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Local Bifurcations, Center Manifolds, and Normal Forms in Infinite-Dimensional Dynamical Systems

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Preface

This book is an extension of different lectures given by the authors during many years at the University of Nice, at the University of Stuttgart in 1990, and the University of Bordeaux in 2000 and 2001. Large parts of the first four chapters are of master level and contain various examples and exercises, partly posed at exams. However, the infinite-dimensional set-up in Chapter 2 requires several tools and results from the theory of linear operators. A brief description of these tools and results is given in Appendix A.

Bifurcation theory forms the object of many different books over the past 30 years. We refer, for instance, to [4, 58, 17, 38, 29, 30, 39, 51, 110, 84, 16, 10, 79] for some references covering various topics, going from elementary local bifurcations to global bifurcations and applications to partial differential equations. In this book we restrict our attention to the study of local bifurcations. Starting with the simplest bifurcation problems arising for ordinary differential equations in one and two dimensions, the purpose of this book is to describe several tools from the theory of infinite-dimensional dynamical systems, allowing to treat more complicated bifurcation problems, as for instance bifurcations arising in partial differential equations. Such tools are extensively used to solve concrete problems arising in physics and natural sciences.

In a parameter-dependent physical system, for example, modelized by a differential equation, the presence of a bifurcation corresponds to a topological change in the structure of the solution set (which may break its symmetry in the case of a system invariant under some symmetry group). Such a change may imply the occurrence of new solutions, or the disappearance of certain solutions, or may indicate a change of stability of certain solutions. Local bifurcation theory allows one to detect solutions and to describe their geometric (including symmetries) and dynamic properties. During the last decades the use of bifurcation theory, and in particular of the methods presented in this book, led to significant progress in the understanding of nonlinear phenomena in partial differential equations, including hydrodynamic problems, structural mechanics, but also pattern formation, population dynamics, or questions in biophysics. For instance, in the classical Couette–Taylor problem describing flows between two coaxial rotating cylinders (briefly presented in Section 5.1.2), the theory was not only a qualitative one, but also sufficiently quantitative to allow prediction of numerical values of the parameters, where new flows, such as "ribbons," were expected to be observed. These were indeed later observed experimentally [117]. This predictive power of the local theory appeared again in water wave theory, where new forms of "solitary waves," with damping oscillations at infinity, were found (see Section 5.2.1), or in the propagation of interfaces between metastable states, where new types of fronts were constructed (see Section 5.2.2).

In this book we focus on two specific methods that arise in the analysis of local bifurcations in infinite-dimensional systems, namely the center manifold reduction and the normal form theory. Center manifolds provide a powerful method of analysis of such systems, as they allow one to reduce, under certain conditions, the infinite-dimensional dynamics near a bifurcation point to a finite-dimensional dynamics, described by a system of ordinary differential equations. An efficient way of studying the resulting reduced systems is with the help of normal form theory, which consists in suitably transforming a nonlinear system, in order to keep only the relevant nonlinear terms and to allow easier recognition of its dynamics. The combination of these two methods led over the recent years to significant progress in the understanding of various problems arising in applied sciences, and in particular in the study of nonlinear waves. A common feature of many of these problems is the presence of symmetries, as for instance reversibility symmetries. It turns out that both the center manifold reduction and the normal form transformations preserve symmetries, allowing then an efficient treatment of such problems. In addition, they provide a detailed comprehensive study near a singularity in the solution set of the system, which might also orient a numerical treatment of such problems.

The book is organized as follows. We start in Chapter 1 with a presentation of the simplest bifurcations for one- and two-dimensional ordinary differential equations: saddle-node, pitchfork, Hopf, and steady bifurcations in the presence of a simple symmetry group. The purpose of this particular choice is to also introduce the reader to some of the techniques and notations used in the next chapters. Chapter 2 is devoted to the center manifold theory. This is the core tool used all throughout this book. We present the center manifold reduction for infinite-dimensional systems, together with simple examples and exercises illustrating the variety of possible applications. The aim is to allow readers who are not familiar with the subject to use this reduction method simply by checking some clear assumptions. Chapter 3 is concerned with the normal form theory. In particular, we show how to systematically compute the normal forms in concrete situations. We illustrate the general theory on different bifurcation problems, for which we provide explicit formulas for the normal form, allowing one to obtain quantitative results for the resulting systems. In Chapter 4 the normal form theory is applied to the study of reversible bifurcations, which appear to be of particular importance in applications, as this is shown in Chapter 5. We focus on bifurcations of codimension 1, i.e., bifurcations involving a single parameter, which arise generically for systems in dimensions 2, 3, and 4. In all cases, we give the normal forms and collect some known facts on their dynamics. Finally, in Chapter 5 we present some applications of the methods described in the previous chapters. Without going into detail, for which we refer to the literature, we discuss hydrodynamic instabilities arising in the Couette–Taylor and the Bénard–Rayleigh convection problems and the questions of existence of traveling water waves, of almost planar waves in reaction-diffusion systems, and of traveling waves in lattices. The proofs (few being original) of some of the results in Chapters 2 and 3, and some of the normal form calculations in Chapters 3 and 4, are given in the Appendix. The Appendix is completed by a brief collection of results from the theory of linear operators used in Chapters 2, 3, and 5, and a short introduction to basic Sobolev spaces.

Historical Remark

Many authors refer to the work of C. G. J. Jacobi from 1834, on equilibria of self-gravitating rotating ellipsoids [71], as a first reference in the field of bifurcation theory. However, it seems that the first serious works on bifurcation problems were by Archimedes and Apollonios over 200 years BCE. Archimedes studied the equilibria of a floating paraboloid of revolution [107]. In today's terminology his results would correspond to a pitchfork bifurcation which breaks a flip symmetry, or to a steady bifurcation with $O(2)$ *symmetry, when taking into account the invariance under rotations about the paraboloid axis. Apollonios studied the extrema of the length of segments joining a point of the plane to a given conic [74]. The number of solutions changes from one to three in crossing the envelope of the normals to the conic. Here again, due to the symmetry of the conic, we have an example of a pitchfork bifurcation. Finally, it seems that the French word "bifurcation" was introduced by Poincaré in 1885 [103].*

Notational Remark

We adopt Arnold's notation [4] to distinguish classes of real matrices $\mathbf{L}$ *with the same Jordan form by indicating the eigenvalues of* $\mathbf{L}$ *and the length of their Jordan chain (e.g.,* $i\omega$ *when* $\mathbf{L}$ *has a pair of simple complex eigenvalues* $\pm i\omega$*,* 0^2 *when* $\mathbf{L}$ *has a double zero eigenvalue with a Jordan block of length 2,* $(i\omega_1)(i\omega_2)$ *when* $\mathbf{L}$ *has two pairs of complex eigenvalues* $\pm i\omega_1$ *and* $\pm i\omega_2$*, and so on).*

Remark on Numbering

Each of the five chapters of this book is numbered with Arabic numerals. Sections and subsections are numbered within chapters*. The sections are identified by two numbers, the number of the chapter and the number of the section in the chapter (e.g., Section 1.2 is the second section in Chapter 1). The subsections are identified by three numbers, the number of the chapter, the number of the section, and the* *number of the subsection (e.g., Section 1.2.1 is the first subsection in Section 1.2 of Chapter 1).*

Equations are numbered within sections *and identified by only two numbers: the number of the section inside the chapter (omitting the number of the chapter), and the number of the equation inside the section (e.g., equation (2.1) is the first equation in the second section of the current chapter). When referring to an equation, we only give the number, e.g., equation (2.1), if the equation is in the current chapter, but also mention the number of the chapter if the equation is in a different chapter, e.g., equation (2.1) in Chapter 2.*

Definitions, hypotheses, theorems, lemmas, corollaries, remarks, and exercises are numbered together within sections, *and identified by two numbers, just as the equations. Figures are numbered independently* within sections *and identified also by two numbers, just as equations.*

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