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A SIMPLE GLOBAL CHARACTERIZATION FOR NORMAL FORMS OF SINGULAR VECTOR FIELDS

C. ELPHICK[1]*, E. TIRAPEGUI[2], M.E. BRACHET[3,4], P. COULLET[1,3] and G. IOOSS[5]
[1]*Physique Théorique, Université de Nice, Parc Valrose, 06034 Nice, France. Unité Associée au CNRS*
[2]*Faculté des Sciences, Université Libre de Bruxelles, CP 226, 1050 Bruxelles, Belgique and Universidad de Chile, Facultad de Ciencias Fisicas y Matematicas, Departamento de Fisica, Santiago, Chile*
[3]*Observatoire de Nice, 06003 Nice, France*
[4]*Groupe de Physique des Solides, ENS, 24 rue Lhomond, 75231 Paris, France*
[5]*Laboratoire de Mathématiques, Université de Nice, Parc Valrose, 06034 Nice Cedex, France*

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We derive a new global characterization of the normal forms of amplitude equations describing the dynamics of competing order parameters in degenerate bifurcation problems. Using an appropriate scalar product in the space of homogeneous vector polynomials, we show that the resonant terms commute with the group generated by the adjoint of the original critical linear operator. This leads to a very efficient constructive method to compute both the nonlinear coefficients and the unfolding of the normal form. Explicit examples, and results obtained when there are additional symmetries, are also presented.

1. Introduction

The dynamics at the onset of several instabilities in a physical system undergoing a degenerate bifurcation near an equilibrium point can often be reduced to the temporal evolution of a simple set of ordinary differential equations. These so-called "amplitude equations" or "normal forms" describe the behavior of only those normal modes which are mildly unstable or slightly damped in linear theory. All the other, strongly damped, normal modes are eliminated from the description. This type of reduction, besides allowing an obvious simplification of the original problem, is also very useful for classification purposes. An identical set of amplitude equations is obtained for a class of original problems, it will thus display any dynamical behavior common to this class in a prototypical way. In this context, note that although the practical interest of the study of multiple bifurcations decreases with the degree of degeneracy, the study per se of the corresponding normal forms nevertheless presents a great interest because of the possibility of picking up very rich nonlinear behavior (see Arneodo et al. [1] and references therein). The present work deals with the derivation and the characterization of such degenerate normal forms.

Of course the problem of computing a normal form is not new; mathematical references can be found in, e.g., Arnol'd [3] or Guckenheimer and Holmes [11]. There are basically two methods to derive a normal form. With the first method [15, 12, 7] one first computes a locally invariant and attractive small dimensional manifold, the so-called *center manifold*, on which the dynamics reduces for large times. Then a nonlinear change of variables is done to put the small dimensional system into normal form. The second method (which has been used in nonlinear hydrodynamics since a long time, see [17]) has recently been greatly clarified, see [8] and references therein. With this method, one systematically expands the original fields in power of the amplitudes of linearly marginal modes, yielding both the normal form and the center

*Current address: Astronomy Department, Columbia University, New York, NY 10027, USA.

manifold. The crucial point in normal form computations is to find a homogeneous polynomial vector field of degree k in a space complementary to the range of the so-called "homological operator". The dimension of the vector space strongly increases with k and direct computations soon become impracticable [11]. Very recently Cushman and Sanders [9] used representation theory of the group $sl(2,\mathbb{R})$ which leads to the normal form in a constructive way.

As we shall see below, our approach is elementary and more direct since no splitting of the linear operator L_0 (see (13)) in the critical subspace is needed.

Let us recall that the resonant nonlinear terms of a normal form are those ones that cannot be eliminated by a nonlinear polynomial change of variable. Technically this means that they are in the kernel of the adjoint of the homological operator. This is just the Fredholm alternative which, as is well known, is independent of the nondegenerate scalar product used to define "adjoint" and "orthogonal" (see section 2.1). In this work we define a scalar product in the space of homogeneous polynomials such that the adjoint of the homological operator is the homological operator associated with the adjoint of the critical linear part of the original equation. Nonlinear terms of the normal form are thus equivariant under (i.e. commuting with) the group G generated by the adjoint of the original linear operator, this is our main result, given in section 2.2. In particular if the linear critical operator L_0 is diagonal then the whole normal form is found to be equivariant under G. For example in the case of the Hopf bifurcation G is the group of rotations in two dimensions and therefore the corresponding normal form is invariant under arbitrary rotations in the complex plane, property which in turn justifies the widely used argument that the invariance of the original system under time translations implies the invariance of the Hopf-normal form under rotations in the complex plane.

The general characterization obtained in section 2.1 allow us to interpret the resonant terms (i.e. equivariant under G) in the normal form as scalar renormalizations of the linear Arnol'd–Jordan unfolding and to derive a partial differential equation obeyed by these resonant terms. This gives us a very fast computational method that we apply to some classical examples in section 2.4 and to a more complicated one in the appendix. The rest of the paper is organized as follows: In section 2.3 we treat the case where the original problem has an additional symmetry. In section 3.1 we show how to use our method for the computation of the unfolding of the normal form. Finally, in section 3.2, we give the unfolding of the classical examples treated in section 2.4. Results for the special case where none of the critical eigenvalues has zero imaginary part are also given in the appendix.

2. Normal form of the unperturbed vector field

2.1. *The homological equation*

We study here evolution problems of the form

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \mathscr{F}(Z), \tag{1}$$

where Z belongs to the phase space $E = \mathbb{R}^n$ and $\mathscr{F}(0) = 0$. We write (1) in a finite dimensional space, with the understanding that much of the following analysis can be easily adapted for evolution partial differential systems, such as those occurring in non-linear hydrodynamical problems. In fact, all we need to assume is that there exists a stationary solution. In what follows we shall avoid, as much as possible, the

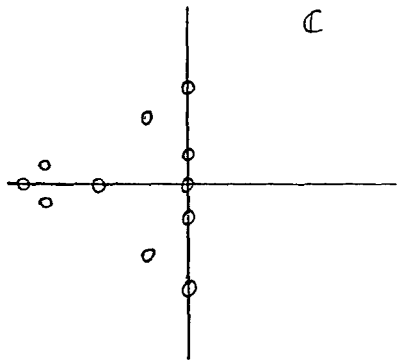

Fig. 1. A typical distribution of the spectrum of $\mathscr{L}$.

use of explicit coordinates such as $\mathscr{F}_i(z_1, \ldots, z_n)$ or $\mathscr{L}_{ij} = \partial\mathscr{F}_i/\partial z_j$. To wit, let us define

$$\mathscr{L} = \mathrm{D}_Z\mathscr{F}(0) \tag{2}$$

to be the linear operator $E \to E$ whose matrix elements are given by $\partial\mathscr{F}_i/\partial z_j$.

We assume that the set of eigenvalues of $\mathscr{L}$ is composed of two pieces, one on the imaginary axis and the other with a strictly negative real part (fig. 1). The decomposition of the space E, associated with this decomposition of the spectrum of $\mathscr{L}$, is written [14] as

$$E = E_0 \oplus E_- \tag{3}$$

and the restrictions of $\mathscr{L}$ to these invariant subspaces are denoted by L_0 and L_-.

The problem of finding a normal form for (1) reads: "Find a polynomial Φ in (X, Y) taking values in E, and a polynomial F in X taking values in E_0, *as simple as possible*, such that we can write

$$Z = X + Y + \Phi(X, Y), \quad X \in E_0, Y \in E_-, \tag{4}$$

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t} &= L_0 X + F(X) + \mathcal{O}\big((|X| + |Y|)^P\big), \\ \frac{\mathrm{d}Y}{\mathrm{d}t} &= L_- Y + N(X, Y) + \mathcal{O}(|X|^P), \end{aligned} \tag{5}$$

where

$$N(X, Y) = \mathcal{O}(|Y|(|X| + |Y|)), \qquad F(X) = \mathcal{O}(|X|^2), \qquad \Phi(X, Y) = \mathcal{O}\big((|X| + |Y|)^2\big) \tag{6}$$

and P is arbitrarily large (but fixed)". The estimate for N in (6) means that the set $Y = 0$ is invariant under the dynamics of (5) up to order P. This is a direct consequence of the form for N in $(10)_4$.

Remark 0. More precisely our program is to find a set of polynomials P_j which generate the invariant algebra of the group given by the flow of the linear vector field $L^*{}_0X$ (* denotes the usual adjoint operation in E_0) such that we can write (see Theorem 3)

$$F(X) = \sum_{i=1}^{\dim(E_0)} \alpha_i(p_j)v_i,$$

where the α_i are rational fractions of the p_j and the v_i are fixed vector valued polynomials.

It is worth noting that in all the examples treated here we can in fact write [9]

$$F(X) = \sum_{i=1}^{n(\dim(E_0))} \tilde{\alpha}_i(p_j)\tilde{v}_i,$$

where $\tilde{\alpha}_i$ are polynomials in the p_j and $\tilde{v}_i$ are fixed vector valued polynomials.

We will say that a singular vector field is in normal form if its non-linear part can be written in any of the two forms for F given above. This will be equivalent (see section 2) to the fact that the Lie derivative of F (in particular of v_i, $\tilde{v}_i$) along L_0^*X vanishes. Let us note that if L_0 is diagonal and F is in normal form then also the Lie derivative of $L_0X + F$ along L_0^*X vanishes.

In what follows we often refer to the normal form of (1) while in fact considering the truncated equation $(5)_1$:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = L_0X + F(X), \tag{7}$$

which in fact is the object of fundamental interest for further analysis of the dynamics of system (1).

Remark 1. The manifold in E given by the equation

$$Z = X + \Phi(X,0) \tag{8}$$

is the approximation, up to order $|X|^P$, of a *center manifold* [15, 12, 7] for (1). It is tangent to the subspace E_0 at 0 and it has the dimension of E_0. It is clear that, if we use (7) instead of $(5)_1$ (neglecting $\mathcal{O}((|X| + |Y|)^P)$, this manifold is locally invariant since Y remains equal to 0. It is also locally attracting, due to the negativeness of the real parts of the eigenvalues of L_-.

Remark 2. The manifold in E given by the equation

$$Z = Y + \Phi(0,Y), \quad Y \in E_- \tag{9}$$

is the approximation up to the order $|Y|^P$ of the *stable manifold* of 0. It is tangent to E_- at 0, and for any initial data on it, $Z(t)$ will relax to 0 exponentially when $t \to \infty$ with the truncated equation $(5)_2$ (without $\mathcal{O}(|X|^P)$).

Let us start by defining the Taylor expansions of $\mathcal{F}$, Φ, F, N as follows:

$$\begin{aligned}
\mathcal{F}(Z) &= \mathcal{L}Z + \sum_{k\geq 2} \mathcal{F}_k[Z^{(k)}],\\
\Phi(X,Y) &= \sum_{p+q\geq 2} \Phi_{pq}[X^{(p)}, Y^{(q)}],\\
F(X) &= \sum_{p\geq 2} F_p[X^{(p)}],\\
N(X,Y) &= \sum_{\substack{p+q\geq 2\\ q>0}} N_{pq}[X^{(p)}, Y^{(q)}],
\end{aligned} \tag{10}$$

where F_p, $\mathcal{F}_p$ are p-linear symmetric in its arguments, and Φ_{pq}, N_{pq} are p-linear symmetric in the X variable, and q-linear symmetric on Y. $Z^{(k)}$ stands for the repetition of k identical arguments Z, equivalent notations are used for $X^{(p)}$ and $Y^{(q)}$. $\mathcal{F}_k[Z^{(k)}]$ is thus a vector valued homogeneous polynomial of degree k in the components of Z.

In order to obtain equations for Φ_{pq}, F_p, N_{pq} we need to identify the expansions in (X, Y) of $\mathcal{F}(X+Y+\Phi(X,Y))$ and of $\mathrm{d}X/\mathrm{d}t + \mathrm{d}Y/\mathrm{d}t + \mathrm{D}_X\Phi(X,Y)\cdot \mathrm{d}X/\mathrm{d}t + \mathrm{D}_Y\Phi(X,Y)\cdot \mathrm{d}Y/\mathrm{d}t$ where $\mathrm{d}X/\mathrm{d}t$ and $\mathrm{d}Y/\mathrm{d}t$ are replaced by the right-hand side of (5) (the notation $\mathrm{D}_X\Phi(X,Y)$ stands for the linear operator acting from E_0 into E, whose matrix is given by $(\partial\Phi_i/\partial X_j)$).

The identification at the order 1 in X and Y is just a verification:

$$\mathcal{L}(X+Y) = L_0X + L_-Y. \tag{11}$$

Next, higher orders $p+q\geq 2$ lead to identities of the form

$$\mathcal{L}\Phi_{pq}[X^{(p)}, Y^{(q)}] - \mathrm{D}_X\Phi_{pq}[X^{(p)}, Y^{(q)}]\cdot L_0X - \mathrm{D}_Y\Phi_{pq}[X^{(p)}, Y^{(q)}]\cdot L_-Y$$
$$= \begin{cases} F_p[X^{(p)}] & q=0 \\ N_{pq}[X^{(p)}, Y^{(q)}] & q>0 \end{cases} + R_{pq}[X^{(p)}, Y^{(q)}], \tag{12}$$

where R_{pq} only depends on $\{\Phi_{p'q'}, F_{p'}, N_{p'q'};\ p'+q' \leq p+q-1\}$. The strategy is to solve (12) step by step, starting with $p+q=2$, and then increasing $p+q$ by 1 at each step.

In what follows we denote by P_0 and P_- the two projections on E_0 and E_- which commute with $\mathcal{L}$ [14]. We have

$$P_0 + P_- = \mathrm{Id}, \qquad P_0L_- = 0, \qquad P_-L_0 = 0, \qquad \mathcal{L}P_0 = L_0, \qquad \mathcal{L}P_- = L_-. \tag{13}$$

Eq. (12) may be decomposed, yielding for $q=0$,

$$L_0P_0\Phi_{p0}[X^{(p)}] - \mathrm{D}_XP_0\Phi_{p0}[X^{(p)}]\cdot L_0X = F_p[X^{(p)}] + P_0R_{p0}[X^{(p)}], \tag{14}$$

$$L_-P_-\Phi_{p0}[X^{(p)}] - \mathrm{D}_XP_-\Phi_{p0}[X^{(p)}]\cdot L_0X = P_-R_{p0}[X^{(p)}]. \tag{15}$$

For $q \neq 0$, eq. (12) gives

$$L_0 P_0 \Phi_{pq}[X^{(p)}, Y^{(q)}] - \mathrm{D}_X P_0 \Phi_{pq}[X^{(p)}, Y^{(q)}] \cdot L_0 X - \mathrm{D}_Y P_0 \Phi_{pq}[X^{(p)}, Y^{(q)}] \cdot L_- Y$$
$$= P_0 R_{pq}[X^{(p)}, Y^{(q)}], \tag{16}$$
$$L_- P_- \Phi_{pq}[X^{(p)}, Y^{(q)}] - \mathrm{D}_X P_- \Phi_{pq}[X^{(p)}, Y^{(q)}] \cdot L_0 X - \mathrm{D}_Y P_- \Phi_{pq}[X^{(p)}, Y^{(q)}] L_- Y$$
$$= N_{pq}[X^{(p)}, Y^{(q)}] + P_- R_{pq}[X^{(p)}, Y^{(q)}], \tag{17}$$

where R_{pq} is known. Since in (17) there is no restriction on N_{pq}, we can choose any $P_- \Phi_{pq}$, for instance 0, and then (17) gives directly N_{pq}. Now the eqs. (15) and (16) are explicitly soluble. They lead to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{L_- t} P_- \Phi_{p0}\left[(\mathrm{e}^{-L_0 t} X)^{(p)}\right]\right] = \mathrm{e}^{L_- t} P_- R_{p0}\left[(\mathrm{e}^{-L_0 t} X)^{(p)}\right], \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-L_0 t} P_0 \Phi_{pq}\left[(\mathrm{e}^{L_0 t} X)^{(p)}, (\mathrm{e}^{L_- t} Y)^{(q)}\right]\right] = -\mathrm{e}^{-L_0 t} P_0 R_{pq}\left[(\mathrm{e}^{L_0 t} X)^{(p)}, (\mathrm{e}^{L_- t} Y)^{(q)}\right], \end{aligned} \tag{18}$$

hence

$$\begin{aligned} &P_- \Phi_{p0}[X^{(p)}] = -\int_0^\infty \mathrm{d}t\, \mathrm{e}^{L_- t} P_- R_{p0}\left[(\mathrm{e}^{-L_0 t} X)^{(p)}\right], \\ &P_0 \Phi_{pq}[X^{(p)}, Y^{(q)}] = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-L_0 t} P_0 R_{pq}\left[(\mathrm{e}^{L_0 t} X)^{(p)}, (\mathrm{e}^{L_- t} Y)^{(q)}\right], \quad q > 0, \end{aligned} \tag{19}$$

where the integrals are convergent, due to the exponential decay of $\mathrm{e}^{L_- t}$ as $t \to \infty$.

Next we need to solve eq. (14) with F_p as "simple as possible". This equation takes the form

$$\left[P_0 \Phi_{p0}[X^{(p)}], L_0 X\right] = P_0 R_{p0}[X^{(p)}] + F_p[X^{(p)}], \tag{20}$$

where the left-hand side is the Poisson–Lie bracket of two vector fields in E_0. Note that (20) is called the "homological equation" in the standard mathematical literature [3, 10].

Remark 3. Looking at (17) we might think about the possibility of choosing N_{pq} as simple as possible, by choosing suitably $P_- \Phi_{pq}$. This equation is also a "homological equation", slightly more complicated than (20). It is shown in [3, 10] that the linear operator acting on $P_- \Phi_{pq}$ has eigenvalues given by the following combinations:

$$\lambda_k^{(-)} - \sum_r p_r \lambda_r^{(0)} - \sum_l q_l \lambda_l^{(-)},$$

where $\lambda_k^{(0)}$ and $\lambda_k^{(-)}$ are the eigenvalues of L_0 and L_-, respectively, p_r and q_l are integers such that $\Sigma_r p_r = p$, $\Sigma_l q_l = q$.

If 0 is not an eigenvalue, we can choose $N_{pq} = 0$. We cannot avoid 0 to be an eigenvalue for $q = 1$, but if the following "non-resonant conditions" are realized:

$$\lambda_k^{(-)} \neq \sum_r p_r \lambda_r^{(0)} + \sum_l q_l \lambda_l^{(-)} \tag{21}$$

for any k, r, l such that

$$2 \leq \sum_r p_r + \sum_l q_l \leq P-1, \quad \sum_l q_l \neq 1,$$

then we can find

$$N(X, Y) \text{ linear in } Y, \text{ up to order } (|X| + |Y|)^P. \tag{22}$$

Nevertheless, we have to stress that these conditions are impossible to check in the infinite dimensional case, since L_- has in general infinitely many eigenvalues.

2.2. *Simple characterization of the normal form*

In this section we solve (14) (or (20)) with the simplest possible F_p. Let us introduce some useful notations. We denote by H_k the vector space of homogeneous polynomials of degree k in $X \in E_0$, taking values in E_0. We denote by $\mathscr{H}_k$ the space and scalar homogeneous polynomials of degree k in $X \in E_0$. We can then write

$$H_k = \mathscr{H}_k \otimes E_0. \tag{23}$$

Note that the eq. (14) is in fact a linear equation of the form

$$\mathscr{A}^{(k)}(P_0 \Phi_{k0}) = P_0 R_{k0} + F_k \tag{24}$$

in H_k. Thus the right-hand side of (24) has to belong to the image of $\mathscr{A}^{(k)}$ in H_k. This gives a condition for F_k which may be chosen in a complementary space of this image. The idea in what follows is to choose a suitable scalar product in the space H_k, allowing a simple computation of the adjoint operator of $\mathscr{A}^{(k)}$. Since the kernel of the adjoint $\mathscr{A}^{(k)*}$ is a complementary space of image $(\mathscr{A}^{(k)})$, a good choice is to take F_k belonging to this space.

In what follows we choose a real basis of E_0 and write $X = (x_1, \ldots, x_N)$. For two scalar polynomials P and Q, let us define

$$\langle P | Q \rangle = P(\partial) Q(X)|_{X=0}. \tag{26}$$

where ∂_j means $\partial / \partial x_j$. For instance we have

$$\langle x^\alpha | x^\beta \rangle = \frac{\partial^\alpha}{\partial x^\alpha}(x^\beta) = \alpha! \delta_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol. Now $\langle P | Q \rangle$ defines a *scalar product* on the space of scalar polynomials in X. We write it symbolically

$$\langle P | Q \rangle = P(\partial) Q(X)|_{X=0}. \tag{26}$$

This scalar product was introduced in quantum mechanics and studied in particular by Bargmann [4], explicited by using the standard L^2 scalar product. This basic property is that

$$\langle QR|P\rangle = Q(\partial)R(\partial)P(X)|_{X=0} = R(\partial)Q(\partial)P(X)|_{X=0} = \langle R|Q(\partial)P\rangle, \tag{27}$$

i.e. multiplication by the polynomial Q is the adjoint of the differentiation by $Q(\partial)$.

The corresponding scalar product in $\mathscr{H}_k$ is obtained by taking homogeneous polynomials P and Q of degree k. We can now endow the space H_k of vector polynomials with the natural scalar product

$$(V|W)_{H_k} = \sum_{j=1}^{N} \langle V_j|W_j\rangle, \tag{28}$$

where $V=(V_1,\dots,V_N)$ and $W=(W_1,\dots,W_N)\in H_k$ and V_j, W_j are homogeneous polynomials of degree k in $X=(x_1,\dots,x_N)$.

Let us consider a linear invertible operator A in E_0; we then have for any polynomials P and Q,

$$\langle P(AX)|Q(X)\rangle = \langle P(X)|Q(A^*X)\rangle. \tag{29}$$

In fact, if we set $Y=A^*X$, it is easy to see that $\partial_X = A\partial_Y$, hence the right-hand side of (29) is just

$$P(A\partial_Y)Q(Y)|_{Y=0} = \langle P_0 A|Q\rangle.$$

Now, by construction of the scalar product in H_k, we have for any V and W in H_k,

$$\left(A^{-1}V(AX)|W(X)\right)_{H_k} = \left(V(AX)|A^{*-1}W(X)\right)_{H_k} = \left(V(X)|A^{*-1}W(A^*X)\right)_{H_k}, \tag{30}$$

where we used (29) to get the last identity.

Let us choose $A=e^{L_0 t}$ in (30), then we have

$$\left(e^{-L_0 t}V(e^{L_0 t}X)|W(X)\right)_{H_k} = \left(V(X)|e^{-L_0^* t}W\left(e^{L_0^* t}X\right)\right)_{H_k}, \tag{31}$$

where we have used the elementary result

$$(e^{L_0 t})^* = e^{L_0^* t}.$$

Now differentiating (31) with respect to t, at $t=0$, we readily obtain

$$(\mathscr{A}^{(k)}V|W)_{H_k} = (V|\mathscr{A}_*^{(k)}W)_{H_k}, \tag{32}$$

where

$$\begin{aligned}[\mathscr{A}_*^{(k)}W][X^{(k)}] &= L_0^* W[X^{(k)}] - \mathrm{D}_X W[X^{(k)}]\cdot L_0^* X \\ &= \left[W[X^{(k)}], L_0^* X\right].\end{aligned} \tag{33}$$

This identity shows that the adjoint of the homological operator $\mathscr{A}^{(k)}$ in H_k, built with L_0, is the homological operator in H_k built with the adjoint L_0^*. So we have proved the following:

Theorem 1. The following decomposition holds for the space H_k of homogeneous vector polynomials of degree k in $X \in E_0$:

$$H_k = \operatorname{Im} \mathscr{A}^{(k)} \oplus \operatorname{Ker} \mathscr{A}_*^{(k)},$$

where

$$\operatorname{Ker} \mathscr{A}_*^{(k)} = \left\{ V \in H_k; \mathrm{e}^{-L_0^* t} V\left[\left(\mathrm{e}^{L_0^* t} X\right)^{(k)}\right] = V[X^{(k)}], \quad \forall t \in \mathbb{R}, \forall X \in E_0 \right\}.$$

Summing up this result for all p, we obtain:

Theorem 2. *General characterization of the normal form.* A normal form $F(X)$ of the nonlinear terms of (7) can be found such that F commutes with $\exp(L_0^* t)$, $t \in \mathbb{R}$, where L_0^* is the adjoint of L_0 in the invariant subspace E_0. An equivalent characterization is the partial differential system

$$\mathrm{D}_X F(X) \cdot L_0^* X - L_0^* F(X) = 0. \tag{34}$$

Remark. We note, after (31), (32), that

$$\left[\exp\left(\mathscr{A}_*^{(k)} t\right) \cdot V\right][X^{(k)}] = \mathrm{e}^{L_0^* t} V\left[\left(\mathrm{e}^{-L_0^* t} X\right)^{(k)}\right]. \tag{35}$$

It is an elementary result that

$$\left\{\exp\left(\mathscr{A}_*^{(k)} t\right) \cdot V \text{ independent of } t\right\} \sim \left\{V \in \operatorname{Ker} \mathscr{A}_*^{(k)}\right\}.$$

Corollary 2. If L_0 is *diagonalizable*, then

$$\operatorname{Ker}\left(\mathscr{A}_*^{(k)}\right) = \operatorname{Ker}\left(\mathscr{A}^{(k)}\right)$$

and a normal form may be found such that it commutes with $\exp(L_0 t)$, $t \in \mathbb{R}$.

Proof. If L_0 is diagonal in a suitable basis (complexified E_0), then $L_0^* = -L_0$. Hence the result is obvious.

Example. Let us assume that L_0 has two pairs of simple pure imaginary eigenvalues $\pm \mathrm{i}\omega_0$, $\pm \mathrm{i}\omega_1$, and let us define $X = (z_0, \bar{z}_0, z_1, \bar{z}_1)$. The components of the normal form are sums of monomials of the form $z_0^p \bar{z}_0^q z_1^r \bar{z}_1^s$. It is easy to see that

$$\mathrm{e}^{L_0^* t} X = \left(\mathrm{e}^{-\mathrm{i}\omega_0 t} z_0, \mathrm{e}^{\mathrm{i}\omega_0 t} \bar{z}_0, \mathrm{e}^{-\mathrm{i}\omega_1 t} z_1, \mathrm{e}^{\mathrm{i}\omega_1 t} \bar{z}_1\right),$$

hence the commutation property of F,

$$\mathrm{e}^{-L_0^* t} F\left(\mathrm{e}^{L_0^* t} X\right) = F(X), \quad \forall t \in \mathbb{R}, \forall X \in E_0, \tag{36}$$

leads, for its first component, to coefficients such that

$$\omega_0(p-q-1)+\omega_1(r-s)=0 \tag{37}$$

and for the third component,

$$\omega_0(p-q)+\omega_1(r-s-1)=0. \tag{38}$$

We immediately see that if ω_0/ω_1 is irrational, we only need to keep in the first component the coefficients such that $p=q+1$, $r=s$ and in the third one $r=s+1$, $p=q$. This shows that in this case the normal form is equivariant under the group T^2, and the trajectories $\{e^{L_0^* t}X;\ t\in\mathbb{R}\}$ are dense on the torus T^2.

Now, if $\omega_0/\omega_1=m/n$, where $(m,n)=1$ (m and n have no common divisor), then the coefficients of the first component of the normal form are such that

$$p=q+1+kn,\quad r=s-km,\quad k\in\mathbb{Z}. \tag{39}$$

This shows that the normal form (7) is (2 complex equations)

$$\begin{aligned}\frac{\mathrm{d}z_0}{\mathrm{d}t}&=\mathrm{i}\omega_0 z_0+z_0P_0(z_0\bar z_0,\ z_1\bar z_1,\ z_0^n\bar z_1^m)+\bar z_0^{n-1}z_1^mP_1(z_0\bar z_0,\ z_1\bar z_1,\ \bar z_0^n z_1^m),\\ \frac{\mathrm{d}z_1}{\mathrm{d}t}&=\mathrm{i}\omega_1 z_1+z_1Q_0(z_0\bar z_0,\ z_1\bar z_1,\ \bar z_0^n z_1^m)+z_0^n\bar z_1^{m-1}Q_1(z_0\bar z_0,\ z_1\bar z_1,\ z_0^n\bar z_1^m),\end{aligned} \tag{40}$$

where P_0, P_1, Q_0, Q_1 are polynomials in their arguments.

Hence the normal form is in this case invariant under the subgroup of T^2: $(\theta_0,\theta_1)\to(\theta_0+ms,\theta_1+ns)$, $s\in\mathbb{R}$ ($\sim\mathrm{T}^1$).

Remark. In this example, corollary 2 applies.

Another very fruitful way to compute the normal form $F(X)$ is to use the linear partial differential equation (34) satisfied by F. Choosing a basis in E_0, complex in general when we use the Jordan decomposition of L_0, this equation may be written more explicitely as follows:

$$\sum_{j,l=1}^{N}\frac{\partial F_i}{\partial x_j}(X)\bar L_{0lj}x_l-\sum_{j=1}^{N}\bar L_{0ji}F_j(X)=0,\quad i=1,\dots,N. \tag{41}$$

The characteristic system [20] associated with (41) is the following:

$$\frac{\mathrm{d}x_j}{\sum_l \bar L_{0lj}x_l}=\frac{\mathrm{d}F_i}{\sum_l \bar L_{0li}F_l},\quad i,j=1,\dots,N \tag{42}$$

and the characteristic curves are the trajectories in E_0 defined by

$$\left\{e^{L_0^* t}X;\ t\in\mathbb{R}\right\}.$$

So that, if we write (42) $=\mathrm{d}t$, we immediately recover (36).

Now, instead of computing $\exp(L_0^* t)$ and using (36), we can use the *independent first integrals* of (42). For this we need to put L_0^* into Jordan form (this also facilitates the computation of $\exp(L_0^* t)$). Then to compute the normal form we can use the following result:

Theorem 3. *Representation of the normal form.* The normal form $F(X)$ of theorem 2 may be made explicit as follows:

$$F(X) = \sum_{j=1}^{N} \alpha_j(X) \mathscr{L}_j X, \tag{43}$$

where $\mathscr{L}_j$, $j = 1, \ldots, N$ are linear operators commuting with L_0^* in E_0, defined by (44), and the scalar functions α_j, $j = 1, \ldots, N$ are rational fractions, first integrals of the characteristic system $\mathrm{d}X/\mathrm{d}t = L_0^* X$.

Remark. It is easy to give a little more precise characterization of the rational fractions α_j. See the proof of the theorem above and Appendix A.2 for the explicit form of the α_j.

Proof of theorem 3. Let us choose N linear operators $\mathscr{L}_1, \ldots, \mathscr{L}_N$ commuting with L_0^* (they then belong to $\operatorname{Ker} \mathscr{A}_*^{(1)}$) such that for almost all X, the system $\{\mathscr{L}_j X;\ j = 1, \ldots, N\}$ forms a basis of E_0. Assume that L_0^* is in Jordan form, with r blocks, each block L_{0j}^* corresponds to an eigenvalue λ_j ($\operatorname{Re} \lambda = 0$) and to an invariant subspace E_{0j}. Define the projection P_j on E_{0j} such that

$$\sum_{j=1}^{r} P_j = \operatorname{Id}_{E_0}$$

and denote by ν_j the dimension of E_{0j}. Then, the linear operators

$$P_j, (L_{0j}^* - \lambda_j) P_j, \ldots, (L_{0j}^* - \lambda_j)^{\nu_j - 1} P_j, \quad j = 1, \ldots, r \tag{44}$$

are N linearly independent linear operators $\mathscr{L}_k$ commuting with L_0^*. It is easy to check, that if X has a nonzero first component in each E_{0j}, then the system $\{\mathscr{L}_j X;\ J = 1, \ldots, N\}$ space E_0. We can then write (43) for almost all X in E_0.

Now, since

$$L_0^* \mathscr{L}_j = \mathscr{L}_j L_0^*, \quad j = 1, \ldots, N,$$

we also have

$$\mathrm{e}^{L_0^* t} \mathscr{L}_j = \mathscr{L}_j\, \mathrm{e}^{L_0^* t}, \quad j = 1, \ldots, N,$$

hence the commutation relation of F (36) with $\mathrm{e}^{L_0^* t}$ leads to

$$\alpha_j\left(\mathrm{e}^{L_0^* t} X\right) = \alpha_j(X), \quad j = 1, \ldots, N.$$

So, the functions α_j are first integrals of the characteristic system. Now, $F(X)$ is a vector polynomial in X. Computing the decomposition (43) of $F(X)$ on the basis $\{\mathscr{L}_j X;\ j = 1, \ldots, N\}$ we easily observe that the coefficients are rational fractions of the components of X whose denominators are powers $\leq \nu_j$ of the first component of each E_{0j} space (see Appendix A.2).

*Remark A**. The simple examples treated in section 2.4, all lead to the following type of normal form:

$$F(X) = \sum_j \beta_j(X)V_j + \sum_j \alpha_j(X)\mathscr{L}_j X, \tag{45}$$

where the scalar functions α_j and β_j are *polynomial* first integrals of the characteristic system, and where $\{V_j\}$ are eigenvectors of L_0^* while $\{\mathscr{L}_j\}$ are all the linear operators commuting with L_0^* ($\mathrm{Ker}\,\mathscr{A}_*^{(1)}$). In fact, theorem 3 shows that we can always find the α_j's as rational fractions. For instance, in the normal form (40) the possible denominators are z_0 and z_1. Note that if we want to express in this example the α_j's as *polynomials of the scalar invariants* $z_0\bar{z}_0$, $z_1\bar{z}_1$, $z_0^n\bar{z}_1^m$, $\bar{z}_0^n z_1^m$ then the possible denominators are $z_0\bar{z}_0$ and $z_1\bar{z}_1$. We shall see a more complicated example in Appendix A.2.

When (45) holds with polynomials α_j and β_j, it is possible to give an obvious physical interpretation: the effect of the nonlinearities is simply to renormalize the increasing rates, and linear coupling terms (see Arnol'd [3] for the affine normal form). This renormalization is done via scalars of the group $\exp(L_0^* t)$. It is nice that this holds for all low codimension examples given in section 2.4.

2.3. *Case of an additional symmetry*

Let us assume that the system (1) is invariant under the representation of some symmetry group. The simplest case is when we only have a symmetry $S \neq \mathrm{Id}$ such that $S^2 = \mathrm{Id}$.

For the moment let us assume that we have a linear invertible operator T in E such that

$$\mathscr{F}(TZ) = T\mathscr{F}(Z). \tag{46}$$

Then T commutes with $\mathscr{L}$, L_0, L_-, and with any derivative of $\mathscr{F}$ at the origin. Hence it is clear that T_0 and $\mathscr{A}^{(k)}$ commutes in the following sense: let us define for any V in H_k,

$$T_{0*}V[X^{(k)}] = T_0^{-1}V\left[(T_0X)^{(k)}\right], \quad T_0 = T|_{E_0},$$

then it is easy to verify that

$$T_{0*}\mathscr{A}^{(k)}(V) = \mathscr{A}^{(k)}T_{0*}(V).$$

It follows that the *image of $\mathscr{A}^{(k)}$ is invariant under T_{0*}*, as well as the kernel of $\mathscr{A}^{(k)}$.

Let us moreover assume that

$$T_0 = T|_{E_0} \quad \text{is a unitary operator on } \mathrm{E}_0, \tag{47}$$

i.e. $T_0^* = T_0^{-1}$ on E_0, then T_0 computes with L_0^* since

$$T_0^{-1}L_0 = L_0T_0^{-1} \quad \text{leads to} \quad L_0^*T_0 = T_0L_0^*.$$

As a consequence, on H_k the linear operator T_{0*} commutes with $\mathscr{A}_*^{(k)}$, hence the kernel of $\mathscr{A}_*^{(k)}$ as well as the image of $\mathscr{A}^{(k)}$ are invariant under T_{0*}, which commutes with the projection operators defined in theorem 1.

It results that the normal form defined in theorem 2 commutes also with T_0. We have proved the following theorem:

Theorem 4. If there exists a linear invertible operator T which commutes with the vector field $\mathscr{F}$, and if its representation T_0 on the invariant subspace E_0 belonging to the eigenvalues of zero real part is a unitary operator, then a normal form can be found which commutes with T_0 as well as with $e^{L_0^* t}$, $t \in \mathbb{R}$.

If all eigenvalues of L_0 are semi-simple (L_0 diagonalizable), the additional assumption on T_0 is useless, because we can avoid to project orthogonally onto $\operatorname{Ker} \mathscr{A}_*^{(k)}$, by projecting in a more natural way on $\operatorname{Ker} \mathscr{A}^{(k)}$ which, in this case, is complementary to $\operatorname{Image}(\mathscr{A}^{(k)})$ (see [13]). The natural projection is then defined for any V in H_k by

$$P(V) = \lim_{T \to \infty} \frac{1}{T} \int_0^T (e^{L_0 t})_* V \, dt. \tag{48}$$

We have just showed the following:

Corollary 4. In the case when L_0 is diagonalizable, and if there exists a linear invertible operator T commuting with the vector field $\mathscr{F}$, then a normal form may be found which commutes with T_0 as well as with the group $e^{L_0 t}$, $t \in \mathbb{R}$.

For an example with a symmetry and a non semi-simple eigenvalue see section 2.4.3.

2.4. *Examples*

Note that examples 2.4.1, 2.4.2, 2.4.3, 2.4.4 are also treated by Cushman and Sanders in [9], using a different, very elegant method. Our method is seen to lead to more elementary and shorter computations.

2.4.1. ζ^2 *singularity* (*see* [2, 8])

We have

$$L_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

hence $X = (x, y)$ and $F(X) = (F_1(x, y), F_2(x, y))$. Eq. (34) may be written as

$$x \frac{\partial F_1}{\partial y} = 0, \quad x \frac{\partial F_2}{\partial y} = F_1.$$

This leads to the solution

$$F_1(x, y) = x\varphi_1(x), \quad F_2(x, y) = y\varphi_1(x) + \varphi_2(x), \tag{49}$$

where φ_1 and φ_2 are polynomials (see Appendix A.3 for the proof).

This normal form is in the frame of remark A* of section 2.2, (see (45)) with

$$F(X)=\beta(x)\begin{pmatrix}0\\1\end{pmatrix}+\alpha_1(x)\begin{pmatrix}x\\y\end{pmatrix}+\alpha_2(x)\begin{pmatrix}0\\x\end{pmatrix}$$

and

$$\varphi_1(x)=\alpha_1(x),\quad \varphi_2(x)=\beta(x)+\alpha_2(x)$$

For terms of degree k, we obtain a subspace of H_k which is 2-dimensional:

$$(ax^k,\ ayx^{k-1}+bx^k) \tag{50}$$

Note that we can change the normal form by choosing another projection (no longer orthogonal in H_k). In fact if we add to (50) the term $(-ax^k, kax^{k-1}y)$, which is orthogonal to $\operatorname{Ker}\mathscr{A}_*^{(k)}$ (hence it is in $\operatorname{Image}(\mathscr{A}_*^{(k)})$), this leads to a simpler normal form:

$$(0,\ a'yx^{k-1}+bx^k). \tag{51}$$

Hence it is possible to write (7) for this example in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=xyP_1(x)+x^2P_2(x), \tag{52}$$

where P_1 and P_2 are polynomials in x.

2.4.2. *ζ^3 singularity*

This example is also treated in [9]. Here we have

$$L_0=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}.$$

Let us denote by x_1, x_2, x_3 the three components of X and by F_1, F_2, F_3 the 3 components of F. Then (34) is written as follows:

$$\begin{aligned}&x_1\frac{\partial F_1}{\partial x_2}+x_2\frac{\partial F_1}{\partial x_3}=0,\\&x_1\frac{\partial F_2}{\partial x_2}+x_2\frac{\partial F_2}{\partial x_3}=F_1,\\&x_1\frac{\partial F_3}{\partial x_2}+x_2\frac{\partial F_2}{\partial x_3}=F_2.\end{aligned} \tag{53}$$

Here, the characteristic system is

$$\frac{\mathrm{d}x_1}{0}=\frac{\mathrm{d}x_2}{x_1}=\frac{\mathrm{d}x_3}{x_2}=\frac{\mathrm{d}F_1}{0}=\frac{\mathrm{d}F_2}{F_1}=\frac{\mathrm{d}F_3}{F_2} \tag{54}$$

and the first integrals are

$$x_1, \quad x_2^2 - 2x_1x_3, \quad F_1, x_1F_2 - x_2F_1, \quad x_1F_3 + x_3F_1 - x_2F_2. \tag{55}$$

Hence, the general solution of (53) takes the form

$$\begin{aligned} F_1(x_1, x_2, x_3) &= x_1\varphi_1(x_1, x_2^2 - 2x_1x_3), \\ F_2(x_1, x_2, x_3) &= x_2\varphi_1(x_1, x_2^2 - 2x_1x_3) + x_1\varphi_2(x_1, x_2^2 - 2x_1x_3), \\ F_3(x_1, x_2, x_3) &= x_3\varphi_1(x_1, x_2^2 - 2x_1x_3) + x_2\varphi_2(x_1, x_2^2 - 2x_1x_3) + \varphi_3(x_1, x_2^2 - 2x_1x_3). \end{aligned} \tag{56}$$

It is not very hard to show that φ_j, $j = 1,2,3$ are polynomials in their arguments (Appendix A.3), so we remark that (56) enters into the framework of remark A* of section 2.2, provided that we write

$$\begin{pmatrix} 0 \\ 0 \\ \varphi_3 \end{pmatrix} = \beta(X) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha_3(X)\mathscr{L}_3X, \quad \mathscr{L}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where α_3 and β are polynomials in X, ands $\mathscr{L}_3L_0^* = L_0^*\mathscr{L}_3$.

Here, like in example 2.4.1, we can change the projection, and choose a normal form such that (7) becomes

$$\begin{aligned} &\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3, \\ &\frac{dx_3}{dt} = x_3P_1(x_1, x_2^2 - 2x_1x_3) + x_2P_2(x_1, x_2^2 - 2x_1x_3) + P_3(x_1, x_2^2 - 2x_1x_3), \end{aligned} \tag{57}$$

where P_j, $j = 1,2$, are polynomials in their arguments starting at degree 1, while P_3 is a polynomial starting at degree 2.

2.4.3. $\zeta^2\zeta^2$ *singularity*

This example is also treated in [9]. Their result, once simplified as in (62), is the same as ours. Here we have

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us denote by x_j and F_j, $j = 1,2,3,4$ the components of X and of F in the normal form (7). Then the characteristic system (20) becomes

$$\frac{dx_1}{0} = \frac{dx_2}{x_1} = \frac{dx_3}{0} = \frac{dx_4}{x_3} = \frac{dF_1}{0} = \frac{dF_2}{F_1} = \frac{dF_3}{0} = \frac{dF_4}{F_3}. \tag{58}$$

First integrals are given by

$$x_1,\quad x_3,\quad v=x_2x_3-x_1x_4,\quad F_1,\quad F_3,\quad x_1F_2-x_2F_1,\quad x_3F_4-x_4F_3. \tag{59}$$

Hence, the general solution of (41) is here

$$\begin{aligned}
F_1(x_1,x_2,x_3,x_4)&=x_1\varphi_1(x_1,x_3,v)+x_3\varphi_2(x_1,x_3,v),\\
F_2(x_1,x_2,x_3,x_4)&=x_2\varphi_1(x_1,x_3,v)+x_4\varphi_2(x_1,x_3,v)+\varphi_3(x_1,x_3),\\
F_3(x_1,x_2,x_3,x_4)&=x_3\varphi_4(x_1,x_3,v)+x_1\varphi_5(x_1,x_3,v),\\
F_4(x_1,x_2,x_3,x_4)&=x_4\varphi_4(x_1,x_3,v)+x_2\varphi_5(x_1,x_3,v)+\varphi_6(x_1,x_3).
\end{aligned} \tag{60}$$

Since $\{F_j;\ j=1,2,3,4\}$ are polynomials in x_1, x_2, x_3, x_4, it is not difficult to show that $\varphi_j;\ j=1,\dots,6$ are polynomials in their arguments, i.e. in x_1, x_3, v or in x_1, x_3. This means that this normal form enters into the frame of remark A* of section 2.2, since

$$\begin{aligned}
F(X)=\beta_1(X)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\beta_2(X)\begin{pmatrix}0\\0\\0\\1\end{pmatrix}+\alpha_1(X)\begin{pmatrix}x_1\\x_2\\0\\0\end{pmatrix}+\alpha_2(X)\begin{pmatrix}0\\x_1\\0\\0\end{pmatrix}\\
+\alpha_3(X)\begin{pmatrix}0\\0\\x_3\\x_4\end{pmatrix}+\alpha_4(X)\begin{pmatrix}0\\0\\0\\x_3\end{pmatrix}+\alpha_5(X)\begin{pmatrix}x_3\\x_4\\0\\0\end{pmatrix}+\alpha_6(X)\begin{pmatrix}0\\x_3\\0\\0\end{pmatrix}+\alpha_7(X)\begin{pmatrix}0\\0\\x_1\\x_2\end{pmatrix}+\alpha_8(X)\begin{pmatrix}0\\0\\0\\x_1\end{pmatrix},
\end{aligned} \tag{61}$$

where

$$\alpha_1(X)=\varphi_1,\quad \alpha_5(X)=\varphi_2,\quad \alpha_3(X)=\varphi_4,\quad \alpha_7(X)=\varphi_5,$$
$$\beta_1(X)+x_1\alpha_2(X)+x_3\alpha_6(X)=\varphi_3,\quad \beta_2(X)+x_1\alpha_8(X)+x_3\alpha_4(X)=\varphi_6.$$

Now, as in examples 2.4.1 and 2.4.2, we can change the projection, and choose the following normal form for (7):

$$\begin{aligned}
\frac{dx_1}{dt}&=x_2,\\
\frac{dx_2}{dt}&=x_2P_2(x_1,x_3,x_2x_3-x_1x_4)+x_4P_2(x_1,x_3,x_2x_3-x_1x_4)+Q_1(x_1,x_3),\\
\frac{dx_3}{dt}&=x_4,\\
\frac{dx_4}{dt}&=x_4P_3(x_1,x_3,x_2x_3-x_1x_4)+x_2P_4(x_1,x_3,x_2x_3-x_1x_4)+Q_2(x_1,x_3),
\end{aligned} \tag{62}$$

where $P_1, P_2, P_3, P_4, Q_1, Q_2$ are polynomials in their arguments, P_j starting at degree 1, and Q_j at degree 2.

Let us assume now that the vector field $\mathscr{F}$ commutes with a symmetry S. Let us also assume that S is not trivial on E_0 in the following sense: we can choose the eigenvectors of L_0 such that they are exchanged by S, as well as the two generalized eigenvectors. The matrix of S in the same basis as for L_0, is now

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

S is clearly a unitary operator on E_0, hence the results of section 2.3 apply. We can then find a normal form invariant under S by taking

$$\begin{aligned} &P_3(x_1, x_3, x_2x_3 - x_1x_4) = P_1(x_3, x_1, x_1x_4 - x_2x_3), \\ &P_4(x_1, x_3, x_2x_3 - x_1x_4) = P_2(x_3, x_1, x_1x_4 - x_2x_3), \\ &Q_2(x_1, x_3) = Q_1(x_3, x_1). \end{aligned} \tag{63}$$

2.4.4. *ω^2 singularity*

This example is also treated in [9], where the normal form is written differently. The physical motivation of such a singularity is discussed in section 3.3.4. Here we have ($\omega \neq 0$)

$$L_0 = \begin{pmatrix} \mathrm{i}\omega & 1 & 0 & 0 \\ 0 & \mathrm{i}\omega & 0 & 0 \\ 0 & 0 & -\mathrm{i}\omega & 1 \\ 0 & 0 & 0 & -\mathrm{i}\omega \end{pmatrix}.$$

Let us write $X = (z_1, z_2, \bar{z}_1, \bar{z}_2)$ and $F = (F_1, F_2, \bar{F}_1, \bar{F}_2)$, then the characteristic system (42) becomes

$$\frac{\mathrm{d}z_1}{-\mathrm{i}\omega z_1} = \frac{\mathrm{d}z_2}{z_1 - \mathrm{i}\omega z_2} = \frac{\mathrm{d}\bar{z}_1}{\mathrm{i}\omega\bar{z}_1} = \frac{\mathrm{d}\bar{z}_2}{\bar{z}_1 + \mathrm{i}\omega\bar{z}_2} = \frac{\mathrm{d}F_1}{-\mathrm{i}\omega F_1} = \frac{\mathrm{d}F_2}{F_1 - \mathrm{i}\omega F_2}. \tag{64}$$

Now, first integrals are

$$z_1\bar{z}_1, \quad z_1\bar{z}_2 - \bar{z}_1 z_2, \quad \mathrm{i}\omega z_2/z_1 + \ln z_1, \quad \bar{z}_1 F_1, \quad \bar{z}_1 F_2 - \bar{z}_2 F_1. \tag{65}$$

The general solution for F_1 may then be written as follows:

$$F_1(z_1, z_2, \bar{z}_1, \bar{z}_2) = z_1\varphi_1(z_1\bar{z}_1, z_1\bar{z}_2 - \bar{z}_1 z_2, \mathrm{i}\omega z_2/z_1 + \ln z_1). \tag{66}$$

Using the fact that F_1 is a polynomial, it is not very difficult to deduce that φ_1 is a polynomial of $z_1\bar{z}_1$, $z_1\bar{z}_2 - \bar{z}_1 z_2$, moreover independent of its last argument. By an easy argument it can be shown that F_2 takes the form predicted by remark A* of section 2.2:

$$F_2(z_1, z_2, \bar{z}_1, \bar{z}_2) = z_2\varphi_1(z_1\bar{z}_1, z_1\bar{z}_2 - \bar{z}_1 z_2) + z_1\varphi_2(z_1\bar{z}_1, z_1\bar{z}_2 - \bar{z}_1 z_2), \tag{67}$$

where φ_2 is again a polynomial in its arguments. Finally, adapting the projection on a suitable space as we did in previous examples, we obtain

$$\begin{aligned}\frac{dz_1}{dt} &= i\omega z_1 + z_2,\\ \frac{dz_2}{dt} &= i\omega z_2 + z_1\varphi_1(z_1\bar{z}_1, z_1\bar{z}_2 - \bar{z}_1 z_2) + z_2\varphi_2(z_1\bar{z}_1, z_1\bar{z}_2 - \bar{z}_1 z_2),\end{aligned} \tag{68}$$

with φ_1 and φ_2 polynomials in their 2 arguments (see Appendix A.3). On this system it is tempting to make the following change of variables:

$$y_j = e^{-i\omega t} z_j, \quad j = 1, 2.$$

Then we get in $\mathbb{C}^2$,

$$\begin{aligned}\frac{dy_1}{dt} &= y_2,\\ \frac{dy_2}{dt} &= y_1\varphi_1(y_1\bar{y}_1, y_1\bar{y}_2 - \bar{y}_1 y_2) + y_2\varphi_2(y_1\bar{y}_1, y_1\bar{y}_2 - \bar{y}_1 y_2),\end{aligned} \tag{69}$$

which is a simpler autonomous second order differential equation.

3. Normal form of a vector field perturbed near a singularity

3.1. *General computation*

We consider now a system depending on a parameter $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$, of the form

$$\frac{dZ}{dt} = \mathscr{F}(\mu, Z) \tag{70}$$

where $\mathscr{F}$ is supposed to be regular with respect to (μ, Z) in a neighbourhood of 0 in $\mathbb{R}^m \times E$. We assume, as before, that

$$\mathscr{F}(0,0) = 0 \tag{71}$$

and we write as in section 2.2

$$\mathscr{L} = D_Z\mathscr{F}(0,0)$$

where $\mathscr{L}$ satisfies the same properties as in section 2. We want to obtain a *normal form for (70)*. The idea is to find *polynomials* $\Phi(\mu, X, Y)$, $N(\mu, X, Y)$, $F(\mu, X)$ such that F is as simple as possible and where we can write

$$Z = X + Y + \Phi(\mu, X, Y), \quad X \in E_0, Y \in E_-; \tag{72}$$

$$\begin{aligned}\frac{dX}{dt} &= L_0 X + F(\mu, X) + \mathscr{O}\left[(|\mu| + |X| + |Y|)^p\right],\\ \frac{dY}{dt} &= L_- Y + N(\mu, X, Y) + \mathscr{O}\left[(|\mu| + |X|)^p\right];\end{aligned} \tag{73}$$

$$\begin{aligned}&\Phi(\mu, X, Y) = \mathscr{O}\left(|\mu| + (|X| + |Y|)^2\right),\\ &N(\mu, X, Y) = \mathscr{O}\left[|Y|(|\mu| + |X| + |Y|)\right],\\ &F(\mu, 0) = \mathscr{O}(|\mu|), \quad D_X F(0,0) = 0.\end{aligned} \tag{74}$$

The estimate for Φ in $(74)_1$ is due to the absence of linear terms in Φ and to the fact that $\Phi(\mu,0,0)=\mathcal{O}(|\mu|)$. The estimate for N in $(74)_2$ means that the set $Y=0$ is invariant under the dynamics of (73) up to order $\mathcal{O}[(|\mu|+|X|)^P]$ and is an immediate consequence of $(80)_4$. In what follows we show that $F(\mu, X)$ can be chosen such that it commutes with $\exp(L_0^* t)$, just as in the case where there is no μ. In particular,

$$F(\mu,0) \in \operatorname{Ker} L_0^* \tag{75}$$

and

$$\mathrm{D}_X F(\mu,0) \text{ commutes with } L_0^*. \tag{76}$$

In the following we shall often refer to the "normal form"

$$\frac{\mathrm{d}X}{\mathrm{d}t} = L_0 X + F(\mu, X). \tag{77}$$

Remark 1. The manifold in E given by the equation

$$Z = X + \Phi(\mu, X, 0), \quad X \in E_0 \tag{78}$$

is the approximation at order $(|\mu|+|X|)^P$ of a *center manifold* for (70) (see remark 1 in section 2.1).

Remark 2. If $F(\mu,0)=0$, the manifold in E given by

$$Z = Y + \Phi(\mu, 0, Y), \quad Y \in E_- \tag{79}$$

is the approximation at order $(|\mu|+|Y|)^P$ of the *stable manifold* of the fixed point $X=Y=0$ for the system (73) without the terms of order P. If $F(\mu,0)\neq 0$, this is not a fixed point and the manifold (79) has no special meaning for (70).

Let us now start by defining the Taylor expansions of all the functions introduced above, as in section 2.1 (same notations):

$$\begin{aligned}
\mathcal{F}(\mu, z) &= \sum_{p+q\geq 1} \mathcal{F}_{pq}[\mu^{(p)}, Z^{(q)}], \quad \mathcal{F}_{01} = \mathcal{L}, \\
\Phi(\mu, X, Y) &= \Phi_{100}[\mu] + \sum_{p+q+r\geq 2} \Phi_{pqr}[\mu^{(p)}, X^{(q)}, Y^{(r)}], \\
F(\mu, X) &= \sum_{\substack{p+q\geq 1\\ (p,q)\neq(0,1)}} F_{pq}[\mu^{(p)}, X^{(q)}], \\
N(\mu, X, Y) &= \sum_{\substack{p+q\geq 1\\ r\geq 1}} N_{pqr}[\mu^{(p)}, X^{(q)}, Y^{(r)}].
\end{aligned} \tag{80}$$

We now identify terms of the same degree in (μ, X, Y) in (70) where we replace Z by (72) and $\mathrm{d}X/\mathrm{d}t$, $\mathrm{d}Y/\mathrm{d}t$ by (73). For $r=0$ we obtain

$$\mathcal{L}\Phi_{pq0}[\mu^{(p)}, X^{(q)}] - \mathrm{D}_X\Phi_{pq0}[\mu^{(p)}, X^{(q)}]\cdot L_0 X = F_{pq}[\mu^{(p)}, X^{(q)}] + R_{pq0}[\mu^{(p)}, X^{(q)}], \tag{81}$$

where R_{pq0} is a known function of $\Phi_{p'-1,q'+1,0}, \Phi_{p',q',0}, F_{p'q'}$ with $p'\leq p$, $q'\leq q$ and $p'+q'\leq p+q-1$.

For $r \geq 1$, we obtain

$$\mathscr{L}\Phi_{pqr}[\mu^{(p)}, X^{(q)}, Y^{(r)}] - D_X\Phi_{pqr}[\mu^{(p)}, X^{(q)}, Y^{(r)}] \cdot L_0 X - D_Y\Phi_{pqr}[\mu^{(p)}, X^{(q)}, Y^{(r)}] L_- Y$$
$$= N_{pqr}[\mu^{(p)}, X^{(q)}, Y^{(r)}] + R_{pqr}[\mu^{(p)}, X^{(q)}, Y^{(r)}], \tag{82}$$

where R_{pqr} is a known function of $\Phi_{p'-1,q'+1,r'}$, $\Phi_{p'q'r'}$, $F_{p'q'}$, $N_{p'q'r'}$ with $p' \leq p, q' \leq q, r' \leq r, p'+q'+r' \leq p+q+r-1$.

The strategy is first to compute the coefficients for $p=0$, and then to increase the values of $q+r$, starting with $q+r=2$, as in section 2.1.

We then compute for $p=1$, starting with $q+r=0$, and so forth computing for a fixed value of p all the needed coefficients by increasing the values of $q+r$, starting at 0.

We note that $p=0$ gives the same computations as the ones of section 2.1. This determines F_{0q}, Φ_{0qr}, N_{0qr} for any (q, r). Now, when $p \neq 0$, we remark that eqs. (81) and (82) have the same structure as (12), hence F_{pq} will have the same structure as F_{0q}, and the determination of Φ_{pqr}, $p \geq 1$, is the same as for Φ_{0qr}, the arbitrariness on N_{pqr} being the same as on N_{0qr}.

Let us consider the special cases when $r=0$ and $q=0$ or $q=1$. If $q=r=0$, then (81) reduces to

$$\mathscr{L}\Phi_{p00}[\mu^{(p)}] = F_{p0}[\mu^{(p)}] + R_{p00}[\mu^{(p)}]. \tag{83}$$

We saw in section 2.2 that F_{p0} may be chosen in $\operatorname{Ker}\mathscr{A}_*^{(0)}$, and since $W \in H_0$ is independent of X,

$$\operatorname{Ker}\mathscr{A}_*^{(0)} = \{W \in E_0;\ L_0^* W = 0\} = \operatorname{Ker} L_0^* = \operatorname{Ker}\mathscr{L}^*.$$

Then, we recover a well-known result of linear algebra, i.e.

$$F_{p0} \in \operatorname{Ker} L_0^* \Leftrightarrow e^{L_0^* t} F_{p0} = F_{p0}.$$

If $q=1$, $r=0$, then (81) reduces, once projected on E_0, to

$$L_0 P_0 \Phi_{p10}[\mu^{(p)}, X] - P_0\Phi_{p10}[\mu^{(p)}, L_0 X] = F_{p1}[\mu^{(p)}, X] + P_0 R_{p10}[\mu^{(p)}, X]. \tag{84}$$

We solve this equation by choosing F_{p1} in $\operatorname{Ker}\mathscr{A}_*^{(1)}$, and since $W \in H_1$ is a linear operator in E_0:

$$\operatorname{Ker}\mathscr{A}_*^{(1)} = \{W \in \mathscr{L}(E_0),\ L_0^* W - W L_0^* = 0\},$$

i.e. F_{p1} commutes with the operator L_0^* (hence with $\exp L_0^* t$). We recover a known result (Arnol'd [3]). In this case for $V, W \in H_1$ we have in fact

$$(V|W)_{H_1} = \operatorname{Tr}(VW^*) = \sum_{ij} V_{ij} W_{ij}$$

and it is clear that

$$(L_0 V - V L_0 | W)_{H_1} = (V | L_0^* W - W L_0^*).$$

We have then proved the following:

Theorem 5. *Normal form of the perturbed vector field.* A normal form $F(\mu, X)$ in (77) can be found such that (72)–(74) are satisfied and

$$F\left(\mu, e^{L_0^* t} X\right) = e^{L_0^* t} F(\mu, X), \quad X \in E_0, t \in \mathbb{R}, \tag{85}$$

where L_0^* is the adjoint of L_0 in E_0. In particular $F(\mu, 0)$ is in the kernel of L_0^* and $D_X F(\mu, 0)$ commutes with L_0^*.

Remark 3. If L_0 is diagonalizable then we can find a normal form $F(\mu, X)$ which commutes with $\exp(L_0 t)$. (See corollary 2.)

Remark 4. In the case of an additional symmetry, the result of theorem 4 still holds for $F(\mu, X)$.

Remark 5. The form (73) of the system is very useful to study in a simpler way the dynamics of (70). Nevertheless the terms $\mathcal{O}[(|\mu| + |X| + |Y|)^P]$ may give rise to serious problems even for very large P. It is fortunate that these terms can be simplified in the case where 0 is not an eigenvalue of $\mathcal{L}$. We already know that $F(\mu, 0) = 0$ by construction, but in fact we can prove the following:

Theorem 6. If 0 is not an eigenvalue of $\mathcal{L}$, then we can find a normal form of the perturbed vector field, of the same type as (72)–(74), but where $\mathcal{O}[(|\mu| + |X| + |Y|)^P]$ in dX/dt is replaced by $\mathcal{O}[(|X| + |Y|)^P]$, and $F(\mu, 0) = 0$.

The proof of this theorem is given in Appendix A.1.

3.2. *Examples*

Hereafter we consider the same examples as in section 2.4, with an additional parameter $\mu = (\mu_1, \ldots, \mu_m) \in \mathbb{R}^m$.

3.2.1. ζ^2 *singularity*

The notations are the ones of section 2.4.1. The kernel of L_0^* is one-dimensional so we can redefine μ_1 by setting

$$F(\mu, 0) = \begin{pmatrix} 0 \\ \mu_1 \end{pmatrix}.$$

The linear operators which commute with L_0^* are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

hence we can redefine the parameters in such a way that

$$F(\mu, X)=\begin{pmatrix} \mu_2 x \\ \mu_1+\mu_3 x+\mu_2 y \end{pmatrix}+\text{h.o.t. in } X. \tag{86}$$

Remark 6. Here μ_1, μ_2, μ_3 are in fact functions of the original $\mu \in \mathbb{R}^m$ not necessarily its three first components. These functions are at the leading order linear combinations of the components of μ. Finally changing the projection as in section 2.4.1, we obtain the normal form (77)

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t}&=y, \\ \frac{\mathrm{d}y}{\mathrm{d}t}&=\mu_1+\mu_3 x+\mu_2 y+xyP_1(\mu, x)+x^2 P_2(\mu, x), \end{aligned} \tag{87}$$

where $\mu=(\mu_1, \mu_2, \ldots)$ and P_1 and P_2 are polynomials in their arguments. Now, making a small translation in x, we can generically *suppress the term* $\mu_3 x$: it is sufficient for that to have a non-small coefficient of x^2 in (87). We then obtain the classical normal form [18, 2, 3]. Since there are two fundamental parameters μ_1 and μ_2 here, one says that this is a codimension 2 singularity. Other components of μ play a minor role, changing slightly the non linear coefficients.

3.2.2. ζ^3 *singularity* (*see* [3] *for the linear terms in* X)

In the same way as above, it can be easily shown that the normal form is obtained by adding to $(57)_3$ the affine terms (see Remark 6 for the redefinition of the μ_j).

$$\mu_0+\mu_1 x_1+\mu_2 x_2+\mu_3 x_3, \tag{88}$$

and by considering that the coefficients of polynomials P_j, $j=1,2,3$ are functions of μ. Moreover, as in 3.2.1, we can generically suppress $\mu_1 x_1$ by making a small translation on x_1. In the same order of idea as in 3.1.1, we shall say that the ζ^3 singularity has a *codimension 3* (main parameters: μ_0, μ_2, μ_3).

3.2.3. $\zeta^2\zeta^2$ *singularity with an additional symmetry* S

The kernel of L_0^* is two-dimensional, but due to the invariance under S we have here (redefining μ_0)

$$F(\mu,0)=\mu_0\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

The kernel of $\mathscr{A}_*^{(1)}$ is 8-dimensional, but thanks to the invariance under S we will have to add in $(62)_2$ and $(62)_4$ respectively (see [3] for the case of a non-symmetric linear part):

$$\begin{aligned} &\mu_0+\mu_1 x_1+\mu_2 x_2+\mu_3 x_3+\mu_4 x_4, \\ &\mu_0+\mu_3 x_1+\mu_4 x_2+\mu_1 x_3+\mu_2 x_4. \end{aligned} \tag{89}$$

In fact we can here again *generically suppress* μ_1 *and* μ_3, by making a small translation in x_1 and x_3, which commutes with S. Hence we have here a *codimension 3 singularity, since the role of the other* *components of* μ just slightly modify the coefficients of polynomials P_1, P_2 and Q_1. Note that without symmetry this singularity is of codimension 8.

3.2.4. ω^2 *singularity see* [5, 9, 10, 16]

We use the notations of section 2.4.4. The kernel of $\mathscr{A}^{(1)}{}_*$is 2-dimensional with complex coefficients, hence 4-dimensional with real coefficients. We have to add to $(68)_2$ the term

$$\mu_1 z_1 + \mu_2 z_2 \tag{90}$$

on the right-hand side, where μ_1 and μ_2 are complex. Other components of μ occur in higher order terms, at least cubic, in the polynomials φ_1 and φ_2.

In fact, the relevant parameters are $\mathrm{Re}\,\mu_2$ and $\mu_2^2 + 4\mu_1$ (complex), hence we shall say that this singularity is of *codimension 3*: two parameters are necessary for having two pairs of eigenvalues crossing the imaginary axis simultaneously, and one parameter to have them crossing at the same point.

A special case is when we have a *conservative system*. This is well known in mechanical problems like the forced oscillations of a wing under the aerodynamical effect of the wind. Here $\mathrm{Re}\,\mu_2 = 0$ since volumes are conserved. Now, the interesting situation (Hamiltonian system) is when two pairs of pure imaginary eigenvalues, moving as a function of a real parameter, meet together at $\mu = 0$. Then, they escape orthogonally from the imaginary axis. A unique real parameter μ is sufficient to describe such a singularity, which in this context is only of *codimension one*.

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Appendix

A.1. *Case when* 0 *is not an eigenvalue*

Here we want to prove theorem 6. So, we assume that 0 is not an eigenvalue of L_0. A first consequence is the existence of a persisting fixed point $Z_0(\mu)$ regular function of μ. To compute it, it is sufficient to identify the powers of μ in

$$\mathscr{F}(\mu, Z) = 0, \tag{91}$$

where Z is replaced by

$$Z_0(\mu) = \sum_{p \geq 1} Z_p[\mu^{(p)}] \tag{92}$$

and $\mathscr{F}$ by $(80)_1$. With the notations of (72) we have

$$Z_0(\mu) = \Phi(\mu, 0, 0). \tag{93}$$

Let us now set $Z = Z_0(\mu) + \tilde{Z}$, then

$$\frac{\mathrm{d}\tilde{Z}}{\mathrm{d}t} = \tilde{\mathscr{F}}(\mu, \tilde{Z}) \equiv \mathscr{F}(\mu, Z_0(\mu) + \tilde{Z}), \quad \tilde{\mathscr{F}}(\mu, 0) = 0. \tag{94}$$

Let us denote by

$$\mathscr{L}_\mu = \mathrm{D}_{\tilde{Z}}\tilde{\mathscr{F}}(\mu,0) = \mathrm{D}_Z\mathscr{F}(\mu, Z_0(\mu)). \tag{95}$$

the derivative of $\mathscr{F}$ at the fixed point. We want first to show how to decouple $X \in E_0$ and $Y \in E_-$ in the linear terms in $\tilde{Z}$. We remark that we want more than (73) since we do not want any more $\mathscr{O}(|\mu|^{P-1}|X|)$ terms in the right-hand side of $\mathrm{d}X/\mathrm{d}t$ and $\mathrm{d}Y/\mathrm{d}t$ (we already have no terms of $\mathscr{O}(|\mu|^P)$).

In fact we want to find linear operators $\Phi_{10}(\mu) \in \mathscr{L}(E_0, E)$ and $\Phi_{01}(\mu) \in \mathscr{L}(E_-, E)$ such that for any $X \in E_0$ and $Y \in E_-$ we have

$$\mathscr{L}_\mu(X + Y + \Phi_{10}(\mu)X + \Phi_{01}(\mu)Y) = L_\mu^{(0)}X + L_\mu^{(-)}Y + \Phi_{10}(\mu)L_\mu^{(0)}X + \Phi_{01}(\mu)L_\mu^{(-)}Y, \tag{96}$$

where

$$L_\mu^{(0)} = L_0 + \mathscr{O}(|\mu|) \in \mathscr{L}(E_0)$$

and

$$L_\mu^{(-)} = L_- + \mathscr{O}(|\mu|) \in \mathscr{L}(E_-).$$

After projecting (96) on E_0 and E_-, we obtain

$$\begin{aligned} &P_0\Phi_{01}L_\mu^{(-)} - P_0\mathscr{L}_\mu P_0\Phi_{01} = P_0\mathscr{L}_\mu P_- + P_0\mathscr{L}_\mu P_-\Phi_{01},\\ &L_\mu^{(-)} + P_-\Phi_{01}L_\mu^{(-)} - P_-\mathscr{L}_\mu P_-\Phi_{01} = P_-\mathscr{L}_\mu P_- + P_-\mathscr{L}_\mu P_0\Phi_{01} \end{aligned} \tag{97}$$

and

$$\begin{aligned} &P_-\Phi_{10}L_\mu^{(0)} - P_-\mathscr{L}_\mu P_-\Phi_{10} = P_-\mathscr{L}_\mu P_0 + P_-\mathscr{L}_\mu P_0\Phi_{10},\\ &L_\mu^{(0)} + P_0\Phi_{10}L_\mu^{(0)} - P_0\mathscr{L}_\mu P_0\Phi_{10} = P_0\mathscr{L}_\mu P_0 + P_0\mathscr{L}_\mu P_-\Phi_{10}. \end{aligned} \tag{98}$$

Since there is no restriction on $L_\mu^{(-)} \in \mathscr{L}(E_-)$, we can choose $P_-\Phi_{01} = 0$, hence the system (97) reduces to

$$P_0\Phi_{01}(P_-\mathscr{L}_\mu P_- + P_-\mathscr{L}_\mu P_0\Phi_{01}) - P_0\mathscr{L}_\mu P_0\Phi_{01} = P_0\mathscr{L}_\mu P_- \quad \text{in } \mathscr{L}(E_-, E_0), \tag{99}$$

where the only unknown is $P_0\Phi_{01}(\mu) \in \mathscr{L}(E_-, E_0)$.

We now observe that

$$\begin{aligned} &P_-\mathscr{L}_\mu P_- = L_- + \mathscr{O}(|\mu|), \quad P_0\mathscr{L}_\mu P_0 = L_0 + \mathscr{O}(|\mu|),\\ &P_-\mathscr{L}_\mu P_0 = \mathscr{O}(|\mu|), \quad P_0\mathscr{L}_\mu P_- = \mathscr{O}(|\mu|), \end{aligned} \tag{100}$$

hence (99) takes the form

$$g(P_0\Phi_{01}, \mu) = 0 \quad \text{in } \mathscr{L}(E_-, E_0),$$

where

$$g(0,0)=0, \quad \mathrm{D}_1 g(0,0)\cdot A = AL_- - L_0 A \quad \text{in } \mathscr{L}(E_-, E_0)$$

for any $A \in \mathscr{L}(E_-, E_0)$. We already saw that the linear operator $\mathrm{D}_1 g(0,0)$ which acts in $\mathscr{L}(E_-, E_0)$ is invertible, since we computed explicitely its inverse in $(19)_2$ with $p=0$, $q=1$. Hence the implicit function theorem applies to solve (99), which means that we can compute, by identifying powers of μ, the solution $P_0\Phi_{01}(\mu)$.

Now, let us consider the system (98) where the unknown are $P_-\Phi_{10}$ and $P_0\Phi_{10}$ and where we want to find $(L_\mu^{(0)} - L_0)$ commuting with L_0^*. We use theorem 1 to define the projections Π and $(\mathrm{Id} - \Pi)$, respectively, on Image $\mathscr{A}^{(1)}$ and Ker $\mathscr{A}_*^{(1)}$. In fact we want

$$\Pi\left(L_\mu^{(0)} - L_0\right) = 0, \tag{101}$$

then the system 98) reduces to

$$\begin{aligned} &P_-\Phi_{10}(\mathrm{Id} + P_0\Phi_{10})^{-1}\left(P_0\mathscr{L}_\mu P_0 + P_0\mathscr{L}_\mu P_0\Phi_{10} + P_0\mathscr{L}_\mu P_-\Phi_{10}\right) \\ &\quad - P_-\mathscr{L}_\mu P_-\Phi_{10} - P_-\mathscr{L}_\mu P_0 - P_-\mathscr{L}_\mu P_0\Phi_{10} = 0, \\ &\Pi\left\{(\mathrm{Id} + P_0\Phi_{10})^{-1}\left(P_0\mathscr{L}_\mu P_0 + P_0\mathscr{L}_\mu P_0\Phi_{10} + P_0\mathscr{L}_\mu P_-\Phi_{10}\right) - L_0\right\} = 0. \end{aligned} \tag{102}$$

We again solve (102) by the implicit function theorem, by choosing $P_0\Phi_{10}$ in a supplementary space of Ker$\mathscr{A}^{(1)}$ in $\mathscr{L}(E_0)$. It is not hard to check that the differential of the left-hand side of (102) with respect to $(P_-\Phi_{10}, P_0\Phi_{10})$ at the point 0, $\mu=0$, is for any $(A, B) \in \mathscr{L}(E_0, E_-) \times \mathscr{L}(E_0)$:

$$(A, B) \mapsto (AL_0 - L_- A, L_0 B - BL_0). \tag{103}$$

The linear operator (103) is invertible since for the first component we already computed the inverse in $(19)_1$ with $p=1$, and since for the second component we look for B in a supplementary space of Ker $\mathscr{A}^{(1)}$. Hence we can obtain $P_0\Phi_{10}(\mu)$ and $P_-\Phi_{10}(\mu)$ by identification of powers of μ in (102), so $L_\mu^{(0)}$ follows directly.

Having solved the problem for the linear terms, we can make the same analysis as in section 2.1, keeping μ at each step. We then obtain equations like (12), but with $\mathscr{L}_\mu$, $L_\mu^{(0)}$, $L_\mu^{(-)}$ instead of $\mathscr{L}$, L_0, L_-, and Φ_{pq} depending on μ. We observe that (19) solves again the equations corresponding to (15), (16) since $\exp(\pm L_\mu^{(0)} t)$ increases slower than $\exp(L_\mu^{(-)} t)$ decreases, when $t \to +\infty$. The only remaining problem is with the homological equation

$$L_\mu^{(0)} P_0\Phi_{p0}[\mu, X^{(p)}] - \mathrm{D}_X P_0\Phi_{p0}[\mu, X^{(p)}] \cdot L_\mu^{(0)} X = F_p[\mu, X^{(p)}] + P_0 R_p[\mu, X^{(p)}]. \tag{104}$$

Introducing the projections Π and $\mathrm{Id} - \Pi$, respectively, on Image $\mathscr{A}^{(p)}$ and on Ker $\mathscr{A}_*^{(p)}$, and defining

$$L_\mu^{(0)} = L_0 + L_\mu^{(1)} \in \mathscr{L}(E_0) \quad \text{where } L_\mu^{(1)} = \mathscr{O}(|\mu|),$$

eq. (104) reduces to

$$\mathscr{A}^{(p)} P_0\Phi_{p0} + \Pi\left(\left[P_0\Phi_{p0}[\mu, X^{(p)}], L_\mu^{(1)} X\right]\right) = \Pi P_0 R_p. \tag{105}$$

Choosing a supplementary space of Ker $\mathscr{A}^{(p)}$, this equation is uniquely solvable in $P_0\Phi_{p0}$ since for $|\mu|$ small the linear operator acting on $P_0\Phi_{p0}$ is a small perturbation of the now invertible operator $\mathscr{A}^{(p)}$. The other part of (104) leads to an F_p regular in μ:

$$F_p = (\Pi - \mathrm{Id})\left\{ P_0 R_p - \left[P_0\Phi_{p0}\left[\mu, X^{(p)}\right], L_\mu^{(1)}X \right] \right\} \quad \text{in Ker}\, \mathscr{A}_*^{(1)}. \tag{106}$$

Hence theorem 6 is proved.

A.2 $\zeta^3\zeta^2$ singularity

In this appendix we consider an example which is less elementary than the one presented in section 2.4. Even though its codimension is 9, (hence it is very improbable physically) it has the interest to give a counter-example to some a priori "reasonable" conjectures, for instance see remark A* in section 2.2.

Here we have

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{107}$$

If we note $(x_1, x_2, x_3, x_4, x_5) = X$ and $(F_1, F_2, F_3, F_4, F_5) = F$, and $\mathscr{D}^*$ the differential operator defined by

$$\mathscr{D}^* = x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_5}, \tag{108}$$

then the partial differential system (41) becomes

$$\mathscr{D}^*F_1 = 0, \quad \mathscr{D}^*F_2 = F_1, \quad \mathscr{D}^*F_3 = F_2, \quad \mathscr{D}^*F_4 = 0, \quad \mathscr{D}^*F_5 = F_4. \tag{109}$$

The characteristic system associated with (108) leads to the following 4 first integrals:

$$Z_1 = x_1, \quad Z_2 = x_4, \quad Z_3 = x_2^2 - 2x_1x_3, \quad Z_4 = x_2x_4 - x_1x_5. \tag{110}$$

Then F_1 has to be a function of Z_1, Z_2, Z_3, Z_4. We want F_1 to be a polynomial in $(x_1, x_2, x_3, x_4, x_5)$. This does not imply that it is a polynomial in Z_1, Z_2, Z_3, Z_4. A first nontrivial result is that F_1 is in fact a polynomial in $(Z_1, Z_2, Z_3, Z_4, Z_5)$ where [6, 21],

$$Z_5 = x_1x_5^2 + 2x_3x_4^2 - 2x_2x_4x_5. \tag{111}$$

We remark that $Z_5 = (Z_4^2 - Z_2^2Z_3)/Z_1$ is hence a first integral, polynomial of degree 3 in X but not polynomial in (Z_1, Z_2, Z_3, Z_4).

Eq. $(109)_1$ can then be solved by setting

$$F_1 = \varphi_1(Z_1, Z_2, Z_3, Z_4, Z_5), \tag{112}$$

where φ_1 is a polynomial in its arguments.

We want now to solve $(109)_2$. To wit let us write

$$F_1 = x_1\psi_1(x_1, x_4, Z_3, Z_4, Z_5) + x_4\psi_2(x_4, Z_3, Z_4, Z_5) + Z_4\psi_3(Z_3, Z_4, Z_5) + Z_5\psi_4(Z_3, Z_5) + \psi_5(Z_3), \tag{113}$$

where $\{\psi_j;\ j = 1, \ldots, 5\}$ are *polynomials in their arguments*. The fact that F_1 is in the image of $\mathscr{D}^*$ will lead to $\psi_4 = \psi_5 = 0$.

Remark. Working with the scalar product defined in (25), it is clear that the adjoint $\mathscr{D}$ of $\mathscr{D}^*$ is the differential operator

$$\mathscr{D} = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + x_5\frac{\partial}{\partial x_4},$$

whose kernel is formed by polynomials of the following type:

$$P\left(x_3, x_5, x_2^2 - 2x_1x_3, x_2x_5 - x_3x_4, x_3x_4^2 + 2x_1x_5^2 - 2x_2x_4x_5\right).$$

So, the fact that F_1 is orthogonal to any such polynomial leads to $\psi_4 = \psi_5 = 0$.

The proof is not direct but also not very hard: let us write the solution F_2 of $\mathscr{D}^*F_2 = F_1$ under the form

$$F_2 = x_2\psi_1 + x_5\psi_2 + q_1\psi_3 + \frac{x_2Z_5}{x_1}\psi_4 + \frac{x_2}{x_1}\psi_5 + \varphi_2(x_1, x_4, Z_3, Z_4), \tag{114}$$

where we only know that F_2 is a polynomial in $(x_1, \ldots, x_5)$, $\psi_1, \ldots, \psi_5$ are polynomials in their arguments and φ_2 is a rational function of its 4 arguments. We use the properties

$$\mathscr{D}^*x_2 = x_1, \quad \mathscr{D}^*x_5 = x_4, \quad \mathscr{D}^*q_1 = Z_4, \quad \mathscr{D}^*\left(\frac{x_2Z_5}{x_1}\right) = Z_5, \quad \mathscr{D}^*\left(\frac{x_2}{x_1}\right) = 1, \tag{115}$$

with $q_1 = 2x_3x_4 - x_2x_5$.

As a result of (114), we know that

$$Q = x_2Z_5\psi_4(Z_3, Z_5) + x_2\psi_5(Z_3) + x_1\varphi_2(x_1, x_4, Z_3, Z_4) \tag{116}$$

is a polynomial, vanishing at $x_1 = 0$.

This leads to

$$x_2Z_5\psi_4\left(x_2^2, Z_5\right) + x_2\psi_5\left(x_2^2\right) + \varphi_2'\left(0, x_4, x_2^2, x_2x_4, Z_5\right) \equiv 0, \tag{117}$$

where we define $\varphi_2'(Z_1, Z_2, Z_3, Z_4, Z_5)$ to be polynomial of its arguments equal to $x_1\varphi_2$ (we now know that it is a polynomial). Inspection of the monomials occurring in (117) immediately shows that $\psi_4 = \psi_5 = 0$ and that Z_1 is in factor in φ_2'. We arrive at

$$\begin{aligned} F_1 &= x_1\psi_1 + x_4\psi_2 + Z_4\psi_3, \\ F_2 &= x_2\psi_1 + x_5\psi_2 + q_1\psi_3 + x_1\chi_1 + x_4\chi_2 + Z_4\chi_3 + Z_5\chi_4 + \chi_5(Z_3), \end{aligned} \tag{118}$$

where $\{\chi_j,\ j = 1, \ldots, 5\}$ are polynomials of the same arguments as $\{\psi_j\}$.

Let us now consider eq. $(109)_3$ and set as for F_2,

$$F_3 = x_3\psi_1 + \frac{x_5^2}{2x_4}\psi_2 + \frac{q_1^2}{Z_4}\psi_3 + x_2\chi_1 + x_5\chi_2 + q_1\chi_3 + \frac{x_2 Z_5}{x_1}\chi_4 + \frac{x_2}{x_1}\chi_5 + \varphi_3(x_1, x_4, Z_3, Z_4). \tag{119}$$

We wish to show that ψ_2 is divisible by x_4, ψ_3 is divisible by Z_4, and $\chi_4 = \chi_5 = 0$, and that φ_3 is a polynomial. We observe that

$$\begin{aligned} Q = {} & \frac{x_5^2}{2x_4}\psi_2(x_4, Z_3, Z_4, Z_5) + \frac{q_1^2}{Z_4}\psi_3(Z_3, Z_4, Z_5) + \frac{x_2 Z_5}{x_1}\chi_4(Z_3, Z_5) \\ & + \frac{x_2}{x_1}\chi_5(Z_3) + \varphi_3(x_1, x_4, Z_3, Z_4) \end{aligned} \tag{120}$$

is a polynomial in $(x_1, \ldots, x_5)$. We deduce immediately that $x_1 x_4 Z_4 \varphi_3(x_1, x_4, Z_3, Z_4)$ is a polynomial $\varphi_3'(x_1, x_4, Z_3, Z_4, Z_5)$. Multiplying (120) by x_4 and making $x_4 = 0$ shows that

$$\psi_2(0, Z_3, -x_1 x_5, x_1 x_5^2) = 0, \tag{121}$$

hence x_4 is in factor in ψ_2.

Multiplying (120) by Z_4 and making $Z_4 = 0$ shows that

$$\psi_3(Z_3, 0, Z_5) = 0 \quad \text{(since } q_1 \neq 0, \text{ and } Z_3, Z_5 \text{ are still independent)},$$

hence Z_4 is in factor in ψ_3. Now we just make the same proof as above for ψ_4 and ψ_5. Finally, we easily obtain (changing notations)

$$\begin{aligned} F_1 &= x_1\psi_1 + x_4^2\psi_2 + Z_4^2\psi_3, \\ F_2 &= x_2\psi_1 + x_4 x_5\psi_2 + q_1 Z_4\psi_3 + x_1\chi_1 + x_4\chi_2 + Z_4\chi_3, \\ F_3 &= x_3\psi_1 + \frac{x_5^2}{2}\psi_2 + \frac{q_1^2}{2}\psi_3 + x_2\chi_1 + x_5\chi_2 + q_1\chi_3 + \varphi_3, \end{aligned} \tag{122}$$

where ψ_1, χ_1 are polynomials in $(x_1, x_4, Z_3, Z_4, Z_5)$, ψ_2 and χ_2 are polynomials in (x_4, Z_3, Z_4, Z_5), ψ_3 and χ_3 polynomials in (Z_3, Z_4, Z_5), φ_3 is a polynomial in $(x_1, x_4, Z_3, Z_4, Z_5)$ and $q_1 = 2x_3 x_4 - x_2 x_5$.

We can compute in the same way the two last components of F:

$$\begin{aligned} F_4 &= x_1\theta_1 + x_4\theta_2 + Z_4\theta_3, \\ F_5 &= x_2\theta_1 + x_5\theta_2 + q_1\theta_3 + \varphi_5, \end{aligned} \tag{123}$$

where $\{\theta_j, j = 1,2,3\}$ are polynomials of the same arguments as ψ_j and φ_5 is a polynomial in $(x_1, x_4, Z_3, Z_4, Z_5)$.

Note that the normal form *does not enter into the frame of remark A** of section 2.2 since

$$\psi_3 \begin{pmatrix} (x_2x_4 - x_1x_5)^2 \\ (2x_3x_4 - x_2x_5)(x_2x_4 - x_1x_5) \\ \frac{1}{2}(2x_3x_4 - x_2x_5)^2 \\ 0 \\ 0 \end{pmatrix} + \psi_2 \begin{pmatrix} x_4^2 \\ x_4x_5 \\ x_5^2/2 \\ 0 \\ 0 \end{pmatrix} + \chi_3 \begin{pmatrix} 0 \\ x_2x_4 - x_1x_5 \\ 2x_3x_4 - x_2x_5 \\ 0 \\ 0 \end{pmatrix} + \theta_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_2x_4 - x_1x_5 \\ 2x_3x_4 - x_2x_5 \end{pmatrix}$$

is not contained in (45).

Finally let us give the explicit form for the rational fractions α_j, $j=1,\dots,5$ characterizing the normal form of $\zeta^3\zeta^2$. From (43) we obtain

$$\begin{aligned}
F_1 &= x_1\alpha_1,\\
F_2 &= x_2\alpha_1 + x_1\alpha_2,\\
F_3 &= x_3\alpha_1 + x_2\alpha_2 + x_1\alpha_3,\\
F_4 &= x_4\alpha_4,\\
F_5 &= x_5\alpha_4 + x_4\alpha_5.
\end{aligned}\tag{124}$$

By equating (124) to (122) and (123) we obtain after some algebra the following expressions for the α_j, $j=1,\dots,5$:

$$\begin{aligned}
\alpha_1 &= \frac{1}{x_1}\left(x_4^2\psi_2 + Z_4^2\psi_3\right) + \psi_1,\\
\alpha_2 &= -\frac{1}{x_1^2}\left(x_4Z_4\psi_2 + x_4Z_4Z_3\psi_3\right) + \frac{1}{x_1}\left(x_4\chi_2 + Z_4\chi_3\right),\\
\alpha_3 &= \frac{1}{2x_1^3}\left[\left(Z_4^2 + Z_3x_4^2\right)\psi_2 + x_4^2Z_3^2\psi_3\right] - \frac{1}{x_1^2}Z_4\chi_2 + \frac{1}{x_1}\left(x_4Z_3\chi_3 + \varphi_3\right),\\
\alpha_4 &= \frac{1}{x_4}\left(x_1\theta_1 + Z_4\theta_3\right) + \theta_2,\\
\alpha_5 &= \frac{1}{x_4^2}\left(Z_4\theta_1 + Z_5\theta_3\right) + \frac{1}{x_4}\varphi_5.
\end{aligned}$$

Therefore in accordance with theorem 3 the rational fractions for $\zeta^3\zeta^2$ are characterized by denominators x_1^p, $p\le 3$, and x_4^q, $q\le 2$, where the highest $p(q)$ is the dimension of the critical subspace associated with the $\zeta^3(\zeta^2)$ Jordan block.

4.3. (a) ζ^2 singularity. Proof of (49)

Since eq. (34) leads to

$$x\frac{\partial F_1}{\partial y} = 0, \qquad x\frac{\partial F_2}{\partial y} = F_1, \tag{125}$$

we obtain that $F_1(x,y)=\varphi(x)$. Since F_1 is a polynomial in (x,y) then φ is a polynomial in x. From $(125)_2$ we obtain

$$\frac{\partial F_2}{\partial y} = \frac{\varphi(x)}{x}, \tag{126}$$

which is a polynomial. Hence φ is divisible by x and can be written as

$$\varphi(x) = x\varphi_1(x), \tag{127}$$

where φ_1 is a polynomial in x. On solving (126) we obtain

$$F_2(x, y) = y\varphi_1(x) + \varphi_2(x)$$

and since F_2 and $y\varphi_1(x)$ are polynomials, also φ_2 is a polynomial.

(b) ζ^3 singularity. Proof of (56)

Let us choose the new variables

$$u_1 = x_1, \quad u_2 = x_2^2 - 2x_1x_3, \quad u_3 = x_2 \tag{128}$$

and define $F_j(x_1, x_2, x_3) = \tilde{F}_j(u_1, u_2, u_3)$ is an obvious way. Then the partial differential system (53) can be written as

$$u_1 \frac{\partial}{\partial u_3}\tilde{F}_1 = 0, \quad u_1 \frac{\partial}{\partial u_3}\tilde{F}_2 = \tilde{F}_1, \quad u_1 \frac{\partial}{\partial u_3}\tilde{F}_3 = \tilde{F}_2. \tag{129}$$

Eq. $(129)_1$ gives

$$F_1(x_1, x_2, x_3) = \varphi(u_1, u_2).$$

Since F_1 is a polynomial in (x_1, x_2, x_3) there exists $n \in \mathbb{Z}^+$ such that

$$\frac{\partial^n F_1}{\partial x_3^n} \equiv (-2x_1)^n \frac{\partial^n \varphi}{\partial u_2^n} = 0.$$

Hence it follows that φ is a polynomial in u_2. Therefore we can write

$$F_1(x_1, x_2, x_3) = \sum_k F_{1k}(x_1, x_2)x_3^k = \sum_k \varphi_k(u_1)u_2^k$$

from which it follows trivially that $\varphi_k(u_1)$ is a polynomial in u_1. This proves that φ is a polynomial in (u_1, u_2) and can be written as $\varphi = u_1\varphi_1(u_1, u_2) + \psi_1(u_2)$ where $\psi_1(u_2) = \varphi(0, u_2)$.

On solving $(129)_2$ for $\tilde{F}_2$ we obtain

$$F_2(x_1, x_2, x_3) = x_2\varphi_1(u_1, u_2) + \frac{x_2}{u_1}\psi_1(u_2) + \frac{1}{u_1}\psi(u_1, u_2), \tag{130}$$

where ψ is a polynomial in u_1, u_2 (the proof is the same as the one given for φ). Multiplying (130) by u_1 and making $u_1 = 0$ we obtain that ψ_1 vanishes and that $(1/u_1)\psi$ is a polynomial in (u_1, u_2). By writing $\psi/u_1 = u_1\varphi_2(u_1, u_2) + \psi_2(u_2)$ we obtain

$$F_2(x_1, x_2, x_3) = x_2\varphi_1(u_1, u_2) + u_1\varphi_2(u_1, u_2) + \psi_2(u_2), \tag{131}$$

where $\varphi_2(\psi_2)$ is a polynomial in $(u_1, u_2)((u_2))$.

Finally, eq. $(129)_3$ leads to

$$F_3(x_1, x_2, x_3) = x_3\varphi_1(u_1, u_2) + x_2\varphi_2(u_1, u_2) + \frac{x_2}{u_1}\psi_2(u_2) + \frac{1}{u_1}\chi(u_1, u_2), \tag{132}$$

where χ is a polynomial in (u_1, u_2) (see the proof of φ). Multiplying (132) by u_1 and making $u_1 = 0$ we obtain $\psi_2 = 0$. Hence (56) has been proved. For the $\zeta^2\zeta^2$ singularity the proof is analogous to this one.

(*c*) *ω^2 singularity. Proof of* (68)

Let us consider the new variables

$$u_1 = z_1\bar{z}_1, \quad u_2 = z_1\bar{z}_2 - \bar{z}_1 z_2, \quad u_3 = \mathrm{i}\omega\frac{z_2}{z_1} + \log z_1,$$

which are independent first integrals of the characteristic system (64).

From (64) we obtain [20]

$$F_1(z_1, z_2, \bar{z}_1, \bar{z}_2) = z_1\varphi(u_1, u_2, u_3), \tag{133}$$

since $\bar{z}_1 F_1$ is also a first integral. Let us prove that φ is a polynomial in (u_1, u_2) and independent of u_3.

First we note that

$$\begin{aligned} z_1\frac{\partial^n\varphi}{\partial u_1^n} &= \left(\frac{z_2}{z_1^2}\frac{\partial}{\partial\bar{z}_2} + \frac{1}{z_1}\frac{\partial}{\partial\bar{z}_1}\right)^n F_1, \\ z_1\frac{\partial^n\varphi}{\partial u_2^n} &= \left(\frac{1}{z_1}\frac{\partial}{\partial\bar{z}_2}\right)^n F_1, \\ z_1\frac{\partial^n\varphi}{\partial u_3^n} &= \left(\frac{z_1}{\mathrm{i}\omega}\frac{\partial}{\partial z_2} + \frac{\bar{z}_1}{\mathrm{i}\omega}\frac{\partial}{\partial\bar{z}_2}\right)^n F_1. \end{aligned} \tag{134}$$

Since F_1 is a polynomial in $(z_1, z_2, \bar{z}_1, \bar{z}_2)$ it follows from (134) that φ is a polynomial in (u_1, u_2, u_3). Therefore φ is a sum of monomials $u_1^{\alpha_1}u_2^{\alpha_2}u_3^{\alpha_3}$ which behaves as $z_1^{2\alpha_1+\alpha_2}(\log z_1)^{\alpha_3}$ for $z_1 \to \infty$ in $\mathbb{R}^+$. This is not possible for a polynomial in z_1 except if $\alpha_3 = 0$. Finally we can write

$$F_1(z_1, z_2, \bar{z}_1, \bar{z}_2) = z_1\varphi_1(u_1, u_2), \tag{135}$$

where φ_1 is a polynomial in (u_1, u_2).

Since $F_2 - z_2 \cdot \varphi_1(u_1, u_2)$ and F_1 satisfy the same partial differential equation we immediately obtain (67) and therefore (68).

Note added in proof*

It is worthwhile remarking that the normal form for $\zeta^3\zeta^2$ instability (eqs. (122) and (123)) although is complete it is not written in its minimal form, that means that the coefficient of a given term is not uniquely determined (in other words the normal form contains some repeated terms). The non-minimality of our normal form comes from the fact that Z_1, Z_2, Z_3, Z_4, Z_5 satisfy the relation $Z_4^2 = Z_1Z_5 + Z_2^2Z_3$. Therefore in order to obtain a minimal normal form we have to allow the functions F_1, F_2, F_3, F_4, F_5 to be

*We thank one of the referees for having called our attention about the non-minimality of the normal form given by (122) and (123). We also thank the same referee for providing us the property (130) which was essential in the proof of minimality.

at most linear in Z_4. The minimal normal form is obtained as follows: we write

$$F_4 = \phi_1(Z_1, Z_2, Z_3, Z_4, Z_5) + Z_4\phi_2(Z_1, Z_2, Z_3, Z_4, Z_5). \tag{136}$$

We easily find that ϕ_1 is of the form $x_1\varphi_1(Z_1, Z_2, Z_3, Z_5) + x_4\varphi_2(Z_1, Z_2, Z_3, Z_5)$ since F_4 must be in Image$(\mathscr{D}^*)$. Hence it follows from (115) that

$$\begin{aligned} F_4 &= x_1\varphi_1 + x_4\varphi_2 + Z_4\phi_2, \\ F_5 &= x_2\varphi_1 + x_5\varphi_2 + q_1\phi_2 + \varphi_3(Z_1, Z_2, Z_3, Z_5). \end{aligned} \tag{137}$$

Similarly we write

$$F_1 = \phi_0(Z_1, Z_2, Z_3, Z_5) + Z_4\psi_0(Z_1, Z_2, Z_3, Z_5). \tag{138}$$

We note that we can always write

$$\psi_0 = x_1\psi_1(Z_1, Z_2, Z_3, Z_5) + \psi_2(Z_2, Z_3, Z_5).$$

By imposing that $F_1 \in$ Image$(\mathscr{D}^*)$ and $F_1 \in$ Image$((\mathscr{D}^*)^2)$ we easily conclude that ψ_2 is divisible by x_4. Similar arguments show that ϕ_0 has the form

$$\phi_0 = x_1\psi_3(Z_1, Z_2, Z_3, Z_5) + x_1x_4\psi_4(Z_1, Z_2, Z_3, Z_5) + x_4^2\psi_5(Z_2, Z_3, Z_5). \tag{139}$$

Hence

$$F_1 = x_1z_4\psi_1 + x_4z_4\psi_2 + x_1\psi_3 + x_1x_4\psi_4 + x_4^2\psi_5. \tag{140a}$$

Using (115) and imposing $F_2 \in$ Image$(\mathscr{D}^*)$ we obtain

$$\begin{aligned} F_2 &= x_2Z_4\psi_1 + \left(x_3x_4^2 - \tfrac{1}{2}x_1x_5^2\right)\psi_2 + x_2\psi_3 + \tfrac{1}{2}(x_2x_4 + x_1x_5)\psi_4 + x_5x_4\psi_5 \\ &\quad + x_1\psi_6(Z_1, Z_2, Z_3, Z_5) + x_4\psi_7(Z_1, Z_2, Z_3, Z_5), \end{aligned} \tag{140b}$$

$$\begin{aligned} F_3 &= x_3Z_4\psi_1 + x_5\left(x_3x_4 - \tfrac{1}{2}x_2x_5\right)\psi_2 + x_3\psi_3 + \tfrac{1}{2}(x_2x_5)\psi_4 + \tfrac{1}{2}x_5^2\psi_5 \\ &\quad + x_2\psi_6 + x_5\psi_7 + \psi_8(Z_1, Z_2, Z_3, Z_5). \end{aligned} \tag{140c}$$

Although by construction the normal form defined by eqs. (137) and (140a, b, c) is minimal let us give an explicit proof of minimality. First we note that the normal form has the form $\Sigma_{j=1}^{12} f_j \boldsymbol{v}_j \equiv N$, where $f_j \in$ Ker$(\mathscr{D}^*)$ and $\boldsymbol{v}_j$, $j = 1, 12$ are vectors satisfying eqs. (109). The normal form will be minimal if $N = 0$ implies $f_j = 0$, $\forall j = 1, 12$. To prove this statement we observe that if P, Q are polynomials in Z_1, Z_2, Z_3, Z_5, then*

$$P + Z_4Q = 0 \Rightarrow P = Q = 0. \tag{141}$$

Writing the normal form in vector form we see that we have to prove that

$$\varphi_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varphi_2\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} + \phi_2\begin{pmatrix} Z_4 \\ q_1 \end{pmatrix} + \varphi_3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{142}$$

implies $\varphi_1 = \varphi_2 = \varphi_3 = \phi_2 = 0$. Using (141) the first component of (142) gives $\phi_2 = 0$, $\varphi_1 = x_4\varphi$, $\varphi_2 = -x_1\varphi$ for some φ. Therefore the second component of (142) reads $\varphi_3 + Z_4\varphi = 0$ which by (141) leads to $\varphi_3 = 0$, $\varphi = 0$. We also have to prove that $F_1 = F_2 = F_3 = 0$ implies $\psi_j = 0$, $j = 1, \ldots, 8$. Using (141) we obtain from

$F_1 = 0$ that

$$x_1\psi_1 + x_4\psi_2 = 0, \tag{143a}$$

$$x_1\psi_3 + x_1x_4\psi_4 + x_4^2\psi_5 = 0. \tag{143b}$$

Since ψ_2 does not depend on x_1 (143a) gives $\psi_1 = \psi_2 = 0$. Similarly since ψ_5 does not depend on x_1 (143b) gives $\psi_5 = 0$ and

$$\psi_3 = -x_4\psi_4. \tag{144}$$

Using (144) the condition $F_2 = 0$ gives

$$x_1\psi_6 + x_4\psi_7 - Z_4\psi_4 = 0$$

which by (141) leads to

$$\psi_4 = 0, \quad \psi_6 = -x_4\phi, \quad \psi_7 = x_1\phi \tag{145}$$

for some ϕ. Using (145) we obtain that $F_3 = 0$ reads

$$\psi_8 - Z_4\phi = 0 \tag{146}$$

and therefore by (141) $\psi_8 = \phi = 0$ which finally proves the minimality of the normal form given by eqs. (137) and (140a, b, c).

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