde{\mu},V}\Pi_N)^{-1}$ in $\Pi_N\pi_0\mathbf{Q}_0\mathcal{K}_s$ for all $s\in[s_0,\overline{s}]$

Lemma

Let $d=2(l_0+1)$ be the dimension of the $\mathbb{Q}$- vector space spanned by the wave vectors $k_j, j=1,...,2q$, and $\tau>d+2+24l_0$ as in previous Lemma. Assume moreover that $0<\gamma\leq\widetilde{\gamma}=\frac{c'}{c2^{2l_0+1}}$, and $(\epsilon,\widetilde{\mu},V)\in[0,\epsilon_1]\times[-\varepsilon,\varepsilon]\times\mathcal{U}_M^{(N)}$, $\widetilde{\mu}\in\mathcal{G}_{\epsilon,\gamma}^{(N)}(V)=\cap_{K\leq N}G_{\varepsilon,\gamma}^{(K)}(V)$, ϵ_1 small enough. There exists $s_0(d,\delta,\tau)>\frac{d}{2}$ where δ is the number introduced in separation property (H2), and let $\overline{s}>s_0$. There exists $m(d,\delta,\tau)$ such that for all $s\in[s_0,\overline{s}]$ there exists $K(s)>0$ such that for any $h\in\Pi_N\pi_0Q_0\mathcal{K}_{0,s}$, we have

$$||(\Pi_N\mathfrak{L}'_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}h||_s\leq K(s)\frac{N^m}{\gamma}(||h||_s+||V(\epsilon,\widetilde{\mu})||_s||h||_{s_0}),$$

and the same estimate holds for $(\Pi_N\mathfrak{L}_{\varepsilon,\widetilde{\mu},V(\varepsilon,\widetilde{\mu})}\Pi_N)^{-1}$.

Main ingredient for applying the Nash-Moser iteration process.

Resolution of the Range equation

Uses Nash-Moser method, following Berti-Bolle-Procesi 2010 with a complement for having a solution $v \in \mathcal{K}_{s_0}$ of the range equation, which is C^2.

$$\mathcal{F}(\epsilon, \tilde{\mu}, v) =: \mathfrak{L}_{\varepsilon,\tilde{\mu}} v + g(\varepsilon, \tilde{\mu}) - \varepsilon^4 \mathbf{Q}_0 \mathcal{B}(v, v) = 0,$$

Theorem

Let s_0 and $\tilde{\gamma}$ be as above. Then for all $0 < \gamma < \tilde{\gamma}$ there exist $\epsilon_2(\gamma) \in [0, \epsilon_0]$ and a C^2-map $V : (0, \epsilon_2(\gamma)) \times [-\varepsilon, \varepsilon] \to \Pi_N \pi_0 Q_0 \mathcal{K}_{s_0}$, such that $V(0, \widetilde{\mu}) = 0$, $||V||_{s_0} \leq 1$, $||\partial_{\widetilde{\mu}} V||_{s_0} \leq M$, $||\partial^2_{\widetilde{\mu}} V||_{s_0} \leq M$, and if $\epsilon \in (0, \epsilon_2(\gamma))$, $\widetilde{\mu} \in ([-\varepsilon, \varepsilon] \setminus C_{\epsilon,\gamma})$, the function $V(\epsilon, \widetilde{\mu})$ is solution of the range equation $\mathcal{F}(\epsilon, \tilde{\mu}, v) = 0$. Here $C_{\epsilon,\gamma}$ is a subset of $[-\varepsilon, \varepsilon]$ which is Hlder continuous in ε, and has Lebesgue-measure less than $C\gamma\epsilon^6$ for some constant $C > 0$ independent of ϵ and γ.

Hint: Proof adapted from Berti, Bolle, Procesi 2010.

Resolution of the Bifurcation equation

$$\widetilde{\mu} - \varepsilon^4 \mu_4 + \mathcal{O}[\varepsilon^3(\varepsilon + ||\tilde{v}||)^2] = 0$$

$$\tilde{v} = V(\varepsilon, \widetilde{\mu}) - h(\varepsilon, \widetilde{\mu})$$

We solve this bifurcation equation with respect to $\widetilde{\mu}$:

$$\widetilde{\mu} = \epsilon^4 \mu_4 + \varepsilon^5 \tilde{h}(\varepsilon), \text{ "curve" (H) }, \tilde{h} \in C^1$$

We need to satisfy that $(\varepsilon, \tilde{\mu})$ lies in the good set, defined in the Theorem above.

Bad Strips

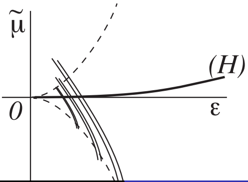
Definition

For N and V fixed, a set of "bad intervals" is defined by

$$BS_N(V) = \{(\varepsilon, \tilde{\mu}) \in [0, \varepsilon_2] \times [-\varepsilon^3, \varepsilon^3]; \tilde{\mu} \in I_\varepsilon^{(N)}\},$$

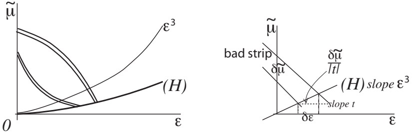
where $I_\varepsilon^{(N)}$ is one of the intervals $(\tilde{\mu}_j^-(\varepsilon), \tilde{\mu}_j^+(\varepsilon))$,or with one of the bounds replaced by $\pm\varepsilon^3$, as defined above.

$BS_N(V)$ is a union of thin Hölder strips in the plane $(\varepsilon, \widetilde{\mu})$. For the proof of the range theorem, we choose $\widetilde{\mu}$ outside of $\cup_{n\in\mathbb{N}} BS_{N_n}(V_{n-1})$ where $N_n = [N_0(\gamma)]^{2^n}$, and V_n are the successive points in the Newton iteration process.



Resolution of the Bifurcation equation- Transversality condition

We need a "transversality condition" to obtain a bound for the bad set of ε corresponding to the intersection of $\cup_{\varepsilon\in(0,\varepsilon_2)} C_{\varepsilon,\gamma}$ with (H):
The slopes $t(\varepsilon)$ of the curves $\tilde{\mu}_j^{\pm}(\varepsilon)$, are such that there exists $c > 0$ independent of (N, ε), with $c\varepsilon^2 < |t(\varepsilon)|$.



Sketch of the "bad set" in the plane $(\varepsilon, \tilde{\mu})$ and its intersection by the "line" (H) given by the bifurcation equation.
The drawing on the right side explains the bound for the measure of $\delta\varepsilon \leq \delta\tilde{\mu}/c\varepsilon^2 \leq C\varepsilon^4$.
Notice that, as for the Swift-Hohenberg equation, we can weaken this transversality condition, so that the true hypothesis is that the curves $\tilde{\mu}_j^{\pm}(\varepsilon)$ are not flat.

Final result

Theorem

Let $q \geq 4$ be an integer. Assume that Hypothesis $\lambda_0'' \neq 0$ holds and that transversality condition above is verified. Moreover assume the convexity condition above, on small eigenvalues σ_j. Then, there exists $s_0 > d/2$, $\varepsilon_0 > 0$, such that, for any $s \geq s_0$, there exists a 1-dimensional set $\overline{\Lambda}_\varepsilon$ centered on μ_4, with the following property : for any $|\varepsilon| < \varepsilon_0$, belonging to a set of asymptotically full measure as $\varepsilon \to 0$ there exist $\overline{\mu}_\varepsilon \in \overline{\Lambda}_\varepsilon$, such that the steady Bénard - Rayleigh system admits a quasipattern solution (u, λ), C^1 in ε, $u \in \mathcal{K}_s$, $\lambda = \lambda_0 - \mu_2\varepsilon^2 - \mu_3\varepsilon^3 - \varepsilon^4\overline{\mu}_\varepsilon$ invariant under rotations of angle π/q of the form

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \varepsilon^4 u_4 + \mathcal{O}(\varepsilon^5),$$

where $\mu_2 > 0$, and coefficients μ_2, μ_4, u_j occurring in formulae above, are the ones defined in the truncated asymptotic expansion of the solution.

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