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Capillary Gravity Waves on the Free Surface of an Inviscid Fluid of Infinite Depth. Existence of Solitary Waves

GÉRARD IOOSS & PIUS KIRRMANN

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Abstract

Permanent capillary gravity waves on the free surface of a two dimensional inviscid fluid of infinite depth are investigated. An application of the hodograph transform converts the free boundary-value problem into a boundary-value problem for the Cauchy-Riemann equations in the lower halfplane with nonlinear differential boundary conditions. This can be converted to an integro-differential equation with symbol $-k^2 + 4|k| - 4(1 + \mu)$, where μ is a bifurcation parameter. A normal-form analysis is presented which shows that the boundary-value problem can be reduced to an integrable system of ordinary differential equations plus a remainder term containing nonlocal terms of higher order for $|\mu|$ small. This normal form system has been studied thoroughly by several authors (IOOSS & KIRCHGÄSSNER [8], IOOSS & PÉROUÈME [10], DIAS & IOOSS [5]). It admits a pair of solitary-wave solutions which are reversible in the sense of KIRCHGÄSSNER [11]. By applying a method introduced in [11], it is shown that this pair of reversible solitary waves persists for the boundary-value problem, and that the decay at infinity of these solitary waves is at least like $1/|x|$.

1. Statement of the problem

One of the open problems in the area of two-dimensional water-wave problems is the question of existence of steady capillary gravity waves of solitary type on deep water. It was conjectured by LONGUET-HIGGINS [12] that such waves indeed exist. Steady capillary gravity solitary waves were calculated numerically by the same author in [13]. He observed that these waves do not decay exponentially but only quadratically with the inverse of the distance from the origin in a moving frame of reference. A boundary-integral-equation technique was used by Vander-Broeck & Dias [17] to compute both free and forced capillary gravity waves numerically.

There are also numerous papers dealing with solitary waves in a two-fluid system where the lower layer is infinitely deep. A model equation for permanent waves in a system of stratified fluids when capillarity is negligible was derived by Benjamin [3]. The corresponding time-dependent equation was derived later by Ono [15]. This model equation, which is known as the Benjamin-Ono equation in the literature, admits an explicit solitary-wave solution decaying algebraically.

The full Euler equations for a two-fluid system, with one infinite layer, were considered by Amick [1] and Sun [16]. They independently proved, by a fixed-point technique, the existence of a solitary wave, near a solitary solution of the Benjamin-Ono equation. Recently, Benjamin [4] proposed an approximate model equation for the interface problem of a two-fluid system, the lower being of infinite depth, with the interface being subject to capillarity. The linear singularity in this problem is the same as the one we are treating below, but a mathematical proof of the existence of solitary waves has still to be given.

The existence of solitary capillary gravity waves on the free surface of a fluid of large but finite depth (i.e., for small Bond number and Froude number less than 1) was proved by Iooss & Kirchgässner [8]. Their analysis was extended by Dias & Iooss [5], who also considered the limit from finite depth to infinite depth.

The subject of the present work is to provide a rigorous existence proof for solitary capillary gravity waves on deep water.

The investigation is confined to waves of permanent form moving with constant velocity c from the right to the left on the free surface of an inviscid, incompressible fluid of uniform density $\rho = \rho_0$.

In a moving frame of reference with coordinates

$$(\xi, \eta) = (X + ct, Y), \quad \xi \in \mathbb{R}, -\infty < \eta < Z(\xi),$$

where $Z(\xi)$ is the free surface, the flow is steady and the velocity field for the undisturbed fluid is the uniform flow $(c,0)$. We consider flows which are close to this uniform flow and denote the perturbation by (cU, cV) so that U and V are dimensionless quantities, i.e.,

$$u(t,X,Y) = cU(\xi,\eta) - c, \quad v(t,X,Y) = cV(\xi,\eta).$$

From the equation of continuity and the assumption that the flow is irrotational it follows that the flow has a potential ϕ. The potential ϕ and the stream function ψ are given by

$$(U, V) = (\phi_\xi, \phi_\eta) = (\psi_\eta, -\psi_\xi). \tag{1.1}$$

Bernoulli's equation ensures that

$$\tfrac{1}{2}\rho_0 c^2(U^2 + V^2) + p + \rho_0 g\eta = \text{const}, \tag{1.2}$$

where p is the pressure and g stands for the acceleration of gravity. At large depth $|\eta| \gg 1$, the pressure p is proportional to $\rho_0 g\eta$.

The free surface $\eta = Z(\xi)$ is a streamline corresponding to $\psi = 0$, say; therefore, it satisfies

$$UZ_\xi - V = 0, \quad \eta = Z(\xi), \quad -\infty < \xi < \infty. \tag{1.3}$$

On the free surface the jump in the pressure is proportional to the curvature

$$p(\xi) = -T\frac{Z_{\xi\xi}(\xi)}{\left(1 + Z_\xi^2(\xi)\right)^{3/2}},$$

where the factor of proportionality is the constant surface tension T. The problem is studied in a hodograph form (cf. [2]) where the coordinates (ϕ, ψ) are used to map the unknown domain

$$\left\{(\xi, \eta) \in \mathbb{R}^2 : -\infty < \eta < Z(\xi), -\infty < \xi < \infty\right\}$$

into the lower half plane. We set $\zeta = \xi + i\eta$, introduce the analytic complex function $w(\zeta) = \phi(\zeta) + i\psi(\zeta)$ and define two real functions α and β of the complex variable w by

$$w'(\zeta) = U(\zeta) - iV(\zeta) = \mathrm{e}^{\beta(w(\zeta)) - \mathrm{i}\alpha(w(\zeta))}.$$

Thus, the magnitude of the velocity is given by e^β, and α is the angle between the velocity and the horizontal line measured in the counterclockwise direction. The independent variable ζ is replaced by w, so that the new domain for w is the lower half plane. As is shown in [2], the equations in the transformed variables become

$$\begin{aligned} \alpha_{\phi\phi} + \alpha_\phi\beta_\phi - \frac{c^2\rho_0}{T}\beta_\phi\mathrm{e}^\beta - \frac{g\rho_0}{T}\mathrm{e}^{-2\beta}\sin\alpha = 0, \quad \psi = 0,\\ \alpha_\phi = \beta_\psi, \quad \beta_\phi = -\alpha_\psi, \quad \psi < 0. \end{aligned}$$

In order to make all quantities dimensionless, we scale the independent variables by introducing a new unit of length:

$$(\phi, \psi) = (lx, ly), \quad l = \frac{4T}{c^2\rho_0}.$$

A further condition at $\eta = -\infty$ has to be added to the boundary-value problem. In order to obtain solutions with physical relevance, we require that both α and β vanish at infinite depth. The equations governing the problem are

$$\begin{aligned} \alpha_{xx} + \alpha_x\beta_x - 4\beta_x\mathrm{e}^\beta - 4(1+\mu)\mathrm{e}^{-2\beta}\sin\alpha = 0, \quad y = 0\\ \alpha_x = \beta_y, \quad \beta_x = -\alpha_y, \quad y < 0,\\ \alpha, \beta \to 0, \quad y \to -\infty, \end{aligned} \tag{1.4}$$

where the dimensionless parameter μ is given by $\mu = 4\frac{gT}{\rho_0 c^4} - 1 \approx 0$. Formally, the fraction coincides with the product $b\lambda$ of the Bond number and the inverse of the square of the Froude number,

$$b = \frac{T}{\rho_0 h c^2}, \quad \lambda = \frac{gh}{c^2}.$$

These two parameters are defined for a fluid of finite depth h. It was shown in [8] that, in the limit $h \to \infty$, the product $b\lambda$ approaches the constant value $\frac{1}{4}$ along a certain branch in the (b, λ) bifurcation diagram where a 1:1 resonance bifurcation occurs. The assumption that $|\mu| \ll 1$ agrees with the range of parameter values one obtains in that limit. We remark that the first equation in (1.4) represents the differentiated form of

$$\alpha_x e^{\beta} - 2e^{2\beta} - 4(1+\mu) \int_{-\infty}^{0} \left(e^{-\beta} \cos\alpha - 1\right) dy = \text{const.}$$

Differentiating this equation with respect to x, one obtains the first equation of (1.4). The trivial solution $(\alpha, \beta) = (0, \text{const})$ is ruled out by the decay condition at $y = -\infty$.

This equation can be viewed as an integro-differential equation in x with the trace of α on $y = 0$ as the unknown function. Since $\alpha + i\beta$ is an analytic function in the lower complex half plane, the trace of β on $y = 0$ is the Hilbert transform of the trace of α on $y = 0$. For the proof of the last statement, we consider the linear boundary value problem

$$\begin{aligned} &\alpha_x = \beta_y, \ \beta_x = -\alpha_y, \quad (x, y) \in \mathbb{R} \times (-\infty, 0), \\ &\lim_{y \uparrow 0} \alpha(x, y) = \alpha^0(x), \quad \alpha, \beta \to 0, \quad y \to -\infty, \end{aligned} \tag{1.5}$$

where the trace α^0 of α is prescribed. If α^0 belongs to $L_2(\mathbb{R})$, for example, then the solution of (1.5) can be found by applying the Fourier transform, solving the transformed system and applying the inverse Fourier transform. One finds that α and β are given by convolution integrals:

$$\begin{aligned} \alpha(x, y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{y^2 + (x-\xi)^2} \alpha^0(\xi)\, d\xi, \\ \beta(x, y) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x-\xi}{y^2 + (x-\xi)^2} \alpha^0(\xi)\, d\xi. \end{aligned} \tag{1.6}$$

The first equation represents Poisson's formula in the half plane. Taking the limit $y \uparrow 0$ in the second equation we obtain

$$\beta(x, 0) = (\mathcal{H}\alpha(\cdot, 0))(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\alpha^0(\xi)}{x - \xi} d\xi, \tag{1.7}$$

where $\mathcal{H}$ denotes the Hilbert transform. The Fourier transform of $\beta^0 = \beta(\cdot, 0)$ is given by

$$\hat{\beta}^0(k) = i \,\text{sign}(k) \hat{\alpha}^0(k), \tag{1.8}$$

which shows that the linearization of (1.4) is equivalent to an integro-differential equation with symbol $-k^2+4|k|-4(1+\mu)$. Replacing $\beta(x,0)$ by $\mathscr{H}\alpha(x,0)$, we obtain the integro-differential equation

$$\alpha_{xx}+\alpha_x\beta_x-4\beta_x e^{\beta}-4(1+\mu)e^{-2\beta}\sin\alpha=0, \quad \beta=\mathscr{H}\alpha. \tag{1.9}$$

Since the zeros of the symbol are $2\pm2\sqrt{-\mu}$ and $-2\pm2\sqrt{-\mu}$, a 1:1 resonance bifurcation takes place at $\mu=0$.

In the next section the boundary value problem (1.4) will be put into a form which makes it amenable to the normal form algorithm.

2. Preparatory analysis

We treat the problem in the formulation (1.4). In view of the normal-form algorithm to be applied, it is convenient to formulate the boundary-value problem as a dynamical system in a Banach space. The horizontal variable x plays the role of the evolutionary variable. We define the vector of unknowns

$$\boldsymbol{u}=\boldsymbol{u}(x)=\left(\alpha_0^0(x),\alpha_1^0(x),\alpha_0(x,\cdot),\alpha_1(x,\cdot),\beta_0(x,\cdot),\beta_1(x,\cdot)\right)^t,$$

where the following identifications are made

$$\alpha_0^0(x)=\alpha(x,0), \quad \alpha_1^0=\alpha_x(x,0), \quad \alpha_0(x,\cdot)=\alpha(x,\cdot), \quad \alpha_1(x,\cdot)=\alpha_x(x,\cdot),$$

and similarly for β. Then we can write (1.4) as a differential equation in $X=\mathbb{C}\times\mathbb{C}\times[L_1(-\infty,0)]^4$ of the form

$$\frac{d}{dx}\boldsymbol{u}(x)=\boldsymbol{L}\boldsymbol{u}(x)+\boldsymbol{N}(\mu;\,\boldsymbol{u}(x)), \quad x\in\mathbb{R} \tag{2.1}$$

A solution $\boldsymbol{u}:\ \mathbb{R}\mapsto D(\boldsymbol{L})$ is a mapping from $\mathbb{R}$ into the domain of the linear operator $\boldsymbol{L}$. More precisely we require $\boldsymbol{u}\in C^1(\mathbb{R},\boldsymbol{X})\cap C^0(\mathbb{R},D(\boldsymbol{L}))$. The derivative in (2.1) is the Fréchet derivative. The linear operator $\boldsymbol{L}:D(\boldsymbol{L})\to\boldsymbol{X}$ and the nonlinear mapping $\boldsymbol{N}:\mathbb{R}\times D(\boldsymbol{L})\to\boldsymbol{X}$ are defined as follows

$$D(\boldsymbol{L})=\{\boldsymbol{u}=(\xi_0,\xi_1,\alpha_0,\alpha_1,\beta_0,\beta_1)^t\in\mathbb{C}^2\times[W^{1,1}(-\infty,0)]^4: \\ \xi_0=\alpha_0|_{y=0},\xi_1=\alpha_1|_{y=0}\}, \tag{2.2}$$

$$\boldsymbol{L}\begin{pmatrix}\zeta_0\\ \zeta_1\\ \alpha_0\\ \alpha_1\\ \beta_0\\ \beta_1\end{pmatrix}=\begin{pmatrix}\zeta_1\\ 4\beta_1^0+4\zeta_0\\ \beta_0'\\ \beta_1'\\ -\alpha_0'\\ -\alpha_1'\end{pmatrix}, \quad \boldsymbol{N}(\mu;\,\boldsymbol{u})=\begin{pmatrix}0\\ F(\mu;\alpha_0^0,\alpha_1^0,\beta_0^0,\beta_1^0)\\ 0\\ 0\\ 0\\ 0\end{pmatrix}. \tag{2.3}$$

The prime $'$ stands for the derivative with respect to y, the superscript ^{0} means evaluation on the upper boundary $y=0$. The nonlinear function $F(\mu;\alpha_0^0,\alpha_1^0,\beta_0^0,\beta_1^0)$ is given by

$$\begin{aligned} F(\mu;\alpha_0,\alpha_1,\beta_0,\beta_1) &= -\alpha_1\beta_1 - 4\beta_1 + 4\beta_1 e^{\beta_0} - 4\alpha_0 \\ &\quad + 4e^{-2\beta_0}\sin\alpha_0 + 4\mu e^{-2\beta_0}\sin\alpha_0 \\ &= \mathcal{O}\Big(\big|(\mu;\alpha_0,\alpha_1,\beta_0,\beta_1)\big|^2\Big) \quad \text{as } (\mu;\alpha_0,\alpha_1,\beta_0,\beta_1) \to 0. \end{aligned}$$

We shall construct a "normal form" of the system (2.1), which is determined by the spectrum of the linear operator $\boldsymbol{L}$ and symmetry properties. One of the notable features of equation (2.1) is its reversibility with respect to the isometry

$$\boldsymbol{R} : (\alpha_0^0, \alpha_1^0, \alpha_0, \alpha_1, \beta_0, \beta_1)^t \mapsto (-\alpha_0^0, \alpha_1^0, -\alpha_0, \alpha_1, \beta_0, -\beta_1)^t. \tag{2.4}$$

Let $\boldsymbol{u}(x)$ be a solution of (2.1); then $\boldsymbol{u}_{\mathrm{R}}(x) = \boldsymbol{R}\boldsymbol{u}(-x)$ is also a solution, since both $\boldsymbol{L}$ and $\boldsymbol{N}$ anticommute with $\boldsymbol{R}$.

Now we are going to study the spectrum of the linear operator $\boldsymbol{L}$. First set $\boldsymbol{u}(x) = \boldsymbol{u}_0 e^{ikx}$ and introduce this into the linearized equation. We obtain the relation $-k^2 + 4|k| - 4 = 0$, i.e., there exist two eigenvalues $\pm 2i$ of $\boldsymbol{L}$, which can be shown to be double. The entire real line constitutes the essential spectrum of $\boldsymbol{L}$. In particular, the closure of the range of $(\boldsymbol{L} - \lambda), \lambda \in \mathbb{R}$, has codimension two, while the kernel is zero. The spectrum at $\lambda = 0$ is reflected by the nonsmoothness of the symbol which contains a term $|k|$. The geometric multiplicity of the two eigenvalues is 1 and there exists a Jordan chain of length 2:

$$(\boldsymbol{L} \mp 2i)\,\boldsymbol{\varphi}_\pm^0 = \boldsymbol{0}, \quad (\boldsymbol{L} \mp 2i)\,\boldsymbol{\varphi}_\pm^1 = \boldsymbol{\varphi}_\pm^0, \tag{2.5}$$

$$y \mapsto \begin{pmatrix} \mp i \\ 2 \\ \mp i e^{2y} \\ 2e^{2y} \\ e^{2y} \\ \pm 2i e^{2y} \end{pmatrix} =: \boldsymbol{\varphi}_\pm^0(y) \quad y \mapsto \begin{pmatrix} 0 \\ \mp i \\ -y e^{2y} \\ \mp i(1+2y)e^{2y} \\ \mp i y e^{2y} \\ (1+2y)e^{2y} \end{pmatrix} =: \boldsymbol{\varphi}_\pm^1(y). \tag{2.6}$$

The eigenvectors and generalized eigenvectors of the formal adjoint $\boldsymbol{L}^*$ of $\boldsymbol{L}$ will be shown to exist. They are needed in the normal-form algorithm. $\boldsymbol{L}^*$ is defined by

$$\begin{aligned} D(\boldsymbol{L}^*) := \big\{\boldsymbol{u} = (\xi_0, \xi_1, \alpha_0, \alpha_1, \beta_0, \beta_1)^t \in \mathbb{C}^2 \times [W^{1,1}(-\infty,0)]^4 : \\ \alpha_0|_{y=0} = 0, \alpha_1|_{y=0} = -4\xi_1, \big\}, \end{aligned} \tag{2.7}$$

$$\boldsymbol{L}^* \begin{pmatrix} \xi_0 \\ \xi_1 \\ \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{pmatrix} := \begin{pmatrix} 4\xi_1 - \beta_0^0 \\ \xi_0 - \beta_1^0 \\ \beta_0' \\ \beta_1' \\ -\alpha_0' \\ -\alpha_1' \end{pmatrix}. \tag{2.8}$$

The normalized eigenfunctions of L^* corresponding to the eigenvalues $\pm 2i$ are

$$(L^* \mp 2i)\psi_1^{\pm} = \mathbf{0}, \quad (L^* \mp 2i)\psi_0^{\pm} = \psi_1^{\pm}, \tag{2.9}$$

$$y \mapsto \begin{pmatrix} 2 \\ \pm i \\ 0 \\ \mp 4ie^{2y} \\ 0 \\ 4e^{2y} \end{pmatrix} =: \psi_1^{\pm}(y), \quad y \mapsto \begin{pmatrix} \pm i \\ 0 \\ 0 \\ -4ye^{2y} \\ 0 \\ \mp 4iye^{2y} \end{pmatrix} =: \psi_0^{\pm}(y). \tag{2.10}$$

The normalization implies the biorthogonality conditions

$$(\varphi_{\pm}^j, \psi_k^{\mp}) = \delta_k^j, \quad (\varphi_{\pm}^j, \psi_k^{\mp}) = 0, \tag{2.11}$$

where the scalar product is the one for $\mathbb{C}^2 \times [L^2(-\infty, 0)]^4$, and where these quantities are well defined due to the exponential decay of eigenfunctions as $y \to -\infty$.

An element $u \in X$ can be decomposed into a "central" component u_0 lying in the sub-space X_0 spanned by the generalized eigenvectors $\varphi_{\pm}^0$ and $\varphi_{\pm}^1$, and a "hyperbolic" component u_1 lying in the complementary subspace X_1. The components of u can be calculated by using the generalized eigenvectors of the adjoint operator as follows:

$$\begin{aligned} u_0 &= P_0 u = a\varphi_+^0 + b\varphi_+^1 + \bar{a}\varphi_-^0 + \bar{b}\varphi_-^1, \\ a &= (u, \psi_0^-), \ b = (u, \psi_1^- .), \\ u_1 &= (I - P_0)\, u = u - u_0. \end{aligned}$$

The central component u_0 is identified with the complex vector $v = (a, b, \bar{a}, \bar{b})^t$ as given above. We write $w = P_1 u = (I - P_0)u$ for the hyperbolic component.

According to the decomposition of u, (2.1) splits into a system of coupled differential equations in $\mathbb{C}^2$ and X_1. The projected system is given by

$$\begin{aligned} \frac{d}{dx} v(x) &= L_0 v(x) + N^0(\mu; v(x), w(x)), \\ \frac{d}{dx} w(x) &= L_1 w(x) + N^1(\mu; v(x), w(x)) \end{aligned} \tag{2.12}$$

with

$$L_0 = \begin{pmatrix} 2i & 1 & 0 & 0 \\ 0 & 2i & 0 & 0 \\ 0 & 0 & -2i & 1 \\ 0 & 0 & 0 & -2i \end{pmatrix}, \quad N^0(\mu; v, w) = \begin{pmatrix} (N(\mu; u_0(v) + w), \psi_0^-) \\ (N(\mu; u_0(v) + w), \psi_1^-) \end{pmatrix},$$

$$N^1(\mu; v, w) = P_1 N(\mu; u_0(v) + w).$$

3. Application of a normal-form algorithm

We are adapting a normal-form algorithm which was developed by Elphick et al. [6].

For the problem of capillary gravity waves on the free surface of a fluid of *finite* depth h, a normal form in the 1:1 resonance case was calculated by Dias & Iooss [5] and the limit $h \to \infty$ was considered. In the mathematics, there are substantial differences between the "finite-depth" and the "infinite-depth" problems. In the case of finite depth there is available a center-manifold reduction theorem which guarantees that all small bounded solutions lie on a center manifold over the linear subspace spanned by the eigenelements and generalized eigenelements corresponding to the eigenvalues with zero real part. In the case of a 1:1 resonance bifurcation, center-manifold theory ensures that all small bounded solutions can be written in the form

$$\begin{aligned} \boldsymbol{u}(x) &= A(x)\boldsymbol{\varphi}_0^+ + B(x)\boldsymbol{\varphi}_1^+ + \overline{A(x)}\boldsymbol{\varphi}_0^- + \overline{B(x)}\boldsymbol{\varphi}_1^- \\ &\quad + \boldsymbol{\Phi}\big(\mu; A(x), B(x), \overline{A(x)}, \overline{B(x)}\big). \end{aligned}$$

A normal form corresponding to a 1:1 resonance bifurcation problem with reversibility is given in the book of Iooss & Adelmeyer [7, pp. 58–59]. The normal form is

$$\begin{aligned} A_x &= 2iA + B + iAP\big(\mu; A\bar{A}, \tfrac{i}{2}(A\bar{B} - \bar{A}B)\big), \\ B_x &= 2iB + iBP\big(\mu; A\bar{A}, \tfrac{i}{2}(A\bar{B} - \bar{A}B)\big) + AQ\big(\mu; A\bar{A}, \tfrac{i}{2}(A\bar{B} - \bar{A}B)\big) \end{aligned} \tag{3.1}$$

with real polynomials P and Q. In the finite-depth problem, the full boundary-value problem is, by center-manifold theory, locally equivalent to a system of ordinary differential equations which consists of the normal-form system (3.1) plus higher-order perturbation terms in $(\mu, A, B, \bar{A}, \bar{B})$.

In the present situation of a fluid of infinite depth, a center-manifold reduction is not available because of lack of gap in the spectrum of $\boldsymbol{L}_1$ near the imaginary axis. However, normal-form theory is applicable to (2.12) with some modifications and restrictions which are due to the non-invertibility of the operator $\boldsymbol{L}_1$. System (2.12) will be transformed into normal form by introducing new coordinates

$$\boldsymbol{V} = (A, B, \bar{A}, \bar{B})^t \in \mathbb{C}^4, \quad \boldsymbol{W} = (\xi_0, \xi_1, \theta_0, \theta_1, \tau_0, \tau_1) \in D(\boldsymbol{L}_1). \tag{3.2}$$

The central equation of (2.12) can be put into normal form with a remainder term depending on $(\mu, \boldsymbol{V}, \boldsymbol{W})$. For the hyperbolic equation of (2.12), it turns out to be useful to remove certain second-order terms only.$\bar{A}$

Theorem 1. *There exist polynomials*

$$\Phi^0 : \mathbb{R} \times \mathbb{C}^4 \to \mathbb{C}^4, \quad \Phi^1 : \mathbb{C}^4 \to D(\boldsymbol{L}_1)$$

respectively of degrees 3 *and* 2 *with respect to* $\boldsymbol{V}$ *such that, by the change of variables*

$$v = V + \Phi^0(\mu; V), \quad w = W + \Phi^1(V), \tag{3.3}$$

system (2.12) *is transformed into*

$$\begin{aligned} \frac{d}{dx} V(x) &= L_0 V(x) + G(\mu; V(x)) + R^0(\mu; V(x), W(x)), \\ \frac{d}{dx} W(x) &= L_1 W(x) + H(V(x)) + R^1(\mu; V(x), W(x)), \end{aligned} \tag{3.4}$$

where $R^0 \in \mathbb{C}^4, R^1 \in X_1$,

$$R^j(\mu; V, W) = \mathcal{O}\big((|\mu||V|^{1-j} + |\mu|^2 + |V|^{3-j})|V| + (|V| + |\mu| + \|W\|_{D(L)})\|W\|_{D(L)}\big)$$

as $|\mu| + |V| + \|W\|_{D(L)} \to 0$. *Moreover,* H *is a quadratic polynomial with*

$$H(A, 0, \bar{A}, 0) = \mathbf{0}. \tag{3.5}$$

The truncated equation for the central part

$$\frac{d}{dx} V(x) = L_0 V(x) + G(\mu; V(x))$$

takes the normal form (3.1) *with real polynomials* P *and* Q *of degree three.*

Remark. Since L_1 is not invertible, we cannot eliminate the polynomial H completely. However, each monomial of H contains B or $\bar{B}$ as factor. This observation will be crucial in the proof of the existence of solitary waves.

Proof. According to (3.3) we make the ansatz

$$\begin{aligned} u(x) &= A(x)\varphi_0^+ + B(x)\varphi_1^+ + \overline{A(x)}\varphi_0^- + \overline{B(x)}\varphi_1^- \\ &\quad + W + \Phi(\mu; A(x), B(x), \overline{A(x)}, \overline{B(x)}), \end{aligned} \tag{3.6}$$

where $\Phi^1 = (I - P_0)\Phi$ and Φ^0 represents $P_0\Phi$. The transformation Φ is written in the form

$$\Phi(\mu; A, B, \bar{A}, \bar{B}) = \sum_{r+i+j+k+l=2}^{3} \Phi_{r,ijkl}\mu^r A^i B^j \bar{A}^k \bar{B}^l$$

with coefficients $\Phi_{r,ijkl} \in D(L)$. The nonlinearity in (2.12) is replaced by its Taylor polynomial of order three plus remainder term:

$$\begin{aligned} N(\mu; u) &= N_{1,1}(\mu; u) + N_{0,2}(u, u) + N_{1,2}(\mu; u, u) + N_{0,3}(u, u, u) \\ &\quad + \mathcal{O}\Big(|\mu|\ \|u\|_{D(L)}^3 + \|u\|_{D(L)}^4\Big), \quad |\mu| + \|u\|_{D(L)} \to 0. \end{aligned}$$

The terms $N_{r,m}(\mu; \cdot, \dots, \cdot)$ are m-linear *symmetric* mappings from $D(L)$ into X.

The polynomials P and Q of the normal form depend on eight unknown coefficients which will be determined in the normal-form algorithm

$$P(\mu; u, v) = P_{1,00}\mu + P_{2,00}\mu^2 + P_{0,10}\mu + P_{0,01}v,$$
$$Q(\mu; u, v) = Q_{1,00}\mu + Q_{2,00}\mu^2 + Q_{0,10}\mu + Q_{0,01}v.$$

Inserting (3.6) into (2.1) and taking into account the system (3.4) to be satisfied by $(\boldsymbol{V}, \boldsymbol{W})$, we obtain, by collecting equal powers in $(\mu, A, B, \bar{A}, \bar{B})$, linear equations for the Taylor coefficients $\boldsymbol{\Phi}_{r,ijkl}$. Since we are looking for real solutions $\boldsymbol{u}$ of (2.1), $\boldsymbol{\Phi}_{r,ijkl} = \overline{\boldsymbol{\Phi}_{r,klij}}$, so we may restrict ourselves to coefficients with indices $i + j \geqq k + l$. In what follows, $\boldsymbol{\Phi}_{r,m}$ means a m-linear symmetric mapping like $\boldsymbol{N}_{r,m}$ above, and a superscript ^{0} means its projection by $\boldsymbol{P}_0$ on the space $\boldsymbol{X}_0$. We obtain the following equations for $\boldsymbol{\Phi}_{0,2}, \boldsymbol{\Phi}^0_{0,3}, \boldsymbol{\Phi}^0_{1,1}$, in which the compatibility conditions determine the coefficients of polynomials P and Q occuring in $G_{0,3}$ and $G_{1,1}$ as shown in (3.1):

$$2\boldsymbol{\Phi}_{0,2}(\boldsymbol{V}, \boldsymbol{L}_0\boldsymbol{V}) - \boldsymbol{L}\boldsymbol{\Phi}_{0,2}(\boldsymbol{V}, \boldsymbol{V}) = \boldsymbol{N}_{0,2}(\boldsymbol{V}, \boldsymbol{V}) - \boldsymbol{H}(\boldsymbol{V}), \tag{3.7}$$

$$\begin{aligned} \boldsymbol{G}_{0,3}(\boldsymbol{V},\boldsymbol{V},\boldsymbol{V}) + 3\boldsymbol{\Phi}^0_{0,3}(\boldsymbol{V},\boldsymbol{V},\boldsymbol{L}_0\boldsymbol{V}) - \boldsymbol{L}_0\boldsymbol{\Phi}^0_{0,3}(\boldsymbol{V},\boldsymbol{V},\boldsymbol{V}) \\ = \boldsymbol{P}_0\boldsymbol{N}_{0,3}(\boldsymbol{V},\boldsymbol{V},\boldsymbol{V}) + 2\boldsymbol{P}_0\boldsymbol{N}_{0,2}\left[\boldsymbol{V}, \boldsymbol{\Phi}_{0,2}(\boldsymbol{V},\boldsymbol{V})\right], \end{aligned} \tag{3.8}$$

$$\boldsymbol{G}_{1,1}(\mu, \boldsymbol{V}) + \boldsymbol{\Phi}^0_{1,1}(\mu, \boldsymbol{L}_0\boldsymbol{V}) - \boldsymbol{L}_0\boldsymbol{\Phi}^0_{1,1}(\mu, \boldsymbol{V}) = \boldsymbol{P}_0\boldsymbol{N}_{1,1}(\mu, \boldsymbol{V}). \tag{3.9}$$

In this system of three equations, notice that $\boldsymbol{\Phi}_{0,2} = \boldsymbol{\Phi}^0_{0,2} + \boldsymbol{\Phi}^1$ has a component in $\boldsymbol{X}_1$ only containing the quadratic terms A^2 and $\bar{A}^2$, since

$$\boldsymbol{P}_1\boldsymbol{N}_{0,2}(\boldsymbol{V}, \boldsymbol{V}) - \boldsymbol{H}(\boldsymbol{V}) = \boldsymbol{h}_1A^2 + \bar{\boldsymbol{h}}_1\bar{A}^2,$$

$$\boldsymbol{H}(\boldsymbol{V}) = \boldsymbol{h}_2A\bar{B} + \bar{\boldsymbol{h}}_2\bar{A}B + \boldsymbol{h}_3B^2 + \bar{\boldsymbol{h}}_3\bar{B}^2$$

where $\boldsymbol{h}_j, j = 1, 2, 3$ lie in $\boldsymbol{X}_1$.
A very nice and important fact arising in our problem is that there is *no term* $|A|^2$ (or $|B|^2$ or AB) *in* $\boldsymbol{N}_{0,2}(\boldsymbol{V}, \boldsymbol{V})$. Hence there is no problem of resonant terms at the lowest order.

Resolution of system (3.8), (3.9) is classical (see for instance [7]); this allows the determination of the polynomials P and Q appearing in $\boldsymbol{G}(\mu, \boldsymbol{V})$ (see (3.1)). More precisely, the coefficient $Q_{0,10}$ is given by

$$Q_{0,10} = \left(3\boldsymbol{N}_{0,3}\left(\boldsymbol{\varphi}^0_+, \boldsymbol{\varphi}^0_+, \boldsymbol{\varphi}^0_-\right) + 2\boldsymbol{N}_{0,2}\left(\boldsymbol{\varphi}^0_-, \boldsymbol{\Phi}_{0,2}\left(\boldsymbol{\varphi}^0_+, \boldsymbol{\varphi}^0_+\right)\right), \psi^-_1\right)$$

where

$$\boldsymbol{\Phi}_{0,2}\left(\boldsymbol{\varphi}^0_+, \boldsymbol{\varphi}^0_+\right) = (4i - \boldsymbol{L})^{-1}\boldsymbol{N}_{0,2}\left(\boldsymbol{\varphi}^0_+, \boldsymbol{\varphi}^0_+\right).$$

For the calculation of the normal form coefficients and the polynomials $\boldsymbol{\Phi}$, we implemented a symbolic algebra program (MAPLE V). The normal form coefficients are found to be

$$\begin{aligned} P_{1,00} = 0, \quad P_{2,00} = -\tfrac{1}{4}, \quad P_{0,10} = \tfrac{9}{2}, \quad P_{0,01} = -5, \\ Q_{1,00} = 4, \quad Q_{2,00} = -1, \quad Q_{0,10} = -22, \quad Q_{0,01} = 47 \end{aligned} \tag{3.10}$$

The inequality $Q_{1,00}Q_{0,10} < 0$ is crucial for the existence of solitary waves for the normal form system (for $\mu > 0$ here).

4. Reduction to a system of ordinary differential equations involving nonlocal terms

The normal-form algorithm yields a system of differential equations in $(\boldsymbol{V}, \boldsymbol{W})$ which is partially decoupled: in the equation for the central component $\boldsymbol{V}$ the hyperbolic component $\boldsymbol{W}$ enters only in the higher-order terms. It is desirable to derive an equation for $\boldsymbol{V}$ which is entirely independent of $\boldsymbol{W}$. Such a reduced equation can be obtained by solving the equation for $\boldsymbol{W}$ with $\boldsymbol{V}$ as a parameter and inserting $\boldsymbol{W} = \boldsymbol{W}(\mu; \boldsymbol{V})$ into the equation for $\boldsymbol{V}$. This is reminiscent of a Lyapunov-Schmidt reduction. One drawback of this reduction is that the reduced equation is a nonlocal equation.

Our goal is to solve the hyperbolic part of (3.4) for $\boldsymbol{W} = \boldsymbol{W}(\mu; \boldsymbol{V})$ where the central component is regarded as a parameter. Note the special form of the nonlinearity $\boldsymbol{N}(\mu; \boldsymbol{u})$, which depends on the traces of $\boldsymbol{u}$ at $y = 0$ only. Moreover, all components of $\boldsymbol{N}$ except the second vanish. For that purpose we investigate the linearized equation and make use of the specific structure of the inhomogeneous terms. From the normal-form algorithm we can show that it takes the form

$$\frac{d}{dx}\boldsymbol{W}(x) = \boldsymbol{L}_1\boldsymbol{W}(x) + \boldsymbol{P}_1\boldsymbol{f}(x) + \boldsymbol{R}_1(4i)\boldsymbol{P}_1\boldsymbol{g}(x) + \boldsymbol{R}_1(-4i)\boldsymbol{P}_1\bar{\boldsymbol{g}}(x), \qquad (4.1)$$

where we can restrict our analysis to functions $\boldsymbol{f}$ and $\boldsymbol{g}$ of the form

$$\boldsymbol{f} = (0, f, 0, 0, 0, 0), \quad \boldsymbol{g} = (0, g, 0, 0, 0, 0),$$

with scalar functions $f(\mu; \boldsymbol{V}, \boldsymbol{W})$ and $g(\mu; \boldsymbol{V}, \boldsymbol{W}) = g_{\mathrm{re}} + i g_{\mathrm{im}}$ which *only depend on the trace* $\boldsymbol{W}^0$ *of* $\boldsymbol{W}$, *and on* $\boldsymbol{V}$, *and hence do not depend on* y. The symbol $\boldsymbol{R}_1$ denotes the resolvent function, $\boldsymbol{R}_1(ik) = (ik - \boldsymbol{L}_1)^{-1}$. Notice that the last two terms are due to the coefficients $\boldsymbol{\Phi}_{0,2000}$ and $\boldsymbol{\Phi}_{0,0020}$, which have nonzero components in the hyperbolic subspace $\boldsymbol{X}_1$.

There are two different types of functions appearing on the right-hand side of (4.1):

$$\begin{aligned}
\boldsymbol{f}_1 &= \boldsymbol{P}_1 \boldsymbol{f} = f(\mu; \boldsymbol{V}, \boldsymbol{W})(0, -1, 0, -2(1+2y)e^{2y}, -2ye^{2y}, 0),\\
\boldsymbol{f}_2 &= \boldsymbol{R}_1(4i)\boldsymbol{P}_1\boldsymbol{g} + \boldsymbol{R}_1(-4i)\boldsymbol{P}_1\bar{\boldsymbol{g}}\\
&= g_{\mathrm{re}}(\mu; \boldsymbol{V}, \boldsymbol{W})\left(\tfrac{1}{18}, 0, \tfrac{2}{3}ye^{2y} + \tfrac{5}{9}e^{2y} - \tfrac{1}{2}e^{4y}, 0, 0, -\tfrac{4}{3}ye^{2y} - \tfrac{16}{9}e^{2y} + 2e^{4y}\right)\\
&\quad + g_{\mathrm{im}}(\mu; \boldsymbol{V}, \boldsymbol{W})\,\big(0, -\tfrac{2}{9}, 0, -\tfrac{8}{3}ye^{2y} - \tfrac{20}{9}e^{2y} + 2e^{4y},\\
&\quad -\tfrac{4}{3}ye^{2y} - \tfrac{4}{9}e^{2y} + \tfrac{1}{2}e^{4y}, 0\big).
\end{aligned}$$

Since we intend to study solutions decaying to zero at infinity, let us apply the Fourier transform. Our study then excludes periodic waves. The solution of the Fourier-transformed system (4.1) can be expressed with the help of the resolvent function

$$\widehat{\boldsymbol{W}}(k) = \widehat{\boldsymbol{W}}_1(k) + \widehat{\boldsymbol{W}}_2(k) = \boldsymbol{R}_1(ik)\widehat{\boldsymbol{f}}_1(k) + \boldsymbol{R}_1(ik)\widehat{\boldsymbol{f}}_2(k), \quad k \in \mathbb{R}\backslash\{0\}. \tag{4.2}$$

Since the nonlinearity depends only on the traces of the components of $\boldsymbol{W}$, it suffices to derive expressions for these traces. The following notation is introduced (cf. (3.2)):

$$\xi_0 = \theta_0|_{y=0}, \quad \xi_1 = \theta_1|_{y=0}, \quad \eta_0 = \tau_0|_{y=0}, \quad \eta_1 = \tau_1|_{y=0},$$
$$\boldsymbol{W}^0 = (\xi_0, \xi_1, \eta_0, \eta_1).$$

System (4.1) can then be formulated as a system for these variables. The full vector $\widehat{\boldsymbol{W}}$ is completely determined by the traces; therefore, it suffices to calculate $(\xi_0, \xi_1, \eta_0, \eta_1)$. Explicit expressions for $\widehat{\boldsymbol{W}}_1(k)$ and $\widehat{\boldsymbol{W}}_2(k)$ are obtained from (4.2).

$$\widehat{\boldsymbol{W}}_1^0(k): \quad \begin{aligned} \hat{\xi}_0(k) &= (2+|k|)^{-2}\hat{f}(k), \\ \hat{\xi}_1(k) &= ik(2+|k|)^{-2}\hat{f}(k), \\ \hat{\eta}_0(k) &= -i \text{ sign } (k)\hat{\xi}_0(k), \\ \hat{\eta}_1(k) &= |k|\hat{\xi}_0(k), \end{aligned} \tag{4.3}$$

$$\widehat{\boldsymbol{W}}_2^0(k): \quad \begin{aligned} \hat{\xi}_0(k) &= \frac{1}{18}\,\frac{(8+|k|)\big(-ik\hat{g}_{\mathrm{re}}(k) + 4\hat{g}_{\mathrm{im}}(k)\big)}{(4+|k|)(2+|k|)^2}, \\ \hat{\xi}_1(k) &= \frac{2}{9}\,\frac{-(4+5|k|)\hat{g}_{\mathrm{re}}(k) + ik(8+|k|)\hat{g}_{\mathrm{im}}(k)}{(4+|k|)(2+|k|)^2}, \\ \hat{\eta}_0(k) &= \frac{1}{18}\,\frac{4(1-|k|)\hat{g}_{\mathrm{re}}(k)}{(4+|k|)(2+|k|)^2} \\ &\quad + \frac{1}{18}\,\frac{-i\big(k|k| + 8k + 36 \text{ sign } (k)\big)\hat{g}_{\mathrm{im}}(k)}{(4+|k|)(2+|k|)^2}, \\ \hat{\eta}_1(k) &= \frac{2}{9}\,\frac{ik(1-|k|)\hat{g}_{\mathrm{re}}(k) - 4(1-|k|)\hat{g}_{\mathrm{im}}(k)}{(4+|k|)(2+|k|)^2}. \end{aligned} \tag{4.4}$$

We observe from (4.3) and (4.4) that the multipliers on the right-hand sides are bounded and continuous for all $k \in \mathbb{R}$ except at $k = 0$ and are of order $\mathcal{O}(1/|k|)$ for $|\boldsymbol{k}| \to \infty$.

Hence, all multipliers are in $L^2(\mathbb{R})$, and they represent Fourier transforms of $L^2(\mathbb{R})$ functions with which one has to make a convolution product with f or g. In fact, all these functions are continuous on $\mathbb{R}$. The inverse transforms of $\hat{\eta}_0$ in (4.3) and (4.4) decay like $1/x$ at infinity. The other inverse transforms decay like $1/x^2$ at infinity.

For instance, behavior for $|x| \to \infty$ follows from integration by parts

$$\int_{\mathbb{R}} \frac{e^{-ikx}}{(2+|k|)^2} dk = \frac{4}{x} \int_0^\infty \frac{\sin kx}{(2+k)^3} dk = \frac{1}{2x^2} + \mathcal{O}\left(\frac{1}{x^4}\right),$$

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{ike^{-ikx}}{(2+|k|)^2} dk = 4\int_0^\infty \left(\frac{\sin kx}{x^2} - \frac{k\cos kx}{x}\right) \frac{dk}{(2+k)^3} = \frac{1}{x^3} + \mathcal{O}\left(\frac{1}{x^5}\right),$$

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{|k|e^{-ikx}}{(2+|k|)^2} dk = 4\int_0^\infty \left(\frac{k\sin kx}{x} - \frac{2\sin^2\left(\frac{kx}{2}\right)}{x^2}\right) \frac{dk}{(2+k)^3}$$

$$= -\frac{1}{2x^2} + \mathcal{O}\left(\frac{1}{x^4}\right),$$

$$\int_{\mathbb{R}} \frac{-i\ \mathrm{sign}(k) e^{-ikx}}{(2+|k|)^2} dk = -\frac{1}{2x} + \mathcal{O}\left(\frac{1}{x^3}\right).$$

Notice that the second and third functions are defined as L^2 limits in the sense of the Fourier-Plancherel transform. The result is that these L^2 limits may be represented by continuous functions.

Continuity in x follows directly from the dominated convergence theorem, except near $x = 0$ in the second and third integral, where one has to split each integral into two integrals over $(0, 1/x)$ and $(1/x, \infty)$ before applying the dominated convergence theorem. All other multipliers occurring in (4.4) may be treated in the same way.

Let us now introduce spaces of continuous functions on $\mathbb{R}$ with an algebraic decay at infinity depending on a positive number ε, and an integer p

$$C_{\varepsilon,p} = \left\{ u \in C^0 : \mathbb{R} \to \mathbb{C} \,\middle|\, \sup_{x\in\mathbb{R}} (1 + \varepsilon|x|^p)|u(x)| < \infty \right\}.$$

We define the corresponding norms

$$|u|_{\varepsilon,p} = \sup_{x\in\mathbb{R}} (1 + \varepsilon|x|^p)|u(x)|,$$

which give these spaces structures of Banach spaces. Now we study the linear mapping $\mathcal{K}$ defined by

$$(f, g, \bar{g}) \mapsto W^0 = \mathcal{K}(f, g, \bar{g}) \tag{4.5}$$

given by the inverse Fourier transforms of (4.3) and (4.4), where we take (f, g) in $(C_{\varepsilon,2})^2$.

Lemma 1. *For any* (f, g) *in* $(C_{\varepsilon,2})^2$, *there is a number* $c > 0$ *such that*

$$|\xi_0|_{\varepsilon,2} + |\xi_1|_{\varepsilon,2} + |\eta_0|_{\sqrt{\varepsilon},1} + |\eta_1|_{\varepsilon,2} \leqq c_\varepsilon(|f|_{\varepsilon,2} + |g|_{\varepsilon,2}), \tag{4.6}$$

where $c_\varepsilon \leqq c(|\ln\varepsilon| + 1)$, *i.e.,* $\mathcal{K}$ *is continuous from* $(c_{\varepsilon,2})^3$ *into* $(C_{\varepsilon,2})^2 \times C_{\sqrt{\varepsilon},1} \times C_{\varepsilon,2}$. *Moreover, if* η_0 *is omitted from the left-hand side of* (4.6), *then the resulting inequality holds with* $c_\varepsilon \leqq c$.

Proof. Let us first study the convolution product of a $C_{1,2}$ function with a $C_{\varepsilon,2}$ function, leading to a $C_{\varepsilon,2}$ function. This proves the results for ξ_0, ξ_1 and η_1 since the convolution kernels belongs to $C_{1,2}$. We have

$$\int_{\mathbb{R}} \frac{(1+\varepsilon x^2)dt}{(1+t^2)[1+\varepsilon(x-t)^2]} = \pi \frac{(1+\varepsilon x^2)[(1-\varepsilon+\varepsilon x^2)+\sqrt{\varepsilon}\,(-1+\varepsilon+\varepsilon x^2)]}{\varepsilon^2 x^4 + 2\varepsilon(1+\varepsilon)x^2 + (1-\varepsilon)^2}$$
$$\leqq \pi(1+\sqrt{\varepsilon}) \leqq 2\pi \quad \text{for } \varepsilon \leqq 1.$$

This shows that the estimate in lemma has a constant c_ε which is independent of $\varepsilon(\leqq 1)$ as far as ξ_0, ξ_1 and η_1 are concerned. Let us now study the convolution product of a $C_{1,1}$ function with a $C_{\varepsilon,2}$ function, for estimating $|\eta_0|_{\sqrt{\varepsilon},1}$.

$$\int_{\mathbb{R}} \frac{(1+\sqrt{\varepsilon}|x|dt}{(1+|t|)[1+\varepsilon(x-t)^2]} = \frac{2(1+\sqrt{\varepsilon}|x|)(1+\varepsilon+\varepsilon x^2)}{(1+\varepsilon+\varepsilon x^2)^2 - 4\varepsilon^2 x^2}$$
$$\times \left\{ -\frac{1}{2}\ln\,\varepsilon + \frac{1}{2}\ln\,(1+\varepsilon x^2) + \sqrt{\varepsilon}\left(\frac{\pi}{2} + |x|\mathrm{Arctg}(\sqrt{\varepsilon}|x|)\right)\right\}$$
$$- \frac{4\varepsilon\sqrt{\varepsilon}|x|(1+\sqrt{\varepsilon}|x|)}{(1+\varepsilon+\varepsilon x^2)^2 - 4\varepsilon^2 x^2}\left(\frac{\pi}{2} + |x| + \mathrm{Arctg}\,\sqrt{\varepsilon}|x|\right)$$

$$< \left(\frac{\sqrt{2}+1}{1-3\varepsilon}\right)\left\{-\frac{1}{2}\ln\,\varepsilon + \frac{\pi}{2}\sqrt{\varepsilon}\right\}\frac{2}{1-3\varepsilon}c_1 \quad \text{for } \varepsilon < \frac{1}{3},$$

where

$$c_1 = \sup_{u>0} \frac{1+u}{1+u^2}\left(\frac{1}{2}\ln\,(1+u^2) + u\,\mathrm{Arctg}\,u\right) < \frac{4}{3}\left(1+\frac{\pi}{2}\right).$$

Notice that the divergence in $|\ln\,\varepsilon|$ of the estimate is not unexpected because the integral diverges for ε tending to 0 (monotonic convergence theorem).

We are now able to solve the nonlinear equation

$$\frac{d}{dx}\boldsymbol{W}(x) = L_1\boldsymbol{W}(x) + \boldsymbol{H}(\boldsymbol{V}(x)) + \boldsymbol{R}^1(\mu; \boldsymbol{V}(x), \boldsymbol{W}(x)) \tag{4.7}$$

locally in a small neighborhood of zero. The central component $\boldsymbol{V} = (A, B, \overline{A}, \overline{B})$ is treated as a parameter in $(C_{\varepsilon,2})^4$, and we observe that the dependence of $\boldsymbol{R}^1$ on $\boldsymbol{W}$ is only through the trace $\boldsymbol{W}^0$. There we use the linear operator $\mathscr{K}$ defined in (4.5), to solve (4.7) in the form

$$\boldsymbol{W}^0 = \mathscr{K}\mathscr{F}(\mu, \boldsymbol{V}; \boldsymbol{W}^0) \tag{4.8}$$

for $\boldsymbol{W}^0$. A useful observation is now that the nonlinear term $\boldsymbol{F}(\mu; \boldsymbol{u}^0)$ defined by (2.3) is analytic in $(\mu; \boldsymbol{u}^0)$ for

$$\boldsymbol{u}^0 = (\alpha_0^0, \alpha_1^0, \beta_0^0, \beta_1^0) \in C_{\varepsilon,2} \times C_{\varepsilon,2} \times C_{\sqrt{\varepsilon},1} \times C_{\varepsilon,2} \to \boldsymbol{F}(\mu; \boldsymbol{u}^0) \in C_{\varepsilon,2}, \quad |\mu| < \delta.$$

This results in particular, from the fact that the product of two continuous functions, one decaying as $1/(1+\varepsilon x^2)$ and the other as $1/(1+\sqrt{\varepsilon}|x|)$, decays at least as fast as $1/(1+\varepsilon x^2)$. Hence $\boldsymbol{f}$ and $\boldsymbol{g}$ in (4.1) depend analytically on $(\mu, A, B, \bar{A}, \bar{B}, \boldsymbol{W}^0)$, which gives the dependence of $\mathscr{F}(\mu, \boldsymbol{V}; \boldsymbol{W}^0)$ in (4.8). A direct estimation leads to

$$\begin{aligned} &|\mathscr{F}(\mu, \boldsymbol{V}; \boldsymbol{W}^0)|_{\varepsilon,2} \\ &\leqq c\Big\{|A|_{\varepsilon,2}|B|_{\varepsilon,2} + |B|^2_{\varepsilon,2} + |A|^3_{\varepsilon,2} + (|\mu| + \|\boldsymbol{W}^0\|)\Big(|A|_{\varepsilon,2} + |B|_{\varepsilon,2} + \|\boldsymbol{W}^0\|\Big)\Big\} \end{aligned} \tag{4.9}$$

where

$$\|\boldsymbol{W}^0\| \equiv |\xi_0|_{\varepsilon,2} + |\xi_1|_{\varepsilon,2} + |\eta_0|_{\sqrt{\varepsilon},1} + |\eta_1|_{\varepsilon,2}.$$

For (A, B, μ) in a sufficiently small ball in $(C_{\varepsilon,2})^2 \times (-\delta, \delta)$ one can solve (4.8) for $\boldsymbol{W}^0$, by the implicit function theorem and find an estimate

$$\|\boldsymbol{W}^0\| \leqq c'_\varepsilon\Big\{|A|_{\varepsilon,2}|B|_{\varepsilon,2} + |B|^2_{\varepsilon,2} + |A|^3_{\varepsilon,2} + |\mu|\ (|A|_{\varepsilon,2} + |B|_{\varepsilon,2})\Big\}, \tag{4.10}$$

where $c'_\varepsilon = c'(1 + |\ln\ \varepsilon|)$.

Let us now introduce the scaling

$$A(x) = \sqrt{|\mu|}\tilde{A}\Big(\sqrt{|\mu|}x\Big)e^{2ix}, \quad B(x) = |\mu|\,\tilde{B}\Big(\sqrt{|\mu|}x\Big)e^{2ix}, \quad \tilde{x} = \sqrt{|\mu|}x, \tag{4.11}$$

and take $\varepsilon = |\mu|$ in (4.10) Then (4.10) becomes

$$\|\boldsymbol{W}^0\| \leqq c''(1 + |\ln\ |\mu||)|\mu|^{3/2}$$

for any $\tilde{A}(\tilde{x}), \tilde{B}(\tilde{x})$ in a fixed ball of $C_{1,2}$ and $\boldsymbol{W}^0(\mu, \tilde{\boldsymbol{V}})$ is now replaced in the $\boldsymbol{V}$ part of (3.4). For $\mu > 0$ this leads to the following reduced equation:

$$\begin{aligned} \tilde{A}' &= \tilde{B} + \tilde{R}_0\Big(\mu; \tilde{A}, \tilde{B}, \bar{\tilde{A}}, \bar{\tilde{B}}\Big), \\ \tilde{B}' &= Q_{1,00}\tilde{A} + Q_{0,10}\tilde{A}|\tilde{A}|^2 + \tilde{R}_1\Big(\mu; \tilde{A}, \tilde{B}, \bar{\tilde{A}}, \bar{\tilde{B}}\Big). \end{aligned} \tag{4.12}$$

Here the prime denotes differentiation with respect to $\tilde{x}$. The remainder terms $\tilde{R}_j (j = 0, 1)$ are nonlocal functions of $(\tilde{A}, \tilde{B})$ with

$$\Big|\tilde{R}_0\Big(\mu; \tilde{A}, \tilde{B}, \bar{\tilde{A}}, \bar{\tilde{B}}\Big)\Big|_{1,2} = \mathcal{O}(\sqrt{\mu}), \quad \Big|\tilde{R}_1\Big(\mu; \tilde{A}, \tilde{B}, \bar{\tilde{A}}, \bar{\tilde{B}}\Big)\Big|_{1,2} = \mathcal{O}(\sqrt{\mu}|\ln\ \mu|), \quad \mu \to 0.$$

The first equation of (4.12) can be solved for $\tilde{B}$. Insertion into the second equation yields a complex second-order equation for $\tilde{A}$:

$$\tilde{A}'' - Q_{1,00}\tilde{A} - Q_{0,10}\tilde{A}|\tilde{A}|^2 = \tilde{R}\left(\mu;\tilde{A},\tilde{A}',\bar{\tilde{A}},\bar{\tilde{A}}'\right). \tag{4.13}$$

The remainder term $\tilde{R}$ is of order $\mathcal{O}(\sqrt{\mu}|\ln \mu|)$, $\mu \to 0$. Equation (4.13) is studied in the next section.

5. Solitary waves with damped oscillations

The normal-form system (3.1) is an integrable system which admits many different types of solutions, e.g., periodic solutions, quasiperiodic solutions, homoclinic solutions, etc.; see Iooss & Pérouème [10]. This paper also treats the subtle problem of persistence of normal-form solutions under (reversible) perturbations.

It was shown in [8] and [10] that the normal-form system (3.1) has — under certain sign conditions for the coefficients — a pair of solitary waves (homoclinic solutions) with damped oscillations which are reversible and which persist under reversible perturbations of the vector field. It follows from (3.10) that these sign conditions are fulfilled for the present problem ($Q_{1,00} = 4 > 0$ $Q_{0,10} = -22 < 0$). For $\mu > 0$ a pair of reversible solitary waves exists for the normal-form system (3.1) and has the explicit representation

$$A(x) = r_0(x)e^{i(2x+\theta_0(x))}, \quad B(x) = r_1(x)e^{i(2x+\theta_1(x))}, \tag{5.1}$$

with the asymptotic expressions

$$\begin{aligned} r_0(x) &= \pm\sqrt{\frac{2\mu Q_{1,00}}{(-Q_{0,10})}}\,\frac{1}{\cosh\sqrt{\mu Q_{1,00}}\,x}, \\ r_1(x) &= r_0'(x), \\ \theta_0(x) &= P_{1,00}\mu x + \frac{2P_{0,10}}{(-Q_{0,10})}\sqrt{\mu Q_{1,00}}\tanh\left(\sqrt{\mu Q_{1,00}}\,x\right), \\ \theta_1(x) &= \theta_0(x). \end{aligned} \tag{5.2}$$

It is shown in the sequel that these two reversible solitary-wave solutions persist under nonlocal reversible perturbations of the normal-form vector field. We cannot directly apply the results of Iooss & Pérouème [10] because the perturbations are nonlocal in the present case. We shall exploit an argument given by Kirchgässner in [11] which also works for nonlocal reversible perturbations.

The truncated equation (4.13) with right-hand side zero, i.e., the equation

$$\tilde{A}'' - Q_{1,00}\tilde{A} - Q_{0,10}\tilde{A}|\tilde{A}|^2 = 0,$$

has a real homoclinic solution $\tilde{A} = A^* = r_0^*$ which is even. We are looking for a reversible homoclinic solution of the full equations (4.13) which is close to A^*. The replacement $\tilde{A} = A^* + A_p$ leads to nonautonomous equation for the perturbation term A_p:

$$A''_{\mathrm{p}} - Q_{1,00}A_{\mathrm{p}} - Q_{0,10}\left[(A^*(\tilde{x}))^2\bar{A}_{\mathrm{p}} + 2|A^*(\tilde{x})|^2 A_{\mathrm{p}}\right]$$
$$= N\left(\tilde{x}; A_{\mathrm{p}}, A'_{\mathrm{p}}, \bar{A}_{\mathrm{p}}, \bar{A}'_{\mathrm{p}}\right) + R\left(\mu; \tilde{x}; A_{\mathrm{p}}, A'_{\mathrm{p}}, \bar{A}_{\mathrm{p}}, \bar{A}'_{\mathrm{p}}\right) \quad (5.3)$$

We wish to use Banach's fixed-point theorem to prove the existence of a small solution A_{p} of (5.3) which decays at $\tilde{x} = \pm\infty$. Therefore, the linear operator $\mathscr{L}(\tilde{x})$ on the left-hand side of (5.3) has to be inverted in some suitable function space. Since this operator represents the variational equation around the homoclinic solution of the truncated system, it has a zero eigenvalue with eigenfunction $dA^*/d\tilde{x}$. Decomposing $A_{\mathrm{p}} = A_{\mathrm{p,re}} + iA_{\mathrm{p,im}}$ into real and imaginary parts we obtain from (5.3) two linearly decoupled equations:

$$\begin{aligned} &A''_{\mathrm{p,re}} - Q_{1,00}A_{\mathrm{p,re}} - 3Q_{0,10}[A^*(\tilde{x})]^2 A_{\mathrm{p,re}} \\ &\quad = N_{\mathrm{re}}\left(\tilde{x}; A_{\mathrm{p}}, A'_{\mathrm{p}}, \bar{A}_{\mathrm{p}}, \bar{A}'_{\mathrm{p}}\right) + R_{\mathrm{re}}\left(\mu; \tilde{x}; A_{\mathrm{p}}, A'_{\mathrm{p}}, \bar{A}_{\mathrm{p}}, \bar{A}'_{\mathrm{p}}\right), \\ &A''_{\mathrm{p,im}} - Q_{1,00}A_{\mathrm{p,im}} - Q_{0,10}[A^*(\tilde{x})]^2 A_{\mathrm{p,im}} \\ &\quad = N_{\mathrm{im}}\left(\tilde{x}; A_{\mathrm{p}}, A'_{\mathrm{p}}, \bar{A}_{\mathrm{p}}, \bar{A}'_{\mathrm{p}}\right) + R_{\mathrm{im}}\left(\mu; \tilde{x}; A_{\mathrm{p}}, A'_{\mathrm{p}}, \bar{A}_{\mathrm{p}}, \bar{A}'_{\mathrm{p}}\right). \end{aligned} \quad (5.4)$$

Since we are looking for reversible solutions, i.e, solutions such that $A(-x) = \overline{A(x)}$, the real part $A_{\mathrm{p,re}}$ of A_{p} must be even, while the imaginary part $A_{\mathrm{p,im}}$ of A_{p} must be odd. According to the argument given in [11], we can invert the linear operator $\mathscr{L}_{\mathrm{re}}$ in $C_{1,2}$ on the left-hand side of the first equation of (5.4) if we restrict its domain of definition to even functions. Note that this restriction eliminates the zero eigenfunction $dA^*/d\tilde{x}$ of $\mathscr{L}_{\mathrm{re}}$, which is an odd function.

The linear operator $\mathscr{L}_{\mathrm{im}}$ associated with the second equation also has a simple zero eigenvalue with eigenfunction A^* which is even. Therefore, $\mathscr{L}_{\mathrm{im}}$ is also invertible on odd functions in $C_{1,2}$. Everything works in $C_{1,2}$, so $A_{\mathrm{p}} \in C^2_{1,2}$, hence $\tilde{A}$ and $\tilde{B}$ are in $C_{1,2}$. Tracing back to the form of the free surface $\eta = Z(\xi)$, we have after a careful examination that

$$Z_\xi = \tan\alpha(x) = \mathcal{O}\left(\frac{\mu\sqrt{\mu}}{1 + \mu x^2}\right) \text{ at infinity}$$

and since $\partial\xi/\partial x \approx 1$, we finally have

$$Z(\xi) = \mathcal{O}\left(\frac{\sqrt{\mu}}{|\xi|}\right) \quad \text{for } |\xi| \to \infty\,.$$

Summarizing we have proved

Theorem 2. *There exists a pair of reversible solitary wave solutions for the system* (1.1), (1.2), (1.3) *such that* $U - 1 = \mathcal{O}\left(\mu^{3/2}|\ln\mu|/(1 + \mu^{1/2}|\xi|)\right)$, $V = \mathcal{O}(\mu\sqrt{\mu}/(1 + \mu\xi^2))$ *and* $Z = \mathcal{O}(\sqrt{\mu}/|\xi|)$ as $|\xi| \to \infty$.

Remark. It has been pointed out to us by J. C. Saut that these solitary waves do not decay exponentially. This results from the nonsmoothness of the Fourier symbol at the origin, while the Fourier transform of an exponentially decaying function is analytic in a strip containing the real axis. Because of (1.8), α and β cannot both decay exponentially at infinity, hence neither of them decays exponentially due to (1.5).

Let us mention that in the papers of Amick [1] and Sun [16] (the problem is different, but a similiar method should work) it is found that the decay is like $1/x^2$. The problem of the true rate here is different, since the principal part of the solitary wave coming from the normal form has an exponential decay. Here the non–exponential decay comes from high-order terms.

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Institut Universitaire
de France
INLN, UMR CNRS-UNSA
1361 Route des Lucioles
Sophia Antipolis
06560 Valbonne, France

and

Mathematisches Institut A
Universität Stuttgart
Pfaffenwaldring 57
D-70569 Stuttgart, Germany

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