Quasipatterns in steady Bénard-Rayleigh convection

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Abstract

Quasipatterns in the steady Bénard-Rayleigh convection problem are considered. These are two-dimensional patterns, quasiperiodic in any horizontal direction, invariant under horizontal rotations of angle $2\pi/Q$. As with problems involving quasiperiodicity, there is a small divisor problem. In this paper, we consider all cases with an even number $Q \geq 8$. We prove that a formal solution, given by a divergent series, may be used to build a smooth quasiperiodic convection solution which is an approximate solution of the Bénard-Rayleigh system, up to an exponentially small error.

Keywords: Rayleigh-Bénard convection, bifurcations, quasipattern, small divisors, Gevrey series

AMS: 35B32, 35C20, 40G10, 52C23

1 Introduction

Quasipatterns are two-dimensional patterns which have no translation symmetries and are quasiperiodic in any spatial direction (see figure 1). The spatial Fourier transforms of quasipatterns have discrete rotational order (most often, 8, 10 or 12-fold) and were first discovered in nonlinear pattern-forming systems in the Faraday wave experiment [3, 5], in which a layer of fluid is subjected to vertical oscillation. Since their discovery, they have also been in particular observed, in shaken convection [15, 12].

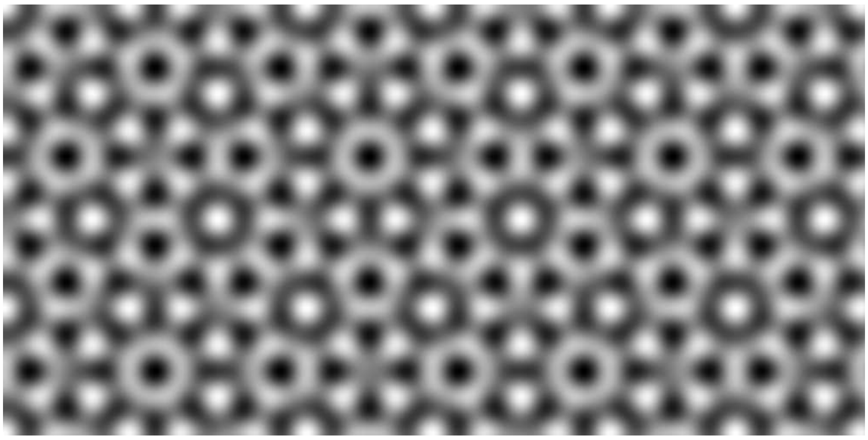

Figure 1: Example 8-fold quasipattern. This is an approximate solution of the Swift–Hohenberg equation, see [7].

In many of these experiments, the domain is large compared to the size of the pattern, and the boundaries appear to have little effect. Furthermore, the pattern is usually formed in two directions (x and y), while the third direction (z) plays little role. Mathematical models of the experiments are therefore often posed with two unbounded directions, and the basic symmetry of the problem is the Euclidean group of rotations, translations and reflections of the (x, y) plane. This is in particular the case for the studies made in the works [13], [14] and [7].

Quasipatterns do not fit into any spatially periodic domain and have Fourier expansions with wavevectors that live on a *quasilattice* (defined below). At the onset of pattern formation, the primary modes have zero growth rate, and there are other modes on the quasilattice which have growth rates arbitrarily close to zero, and techniques (like Lyapunov-Schmidt reduction, or center manifold reduction) which are used for periodic patterns cannot be applied. These small growth rates appear as *small divisors*, as seen below.

This paper strongly relies on the paper [7] dealing with the Swift-Hohenberg PDE. It is known that this PDE is a simple model of Bénard-Rayleigh convection for the bifurcation to a steady convective regime. In the present paper we solve the same problem but ruled by the full Boussinesq equations which are usually taken for the study of Bénard-Rayleigh convection between two horizontal planes. Section 2 establishes the classical Boussinesq system, section 3 defines quasilattices and related useful algebraic results, section 4 defines the function spaces and operators used to rewrite in the suitable form the Boussinesq system in section 5. Also in section 5 we study in details the linearized operator, and the criticality conditions. Section 6 gives the way on how to compute the expansion of the formal series, solution in powers of the amplitude ε. Section 7 provides, in all cases, Gevrey estimates for this series (see Theorem 7.1), and section 8 gives the *equation exactly solved by the Borel transform* of the Gevrey series previously obtained. Then, taking an approximate inverse of this equation, we prove the main result (see Theorem 9.1) which is that for any even order $Q \geq 8$, there exists, above the convection threshold, a bifurcating spatially quasiperiodic pattern of order Q, solution up to an exponentially small term, of the Boussinesq system.

2 The Bénard - Rayleigh convection problem

Consider a viscous fluid filling the region between two horizontal planes. Each planar boundary may be a rigid plane, or a "free" boundary. In addition, we assume that the lower and upper planes are at temperatures T_0 and T_1, respectively, with $T_0 > T_1$. The difference of temperature between the two planes modifies the fluid density, tending to place the lighter fluid below the heavier one. The gravity then induces, through the Archimedian force, an instability of the "conduction regime" where the fluid is at rest, while the temperature depends linearly on the vertical coordinate z. This instability is prevented up to a certain level by viscosity ν, so that there is a critical value of the temperature difference, below which nothing happens and above which a steady "convective regime" bifurcates.

The Navier-Stokes momentum equation needs to be completed by an equation for energy conservation. In the Boussinesq approximation, the dependency of the density ρ in function of the temperature T, reads

$$\rho = \rho_0 \left(1 - \alpha(T - T_0)\right),$$

where α is the (constant) volume expansion coefficient, is taken into account in the momentum equation, only in the external volumic gravity force $-\rho g e_z$, introducing a coupling between the particles velocity, and pressure (V, p) and T. We refer to [8, Vol. II] for a very complete discussion and bibliography on various geometries and boundary conditions in this problem.

Several different scalings are used in literature. We are only considering *steady solutions*, so we adopt here the formulation derived in [9], which leads to the following system

$$\begin{aligned} V\cdot\nabla V+\nabla p &= \mathcal{P}(\theta e_z+\Delta V), \\ V\cdot\nabla\theta &= \Delta\theta+\mathcal{R}(V\cdot e_z), \\ \nabla\cdot V &= 0. \end{aligned} \tag{1}$$

Here θ is the deviation of the temperature from the conduction profile, which satisfies the boundary conditions, and $V=(V^{(H)},v_z)$, $V^{(H)}=(v_1,v_2)$, p, and θ are functions of $X=(\mathbf{x},z)$, with $\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$ the horizontal coordinates and $z\in(0,1)$ the vertical coordinate, e_z being the unitary ascendent vector. There are two dimensionless constant numbers in this problem: the Prandtl number $\mathcal{P}$ and the Rayleigh number $\mathcal{R}$ defined as

$$\mathcal{P}=\frac{\nu}{\kappa},\quad \mathcal{R}=\frac{\alpha g d^3(T_0-T_1)}{\nu\kappa},$$

where d is the distance between the planes, κ is the thermal diffusivity. The system (1) is completed by the boundary conditions

$$v_z=\theta=0,\quad z=0,1,$$

together with either a "rigid surface" condition

$$v_1=v_2=0, \tag{2}$$

or a "free surface" condition (in fact no tangential stress condition)

$$\frac{\partial v_1}{\partial z}=\frac{\partial v_2}{\partial z}=0, \tag{3}$$

on the planes $z=0$ or $z=1$.

Our next task is to formulate the problem ruled by the system (1) in a suitable function space, and find critical values of the parameters, for being able to use a method similar to the one in [7].

3 Quasilattices and Diophantine bounds

Let an even number $Q\geq 8$, be the order of a quasipattern and define wavevectors

$$\mathbf{k}_j=k_c\left(\cos\left(2\pi\frac{j-1}{Q}\right),\sin\left(2\pi\frac{j-1}{Q}\right)\right)=R_{2(j-1)\pi/Q}\mathbf{k}_1,\qquad j=1,2,\dots,Q$$

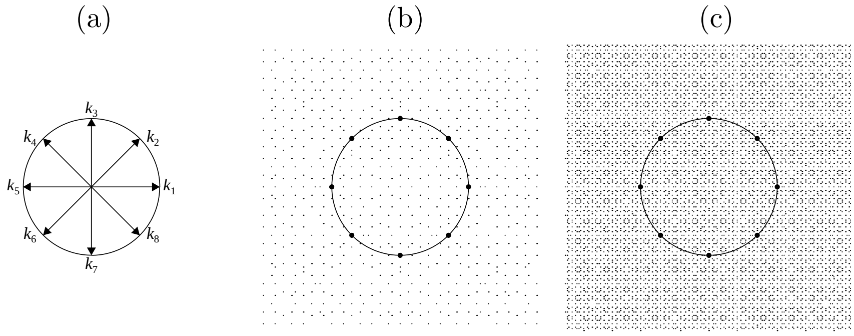

Figure 2: Example of quasilattice with $Q = 8$, after [13]. (a) The 8 wavevectors with $|\mathbf{k}| = 1$ which form the basis of the quasilattice. (b,c) The truncated quasilattices Γ_9 and Γ_{27}. The small dots mark the positions of combinations of up to 9 or 27 of the 8 basis vectors on the unit circle.

where k_c is a positive number which is defined later, and R_θ is the rotation of angle θ around the vertical axis (see figure 2a). We define the *quasilattice* $\Gamma \subset \mathbb{R}^2$ to be the set of points spanned by integer combinations $\mathbf{k_m}$ of the form

$$\mathbf{k_m} = \sum_{j=1}^{Q} m_j \mathbf{k}_j, \qquad \text{where} \qquad \mathbf{m} = (m_1, m_2, \ldots, m_Q) \in \mathbb{N}^Q. \tag{4}$$

The set Γ is dense in $\mathbb{R}^2$. Since Q is even, $\mathbf{k}_j$ and $-\mathbf{k}_j$ belong to Γ, hence $\mathbf{k_m}$ and $-\mathbf{k_m}$ are both in Γ. The evenness of Q allows to obtain real quantities of the form

$$\sum_{\mathbf{k}\in\Gamma} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^2$$

provided that

$$\overline{u_{-\mathbf{k}}} = u_{\mathbf{k}}.$$

Define $|\mathbf{m}| = \sum_j m_j$, then, for a given wavevector $\mathbf{k} \in \Gamma$, we define the order $N_{\mathbf{k}}$ of $\mathbf{k}$ by

$$N_{\mathbf{k}} = \min\{|\mathbf{m}|; \mathbf{k} = \mathbf{k_m}\}.$$

Notice that there is an infinite set of $\mathbf{m}$'s satisfying $\mathbf{k} = \mathbf{k_m}$. For example, we could increase m_j and $m_{j+Q/2}$ by 1: this increases $|\mathbf{m}|$ by 2 but does not change $\mathbf{k_m}$. In general we have

$$N_{\mathbf{k}} \le \sum_{j=1}^{Q/2} |m'_j|, \ m'_j = m_j - m_{j+Q/2}.$$

The above inequality can occur strictly (for example) in the case $Q = 12$, because only 4 of the 12 vectors $\mathbf{k}_j$ are rationally independent in this case. Whenever solutions are computed numerically, it is necessary to use only a finite number of Fourier modes, so we define the *truncated quasilattice* Γ_N to be:

$$\Gamma_N = \{\mathbf{k} \in \Gamma : N_{\mathbf{k}} \leq N\} . \tag{5}$$

Figure 2(b,c) shows the truncated quasilattices Γ_9 and Γ_{27} in the case $Q = 8$.

In the calculations that follow, we will require diophantine bounds on the magnitude of the small divisors in terms of $N_{\mathbf{k}}$. We see below that the small divisors are of the form $\left||\mathbf{k}|^2 - k_c^2\right|$, for $\mathbf{k} \in \Gamma$. To compute the required bound, we start with

$$|\mathbf{k_m}|^2 = \sum_{1\leq j_1<j_2\leq Q} 2m_{j_1}m_{j_2}k_c^2 \cos\frac{2\pi}{Q}(j_1 - j_2) + \sum_{1\leq j\leq Q} m_j^2 k_c^2.$$

Defining the *algebraic number* ω by

$$\omega = 2\cos\frac{2\pi}{Q},$$

the expression $2\cos\frac{2p\pi}{Q}$ is a polynomial in ω with integer coefficients (proof by induction) only depending on Q, for $1 \leq p \leq Q - 1$, given by

$$2\cos\frac{2p\pi}{Q} = \omega^p - p\omega^{p-2} + \frac{p(p-3)}{2}\omega^{p-4}.....$$

The algebraic number ω is solution of a polynomial $P(\omega)$ of degree $l + 1 \leq Q/2$ with integer coefficients. In the cases $Q = 8$, 10 and 12, the irrational numbers $\omega = 2\cos(2\pi/Q)$ are $\sqrt{2}$, $\frac{1+\sqrt{5}}{2}$ and $\sqrt{3}$: these are quadratic algebraic numbers ($l + 1 = 2$), while for $Q = 14$, ω is cubic ($l + 1 = 3$). For an algebraic number ω of order $l+1$, the quantity $|q_0 + \omega q_1 + \cdots + \omega^l q_l|$ may be as small as we want for good choices of large integers q_j. Moreover, dividing (if necessary) by $P(\omega)$, we obtain

$$\frac{|\mathbf{k_m}|^2 - k_c^2}{k_c^2} = q_0 + \omega q_1 + \cdots + \omega^l q_l \tag{6}$$

where $q_0 + 1$ and q_j, $j = 1, \ldots, l$ are integer-valued *quadratic forms* of $\mathbf{m}$.

We have now a diophantine bound valid for any even $Q \geq 8$: there exists $c > 0$ depending only on Q, such that for any $\mathbf{k} \in \Gamma$, with $|\mathbf{k}| \neq 1$, there exists an integer $l \geq 1$ such that

$$\left||\mathbf{k}|^2 - k_c^2\right| \geq \frac{c}{N_{\mathbf{k}}^{2l}}. \tag{7}$$

To show this, we use the following known result (for example see [4]) also proved in Appendix in [7].

Lemma 3.1 *Let ω be an algebraic number of order $l+1$, that is, a solution of $P(\omega)=0$ where P is a polynomial of degree $l+1$ with integer coefficients, which is irreducible on $\mathbb{Q}$. Then, there exists a constant C such that for any $\mathbf{q}=(q_0,q_1,\dots,q_l)\in\mathbb{Z}^{l+1}\backslash\{0\}$, the following estimate*

$$|q_0+q_1\omega+q_2\omega^2+\cdots+q_l\omega^l|\geq\frac{C}{|\mathbf{q}|^l} \tag{8}$$

holds, where $|\mathbf{q}|=\sum_{0\leq j\leq l}|q_j|$.

As an aside, the polynomials P are related to cyclotomic polynomials, and the order $l+1$ of the algebraic number ω is $\varphi(Q)/2$, where $\varphi(Q)$ is Euler's Totient function, the number of positive integers $j<Q$ such that j and Q are relatively prime. For example, $\varphi(14)=6$ since the 6 numbers 1, 3, 5, 9, 11 and 13 have no factors in common with 14, and so $l+1=3$ in the case $Q=14$.

Since in (6) the coefficients q_j are quadratic in $\mathbf{m}$, we have the estimate

$$|\mathbf{q}|\leq c(Q)N_{\mathbf{k}}^2$$

where $c(Q)$ depends only on Q. Estimate (7) is then satisfied by taking

$$c=\frac{Ck_c^2}{[c(Q)]^l}.$$

4 Function spaces and operators

We characterise the functions of interest by their Fourier coefficients on the quasilattice Γ generated by the Q unit vectors $\mathbf{k}_j$:

$$u(\mathbf{x})=\sum_{\mathbf{k}\in\Gamma}u_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}},\quad \mathbf{x}\in\mathbb{R}^2.$$

Recall that for each $\mathbf{k}\in\Gamma$, there exists a vector $\mathbf{m}\in\mathbb{N}^Q$ such that $\mathbf{k}=\mathbf{k}_{\mathbf{m}}=\sum_{j=1}^{Q}m_j\mathbf{k}_j$ and we can choose $\mathbf{m}$ such that

$$|\mathbf{m}|=N_{\mathbf{k}}=\min\{|\mathbf{m}|:\mathbf{k}=\mathbf{k}_{\mathbf{m}}\}.$$

Define now the space of scalar functions

$$\mathcal{H}_s=\left\{u=\sum_{\mathbf{k}\in\Gamma}u_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}\,:\,||u||_s^2=\sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}{}^2)^s|u_{\mathbf{k}}|^2<\infty\right\}, \tag{9}$$

which becomes a Hilbert space with the scalar product

$$\langle w, v\rangle_s = \sum_{\mathbf{k}\in\Gamma}(1+N_{\mathbf{k}}{}^2)^s w_{\mathbf{k}}\overline{v}_{\mathbf{k}}. \tag{10}$$

In the sequel we use the following lemma, proved in the Appendix of [7]:

Lemma 4.1 *The space $\mathcal{H}_s$ is a Banach algebra for $s > Q/4$. In particular there exists $c_s > 0$ such that*

$$||uv||_s \leq c_s ||u||_s ||v||_s. \tag{11}$$

For $\ell \geq 0$ and $s > \ell + Q/4$, $\mathcal{H}_s$ is continuously embedded into $\mathcal{C}^\ell$.

In fact we need more complicate function spaces for our system (1). This is due to the necessity to control in terms of $|\mathbf{k}|$ the gain of regularity provided by the inverse of the linear operator on the complementary space of its kernel (here, contrary to [7], the nonlinear term looses one derivative), hence the inverse of the linear operator is used to regain this loss (for large $|\mathbf{k}|$), while the loss due to the small divisor problem (for $|\mathbf{k}|$ close to k_c) is in terms of $N_{\mathbf{k}}$.

4.1 Projection Π

First we define a projection operator Π on divergence free vector fields. Let consider a vector field $V(\mathbf{x}, z)$ under the form

$$V(\mathbf{x}, z) = \sum_{\mathbf{k}\in\Gamma} V_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}},$$

which, for a fixed z belongs to $(\mathcal{H}_s)^3$. Then we consider the system

$$\begin{aligned} W_{\mathbf{k}}^{(H)} + i\mathbf{k}\phi_{\mathbf{k}} &= V_{\mathbf{k}}^{(H)}, \\ w_{\mathbf{k}}^{(z)} + \frac{d\phi_{\mathbf{k}}}{dz} &= v_{\mathbf{k}}^{(z)}, \\ i\mathbf{k}\cdot W_{\mathbf{k}}^{(H)} + \frac{dw_{\mathbf{k}}^{(z)}}{dz} &= 0, \end{aligned} \tag{12}$$

where $V_{\mathbf{k}} = (V_{\mathbf{k}}^{(H)}, v_{\mathbf{k}}^{(z)})$, $V_{\mathbf{k}}^{(H)}$ being the horizontal component of $V_{\mathbf{k}}$, and where we want to satisfy the boundary condition

$$w_{\mathbf{k}}^{(z)}|_{z=0,1} = 0, \tag{13}$$

for the unknown vector field $W_{\mathbf{k}} = (W_{\mathbf{k}}^{(H)}, w_{\mathbf{k}}^{(z)})$. We then obtain the following equation for $\phi_{\mathbf{k}}$:

$$\begin{aligned} \frac{d^2\phi_{\mathbf{k}}}{dz^2} - |\mathbf{k}|^2\phi_{\mathbf{k}} &= i\mathbf{k}\cdot V_{\mathbf{k}}^{(H)} + \frac{dv_{\mathbf{k}}^{(z)}}{dz}, \\ \frac{d\phi_{\mathbf{k}}}{dz}|_{z=0,1} &= v_{\mathbf{k}}^{(z)}|_{z=0,1}. \end{aligned} \tag{14}$$

For $\mathbf{k} \neq \mathbf{0}$, it is well known that, if $V_{\mathbf{k}}^{(H)} \in \{L^2(0,1)\}^2$, $v_{\mathbf{k}}^{(z)} \in H^1(0,1)$, then there is a unique solution $\phi_{\mathbf{k}} \in H^2(0,1)$ of this Neumann problem, which satisfies the estimates

$$|\mathbf{k}|^2||\phi_{\mathbf{k}}||^2 + \left\|\frac{d\phi_{\mathbf{k}}}{dz}\right\|^2 \leq 2||V_{\mathbf{k}}||^2, \tag{15}$$

$$|\mathbf{k}|^4||\phi_{\mathbf{k}}||^2 + |\mathbf{k}|^2\left\|\frac{d\phi_{\mathbf{k}}}{dz}\right\|^2 + \left\|\frac{d^2\phi_{\mathbf{k}}}{dz^2}\right\|^2 \leq c\left\{\left\|\frac{dv_{\mathbf{k}}^{(z)}}{dz}\right\|^2 + |\mathbf{k}|^2||V_{\mathbf{k}}||^2\right\}. \tag{16}$$

In the case when $\mathbf{k} = \mathbf{0}$, we have $w_{\mathbf{0}}^{(z)} = 0$, $W_{\mathbf{0}}^{(H)} = V_{\mathbf{0}}^{(H)}$, and $\frac{d\phi_{\mathbf{0}}}{dz} = v_{\mathbf{0}}^{(z)}$ defines $\phi_{\mathbf{0}}$ up to a constant. Hence, this remark, with (15) and (16) lead to

$$\begin{aligned} ||W_{\mathbf{k}}||^2 &\leq c||V_{\mathbf{k}}||^2, \\ |\mathbf{k}|^2||W_{\mathbf{k}}||^2 + \left\|\frac{dW_{\mathbf{k}}}{dz}\right\|^2 &\leq c\left\{|\mathbf{k}|^2||V_{\mathbf{k}}||^2 + \left\|\frac{dV_{\mathbf{k}}}{dz}\right\|^2\right\}, \end{aligned} \tag{17}$$

for a constant c independent of $\mathbf{k} \in \Gamma$.

Definition 4.2 *The operator Π is the linear operator defined as*

$$V = \sum_{\mathbf{k}\in\Gamma} V_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}} \overset{\Pi}{\mapsto} W = \sum_{\mathbf{k}\in\Gamma} W_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $W_{\mathbf{k}}$ is solution of (12).

We notice that if V is divergence free and satisfies $v^{(z)}|_{z=0,1} = 0$ then Π acts as the identity. Hence the operator Π is a projection.

Remark 4.3 *Notice that when V is divergence free with $V_{\mathbf{k}} \in \{L^2(0,1)\}^3$, but does not satisfy the boundary condition $v^{(z)}|_{z=0,1} = 0$ (this has a meaning here since the divergence free condition leads to $v_{\mathbf{k}}^{(z)} \in H^1(0,1)$), then we still have $||W_{\mathbf{k}}||^2 \leq c||V_{\mathbf{k}}||^2$, even in just assuming $V_{\mathbf{k}}$ in $\{L^2(0,1)\}^3 \cap \{\nabla\cdot V = 0\}$.*

4.2 Function spaces

Let us define function spaces for the 4-components vector field $U = (V, \theta)$:

$$\mathcal{H}_{r,s} = \left\{ U = (V,\theta)(\mathbf{x},z) = \sum_{\mathbf{k}\in\Gamma} U_{\mathbf{k}}(z) e^{i\mathbf{k}\cdot\mathbf{x}}; \sum_{\mathbf{k}\in\Gamma} \left((1+N_{\mathbf{k}}^2)^s ||U_{\mathbf{k}}||_r^2 \right) < \infty \right\} \tag{18}$$

where

$$||U_{\mathbf{k}}||_r^2 = \sum_{0\le l\le r} |\mathbf{k}|^{2(r-l)} ||U_{\mathbf{k}}||_{H^l}^2.$$

Notice the following equivalence between (squared) norms in (18)

$$\sum_{0\le l\le r} |\mathbf{k}|^{2(r-l)} ||U_{\mathbf{k}}||_{H^l}^2 \sim \sum_{0\le l\le r} (1+|\mathbf{k}|^2)^{(r-l)} ||\frac{d^l U_{\mathbf{k}}}{dz^l}||_{L^2}^2.$$

The space $\mathcal{H}_{r,s}$ has a natural Hilbertian structure. For example, for U, $U' \in \mathcal{H}_{0,s}$, the scalar product reads

$$\langle U, U' \rangle_{0,s} = \sum_{\mathbf{k}\in\Gamma} \left((1+N_{\mathbf{k}}^2)^s \int_0^1 U_{\mathbf{k}} \cdot \overline{U'_{\mathbf{k}}} dz \right),$$

where $U_{\mathbf{k}} \cdot \overline{U'_{\mathbf{k}}}$ is the usual hermitian scalar product in $\mathbb{C}^4$. Now denoting $\Pi U = (\Pi V, \theta)$, we have the following

Lemma 4.4 *The projection Π is bounded in $\mathcal{H}_{r,s}$ for $r \ge 1$, and bounded in the subspace $\mathcal{H}'_{0,s}$ of $\mathcal{H}_{0,s}$ such that $\nabla \cdot V = 0$. For any U, $U' \in \mathcal{H}_{1,s}$, or $\mathcal{H}'_{0,s}$, we have*

$$\langle U, \Pi U' \rangle_{0,s} = \langle \Pi U, \Pi U' \rangle_{0,s}.$$

Remark 4.5 *The above Lemma means that $(\mathbb{I} - \Pi)\mathcal{H}_{1,s}$ is orthogonal to $\Pi\mathcal{H}_{1,s}$ with the scalar product of $\mathcal{H}_{0,s}$. In other words, Π is an orthogonal projection in $\mathcal{H}_{0,s}$ restricted to subspaces $\mathcal{H}_{1,s}$ and $\mathcal{H}'_{0,s}$.*

Proof. The boundedness of Π in $\mathcal{H}_{1,s}$ results immediately from (17), and in $\mathcal{H}'_{0,s}$ from the remark 4.3. For the boundedness in $\mathcal{H}_{r,s}$ for $r > 1$, this follows easily after differentiating (12) and (14). Now assume U, $U' \in \mathcal{H}_{1,s}$ or $\mathcal{H}'_{0,s}$, and define $\Pi U' = (V', \theta')$, then from the form of $V_{\mathbf{k}} - W_{\mathbf{k}}$ indicated in (12), we have

$$
\begin{aligned}
\langle(\mathbb{I}-\Pi)U,\Pi U'\rangle_{0,s} &= \sum_{\mathbf{k}\in\Gamma}\left((1+N_{\mathbf{k}}^2)^s\int_0^1\left(i\mathbf{k}\phi_{\mathbf{k}}\cdot\overline{V_{\mathbf{k}}'^{(H)}}+\frac{d\phi_{\mathbf{k}}}{dz}\overline{v_{\mathbf{k}}'^{(z)}}+0\right)dz\right)\\
&= \sum_{\mathbf{k}\in\Gamma}\left((1+N_{\mathbf{k}}^2)^s\int_0^1\phi_{\mathbf{k}}\left(i\mathbf{k}\cdot\overline{V_{\mathbf{k}}'^{(H)}}-\frac{d\overline{v_{\mathbf{k}}'^{(z)}}}{dz}\right)dz\right)\\
&= 0.
\end{aligned}
$$

■

5 New formulation of the convection problem

5.1 Operators $\mathcal{L}$ and $\mathcal{B}$

Definition 5.1 *We say that U satisfies* Condition ***b.c.*** *if one of the following boundary conditions are satisfied*

(i) $V^{(H)}|_{z=0,1}=0$ *(rigid-rigid),*

(ii) $V^{(H)}|_{z=0}=\frac{dV^{(H)}}{dz}|_{z=1}=0$ *(rigid - free),*

(iii) $\frac{dV^{(H)}}{dz}|_{z=0}=V^{(H)}|_{z=1}=0$ *(free - rigid),*

(iv) $\frac{dV^{(H)}}{dz}|_{z=0,1}=0$ *(free - free).*

Then, we define the following function spaces for r and s non-negative integers

$$
\begin{aligned}
\mathcal{K}_{r,s} &= \Pi\mathcal{H}_{r,s}=\{U=(V,\theta)\in\mathcal{H}_{r,s};\nabla\cdot V=0,v^{(z)}|_{z=0,1}=0\},\\
D_s(\mathcal{L}) &= \mathcal{K}_{2,s}\cap\{U \text{ satisfies Condition } \mathbf{b.c.,}\ \theta|_{z=0,1}=0\},
\end{aligned}
$$

and we put, respectively on these subspaces, the norms of $\mathcal{H}_{r,s}$ and $\mathcal{H}_{2,s}$.

Definition 5.2 *For any $U\in D_s(\mathcal{L})$, we define the linear operator $\mathcal{L}$ and quadratic operator $\mathcal{B}$ by*

$$
\begin{aligned}
\mathcal{L}U &= \left(\Pi(\Delta V+\theta e_z),\frac{1}{\mathcal{R}}\Delta\theta+V\cdot e_z\right),\\
\mathcal{B}(U,U) &= \left(\frac{1}{\mathcal{P}}\Pi(V\cdot\nabla V),\frac{1}{\mathcal{R}}V\cdot\nabla\theta\right).
\end{aligned}
$$

It is clear that $\mathcal{L}$ maps continuously $D_s(\mathcal{L})$ to $\mathcal{K}_{0,s}$. For $s>Q/4$ the quadratic operator maps continuously $D_s(\mathcal{L})$ to $\mathcal{K}_{1,s}$ as this results easily from the fact that $H^1(0,1)$ is an algebra, as well as $\mathcal{H}_s$ for $s > Q/4$ (see Lemma 4.1 and see Appendix A for the rest of the proof). This means that there exists $c(s)$ such that for any $U \in D_s(\mathcal{L})$, we have

$$||\mathcal{B}(U,U)||_{\mathcal{K}_{1,s}} \le c(s)||U||^2_{\mathcal{K}_{2,s}}. \tag{19}$$

Now solving the system (1) reduces to solving the equation

$$\mathcal{L}U = \mathcal{B}(U,U), \quad U \in D_s(\mathcal{L}). \tag{20}$$

Let us show the following useful simple properties of operators $\mathcal{L}$ and $\mathcal{B}$:

Lemma 5.3 *For U and $U' \in D_s(\mathcal{L})$, we have the identity*

$$\langle \mathcal{L}U, U'\rangle_{0,s} = \langle U, \mathcal{L}U'\rangle_{0,s}.$$

For $U, U' \in D_s(\mathcal{L})$ and U, U' real, i.e. $U = \overline{U}$, $U' = \overline{U'}$ we have

$$\langle \mathcal{B}(U,U), U\rangle_{0,0} = 0, \tag{21}$$

$$\langle 2\mathcal{B}(U,U'), U\rangle_{0,0} = -\langle \mathcal{B}(U,U), U'\rangle_{0,0}. \tag{22}$$

Proof. First we have, by using Lemma 4.4

$$\begin{aligned}
\langle \mathcal{L}U, U'\rangle_{0,s} &= \langle \left(\Pi(\Delta V + \theta e_z), \frac{1}{\mathcal{R}}\Delta\theta + V\cdot e_z\right), (V',\theta')\rangle_{0,s} \\
&= \langle \left(\Delta V, \frac{1}{\mathcal{R}}\Delta\theta\right), (V',\theta')\rangle_{0,s} + \langle (\theta e_z, V\cdot e_z), (V',\theta')\rangle_{0,s} \\
&= \langle \Delta V, V'\rangle_{0,s} + \frac{1}{\mathcal{R}}\langle \Delta\theta, \theta'\rangle_{0,s} + \langle \theta, v'^{(z)}\rangle_{0,s} + \langle v^{(z)}, \theta'\rangle_{0,s}.
\end{aligned}$$

Then we observe that $\langle \theta, v'^{(z)}\rangle_{0,s} + \langle v^{(z)}, \theta'\rangle_{0,s}$ is symmetric in (U, U'). Moreover by integrating by parts, since $\theta_{\mathbf{k}}|_{z=0,1} = 0$,

$$\begin{aligned}
\langle \Delta\theta, \theta'\rangle_{0,s} &= \sum_{\mathbf{k}\in\Gamma} (1+N^2_{\mathbf{k}})^s \int_0^1 \left(\frac{d^2\theta_{\mathbf{k}}}{dz^2} - |k|^2\theta_{\mathbf{k}}\right) \overline{\theta'_{\mathbf{k}}} dz \\
&= -\sum_{\mathbf{k}\in\Gamma} (1+N^2_{\mathbf{k}})^s \int_0^1 \left(\frac{d\theta_{\mathbf{k}}}{dz}\frac{d\overline{\theta'_{\mathbf{k}}}}{dz} + |k|^2\theta_{\mathbf{k}}\overline{\theta'_{\mathbf{k}}}\right) dz,
\end{aligned}$$

which is symmetric in (U, U'). The same computation holds by using the boundary conditions satisfied by V for $U \in D_s(\mathcal{L})$, and shows that $\langle \Delta V, V'\rangle_{0,s}$ is symmetric in (U, U').

In the same way we have

$$\langle \mathcal{B}(U,U),U\rangle_{0,s} = \frac{1}{\mathcal{P}}\langle V\cdot\nabla V,V\rangle_{0,s} + \frac{1}{\mathcal{R}}\langle V\cdot\nabla\theta,\theta\rangle_{0,s},$$

and by using $\overline{\theta_{\mathbf{p}+\mathbf{q}}} = \theta_{\mathbf{r}}$, when $\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0}$, since θ is real,

$$\begin{aligned}
\langle V\cdot\nabla\theta,\theta\rangle_{0,0} &= \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left((i\mathbf{q}\cdot V_{\mathbf{p}}^{(H)})\theta_{\mathbf{q}} + v_{\mathbf{p}}^{(z)}\frac{d\theta_{\mathbf{q}}}{dz}\right)\theta_{\mathbf{r}}dz \\
&= \sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left((i\mathbf{r}\cdot V_{\mathbf{p}}^{(H)})\theta_{\mathbf{r}} + v_{\mathbf{p}}^{(z)}\frac{d\theta_{\mathbf{r}}}{dz}\right)\theta_{\mathbf{q}}dz \\
&= \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left((-i\mathbf{p}\cdot V_{\mathbf{p}}^{(H)})\theta_{\mathbf{q}}\theta_{\mathbf{r}} + v_{\mathbf{p}}^{(z)}\frac{d(\theta_{\mathbf{q}}\theta_{\mathbf{r}})}{dz}\right)dz \\
&= \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \left(\frac{dv_{\mathbf{p}}^{(z)}}{dz}\theta_{\mathbf{q}}\theta_{\mathbf{r}} + v_{\mathbf{p}}^{(z)}\frac{d(\theta_{\mathbf{q}}\theta_{\mathbf{r}})}{dz}\right)dz = 0.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
\langle V\cdot\nabla V,V\rangle_{0,0} &= \langle V\cdot\nabla V^{(H)},V^{(H)}\rangle_{0,s} + \langle V\cdot\nabla v^{(z)},v^{(z)}\rangle_{0,s} \\
&= \frac{1}{2}\sum_{\mathbf{p}+\mathbf{q}+\mathbf{r}=\mathbf{0},\ \mathbf{p},\mathbf{q},\mathbf{r}\in\Gamma} \int_0^1 \frac{d(v_{\mathbf{p}}^{(z)}V_{\mathbf{q}}\cdot V_{\mathbf{r}})}{dz}dz = 0,
\end{aligned}$$

which ends the proof of (21). The identity (22) is a consequence of (21): indeed let us consider the identity

$$\langle \mathcal{B}(U+tU',U+tU'),U+tU'\rangle_{0,0} = 0$$

which holds for any $t\in\mathbb{R}$. It results that the coefficient of degree 1 in t of this polynomial is zero, which is exactly the property (22). ■

5.2 Rotationnal Symmetry

The system (1), completed with the boundary conditions included in the definition of $D(\mathcal{L})$, is invariant under horizontal rotations of angle $2\pi/Q$. To make this precise, let us define the linear operator $\mathbf{R}_{2\pi/Q}$, by

$$\mathbf{R}_{2\pi/Q}U(\mathbf{x},z) = \left(R_{2\pi/Q}V(R_{-2\pi/Q}\mathbf{x},z),\theta(R_{-2\pi/Q}\mathbf{x},z)\right),$$

where R_ϕ is the horizontal rotation of angle ϕ. More precisely, by using the identity $\mathbf{k}\cdot R_{-\phi}\mathbf{x} = R_\phi\mathbf{k}\cdot\mathbf{x}$, we have

$$\mathbf{R}_{2\pi/Q}\sum_{\mathbf{k}\in\Gamma}U_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{k}\in\Gamma}\left(R_{2\pi/Q}V_{\mathbf{k}}(z),\theta_{\mathbf{k}}(z)\right)e^{iR_{2\pi/Q}\mathbf{k}\cdot\mathbf{x}}. \qquad (23)$$

Definition 5.4 *We say that U is invariant under $\mathbf{R}_{2\pi/Q}$ if the following holds*

$$R_{2\pi/Q}V_{\mathbf{k}}(z) = V_{R_{2\pi/Q}\mathbf{k}}(z), \quad \theta_{\mathbf{k}}(z) = \theta_{R_{2\pi/Q}\mathbf{k}}(z).$$

Then, we have the following

Lemma 5.5 *The linear operator $\mathcal{L}$ and the quadratic operator $\mathcal{B}$ commute with $\mathbf{R}_{2\pi/Q}$: for $U \in D(\mathcal{L})$*

$$\begin{aligned} \mathbf{R}_{2\pi/Q}\mathcal{L}U &= \mathcal{L}\mathbf{R}_{2\pi/Q}U, \\ \mathbf{R}_{2\pi/Q}\mathcal{B}(U,U) &= \mathcal{B}(\mathbf{R}_{2\pi/Q}U, \mathbf{R}_{2\pi/Q}U). \end{aligned} \tag{24}$$

Proof. This results from the commutation of the original system (1) under any horizontal rotations, and from the commutation property

$$\mathbf{R}_{2\pi/Q}\Pi = \Pi\mathbf{R}_{2\pi/Q}$$

which is easy to check from the construction of projection Π. ■

5.3 Study of the criticality condition for $\mathcal{L}$

Let us consider the linear system

$$\mathcal{L}U = G = (F, g) \tag{25}$$

where $G = (F, g)$ is given in $\mathcal{K}_{0,s}$, and we are looking for $U \in D_{s'}(\mathcal{L})$. Let us define

$$\begin{aligned} U_{\mathbf{k}} &= (V_{\mathbf{k}}^{(H)}, v_{\mathbf{k}}^{(H)}, \theta_{\mathbf{k}}), \\ G_{\mathbf{k}} &= (F, g)_{\mathbf{k}} = (F_{\mathbf{k}}^{(H)}, f_{\mathbf{k}}^{(H)}, g_{\mathbf{k}}), \end{aligned}$$

then we have the following system which holds for all $\mathbf{k} \in \Gamma$

$$\begin{aligned} (D^2 - |\mathbf{k}|^2)v_{\mathbf{k}}^{(z)} + \theta_{\mathbf{k}} - Dq_{\mathbf{k}} &= f_{\mathbf{k}}^{(z)}, \\ (D^2 - |\mathbf{k}|^2)V_{\mathbf{k}}^{(H)} - i\mathbf{k}q_{\mathbf{k}} &= F_{\mathbf{k}}^{(h)}, \\ (D^2 - |\mathbf{k}|^2)\theta_{\mathbf{k}} + \mathcal{R}v_{\mathbf{k}}^{(z)} &= \mathcal{R}g_{\mathbf{k}}, \\ Dv_{\mathbf{k}}^{(z)} + i\mathbf{k}\cdot V_{\mathbf{k}}^{(H)} &= 0, \end{aligned} \tag{26}$$

where $D = d/dz$, $(F, g)_{\mathbf{k}} \in \{L^2(0,1)\}^4$ and $Df_{\mathbf{k}}^{(z)} + i\mathbf{k}\cdot F_{\mathbf{k}}^{(H)} = 0$, $f_{\mathbf{k}}^{(z)}|_{z=0,1} = 0$, and with the boundary conditions depending on Conditions **b.c.** (see Definition 5.1):

$$v_{\mathbf{k}}^{(z)}|_{z=0,1} = \theta_{\mathbf{k}}|_{z=0,1} = 0, \tag{27}$$

and either

$$\begin{aligned} V_{\mathbf{k}}^{(H)}|_{z=0,1} &= 0,\ \text{or}\ V_{\mathbf{k}}^{(H)}|_{z=0} = DV_{\mathbf{k}}^{(H)}|_{z=1} = 0, \\ \text{or}\ DV_{\mathbf{k}}^{(H)}|_{z=0} &= V_{\mathbf{k}}^{(H)}|_{z=1} = 0,\ \text{or}\ DV_{\mathbf{k}}^{(H)}|_{z=0,1} = 0. \end{aligned}$$

For a fixed $\mathbf{k}$, this system is *exactly the same as the one obtained in the periodic case*, described in details for example in Chapter II of [2]. In particular the above system of 6th order, reduces to

$$\begin{aligned} (D^2 - |\mathbf{k}|^2)^2 v_{\mathbf{k}}^{(z)} - |\mathbf{k}|^2 \theta_{\mathbf{k}} &= (D^2 - |\mathbf{k}|^2) f_{\mathbf{k}}^{(z)}, \\ (D^2 - |\mathbf{k}|^2)\theta_k + \mathcal{R} v_{\mathbf{k}}^{(z)} &= \mathcal{R} g_{\mathbf{k}}, \end{aligned}$$

with boundary conditions (27) and either

$$\begin{aligned} Dv_{\mathbf{k}}^{(z)}|_{z=0,1} &= 0,\ \text{or}\ Dv_{\mathbf{k}}^{(z)}|_{z=0} = D^2 v_{\mathbf{k}}^{(z)}|_{z=1} = 0, \\ \text{or}\ D^2 v_{\mathbf{k}}^{(z)}|_{z=0} &= Dv_{\mathbf{k}}^{(z)}|_{z=1} = 0,\ \text{or}\ D^2 v_{\mathbf{k}}^{(z)}|_{z=0,1} = 0. \end{aligned}$$

Then, it is known (see Yudovich [16]) that for a fixed $|\mathbf{k}|$ there is a denumerable sequence of $\mathcal{R}_j$ such that the system (26) has a non trivial solution for $(F,g)_{\mathbf{k}} = 0$, and there is a variational principle for finding $\mathcal{R}_1(|\mathbf{k}|^2) = \min \mathcal{R}_j$. It is also known mathematically (see Yudovich [16]) that the function $\mathcal{R}_1(|\mathbf{k}|^2)$ is analytic, tends towards ∞ as $|\mathbf{k}|^2 \to 0$ and as $|\mathbf{k}|^2 \to \infty$, and that there is a minimum $\mathcal{R}_c$ obtained for a critical value k_c^2. However, it is *only known numerically* (see [2]), that *this minimum is unique* and the kernel for $\mathbf{k} = \mathbf{k}_1 = (k_c, 0)$ is one-dimensional ([16]). It results that the kernel of the linear operator $\mathcal{L}_0 = \mathcal{L}|_{\mathcal{R}=\mathcal{R}_c}$ is Q - dimensional, spanned by

$$\boldsymbol{\xi}_j = \mathbf{R}_{\frac{2\pi(j-1)}{Q}} \left(\widehat{U}_{\mathbf{k}_1}(z) e^{i\mathbf{k}_1 \cdot \mathbf{x}} \right),\ j = 1, 2, ...Q, \tag{28}$$

where

$$\widehat{U}_{\mathbf{k}_1} = (V_{\mathbf{k}_1}^{(h)}, v_{\mathbf{k}_1}^{(z)}, \theta_{\mathbf{k}_1})$$

is solution of the homogeneous system (26) for $\mathbf{k} = \mathbf{k}_1$, and with $(F,g)_{\mathbf{k}} = 0$, and $\mathcal{R} = \mathcal{R}_c$. The classical linear stability theory ([2], [18]) says that

$$\langle \mathcal{L}_0 U, U \rangle_{0,0} < 0 \text{ for all } U \in D(\mathcal{L}) \text{ not in } \ker \mathcal{L}_0. \tag{29}$$

We need now to estimate the inverse of the linear operator defined by the system (26) for $|\mathbf{k}| \neq k_c$. From the now standard study of the resolvent operator for Navier-Stokes type of system (see [19]), as here, but in a periodic frame, we deduce that there is a function $c(|\mathbf{k}|^2)$ bounded as $|\mathbf{k}| \to \infty$ and $|\mathbf{k}| \to 0$ such that

$$\begin{aligned} ||U_{\mathbf{k}}||_2^2 &= ||D^2 U_{\mathbf{k}}||_{L^2}^2 + (1+|\mathbf{k}|^2)||DU_{\mathbf{k}}||_{L^2}^2 + (1+|\mathbf{k}|^2)^2||U_{\mathbf{k}}||_{L^2}^2 \\ &\leq [c(|\mathbf{k}|^2)]^2||G_{\mathbf{k}}||_0^2. \end{aligned} \tag{30}$$

For $|\mathbf{k}|$ near k_c, we know that $c(|\mathbf{k}|^2)$ diverges as $|\mathbf{k}|^2 \to k_c^2$. In fact $c(|\mathbf{k}|^2)^{-1}$ is proportional to the dispersion equation obtained when we look for eigenvectors of the homogeneous system (26) which has constant coefficients (see [2]). This dispersion equation depends analytically on $|\mathbf{k}|^2$ ([16]) and cancels with a double root for $|\mathbf{k}|^2 = k_c^2$ (because of the extremum for $\mathcal{R} = \mathcal{R}_c$). This means that we have in fact

$$c(|\mathbf{k}|^2) = \frac{c_1(|\mathbf{k}|^2)}{(|\mathbf{k}|^2 - k_c^2)^2} \tag{31}$$

where c_1 is bounded for all bounded $|\mathbf{k}|^2$ and is $O(|\mathbf{k}|^4)$ as $|\mathbf{k}| \to \infty$.

For $|\mathbf{k}| = k_c$ and $\mathbf{k} \in \Gamma$, this implies that $\mathbf{k}$ belongs to the basis of the quasipattern. Then, following [2], [16] and [17] the system (26) is solvable provided the compatibility conditions

$$\langle G, \boldsymbol{\xi}_j \rangle_{0,0} = \int_0^1 G_{\mathbf{k}_j} \cdot \overline{\widehat{U}_{\mathbf{k}_j}} dz = \int_0^1 (F_{\mathbf{k}_j} \cdot \overline{\widehat{V}_{\mathbf{k}_j}} + g_{\mathbf{k}_j} \cdot \overline{\widehat{\theta}_{\mathbf{k}_j}}) dz = 0, \ j = 1, ...Q$$

hold.

5.4 Pseudo-inverse of $\mathcal{L}_0$

Let us define a projection $\mathbf{P}_0$ on the kernel of $\mathcal{L}_0$: for any $U \in \mathcal{K}_{0,s}$

$$\mathbf{P}_0 U = \sum_{1 \leq j \leq Q} \gamma_j \boldsymbol{\xi}_j, \quad \gamma_j = \frac{\langle U, \boldsymbol{\xi}_j \rangle_{0,0}}{\langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_1 \rangle_{0,0}}, \tag{32}$$

where we notice that

$$\langle \boldsymbol{\xi}_1, \boldsymbol{\xi}_1 \rangle_{0,0} = \langle \boldsymbol{\xi}_j, \boldsymbol{\xi}_j \rangle_{0,0}, \quad j = 2, ..Q.$$

We denote by $\mathbf{Q}_0 = \mathbb{I} - \mathbf{P}_0$ the projection on the complementary space (of codimension Q). Since the eigenvectors $\boldsymbol{\xi}_j$ belong to $D_s(\mathcal{L})$ for any s, the projection $\mathbf{Q}_0$ is bounded in all $\mathcal{K}_{r,s}$, as well as in $D_s(\mathcal{L})$. Notice that if U is invariant under $\mathbf{R}_{2\pi/Q}$, then $\gamma_j = \gamma_1$ for $j = 2, ..Q$.

Now coming back to the linear system

$$\mathcal{L}_0 U = G,$$

where $G \in \mathcal{K}_{0,s}$ satisfies the compatibility condition $\mathbf{P}_0 G = 0$, the above estimate (30), and the form (31) of $c(|\mathbf{k}|^2)$ show that there is a unique solution U satisfying $\mathbf{P}_0 U = 0$ and there exists a constant $c > 0$ such that

$$||U_{\mathbf{k}}||_2 \leq c \left[\frac{(1-\delta_{k_c}(|\mathbf{k}|))(1+|\mathbf{k}|^2)^2}{(|\mathbf{k}|^2 - k_c^2)^2} + \delta_{k_c}(|\mathbf{k}|) \right] ||G_{\mathbf{k}}||_0,$$

where $\delta_{k_c}(|\mathbf{k}|) = 1$ if $|\mathbf{k}| = k_c$, and $= 0$ otherwise. By using the diophantine inequality (7), this leads to the following

Lemma 5.6 *The linear operator $\mathcal{L}_0$ has a bounded inverse from the subspace $\mathbf{Q}_0\mathcal{K}_{0,s}$ to the subspace $\mathbf{Q}_0 D_{s-4l}(\mathcal{L}) \subset \mathcal{K}_{2,s-4l}$. In other words, there exists $c > 0$ such that for U solution in $\mathbf{Q}_0 D_{s-4l}(\mathcal{L})$ of $\mathcal{L}_0 U = G \in \mathbf{Q}_0\mathcal{K}_{0,s}$, the following estimate holds*

$$||U_{\mathbf{k}}||_2 \leq c N_{\mathbf{k}}^{4l} ||G_{\mathbf{k}}||_0.$$

6 Formal power series computation

Let us rewrite the system (1) as

$$\mathcal{L}_0 U = \mu \mathcal{L}_1 U + \mathcal{B}_0(U,U) + \mu \mathcal{B}_1(U,U), \tag{33}$$

where

$$\begin{aligned} \mu &= \frac{1}{\mathcal{R}_c} - \frac{1}{\mathcal{R}}, \ \mathcal{L}_1 U = (0, \Delta\theta), \\ \mathcal{B}_0(U,U) &= \mathcal{B}(U,U)|_{\mathcal{R}=\mathcal{R}_c}, \ \mathcal{B}_1(U,U) = (0, -V \cdot \nabla\theta). \end{aligned}$$

Let us define $\varepsilon = \sqrt{\mu}$, this choice being justified below. We are looking for a solution of (33) which is invariant under $\mathbf{R}_{2\pi/Q}$ under the form of a formal expansion

$$U = \sum_{n \geq 1} \varepsilon^n U^{(n)}, \tag{34}$$

where $U^{(n)} \in D_s(\mathcal{L})$. Identifying powers of ε at orders ε, ε^2, ε^3, leads to the system

$$\mathcal{L}_0 U^{(1)} = 0, \tag{35}$$

$$\mathcal{L}_0 U^{(2)} = \mathcal{B}_0(U^{(1)}, U^{(1)}) \tag{36}$$

$$\mathcal{L}_0 U^{(3)} = \mathcal{L}_1 U^{(1)} + 2\mathcal{B}_0(U^{(1)}, U^{(2)}). \tag{37}$$

Equation (35) gives

$$U^{(1)} = \beta_1 \widetilde{U}^{(1)},\ \widetilde{U}^{(1)} = \mathbf{R}_{2\pi/Q} \widetilde{U}^{(1)} = \sum_{1 \le j \le Q} \boldsymbol{\xi}_j, \tag{38}$$

where β_1 is a real number still to be determined. Then we observe, thanks to property (21), that

$$\langle \mathcal{B}_0(U^{(1)}, U^{(1)}), U^{(1)} \rangle_{0,0} = 0,$$

and since

$$\mathbf{R}_{2\pi/Q} \mathcal{B}_0(U^{(1)}, U^{(1)}) = \mathcal{B}_0(U^{(1)}, U^{(1)}),$$

this means that

$$\mathbf{P}_0 \mathcal{B}_0(U^{(1)}, U^{(1)}) = 0,$$

hence equation (36) is solvable and we find

$$U^{(2)} = \widetilde{U}^{(2)} + \beta_2 \widetilde{U}^{(1)}, \tag{39}$$

with

$$\widetilde{U}^{(2)} = \widetilde{\mathcal{L}}_0^{-1} \mathcal{B}_0(U^{(1)}, U^{(1)}) = \beta_1^2 \widetilde{\mathcal{L}}_0^{-1} \mathcal{B}_0(\widetilde{U}^{(1)}, \widetilde{U}^{(1)}), \tag{40}$$

and β_2 is a real number still to be determined, and where $\widetilde{\mathcal{L}}_0^{-1}$ is the pseudo-inverse of $\mathcal{L}_0$ as defined by Lemma 5.6. Now, the compatibility condition for solving (37) gives

$$\langle \mathcal{L}_1 U^{(1)} + 2\mathcal{B}_0(U^{(1)}, U^{(2)}), U^{(1)} \rangle_{0,0} = 0. \tag{41}$$

We first observe that

$$\begin{aligned} \langle \mathcal{L}_1 \widetilde{U}^{(1)}, \widetilde{U}^{(1)} \rangle_{0,0} &= Q \langle \mathcal{L}_1 \boldsymbol{\xi}_1, \boldsymbol{\xi}_1 \rangle_{0,0} = Q \int_0^1 (D^2 - k_c^2) \widehat{\theta}_{\mathbf{k}_1} \overline{\widehat{\theta}_{\mathbf{k}_1}} dz \\ &= -Q \int_0^1 (|D\widehat{\theta}_{\mathbf{k}_1}|^2 + k_c^2 |\widehat{\theta}_{\mathbf{k}_1}|^2) dz < 0. \end{aligned}$$

Then we use the identity (22) to obtain (the result below , in the periodic case, was first obtained by V.Yudovich in [17])

$$\begin{aligned} \langle 2\mathcal{B}_0(\widetilde{U}^{(1)}, U^{(2)}), \widetilde{U}^{(1)} \rangle_{0,0} &= -\langle \mathcal{B}_0(\widetilde{U}^{(1)}, \widetilde{U}^{(1)}), U^{(2)} \rangle_{0,0} \\ &= -\langle \mathcal{B}_0(\widetilde{U}^{(1)}, \widetilde{U}^{(1)}), \widetilde{U}^{(2)} \rangle_{0,0} \\ &= -\langle \mathcal{L}_0 \widetilde{U}^{(2)}, \widetilde{U}^{(2)} \rangle_{0,0} > 0. \end{aligned}$$

The last inequality results from the fact that $\mathbf{P}_0\widetilde{U}^{(2)} = 0$, and from the property (29). It results that β_1 is determined by

$$\beta_1^2 = \frac{\langle \mathcal{L}_1\widetilde{U}^{(1)}, \widetilde{U}^{(1)}\rangle_{0,0}}{\langle \mathcal{B}_0(\widetilde{U}^{(1)}, \widetilde{U}^{(1)}), \widetilde{\mathcal{L}}_0^{-1}\mathcal{B}_0(\widetilde{U}^{(1)}, \widetilde{U}^{(1)})\rangle_{0,0}}, \tag{42}$$

and we choose $\beta_1 > 0$, the other choice would correspond to changing ε into $-\varepsilon$. Now $U^{(3)}$ takes the form

$$U^{(3)} = \widetilde{U}^{(3)} + \beta_3\widetilde{U}^{(1)}, \tag{43}$$

$$\widetilde{U}^{(3)} = \widetilde{U}_1^{(3)} + 2\frac{\beta_2}{\beta_1}\widetilde{U}^{(2)}, \tag{44}$$

$$\widetilde{U}_1^{(3)} = \widetilde{\mathcal{L}}_0^{-1}\mathbf{Q}_0\left\{\mathcal{L}_1U^{(1)} + 2\mathcal{B}_0(U^{(1)}, \widetilde{U}^{(2)})\right\} \tag{45}$$

where β_3 is a new number to be determined. At this stage, we have computed $U^{(1)}$ and $\widetilde{U}^{(2)}$. Let us show that β_2 and $\widetilde{U}^{(3)}$ which is the part of $U^{(3)}$ in the complement of $\ker\mathcal{L}_0$, are determined by the identification at order ε^4 in (33). Indeed we have to solve

$$\mathcal{L}_0U^{(4)} = \mathcal{L}_1U^{(2)} + 2\mathcal{B}_0(U^{(1)}, U^{(3)}) + \mathcal{B}_0(U^{(2)}, U^{(2)}) + \mathcal{B}_1(U^{(1)}, U^{(1)}). \tag{46}$$

The compatibility condition gives here

$$\langle \mathcal{L}_1U^{(2)} + 2\mathcal{B}_0(U^{(1)}, U^{(3)}) + \mathcal{B}_0(U^{(2)}, U^{(2)}) + \mathcal{B}_1(U^{(1)}, U^{(1)}), \widetilde{U}^{(1)}\rangle_{0,0} = 0,$$

and by using the form of $U^{(2)}$ and $U^{(3)}$ and property (21), this leads to

$$\begin{aligned} &\beta_2\langle \mathcal{L}_1\widetilde{U}^{(1)}, \widetilde{U}^{(1)}\rangle_{0,0} + 6\beta_2\langle \mathcal{B}_0(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}), \widetilde{U}^{(1)}\rangle_{0,0} \\ = \; &-\langle \mathcal{L}_1\widetilde{U}^{(2)}, \widetilde{U}^{(1)}\rangle_{0,0} - \langle 2\mathcal{B}_0(U^{(1)}, \widetilde{U}_1^{(3)}), \widetilde{U}^{(1)}\rangle_{0,0} - \langle \mathcal{B}_0(\widetilde{U}^{(2)}, \widetilde{U}^{(2)}), \widetilde{U}^{(1)}\rangle_{0,0}, \end{aligned}$$

and since by definition of β_1, we have

$$\langle \mathcal{L}_1\widetilde{U}^{(1)}, \widetilde{U}^{(1)}\rangle_{0,0} + 2\langle \mathcal{B}_0(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}), \widetilde{U}^{(1)}\rangle_{0,0} = 0,$$

it results that β_2 is given by the following formula

$$\begin{aligned} 2\beta_2\langle \mathcal{L}_1\widetilde{U}^{(1)}, \widetilde{U}^{(1)}\rangle_{0,0} = \; &\langle \mathcal{L}_1\widetilde{U}^{(2)}, \widetilde{U}^{(1)}\rangle_{0,0} + \langle 2\mathcal{B}_0(U^{(1)}, \widetilde{U}_1^{(3)}), \widetilde{U}^{(1)}\rangle_{0,0} + \\ &+\langle \mathcal{B}_0(\widetilde{U}^{(2)}, \widetilde{U}^{(2)}), \widetilde{U}^{(1)}\rangle_{0,0}. \end{aligned} \tag{47}$$

This ends the determination of $U^{(2)}$ and $\widetilde{U}^{(3)}$, and $U^{(4)}$ takes the form

$$\begin{aligned} U^{(4)} &= \widetilde{U}^{(4)} + \beta_4\widetilde{U}^{(1)}, \;\; \mathbf{P}_0\widetilde{U}^{(4)} = 0, \\ \widetilde{U}^{(4)} &= \widetilde{U}_1^{(4)} + 2\frac{\beta_3}{\beta_1}\widetilde{U}^{(2)}, \end{aligned} \tag{48}$$

where $\widetilde{U}_1^{(4)}$ is completely known.

Let us show by induction, that for higher orders, the situation is similar. Assume that we have identified orders up to ε^n, and completely determined $U^{(n-2)}$, $\widetilde{U}^{(n-1)}$, and $\widetilde{U}_1^{(n)}$ with

$$\widetilde{U}^{(n)} = \widetilde{U}_1^{(n)} + 2\frac{\beta_{n-1}}{\beta_1}\widetilde{U}^{(2)}, \tag{49}$$

and

$$\widetilde{U}^{(p)} = \mathbf{Q}_0 U^{(p)} \text{ for all } p \geq 2.$$

Then the identification at order ε^{n+1}, $n \geq 4$ leads to

$$\mathcal{L}_0 U^{(n+1)} = \mathcal{L}_1 U^{(n-1)} + 2\mathcal{B}_0(U^{(1)}, U^{(n)}) + 2\mathcal{B}_0(U^{(2)}, U^{(n-1)}) + R^{(n+1)}, \tag{50}$$

where $R^{(n+1)}$ is known in terms of already computed lower orders. The compatibility condition gives now

$$\langle \mathcal{L}_1 U^{(n-1)} + 2\mathcal{B}_0(U^{(1)}, U^{(n)}) + 2\mathcal{B}_0(U^{(2)}, U^{(n-1)}) + R^{(n+1)}, \widetilde{U}^{(1)}\rangle_{0,0} = 0,$$

hence, as above β_{n-1} is determined by

$$2\beta_{n-1}\langle \mathcal{L}_1\widetilde{U}^{(1)}, \widetilde{U}^{(1)}\rangle_{0,0} = \langle \mathcal{L}_1\widetilde{U}^{(n-1)} + 2\mathcal{B}_0(U^{(1)}, \widetilde{U}_1^{(n)}) + 2\mathcal{B}_0(\widetilde{U}^{(2)}, \widetilde{U}^{(n-1)}) + R^{(n+1)}, \widetilde{U}^{(1)}\rangle_{0,0}. \tag{51}$$

This ends the determination of $U^{(n)}$ and $\widetilde{U}^{(n)}$. Moreover we have

$$\widetilde{U}^{(n+1)} = \widetilde{U}_1^{(n+1)} + 2\frac{\beta_n}{\beta_1}\widetilde{U}^{(2)},$$

where $\widetilde{U}_1^{(n+1)}$ is completely known. Hence the induction is proved.

7 Gevrey estimates

In this section we prove a Gevrey estimate of the coefficients $U^{(n)}$ in (34). Recall that a formal power series $\sum_{n=0}^{\infty} u_n \zeta^n$ is *Gevrey - k* [6], where k is a positive integer, if there are constants $\delta > 0$ and $K > 0$ such that

$$|u_n| \leq \delta K^n (n!)^k \qquad \forall n \geq 0. \tag{52}$$

Theorem 7.1 *For any even $Q \geq 8$, assume that $s > Q/4$. Then there exist positive numbers $K(Q, c, s)$ and $\delta(Q, s)$ such that there exists a unique $U(\varepsilon)$ as a power series in $\varepsilon = \mu^{1/2}$, all coefficients belonging to $D_s(\mathcal{L})$ and* *being invariant under $\mathbf{R}_{2\pi/Q}$, which is a formal solution of (20), and which satisfies*

$$\begin{aligned} U(\varepsilon) &= \sum_{n\geq 0} \varepsilon^n U^{(n)}, \\ ||U^{(n)}||_{2,s} &\leq \delta K^n (n!)^{4l}, \quad n \geq 2, \end{aligned}$$

where l is the integer defined in Lemma 3.1 ($l = (1/2)\varphi(Q) - 1$, where $\varphi(Q)$ is Euler's Totient function).

Remark 7.2 *The above Theorem claims that the series U in powers of $\sqrt{\mu}$ is Gevrey - $4l$, its coefficients taking their values in $D_s(\mathcal{L})$.*

Proof We choose $s > Q/4$ since estimate (19) insures that

$$||\mathcal{B}_j(U,V)||_{0,s} \leq c(s)||U||_{2,s}||V||_{2,s}, \quad j = 0, 1,$$

where we denote by $||\cdot||_{r,s}$ the norm in $\mathcal{K}_{r,s}$ for $r = 0$ and in $D_s(\mathcal{L})$ for $r = 2$, and we observe that

$$||\mathcal{L}_1 U||_{0,s} \leq ||U||_{2,s}.$$

The main observation which is needed for estimating $U^{(n)}$ is the fact that in the Fourier expansion of $U^{(n)}$ there occur only harmonics $\mathbf{k_m} \in \Gamma$ with $|\mathbf{m}| \leq n$. Hence, the result of Lemma 5.6 allows to invert $\mathcal{L}_0$ on the complement of its kernel, defined by the projection $\mathbf{Q}_0$.

We start with known $U^{(1)}$, $U^{(2)}$, $U^{(3)}$, $\widetilde{U}^{(4)}$ and $\beta_1, \beta_2, \beta_3$ with $\beta_1 > 0$ defined by (38), (39), (43), (48), and (42), (47), (51) for $n = 4$, and we define

$$a = -\langle \mathcal{L}_1 \widetilde{U}^{(1)}, \widetilde{U}^{(1)} \rangle_{0,0} > 0, \; M = \max_{1\leq j\leq 3} \{||U^{(j)}||_{2,s}, ||\widetilde{U}^{(j+1)}||_{2,s}\}.$$

Then, from (50) for $n = 4$ we obtain $\widetilde{U}_1^{(5)}$ such that

$$\mathcal{L}_0 \widetilde{U}_1^{(5)} = \mathbf{Q}_0 \left(\mathcal{L}_1 U^{(3)} + 2\mathcal{B}_0(U^{(1)}, \widetilde{U}^{(4)}) + 2\mathcal{B}_0(U^{(2)}, U^{(3)}) + 2\mathcal{B}_1(U^{(1)}, U^{(2)}) \right),$$

satisfying the estimate

$$||\widetilde{U}_1^{(5)}||_{2,s} \leq c 5^{4l} M(1 + 6c(s)M), \tag{53}$$

and from (51) for $n = 5$, we obtain β_4 and the estimate

$$|\beta_4| \leq \frac{M^2}{2a\beta_1}\{(1 + 6c(0)M) + 2cc(0)5^{4l}(1 + 6c(s)M)\}. \tag{54}$$

Now, let us assume that for $4 \leq p \leq n-2$ we have determined β_p and $\widetilde{U}_1^{(p+1)}$ defined by (49), and that the following estimates hold

$$\begin{aligned} ||\widetilde{U}_1^{(p)}||_{2,s} &\leq \delta_1 K_0^p (p!)^{4l}, \\ |\beta_p| &\leq \delta_0 K_0^{p+1} ((p+1)!)^{4l}. \end{aligned}$$

For consistancy, we need to satisfy

$$M_1 \leq \delta_1 K_0^5, \tag{55}$$
$$M_2 \leq \delta_0 K_0^5, \tag{56}$$

with

$$\begin{aligned} M_1 &= c(4!)^{-4l} M(1+6c(s)M), \\ M_2 &= \frac{M^2}{2a\beta_1}\{(1+6c(0)M)(5!)^{-4l} + 2cc(0)(4!)^{-4l}(1+6c(s)M)\}. \end{aligned}$$

It results that we have the following estimates for $\widetilde{U}^{(p)}$ for $5 \leq p \leq n-1$ and $U^{(p)}$ for $4 \leq p \leq n-2$:

$$||\widetilde{U}^{(p)}||_{2,s} \leq \delta_2 K_0^p (p!)^{4l}, \quad \delta_2 = \delta_1 + \frac{2M}{\beta_1}\delta_0, \tag{57}$$

$$\begin{aligned} ||U^{(p)}||_{2,s} &\leq \delta_2 K_0^p (p!)^{4l} + \frac{M}{\beta_1}\delta_0 K_0^{p+1}((p+1)!)^{4l} \\ &\leq \delta_3 K_0^{p+1}((p+1)!)^{4l}, \quad \delta_3 = \frac{\delta_2}{5^{4l}K_0} + \frac{M}{\beta_1}\delta_0. \end{aligned} \tag{58}$$

Now we can find a bound for $\widetilde{U}_1^{(n)}$ in using the identities

$$\begin{aligned} \mathcal{L}_0 \widetilde{U}_1^{(n)} &= \mathbf{Q}_0 \left(\mathcal{L}_1 U^{(n-2)} + 2\mathcal{B}_0(U^{(1)}, \widetilde{U}^{(n-1)}) + 2\mathcal{B}_0(U^{(2)}, U^{(n-2)}) + R^{(n)} \right), \\ R^{(n)} &= \sum_{p+q=n,\ \ p,q\geq 3} \mathcal{B}_0(U^{(p)}, U^{(q)}) + \sum_{p+q=n-2,\ \ p,q\geq 1} \mathcal{B}_1(U^{(p)}, U^{(q)}). \end{aligned}$$

Indeed, we first have for $n \geq 6$ (where $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$, and $= 0$ otherwise)

$$\sum_{p+q=n,\ \ p,q\geq 3} \mathcal{B}_0(U^{(p)}, U^{(q)}) = \sum_{p+q=n,\ \ p,q\geq 4} \mathcal{B}_0(U^{(p)}, U^{(q)}) + (2-\delta_{n,6})\mathcal{B}_0(U^{(3)}, U^{(n-3)}),$$

$$
\begin{aligned}
\left\| \sum_{p+q=n-2,\ p,q\geq 1} \mathcal{B}_1(U^{(p)},U^{(q)}) \right\|_{0,s} &\leq \left\| \sum_{p+q=n-2,\ p,q\geq 4} \mathcal{B}_1(U^{(p)},U^{(q)}) \right\|_{0,s} + \\
&+ \left\| \sum_{p+q=n-2,\ 1\leq p\leq 3, q\geq 4} \mathcal{B}_1(U^{(p)},U^{(q)}) \right\|_{0,s} + \\
&+ \left\| \sum_{p+q=n-2,\ 1\leq p,q\leq 3} \mathcal{B}_1(U^{(p)},U^{(q)}) \right\|_{0,s},
\end{aligned}
$$

hence

$$
\begin{aligned}
||R^{(n)}||_{0,s} &\leq c(s)\delta_3^2 K_0^{n+2} \sum_{p+q=n,\ p,q\geq 4} [(p+1)!(q+1)!]^{4l} + 4c(s)M^2 + \\
&+ c(s)\delta_3^2 K_0^n \sum_{p+q=n,\ p,q\geq 3} [p!q!]^{4l} + 8c(s)M\delta_3 K_0^{n-2}((n-2)!)^{4l}.
\end{aligned}
$$

We can use the following estimate, easily proved with the method used in [7]:

$$
\begin{aligned}
\sum_{p+q=n,\ p,q\geq 4} [(p+1)!(q+1)!]^{4l} &\leq C((n-3)!)^{4l} \\
\sum_{p+q=n,\ p,q\geq 3} [p!q!]^{4l} &\leq C((n-3)!)^{4l}
\end{aligned}
$$

valid for a certain $C > 0$. This leads to

$$
||R^{(n)}||_{0,s} \leq \left(Cc(s)(K_0^{-2}+1)\frac{\delta_3^2 K_0^2}{(n-2)^{4l}} + 8c(s)M\delta_3 K_0^{-2} \right) K_0^n((n-2)!)^{4l} + 4c(s)M^2 \tag{59}
$$

and for $n \geq 6$, assuming $K_0 \geq 1$

$$
\begin{aligned}
||\widetilde{U}_1^{(n)}||_{2,s} &\leq cn^{4l} \left\{ (1+2c(s)M)||U^{(n-2)}||_{2,s} + 2c(s)M||\widetilde{U}^{(n-1)}||_{2,s} + ||R^{(n)}||_{0,s} \right\}, \\
&\leq cK_0^n(n!)^{4l} \left\{ \frac{(1+2c(s)M)\delta_3}{K_0} + 2c(s)M\frac{\delta_2}{K_0} + \frac{4c(s)M^2}{K_0^6(5!)^{4l}} + \right. \\
&\left. + \left(Cc(s)(K_0^{-2}+1)\frac{\delta_3^2 K_0^2}{(20)^{4l}} + \frac{8c(s)M\delta_3}{K_0^2 5^{4l}} \right) \right\}.
\end{aligned}
$$

Hence, for verifying the recurrence assumption we need to satisfy the condition

$$\begin{aligned}\delta_1 \quad \geq \quad & \frac{c(1+2c(s)M)\delta_3}{K_0} + 2cc(s)M\frac{\delta_2}{K_0} + Ccc(s)(K_0^{-2}+1)\frac{\delta_3^2 K_0^2}{(20)^{4l}} + \qquad (60)\\ & + \frac{8cc(s)M\delta_3}{K_0^2 5^{4l}} + \frac{4cc(s)M^2}{K_0^6(5!)^{4l}}.\end{aligned}$$

Now, for $n \geq 6$, we can estimate β_{n-1} which is given by (51):

$$|\beta_{n-1}| \leq \frac{MK_0^n(n!)^{4l}}{2a\beta_1}\left\{(1+4c(0)M)\frac{\delta_2}{K_0} + \frac{Cc(s)(K_0^{-2}+1)\delta_3^2K_0^3}{30^{4l}} + 8c(s)M\frac{\delta_3}{K_0 6^{4l}} + \frac{4c(s)M^2}{K_0^6(6!)^{4l}}\right\}$$

and for satisfying the recurrence assumption we need to have

$$\begin{aligned}\delta_0 \quad \geq \quad & \frac{M}{2a\beta_1}\left\{(1+2c(0)M)\frac{\delta_2}{K_0} + 2c(0)M\delta_1 + \frac{Cc(s)(K_0^{-2}+1)\delta_3^2K_0^3}{30^{4l}}\right. \qquad (61)\\ & \left. + 8c(s)M\frac{\delta_3}{K_0 6^{4l}} + \frac{4c(s)M^2}{K_0^6(6!)^{4l}}\right\}.\end{aligned}$$

It remains to choose δ_0, δ_1 and K_0 satisfying conditions (55), (56), (60), (61). For this purpose, it is sufficient to choose

$$\delta_1 = \frac{a\beta_1}{2c(0)M^2}\delta_0, \quad \delta_0 = \frac{1}{K_0^4},$$

and K_0 sufficiently large. Indeed, conditions (55), (56) are easily satisfied with this choice, and we notice that δ_2 and δ_3 are of the same order as δ_0, which implies that conditions (60), (61) reduce to satisfy

$$\delta_0^2 K_0^3 + O(\frac{1}{K_0^6}) << \delta_0,$$

which holds for $\delta_0 = 1/K_0^4$, K_0 large enough. Now, we notice that in choosing

$$K \geq K_0 c_0, \quad c_0 = \sup_{n\geq 1}(n+1)^{4l/n},$$

then

$$||U^{(n)}|| \leq \delta_3 K_0 K^n (n!)^{4l}$$

holds, and the Theorem 7.1 is then proved in choosing $\delta = \delta_3 K_0$.

8 Borel transform of the formal solution

In this section we set

$$\sqrt{\mu} = \varepsilon = \zeta^{4l}.$$

The formal expansion (34) becomes

$$U = \sum_{n \geq 1} \zeta^{4nl} U^{(n)}, \tag{62}$$

and we have the estimate

$$||U^{(n)}||_{2,s} \leq \delta K^n (n!)^{4l} \leq \delta K^n (4nl!).$$

Thus the formal power series (62) is a Gevrey 1 series in ζ.

Let us now consider the function $\zeta \mapsto \widehat{U}(\zeta)$, taking its values in $D_s(\mathcal{L})$, defined by

$$\widehat{U}(\zeta) = \sum_{n \geq 1} \frac{\zeta^{4nl}}{(4nl)!} U^{(n)}.$$

Indeed, by construction, this function is analytic in the disc $|\zeta| < \rho = K^{-1/4l}$, with values in the Hilbert space $D_s(\mathcal{L})$ and invariant under $\mathbf{R}_{2\pi/Q}$ (representation of rotations by $2\pi/Q$). The mapping $U \mapsto \widehat{U}$, where we divide the coefficient of ζ^n by $n!$, is the *Borel transform* [1] applied to the series U. Since U satisfies a Gevrey 1 estimate, the Borel transform $\widehat{U}$ is analytic in a disc.

We now need to show that this function $\widehat{U}(\zeta)$ is solution of a certain partial differential equation. This is based on simple properties of Gevrey 1 series. Consider two scalar Gevrey 1 series u and v

$$\begin{aligned} u &= \sum_{n \geq 1} u_n \zeta^n, \qquad v = \sum_{n \geq 1} v_n \zeta^n, \\ |u_n| &\leq c_1 \frac{n!}{\rho^n}, \qquad |v_n| \leq c_2 \frac{n!}{\rho^n}, \end{aligned}$$

then we have

$$\begin{aligned} (uv)_n &= \sum_{1 \leq k \leq n-1} u_k v_{n-k}, \\ |(uv)_n| &\leq c_1 c_2 \frac{n!}{\rho^n}. \end{aligned}$$

We can then define

$$\widehat{uv} = \widehat{u} *_G \widehat{v} \tag{63}$$

where the *convolution product*, written as $*_G$, is well defined by

$$(\hat{u} *_G \hat{v})(\zeta) = \sum_{n\geq 1} \sum_{1\leq k\leq n-1} \frac{u_k v_{n-k}}{n!}\zeta^n.$$

This convolution product is easily extended for two functions $f(\zeta)$ and $g(\zeta)$, analytic in the disc $|\zeta| < \rho$, and with no zero order term, by

$$(f * g)(\zeta) = \sum_{n\geq 1} \sum_{1\leq k\leq n-1} f_k g_{n-k} \frac{k!(n-k)!}{n!}\zeta^n. \tag{64}$$

It is clear that for $f = \hat{u}$, and $g = \hat{v}$ we have

$$f * g = (\hat{u} *_G \hat{v}) = \widehat{(uv)}.$$

Since we have (63), it is clear from the fact that the linear operators $\mathcal{L}_0$ and $\mathcal{L}_1$ are independent of ζ, that we have

$$(\widehat{\mathcal{L}_j U})(\mathbf{x}, z, \zeta) = \mathcal{L}_j \widehat{U}(\mathbf{x}, z, \zeta), \text{ for any } U \in D_s(\mathcal{L}),\ j = 0, 1.$$

We may define for any $V : \zeta \mapsto V(\zeta) = (W(\cdot, \zeta), \theta(\cdot, \zeta))$ analytic in the disc $|\zeta| \leq \rho$, taking values in $D_s(\mathcal{L})$, the quadratic operators

$$\begin{aligned}
\mathcal{B}_0^*(V, V) &= \left(\frac{1}{\mathcal{P}}\Pi(W *_G \nabla W), \frac{1}{\mathcal{R}_c} W *_G \nabla\theta\right), \\
\mathcal{B}_1^*(V, V) &= (0, -W *_G \nabla\theta),
\end{aligned}$$

Let us also define a bounded linear operator $\mathfrak{K}$ as follows: for any function $\zeta \mapsto V(\zeta)$ analytic in the disc $|\zeta| < \rho$, taking values in $\mathcal{K}_{0,s}$, canceling for $\zeta = 0$, and satisfying

$$V(\zeta) = \sum_{n\geq 1} V_n \zeta^n, \quad ||V_n||_{0,s} \leq \frac{c}{\rho^n},$$

we define

$$(\mathfrak{K}V)(\zeta) = \sum_{n\geq 1} \frac{n!}{(n+8l)!}\zeta^{n+8l} V_n.$$

It is then clear for $V = \widehat{U}$ that

$$(\mathfrak{K}\widehat{U})(\zeta) = \sum_{n\geq 1} \frac{\zeta^{4nl+8l}}{(4nl+8l)!} U^{(n)} = \widehat{(\zeta^{8l} U)},$$

and we see that

$$\partial_\zeta^{8l}(\mathfrak{K}\widehat{U}) = \widehat{U}.$$

We now claim the following:

Theorem 8.1 *The Borel transform $\widehat{U}(\mathbf{x},z,\zeta)$ of the Gevrey series, solution found in Theorem 7.1, is the unique solution, analytic in the disc $|\zeta|<K^{-1/4l}$, cancelling for $\zeta=0$, taking values in $D_s(\mathcal{L})$, and invariant under $\mathbf{R}_{2\pi/Q}$, of the equation*

$$\mathcal{L}_0 V = \mathfrak{K}\mathcal{L}_1 V + \mathcal{B}_0^*(V,V) + \mathfrak{K}\mathcal{B}_1^*(V,V). \tag{65}$$

Proof. We assume $l=1$ in what follows, for the sake of simplicity. The changes needed for larger l's are left to the reader. Let us look for a solution V in the form

$$V=\sum_{n\geq 1}\zeta^n V_n,$$

where $V_n \in D_s(\mathcal{L})$ is invariant under $\mathbf{R}_{2\pi/Q}$. Then defining a formal series

$$U=\sum_{n\geq 1}\zeta^n U_n, \qquad U_n = n!V_n,$$

it is clear that U satisfies formally

$$\mathcal{L}_0 U = \zeta^8\mathcal{L}_1 U + \mathcal{B}_0(U,U) + \zeta^8\mathcal{B}_1(U,U),$$

and by identifying powers of ζ:

$$\begin{aligned}
\mathcal{L}_0 U_1 &= 0,\\
\mathcal{L}_0 U_2 &= \mathcal{B}_0(U_1,U_1),\\
\mathcal{L}_0 U_3 &= 2\mathcal{B}_0(U_1,U_2),
\end{aligned}$$

which leads to $U_1=0$ because of the 3rd equation where the solvability condition cannot be satisfied if $U_1\neq 0$. Then we have

$$U_1=0, \qquad \mathcal{L}_0 U_j = 0, j=2,3.$$

The same argument holds, showing that

$$U_2=U_3=0.$$

Then we obtain

$$\begin{aligned}
\mathcal{L}_0 U_4 &= 0,\ \mathcal{L}_0 U_5 = 0,\ \mathcal{L}_0 U_6 = 0,\ \mathcal{L}_0 U_7 = 0,\\
\mathcal{L}_0 U_8 &= \mathcal{B}_0(U_4,U_4),\\
\mathcal{L}_0 U_9 &= 2\mathcal{B}_0(U_4,U_5),\\
\mathcal{L}_0 U_{10} &= 2\mathcal{B}_0(U_4,U_6)+\mathcal{B}_0(U_5,U_5),\\
\mathcal{L}_0 U_{11} &= 2\mathcal{B}_0(U_4,U_7)+2\mathcal{B}_0(U_5,U_6),\\
\mathcal{L}_0 U_{12} &= \mathcal{L}_1 U_4 + 2\mathcal{B}_0(U_4,U_8)+2\mathcal{B}_0(U_5,U_7)+\mathcal{B}_0(U_6,U_6).
\end{aligned}$$

We observe that U_4 and U_8 satisfy the equations verified by $U^{(1)}$ and $U^{(2)}$ (see (35), (36). Now, $U_5, U_6, U_7 \in \ker \mathcal{L}_0$ hence

$$U_j = \alpha_j U^{(1)},\ j = 5, 6, 7,$$

and we have

$$\langle 2\mathcal{B}_0(U_5, U_7) + \mathcal{B}_0(U_6, U_6), \widetilde{U}^{(1)} \rangle_{0,0} = 0.$$

This determines completely U_4 and U_8 (see section 6), and proves that

$$\begin{aligned}
U_4 &= U^{(1)},\ U_8 = U^{(2)},\ U_{12} = \widetilde{U}^{(3)} + (2\alpha_5\alpha_7 + \alpha_6^2)\widetilde{U}^{(2)} + \alpha_{12}U^{(1)}, \\
U_9 &= \alpha_5\widetilde{U}^{(2)} + \alpha_9 U^{(1)}, U_{10} = (2\alpha_6 + \alpha_5^2)\widetilde{U}^{(2)} + \alpha_{10}U^{(1)}, \\
U_{11} &= (2\alpha_7 + 2\alpha_5\alpha_6)\widetilde{U}^{(2)} + \alpha_{11}U^{(1)}.
\end{aligned}$$

Now at order ζ^{13} we get

$$\begin{aligned}
\mathcal{L}_0 U_{13} &= \mathcal{L}_1 U_5 + 2\mathcal{B}_0(U_4, U_9) + 2\mathcal{B}_0(U_5, U_8) + 2\mathcal{B}_0(U_6, U_7), \\
&= \alpha_5 \mathcal{L}_1 U^{(1)} + 4\alpha_5 \mathcal{B}_0(U^{(1)}, \widetilde{U}^{(2)}) + \\
&\quad +(2\alpha_9 + 2\alpha_5\alpha_8 + 2\alpha_6\alpha_7)\mathcal{B}_0(U^{(1)}, U^{(1)}),
\end{aligned}$$

and the solvability condition gives

$$0 = a\alpha_5$$

hence $\alpha_5 = 0$ and $U_5 = 0$. It is the same for $U_6 = U_7 = 0$, and we obtain $\mathcal{L}_0 U_9 = \mathcal{L}_0 U_{10} = \mathcal{L}_0 U_{11} = 0$, and $U_{12} = \widetilde{U}^{(3)} + \alpha_{12}U^{(1)}$. Then the computation of higher orders follows the same lines, giving $U_n = 0$ if n is not a multiple of 4, and is for U_{4n} exactly as the one for $U^{(n)}$. Coming back to the definition of $U_n = n!V_n$, it is then clear that Theorem 8.1 is proved. ∎

9 Truncated Laplace transform

For the sake of simplicity, here again, we assume $l = 1$ in Lemma 3.1, the general case being left to the reader. Let us take $\rho' < \rho$ and define a linear mapping $U \mapsto \bar{U}$ in the set of Gevrey 1 series taking values in $D_s(\mathcal{L})$

$$\bar{U}(\xi) = (\mathfrak{L}_{\rho'}\widehat{U})(\xi) \overset{def}{=} \frac{1}{\xi}\int_0^{\rho'} e^{-\frac{\zeta}{\xi}}\widehat{U}(\zeta)\, d\zeta, \tag{66}$$

where $\widehat{U}(\zeta)$ is the Borel transform of U as defined above, which is analytic in the disc $|\zeta| < \rho$. The function $\xi \mapsto \bar{U}(\xi)$ is a truncated Laplace transform of the Borel transform of U.

It is clear that $\bar{U}(\xi)$ is a $\mathcal{C}^\infty$ function of ξ in a neighborhood of 0, taking its values in $D_s(\mathcal{L})$, as this results from

$$\bar{U}(\xi) = \int_0^{\frac{\rho'}{\xi}} e^{-z}\widehat{U}(\xi z)\,dz$$

and from the dominated convergence theorem. Moreover $\bar{U}(\xi)$ and $U(\varepsilon)$ have the same asymptotic expansion in powers on ξ, when we set $\varepsilon = \xi^4$, as this results from

$$\frac{1}{\xi}\int_0^{\rho'} e^{-\frac{\zeta}{\xi}}\frac{\zeta^n}{n!}\,d\zeta = \xi^n - e^{-\frac{\rho'}{\xi}}\left(\frac{\xi^n}{1} + \frac{\rho'\xi^{n-1}}{1!} + \cdots + \frac{\xi\acute{\rho}^{n-1}}{(n-1)!} + \frac{\rho'^n}{n!}\right). \quad (67)$$

It is also clear that in a little disc near the origin

$$\widehat{\bar{U}} = \widehat{U},$$

but this does not imply that $\bar{U}(\xi) = U(\xi^4)$ since U has no meaning as a function of ε, and an asymptotic expansion does not define a unique function. The real question is whether or not $\bar{U}(\mu^{1/8})$ is solution of (33) in $D_s(\mathcal{L})$.

By construction, we know that the Gevrey 1 expansion of

$$\mathcal{L}_0\bar{U}(\mu^{1/8}) - \mu\mathcal{L}_1\bar{U}(\mu^{1/8}) - \mathcal{B}_0(\bar{U}(\mu^{1/8}), \bar{U}(\mu^{1/8})) - \mu\mathcal{B}_1(\bar{U}(\mu^{1/8}), \bar{U}(\mu^{1/8})) \quad (68)$$

in powers of $\mu^{1/8}$ is identically 0, but we don't know whether this function (smooth in $\mu^{1/8}$), which is in $\mathcal{K}_{0,s-4}$, is indeed 0. In fact we have the following

Theorem 9.1 *For any even $Q \geq 8$, take $s > \min\{Q/4, 4l\}$. Then, l being defined by Lemma 3.1, the quasiperiodic function $\bar{U}(\mu^{1/4l}) \in D_s(\mathcal{L})$, defined from the series found in Theorem 7.1, is solution of (33) up to an exponentially small term bounded by $C(\rho')e^{-\frac{\rho'}{\mu^{1/4l}}}$ in $\mathcal{K}_{0,s-4l}$, for any $\rho' < K^{-1/4l}$.*

Proof. We assume again that $l = 1$ and let us estimate of the difference beween (68) and the truncated Laplace transform of the equation (65) (which is then 0). We first have for $j = 0, 1$

$$\mathcal{L}_j(\mathfrak{L}_{\rho'}\widehat{U})(\xi) = \mathcal{L}_j\bar{U}(\xi) = \frac{1}{\xi}\int_0^{\rho'} e^{-\frac{\zeta}{\xi}}\mathcal{L}_j\widehat{U}(\zeta)d\zeta = (\mathfrak{L}_{\rho'}\mathcal{L}_j\widehat{U})(\xi),$$

which holds in $\mathcal{K}_{0,s-4}$ for analytic functions $\zeta \mapsto \widehat{U}(\zeta)$ taking values in $D_s(\mathcal{L})$. Now, it is shown, following for example the lines of Appendix E in

[7], or from the more general results in [11] and [10], that for $j = 0, 1$ we have the estimate

$$\left\|(\mathfrak{L}_{\rho'}\mathcal{B}_j^*(\widehat{U},\widehat{U}))(\xi) - \mathcal{B}_j(\mathfrak{L}_{\rho'}\widehat{U})(\xi),(\mathfrak{L}_{\rho'}\widehat{U})(\xi))\right\|_{0,s} \leq C(\rho')e^{-\frac{\rho'}{\xi}}(\frac{\rho'}{\xi})^2 \sup_{|\zeta|<\rho'} ||\widehat{U}(\zeta)||_{2,s}^2,$$

and for analytic functions $\zeta \mapsto \widehat{V}(\zeta)$ taking values in $\mathcal{K}_{0,s}$ the following estimate holds

$$\left\|(\mathfrak{L}_{\rho'}\mathcal{K}\widehat{V})(\xi) - \xi^8(\mathfrak{L}_{\rho'}\widehat{V})(\xi)\right\|_{0,s} \leq C(\rho')e^{-\frac{\rho'}{\xi}} \sup_{|\zeta|<\rho'} ||\widehat{V}(\zeta)||_{0,s}.$$

Then, the results of Theorem 9.1 for $l = 1$, follows immediately from the result of Theorem 8.1 by applying the operator $\mathfrak{L}_{\rho'}$ (notice that ρ' is arbitrary $< K^{-1/4}$. The extension to larger l's is left to the reader. ■

A Proof of the bound for the quadratic term

Let u be a scalar function in $\mathcal{H}_{1,s}$, which means that

$$u(\mathbf{x}, z) = \sum_{\mathbf{k}\in\Gamma} u_{\mathbf{k}}(z)e^{i\mathbf{k}\cdot\mathbf{x}},$$

with

$$\sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s||u_{\mathbf{k}}||_1^2 < \infty, \quad ||u_{\mathbf{k}}||_1^2 = \int_0^1 \left(|Du_{\mathbf{k}}|^2 + (1 + |\mathbf{k}|^2)|u_{\mathbf{k}}|^2\right) dz.$$

Assume now that u and v are scalar functions in $\mathcal{H}_{1,s}$, then

$$||uv||_{\mathcal{H}_{1,s}}^2 = \int_0^1 \sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s \left(|D(uv)_{\mathbf{k}}|^2 + (1 + |\mathbf{k}|^2)|(uv)_{\mathbf{k}}|^2\right) dz$$

$$\leq \int_0^1 \sum_{\mathbf{k}\in\Gamma}(1 + N_{\mathbf{k}}^2)^s \left(2|(vDu)_{\mathbf{k}}|^2 + 2|(uDv)_{\mathbf{k}}|^2 + (1 + |\mathbf{k}|^2)|(uv)_{\mathbf{k}}|^2\right).$$

We now use Lemma 4.1 and $(1+|\mathbf{l}+\mathbf{m}|^2) \le 2((1+|\mathbf{l}|^2)+2(1+|\mathbf{m}|^2)$ which gives

$$
\begin{aligned}
||uv||^2_{\mathcal{H}_{1,s}} \quad \le \quad & c_s^2 \int_0^1 \left(2 \sum_{\mathbf{l}\in\Gamma} (1+N_{\mathbf{l}}^2)^s |Du_{\mathbf{l}}|^2 \right) \left(\sum_{\mathbf{m}\in\Gamma} (1+N_{\mathbf{m}}^2)^s |v_{\mathbf{m}}|^2 \right) dz + \\
& + c_s^2 \int_0^1 \left(2 \sum_{\mathbf{l}\in\Gamma} (1+N_{\mathbf{l}}^2)^s |u_{\mathbf{l}}|^2 \right) \left(\sum_{\mathbf{m}\in\Gamma} (1+N_{\mathbf{m}}^2)^s |Dv_{\mathbf{m}}|^2 \right) dz + \\
& + c_s^2 \int_0^1 \left(2 \sum_{\mathbf{l}\in\Gamma} (1+N_{\mathbf{l}}^2)^s (1+|\mathbf{l}|^2) |u_{\mathbf{l}}|^2 \right) \left(\sum_{\mathbf{m}\in\Gamma} (1+N_{\mathbf{m}}^2)^s |v_{\mathbf{m}}|^2 \right) dz + \\
& + c_s^2 \int_0^1 \left(2 \sum_{\mathbf{l}\in\Gamma} (1+N_{\mathbf{l}}^2)^s |u_{\mathbf{l}}|^2 \right) \left(\sum_{\mathbf{m}\in\Gamma} (1+N_{\mathbf{m}}^2)^s (1+|\mathbf{m}|^2) |v_{\mathbf{m}}|^2 \right) dz.
\end{aligned}
$$

Now we can use

$$
\begin{aligned}
\int_0^1 |Du_{\mathbf{l}}|^2 |v_{\mathbf{m}}|^2 dz \quad & \le \quad c||u_{\mathbf{l}}||^2_{H^1} ||v_{\mathbf{m}}||^2_{H_1} \\
\int_0^1 (1+|\mathbf{l}|^2) |u_{\mathbf{l}}|^2 |v_{\mathbf{m}}|^2 dz \quad & \le \quad c(1+|\mathbf{l}|^2) ||u_{\mathbf{l}}||^2_{L^2} ||v_{\mathbf{m}}||^2_{H_1}
\end{aligned}
$$

and the similar symmetric estimates to show that there is a constant $c(s)$ such that

$$||uv||_{\mathcal{H}_{1,s}} \le c(s) ||u||_{\mathcal{H}_{1,s}} ||v||_{\mathcal{H}_{1,s}}.$$

Now, for $U \in \mathcal{K}_{2,s}$, we have all components of ∇V and $\nabla\theta$ which are in $\mathcal{H}_{1,s}$ and the above property for the scalar case applies to prove (19).

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