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Global Characterization of the Normal Form for a Vector Field Near a Closed Orbit

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1. Introduction

The theory of normal forms originally developed by Poincaré in his thesis is a very powerfull tool in understanding the local dynamical behavior of systems near a singularity. This is extensively shown for instance in the books of Arnold [1] and Guckenheimer and Holmes [10]. Very recently a simple *global characterization* of such normal forms was given by Elphick *et al.* [8], for vector fields near a singularity, in terms of an equivariance under a one parameter group generated by the adjoint of the linearized operator restricted to the critical subspace. In what follows we derive a natural generalization of this global characterization for vector fields near a closed orbit.

It is known that normal form theory runs well near a closed orbit of an autonomous vector field. A first way to make it is to consider the Poincaré map. In most of the cases this is sufficient (see [1, Chap. 5] and see [14] for a global characterization), but by this method we miss one dimension: the phase coordinate along the closed orbit. A second way consists of keeping the continuous time, and then, curiously, there seems to exist no systematic study of such normal forms, even though in principle there is no theoretical difficulty (see [1, Chap. 5, Sect. 26.G]).

A first motivation in keeping this phase is to directly obtain all information on the time law of the movement on the trajectories traced on the Poincaré section.

A second motivation is to prepare Future, since such a phase may cause irregular dynamics once coupled with a spatial phase in an envelope equation dealing with so-called "large scale effects." In fact there is an extensive use by physicists of normal forms for so-called "extended systems," i.e., systems with a translational invariance in space. These systems lead to envelope equations (see Newell–Whitehead and Segel [17]

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and Coullet *et al.* [6]) which are partial differential equations instead of O.D.E. on the center manifold.

The presentation we make here for vector fields in finite dimensions near a closed orbit is in fact suitable for two types of generalizations:

(i) for evolution problems such as the ones which occur in fluid mechanics satisfying Navier–Stokes equations in a bounded or periodic domain;

(ii) for extended systems (such as in [6]) even though at present no mathematical justification of the envelope equations exists yet.

In this paper we start with some results of Floquet-type (part of them are in [11]); they are self contained here and are useful for deriving the three theorems which are the main results of this work. The distinction between different cases rests upon the size of the Jordan block of the monodromy operator belonging to the tangent to the closed orbit, associated with the Floquet multiplier 1.

Our global charaterization separates the time law for the phase along the closed curve and the transverse dynamics in terms of this phase. This is expressed by a natural-extension of the characterization obtained in the autonomous case, again in a constructive way. We give several examples to show how simply normal forms can be derived. Finally we show how to handle these results in the presence of parameters and of additional symmetries in the system.

2. Preliminaries—Floquet Type Results

We consider in $\mathbb{R}^n$ the following differential equation,

$$\frac{dZ}{dt} = \mathscr{F}(Z), \tag{1}$$

such that u_0 is a T-periodic solution,

$$\dot{u}_0(t) = \mathscr{F}[u_0(t)] \qquad \text{(the dot means time differentiation)}, \tag{2}$$

and where $\mathscr{F}$ is C^k, k large enough, in a neighborhood of the closed orbit $\Gamma = \{u_0(t);\ t \in \mathbb{R}\}$. We now suppose that Γ is neutrally stable, i.e., we assume that this is not a hyperbolic situation. More precisely, let us define the linear operator $\mathscr{L}(t)$ in $\mathbb{R}^n$ by

$$\mathscr{L}(t) = D_Z\mathscr{F}[u_0(t)], \qquad \mathscr{L} \in C^{k-1}[\mathbb{R}/T\mathbb{Z}, \mathscr{L}(\mathbb{R}^n)], \tag{3}$$

and let us define the fundamental matrix $S(t, s)$,

$$\frac{\partial}{\partial t} S(t, s) = \mathscr{L}(t)\, S(t, s)$$
$$S(s, s) = \mathrm{Id}, \qquad S \in C^k[\mathbb{R}^2; \mathscr{L}(\mathbb{R}^n)], \tag{4}$$

where we denote by $S(t) \equiv S(t, 0)$. In what follows we make precise assumptions on the spectrum of the monodromy operator $S(T)$: it is the union of a part σ_0 on the unit circle, and a part σ_- strictly inside the unit disk. To the part σ_0 is associated a $m+1$ dimensional subspace, invariant under $S(T)$, which contains the tangent at $u_0(0)$ to the orbit Γ.

More precisely we have the following:

LEMMA 1. *$\dot{u}_0(0)$ is an eigenvector of $S(T)$ belonging to the eigenvalue* 1.

Proof. If we differentiate with respect to t the identity (2), we obtain

$$\left[-\frac{d}{dt} + \mathscr{L}(t)\right] \dot{u}_0(t) = 0 \tag{5}$$

which shows that

$$\dot{u}_0(t) = S(t)\, \dot{u}_0(0),$$

hence the result follows immediately thanks to the T-periodicity of $\dot{u}_0(t)$.

The part σ_0 of the spectrum of $S(T)$ plays a key role in all that follows. Moreover in important evolution problems, like in hydrodynamics, where $\mathbb{R}^n$ has to be replaced by some infinite dimensional Hilbert space, we can define $S(T)$ and, as here, σ_0 is a finite union of discrete eigenvalues of finite multiplicities, while σ_- may be infinite with an accumulation point at 0 (see [16, Chap. V.4]). This is a first motivation in deriving a sort of Floquet theory restricted to the part σ_0 of the spectrum of $S(T)$. A second motivation is that we explicitly give the way to obtain operators occurring in Floquet theory.

Let us consider a Jordan block of $S(T)$ of size v:

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & & & \vdots \\ & & \ddots & \ddots & 0 \\ \vdots & 0 & & & 1 \\ 0 & & & 0 & \lambda \end{pmatrix}. \tag{6}$$

This means that there are ν independent vectors of $\mathbb{C}^n$: $\{\zeta_0, ..., \zeta_{\nu-1}\}$ spanning a subspace E_λ invariant under $S(T)$, and such that

$$[S(T)-\lambda]\zeta_j = \begin{cases} \zeta_{j-1}, & j=1, ..., \nu-1, \\ 0, & j=0. \end{cases} \tag{7}$$

Now we have the following

LEMMA 2. *To each Jordan block of the monodromy operator $S(T)$ corresponds ν independent T-periodic C^k-vector-functions $\zeta_j(t)$ such that*

$$\left[-\frac{d}{dt}+\mathscr{L}(t)-\sigma\right]\zeta_j(t) = \begin{cases} \zeta_{j-1}(t), & j=1, ..., \nu-1, \\ 0, & j=0, \end{cases} \tag{8}$$

where σ satisfies $e^{\sigma T}=\lambda$.

Remark. In infinite dimensional spaces we should add the condition $\lambda \neq 0$.

Proof. First, let us consider the $\zeta_0(t)$ solution of the system

$$\left[-\frac{d}{dt}+\mathscr{L}(t)-\sigma\right]\zeta_0(t)=0,$$

$$\zeta_0(0)=\zeta_0.$$

Then it is clear that $(d/dt)[e^{\sigma t}\zeta_0(t)] = \mathscr{L}(t)[e^{\sigma t}\zeta_0(t)]$, hence $e^{\sigma t}\zeta_0(t) = S(t)\zeta_0$, and finally the T-periodicity of $\zeta_0(t)$ comes from the identities:

$$\lambda\zeta_0(T) = S(T)\zeta_0 = \lambda\zeta_0.$$

Now consider the $\zeta_j(t)$ solution of the system

$$\begin{gathered} \left[-\frac{d}{dt}+\mathscr{L}(t)-\sigma\right]\zeta_j(t)=\zeta_{j-1}(t), \qquad j\geqslant 1, \\ \zeta_j(0)=\alpha_{jj}\zeta_j+\alpha_{jj-1}\zeta_{j-1}+\cdots+\alpha_{j1}\zeta_1, \end{gathered} \tag{9}$$

where $\zeta_{j-1}(t), ..., \zeta_0(t)$ are supposed to be known T-periodic vector functions, and where we look for the coefficients $\alpha_{jj}, \alpha_{jj-1}, ..., \alpha_{j1}$ such that $\zeta_j(t)$ is T-periodic. We obtain immediately by the Duhamel formula

$$e^{\sigma t}\zeta_j(t) = S(t)\,\zeta_j(0) - \int_0^t S(t,s)\,e^{\sigma s}\zeta_{j-1}(s)\,ds, \tag{10}$$

and, since we want a T-periodic solution, we have

$$[S(T)-\lambda]\,\zeta_j(0)=\int_0^T S(T,s)\,e^{\sigma s}\zeta_{j-1}(s)\,ds.$$

Finally, using this identity with (10) and (9) we obtain

$$\alpha_{jj}\zeta_{j-1}+\cdots+\alpha_{j1}\zeta_0=S(T)\left[T\zeta_{j-1}^{(0)}-\frac{T^2}{2!}\zeta_{j-2}(0)+\cdots+(-1)^{j+1}\frac{T^j}{j!}\zeta_0\right],$$

and this gives all the α_{jk} by recurrence. For instance we have

$$\alpha_{jj}=\lambda^j T^j,\qquad \alpha_{jj-1}=\frac{j-1}{2}\lambda^{j-1}T^j,\ \ldots.$$

Let us now complete the previous constructive lemma, by the following:

LEMMA 3. *With the same assumption as in Lemma 2, there exists a T-periodic family of C^k-projection $P(t)$ on the subspaces $E(t)$ spanned by $\{\zeta_0(t),\ldots,\zeta_{\nu-1}(t)\}$. More precisely we have*

$$\left[-\frac{d}{dt}+\mathscr{L}(t)\right]P(t)=P(t)\,\mathscr{L}(t),\qquad P(0)=\bar{P},\tag{11}$$

where $\bar{P}$ is the projection, on the subspace $\{\zeta_0,\ldots,\zeta_{\nu-1}\}$, which commutes with $S(T)$.

Proof. Let us consider the Cauchy problem (11) for P. Since it is linear, this defines $P(t)$ for $t\in\mathbb{R}$.

We first observe easily that we have the identity

$$P(t)\,S(t,s)=S(t,s)\,P(s),\tag{12}$$

since both sides are solutions of the same differential equation, with the same initial data at $t=s$.

Now $P^2(t)$ satisfies

$$\frac{d}{dt}P^2(t)=\mathscr{L}(t)\,P^2(t)-P^2(t)\,\mathscr{L}(t),\qquad P^2(0)=\bar{P}^2=\bar{P},$$

hence, by the uniqueness of the solution, $P(t)$ is a projection

$$P^2(t)=P(t).$$

We now consider one of the vectors $\zeta_j(t)$ defined in Lemma 2. We have

$$\left[-\frac{d}{dt}+\mathscr{L}(t)-\sigma\right]\left[P(t)\,\zeta_j(t)\right]=P(t)\left[-\frac{d}{dt}+\mathscr{L}(t)-\sigma\right]\zeta_j(t)$$

$$=\begin{cases} P(t)\,\zeta_{j-1}(t), & j=1,...,\nu-1,\\ 0, & j=0.\end{cases}$$

Since $P(0)\,\zeta_j(0)=\bar{P}\zeta_j(0)=\zeta_j(0)$, we observe that the uniqueness leads to

$$P(t)\,\zeta_j(t)=\zeta_j(t),$$

and $P(t)$ is the identity on $E(t)$. Since $P(0)=\bar{P}$ is a projection on a ν-dimensional subspace, and since $\zeta_j(t)$ are T-periodic and linearly independent, this shows that $E(t)$ is ν-dimensional and is precisely the image of $P(t)$.

Finally, by (12) and the commutativity of $P(0)$ and $S(T)$ we obtain

$$P(T)\,S(T)=S(T)\,P(0)=P(0)\,S(T),$$

and since $S(T)$ is invertible $P(0)=P(T)$; hence P is T-periodic.

Remark. This last property is weakened in infinite dimensional spaces, since $S(T)$ might not be invertible. In fact $P(t)$ is T-periodic, once it is reduced to an invariant subspace of $S(T)$, where the restricted operator is invertible. In all that follows, we do not use the T-periodicity of P.

We now need a special Floquet operator, defined in the following:

LEMMA 4. *With the same assumption as in Lemma 2, we define a T-periodic family of C^k-linear invertible operators from $\mathbb{C}^\nu$ to $E(t)$:*

$$\forall X=(x_0,...,x_{\nu-1})\in\mathbb{C}^\nu,\qquad Q(t)X=\sum_{j=0}^{\nu-1}x_j\zeta_j(t).$$

These operators $Q(t)$ satisfy

$$\frac{dQ}{dt}=\mathscr{L}(t)\,Q(t)-Q(t)L, \tag{13}$$

where

$$L=\begin{pmatrix}\sigma & 1 & 0 & \cdots & 0\\ 0 & \sigma & 1 & \ddots & \vdots\\ \vdots & & & \ddots & 0\\ \vdots & & \ddots & \ddots & 1\\ 0 & & \cdots & 0 & \sigma\end{pmatrix}$$

is constant.

Proof. This results immediately from (8).

In what follows, we consider *real* solutions, so we wish to find a matrix L representing a real operator. For instance, when we introduce a Jordan block such as the one in Lemma 2, which leads to the results described in Lemmas 2, 3, 4, we shall also introduce the complex conjugate Jordan block and choose the determination $\bar{\sigma}$ of $1/T \operatorname{Log} \bar{\lambda}$, and complex conjugate eigenvectors $\bar{\zeta}_j(t)$. This is always possible when λ is complex. If λ is real positive, we choose σ real and $\zeta_j(t)$ real. If $\lambda < 0$ this becomes impossible, since the Floquet exponents σ are $i\pi/T + 2l\pi i/T$, $l \in \mathbb{Z}$. It is also useless to introduce the complex conjugate since it corresponds to the same eigen-directions for $S(T)$. The standard way to deal with this difficulty is to double the period, since if λ is an eigenvalue of $S(T)$, then λ^2 is an eigenvalue of $S(2T)$. We have the following result:

LEMMA 5. *For a Jordan block (of size ν) of $S(T)$ belonging to a negative eigenvalue λ, there exists ν C^k-vector functions $\zeta_j(t)$ such that*

$$\zeta_j(t+T) = -\zeta_j(t),$$
$$\left[-\frac{d}{dt} + \mathscr{L}(t) - \tilde{\sigma}\right]\zeta_j(t) = \begin{cases} \zeta_{j-1}(t), & j = 1, ..., \nu - 1, \\ 0, & j = 0, \end{cases} \tag{14}$$

where $\tilde{\sigma} = (1/T) \operatorname{Log} |\lambda| \in \mathbb{R}$, and $\{\zeta_0(0), ..., \zeta_{\nu-1}(0)\}$ spans the ν-dimensional subspace invariant under $S(T)$, belonging to our Jordan block. The T-periodic projection $P(t)$ is now real and defined like in Lemma 3, *by Eqs.* (11). *The linear operator $Q(t)$, defined in Lemma* 4, *now satisfies $Q(t+T) = -Q(t)$ and the linear constant operator L defined in* (13) *acts from $\mathbb{R}^\nu$ and corresponds to the choice of $\tilde{\sigma}$.*

Proof. We consider the same construction for $\zeta_j(t)$ as in Lemma 2, but with $2T$-periodic functions. In fact, defining again ζ_j by (7), we obtain for $\zeta_0(t)$

$$e^{\tilde{\sigma}t}\zeta_0(t) = S(t)\zeta_0,$$

and since $e^{\tilde{\sigma}T} = |\lambda| = -\lambda$, this leads to $\zeta_0(T) = -\zeta_0$, hence

$$\zeta_0(t+T) = -\zeta_0(t).$$

Now assuming that the right hand side of (9) satisfies

$$\zeta_{j-1}(t+T) = -\zeta_{j-1}(t),$$

we first observe that Eq. (10), associated with the condition

$$\zeta_j(0) = -\zeta_j(T),$$

leads to

$$[S(T)-\lambda]\,\zeta_j(0)=\int_0^T S(T,s)\,e^{\bar{\sigma}s}\zeta_{j-1}(s)\,ds,$$

which defines $\zeta_j(0)$ like in Lemma 2. Now the property

$$\zeta_j(t+T)=-\zeta_j(t)$$

follows from the uniqueness of the solution of (9). The remaining part of Lemma 5 is now obvious.

3. Simple Case

In this section we make the following assumptions:

A.1. *The Jordan block belonging to the eigenvector* $\dot{u}_0(0)$ *for* $S(T)$ *is one dimensional.*

A.2. *Some eigenvalues of* $S(T)$ *have modulus* 1, *other eigenvalues have moduli* <1.

A.3. -1 *is not an eigenvalue of* $S(T)$.

In fact, 1 may be a multiple eigenvalue with other Jordan blocks besides the one considered in A.1. The last assumption A.3 is just to simplify the analysis, for the moment.

Let us denote by $E_{00}(t)$, $\tilde{E}_0(t)$, and $E_-(t)$ respectively the subspaces spanned by (i) $\dot{u}_0(t)$, (ii) the m vector-functions $\zeta_j(t)$ built in Lemma 2 for all Jordan blocks belonging to eigenvalues of modulus 1 for $S(T)$, except the one dimensional block considered at (i) and, (iii) the complementary subspace built in Lemma 3, belonging to eigenvalues of moduli less than 1 for $S(T)$. Let us denote by $P_{00}(t)$, $P_0(t)$, $P_-(t)$ respectively the projections on $E_{00}(t)$, $\tilde{E}_0(t)$, $E_-(t)$ defined in Lemma 3 and denote by $Q_0(t)$ the Floquet operator which corresponds to $\tilde{E}_0(t)$, defined as in Lemma 4. The corresponding constant linear operator in $\mathbb{R}^m$ is noted L_0. In fact L_0 will be written in general in a complex basis, to have it in Jordan form.

The main result of this paragraph is as follows:

Theorem 1. *If we assume that* A.1, A.2, A.3 *hold, then a center manifold for* (1) *in the neighborhood of* Γ *may be represented as*

$$Z=u_0(\tau)+Q_0(\tau)X+\Phi(\tau,X), \tag{15}$$

where $Q_0(\tau)\colon \mathbb{R}^m \mapsto \tilde{E}_0(\tau)$ *is the T-periodic Floquet operator defined above and* Φ *is T-periodic in* τ, *and at least quadratic in* X. τ *plays the role of a* *phase-coordinate along Γ. A* normal form *for the vector field on the center manifold may be found such that* (1) *becomes*

$$\frac{d\tau}{dt} = 1 + n(\tau, X) \qquad \text{in } \mathbb{R},$$
$$\frac{dX}{d\tau} = L_0 X + N(\tau, X) \qquad \text{in } \mathbb{R}^m, \tag{16}$$

where n and N are T-periodic in τ, are polynomials at least quadratic in X, and satisfy for any $\tau \in \mathbb{R}$, $X \in \mathbb{R}^m$:

$$\frac{d}{d\tau} n[\tau, e^{-L_0^*\tau} X] = 0, \qquad \frac{d}{d\tau} e^{L_0^*\tau} N[\tau, e^{-L_0^*\tau} X] = 0, \tag{17}$$

where L_0^* is the adjoint of L_0 in $\mathbb{R}^m$.

Remark 1. This characterization generalizes the known result for singular vector fields near an equilibrium [8], in a very natural way. If we suppress τ into N we recover this previous case.

Remark 2. We have to understand (16) *up to a certain order in X*, which depends on the regularity of $\mathscr{F}$ in (1). Nevertheless, even though $\mathscr{F}$ is C^∞, we have to stop at some arbitrarily fixed *finite* order to write (16).

EXAMPLE 1. Let us consider the case where $S(t)$ restricted to the invariant subspace $E_{00} \oplus \tilde{E}_0$ takes the following form in a suitable basis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \bar{\lambda} \end{pmatrix}, \qquad \lambda = e^{i\omega T}, \qquad \omega T/\pi \notin \mathbb{Z}. \tag{18}$$

We choose L_0 as

$$L_0 = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}, \qquad 0 < \omega < \pi/T$$

and E_0 is two dimensional. In the basis where L_0 is diagonal, there coordinates of X are written $(z, \bar{z})$ and $N = (m, \bar{m})$. Since n and N depend T-periodically on τ we may write

$$n(\tau, \cdot) = \sum_{p \in \mathbb{Z}} n_p \, e^{2i\pi p\tau/T},$$
$$N(\tau, \cdot) = \sum_{p \in \mathbb{Z}} N_p \, e^{2i\pi p\tau/T}.$$

Now $e^{-L_0^* \tau} X = (e^{i\omega\tau} z, e^{-i\omega\tau} \bar{z})$, and (17) leads to resonant terms in $z^r \bar{z}^s$ such that

$$\begin{aligned} \omega(r-s) + 2\pi l/T = 0 \qquad & \text{for} \quad n_l, \\ \omega(r-s-1) + 2\pi l/T = 0 \qquad & \text{for} \quad m_l. \end{aligned} \tag{19}$$

Let us consider the case when $\omega T/2\pi = p/q$; then the normal form reads

$$\begin{aligned} \frac{d\tau}{dt} &= 1 + P(|z|^2, \bar{z}^q e^{2i\pi p\tau/T}, z^q e^{-2i\pi p\tau/T}) \qquad \text{(real)}, \\ \frac{dz}{d\tau} &= i\omega z + zQ_0[|z|^2, z^q e^{-2i\pi p\tau/T}] + \bar{z}^{q-1} e^{2i\pi p\tau/T} Q_1[|z|^2, \bar{z}^q e^{2i\pi p\tau/T}]. \end{aligned} \tag{20}$$

The z part of (20) is well known in the Poincaré map [16, 14], and we might observe that if we define $y = z e^{-i\omega\tau}$, we obtain an *autonomous* system (up to an arbitrary order) which is equivariant under the rotations of angle $2\pi/q$: $z \mapsto z e^{2i\pi/q}$ (see [1, Chap. 5, sect. 26; 15, Chap. XI]).

Proof of Theorem 1. By the implicit function theorem in the neighborhood of Γ we have for any Z in such a neighborhood

$$Z = u_0(\tau) + Q_0(\tau) X + Y, \tag{21}$$

where $\tau \in \mathbb{R}$, $Y \in E_-(\tau)$, $Q_0(\tau) X \in \tilde{E}_0(\tau)$. Now a center manifold for (1) contains Γ and could be expressed with (21) and Y as a function T-periodic in τ, at least quadratic in X. In fact we allow Φ in (15) to have components in $E_{00}(\tau)$ and $\tilde{E}_0(t)$. This means that we incorporate an eventual nonlinear change of coordinates on τ and X in (21) in such a way that the differential equation (1) written on the center manifold is in its *normal form.*

As usual, the method consists of using (15) and (16) into (1) and identifying powers of X. We have in fact

$$\begin{aligned} [1 + n(\tau, X)] \Big\{ & \dot{u}_0(\tau) + \dot{Q}_0(\tau) X + \frac{\partial \Phi}{\partial \tau}(\tau, X) \\ & + [Q_0(\tau) + D_X \Phi(\tau, X)][L_0 X + N(\tau, X)] \Big\} \\ = {} & \mathscr{F}[u_0(\tau) + Q_0(\tau) X + \Phi(\tau, X)]. \end{aligned} \tag{22}$$

Identification of order 0 in X gives (2). Now at the order 1 we obtain

$$\dot{Q}_0(\tau) + Q_0(\tau) L_0 = \mathscr{L}(\tau) Q_0(\tau) \tag{23}$$

which is just the basic property satisfied by $Q_0(\tau)$ and L_0.

Let us denote by $n_p(\tau, X)$, $\Phi_p(\tau, X)$, $N_p(\tau, X)$ the terms of degree p in X in the Taylor expansions of n, Φ, N in the neighborhood of 0 and denote by $\mathscr{F}_p(\tau, Y)$ the term of degree p in Y in the expansion of $\mathscr{F}[u_0(\tau)+Y]$ near Γ. Now the identification at order p in (22) leads to

$$n_p(\tau, X)\,\dot{u}_0(\tau)+\frac{\partial \Phi_p}{\partial \tau}(\tau, X)+Q_0(\tau)\,N_p(\tau, X)+D_X\Phi_p(\tau, X)\cdot L_0 X$$
$$=\mathscr{L}(\tau)\,\Phi_p(\tau, X)+\mathscr{R}_p(\tau, X), \tag{24}$$

where $\mathscr{R}_p$ only depends on $n_{p'}$, $N_{p'}$, $\Phi_{p'}$, with $p' \leqslant p-1$. We have for instance

$$\mathscr{R}_2(\tau, X)=\mathscr{F}_2[\tau, Q_0(\tau)X].$$

In (24) we look for Φ_p such that n_p and N_p can be found in the simplest way possible. Let us project (24) on $E_{00}(\tau)$, $\tilde{E}_0(\tau)$, $E_-(\tau)$ and use the following decomposition of Φ_p:

$$\Phi_p(\tau, X)=\Phi_p^{00}(\tau, X)\cdot\dot{u}_0(\tau)+Q_0(\tau)\,\Phi_p^0(\tau, X)+\Phi_p^-(\tau, X), \tag{25}$$

where $\Phi_p^{00}\in\mathbb{R}$, $\Phi_p^0\in\mathbb{R}^m$, $\Phi_p^-\in E_-(\tau)$. We use extensively the identity (11) for $P_{00}(\tau)$, $P_0(\tau)$, $P_-(\tau)$ and the property (13) for $Q_0(\tau)$ associated with the constant operator L_0 and $Q_{00}(\tau)$ associated with $L_{00}=0$. We finally obtain the following system:

$$n_p(\tau, X)+\frac{\partial}{\partial\tau}\Phi_p^{00}(\tau, X)+D_X\Phi_p^{00}(\tau, X)\cdot L_0 X$$
$$=Q_{00}^{-1}(\tau)\,P_{00}(\tau)\,\mathscr{R}_p(\tau, X), \tag{26}$$

$$N_p(\tau, X)+\frac{\partial}{\partial\tau}\Phi_p^0(\tau, X)-L_0\Phi_p^0(\tau, X)+D_X\Phi_p^0(\tau, X)\cdot L_0 X$$
$$=Q_0^{-1}(\tau)\,P_0(\tau)\,\mathscr{R}_p(\tau, X), \tag{27}$$

$$\frac{\partial}{\partial\tau}\Phi_p^-(\tau, X)-\mathscr{L}(\tau)\,\Phi_p^-(\tau, X)+D_X\Phi_p^-(\tau, X)\cdot L_0 X$$
$$=P_-(\tau)\,\mathscr{R}_p(\tau, X). \tag{28}$$

Let us first show that (28) is easily solvable with respect to Φ_p^-. For this we recall the identity

$$\frac{\partial}{\partial\tau}S(s, \tau)=-S(s, \tau)\,\mathscr{L}(\tau), \tag{29}$$

and we replace X by $e^{L_0\tau}X$ in (28), while we multiply by $S(s, \tau)$. Then after one integration we obtain

$$\Phi_p^-(\tau, X) = \int_{-\infty}^{\tau} S(\tau, s)\, P_-(s)\, \mathscr{R}_p[s, e^{L_0(s-\tau)}X]\, ds. \tag{30}$$

The integral on the right hand side of (30) converges since $S(\tau, s)\, P_-(s) \to 0$ exponentially when $\tau - s \to +\infty$, while $e^{L_0(s-\tau)}$ behaves polynomially. Moreover T-periodicity in τ is due to the T-periodicity of

$$P_-(s)\, \mathscr{R}_p(s, \cdot) = \mathscr{R}_p(s, \cdot) - [P_0(s) + P_{00}(s)]\, \mathscr{R}_p(s, \cdot)$$

and to the identity

$$S(\tau + T, s + T) = S(\tau, s).$$

Remark. The formula (30) may be easily justified in the infinite dimensional case (notice that P_0 and P_{00} are T-periodic).

Now, we recover, with Eq. (27) the homological equation of Arnold [1, Chap. 5, sect. 26] and the study of Elphick *et al.* [7] for periodically forced autonomous singular systems. By a Fourier analysis of (26) and (27), we obtain for any l in $\mathbb{Z}$ and X in $\mathbb{R}^m$

$$\frac{2i\pi l}{T}\Phi_{pl}^{00}(X) + D_X\Phi_{pl}^{00}(X) \cdot L_0 X = \mathscr{R}_{pl}^{00}(X) - n_{pl}(X), \tag{31}$$

$$\frac{2i\pi l}{T}\Phi_{pl}^{0}(X) - L_0\Phi_{pl}^{0}(X) + D_X\Phi_{pl}^{0}(X) \cdot L_0 X = \mathscr{R}_{pl}^{0}(X) - N_{pl}(X). \tag{32}$$

Equation (31) is in fact a linear equation in the vector space of scalar homogeneous polynomials of degree p in $X \in \mathbb{R}^m$, while Eq. (32) is a linear equation in the space of m-dimensional vector valued homogeneous polynomials of degree p in X. Let us choose the following scalar product in the first space [8],

$$\langle P \mid Q \rangle = P(\partial_X)\, Q(X)|_{X=0},$$

while in the second space we consider

$$\sum_{j=1}^{m} \langle \Phi_j \mid \Psi_j \rangle = \sum_{j=1}^{m} \Phi_j(\partial_X)\, \Psi_j(X)|_{X=0},$$

where Φ_j and Ψ_j are the m components of Φ and Ψ.

It is shown in [8] how to compute the adjoints of the linear operators defined on the left hand side of (31) and (32). We then use the Fredholm alternative, and it is clear that n_{pl} and N_{pl} can be respectively chosen equal to the orthogonal projection of $\mathscr{R}_{pl}^{00}$ and $\mathscr{R}_{pl}^{0}$ on the kernels of the corresponding adjoint operators. Hence we have

$$-\frac{2i\pi l}{T} n_{pl}(X) + D_X n_{pl}(X) \cdot L_0^* X = 0, \tag{33}$$

$$-\frac{2i\pi l}{T} N_{pl}(X) - L_0^* N_{pl}(X) + D_X N_{pl}(X) \cdot L_0^* X = 0, \tag{34}$$

for any l and X. Multiplying now (33) and (34) by $e^{2i\pi l\tau/T}$ and summing up for all l and p, we obtain for any X

$$\frac{\partial n}{\partial \tau}(\tau, X) - D_X n(\tau, X) \cdot L_0^* X = 0, \tag{35}$$

$$\frac{\partial N}{\partial \tau}(\tau, X) + L_0^* N(\tau, X) - D_X N(\tau, X) \cdot L_0^* X = 0, \tag{36}$$

which is equivalent to (17).

4. Nonsimple Case

In this Section we make the following assumptions:

B.1. *The Jordan block belonging to the eigenvector $\dot{u}_0(0)$ for $S(T)$ is more than one dimensional.*

B.2. *Some eigenvalues of $S(T)$ have modulus* 1, *other eigenvalues have moduli* < 1.

B.3. -1 *is not an eigenvalue of* $S(T)$.

Remark. It is well known [15] that generically when two closed orbits collapse into one closed orbit before disappearing (for a one parameter family of vector fields), then at criticality the Floquet multiplier 1 is double non-semi-simple. This shows that B.1 really happens very often.

Let us denote by $E_0(t)$, $E_-(t)$ respectively the subspaces spanned by (i) the $m+1$ vector functions $\zeta_j(t)$ built in Lemma 2 for all Jordan blocks belonging to eigenvalues of modulus 1 for $S(T)$ and (ii) the complementary subspace built in Lemma 3, belonging to eigenvalues of moduli less than 1 for $S(T)$. Let us denote by $P_0(t)$, $P_-(t)$ respectively the projections on $E_0(t)$, $E_-(t)$ defined in Lemma 3 and denote by $Q_0(t)$ the Floquet operator associated with the constant operator L_0 which corresponds to $E_0(t)$, as defined in Lemma 4. If we choose the first vector of the basis of $E_0(t)$ to be $\dot{u}_0(t)$, L_0 takes the following form in $\mathscr{L}(\mathbb{R}^{m+1})$,

$$L_0 = \begin{pmatrix} 0 & 1 \quad 0 \quad \cdots & 0 \\ \cdot & & \\ \cdot & \left(\tilde{L}_0\right) & \\ \cdot & & \\ 0 & & \end{pmatrix}, \tag{37}$$

where $\tilde{L}_0 \in \mathscr{L}(\mathbb{R}^m)$. Moreover we can write

$$E_0(t) = \tilde{E}_0(t) \oplus \{\mathbb{R}\dot{u}_0(t)\},$$

and for any $X \in \mathbb{R}^m$, $\alpha \in \mathbb{R}$, we define linear operators $\tilde{Q}_0(t)$ by

$$Q_0(t)(\alpha, X) = \alpha\dot{u}_0(t) + \tilde{Q}_0(t)X, \tag{38}$$

where $\tilde{Q}_0(t)X \in \tilde{E}_0(t)$. Now, the property (13) for $Q_0(t)$ leads to

$$\left[-\frac{d}{dt} + \mathscr{L}(t)\right]\tilde{Q}_0(t) = \tilde{Q}_0(t)\,\tilde{L}_0 + \dot{u}_0(t)\Pi_1, \tag{39}$$

where $\Pi_1 X = x_1$ (first component of X in $\mathbb{R}^m$).

We can now state the main result of this paragraph:

THEOREM 2. *If we assume that* B.1, B.2, B.3 *hold, then a center manifold for* (1) *in the neighborhood of* Γ *may be represented as*

$$Z = u_0(\tau) + \tilde{Q}_0(\tau)X + \Phi(\tau, X), \tag{40}$$

where Φ *is T-periodic in* τ, *and at least quadratic in* X. τ *plays the role of a phase coordinate along* Γ. *A* normal form *for the vector field on the center manifold may be found such that* (1) *becomes*

$$\begin{aligned} \frac{d\tau}{dt} &= 1 + x_1 + n(\tau, X) \qquad \text{in } \mathbb{R}, \\ \frac{dX}{d\tau} &= \tilde{L}_0 X + N(\tau, X) \qquad \text{in } \mathbb{R}^m, \end{aligned} \tag{41}$$

where n *and* N *are T-periodic in* τ, *are polynomials at least quadratic in* X, *and satisfy for any* $\tau \in \mathbb{R}$, $X \in \mathbb{R}^m$,

$$\frac{d}{d\tau} n[\tau, e^{-\tilde{L}_0^*\tau} X] = 0, \qquad \frac{d}{d\tau} e^{\tilde{L}_0^*\tau} N[\tau, e^{-\tilde{L}_0^*\tau} X] = 0, \tag{42}$$

where $\tilde{L}_0^*$ *is the adjoint of* $\tilde{L}_0$ *in* $\mathbb{R}^m$.

Remark. The only difference with (16) is the presence of x_1 in the phase equation $(41)_1$. We then observe that $\tilde{L}_0$ plays a role equivalent to the one played by L_0 in the simple case.

EXAMPLE 2. Let us consider the case where $S(T)$ restricted to the invariant subspace E_0 takes the following form in a suitable basis:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{43}$$

Then $\tilde{L}_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $X = (x_1, x_2)$ is two dimensional. Now, instead of using (42) we may use the equivalent partial differential equations (35), (36) with $\tilde{L}_0^*$ instead of L_0^*. A Fourier analysis of these equations leads for any p in $\mathbb{Z}$ to the system.

$$\begin{gathered} \frac{2i\pi l}{T} n_l(X) - D_X n_l(X) \cdot L_0^* X = 0, \\ \frac{2i\pi l}{T} N_l(X) + \tilde{L}_0^* N_l(X) - D_X N_l(X) \cdot \tilde{L}_0^* X = 0. \end{gathered} \tag{44}$$

The second equation in (44) means that we are looking for an eigenvector N_l of the homological operator belonging to the eigenvalue $-2i\pi l/T$. Since the only eigenvalue of $\tilde{L}_0^*$ is zero, it is known [1] that the only eigenvalue of the homological operator is 0 too. Moreover the first equation in (44) is the same as the equation for the first component of N_l. Finally, this shows that for $l \neq 0$ we have $n_l = 0$ and $N_l = 0$, hence n and N are *autonomous*, and we recover the case studied in [8]. After a change of variable on x_1 this gives the following normal form which is the classical Takens–Bogdanov normal form for the X components,

$$\begin{aligned} \frac{d\tau}{dt} &= 1 + x_1 + \varphi_0(x_1), \\ \frac{dx_1}{d\tau} &= x_2, \\ \frac{dx_2}{d\tau} &= x_2 \varphi_1(x_1) + \varphi_2(x_1), \end{aligned} \tag{45}$$

where φ_0, φ_1, φ_2 are polynomials, at least of degree 2 for φ_0 and φ_2, and at least of degree 1 for φ_1.

Proof of Theorem 2. We proceed as for Theorem 1. Replacing (40), (41) into (1) we obtain

$$[1+x_1+n(\tau,X)]\left\{\dot{u}_0(\tau)+\dot{\tilde{Q}}_0(\tau)X+\frac{\partial\Phi}{\partial\tau}(\tau,X)\right.$$
$$\left.+[\tilde{Q}_0(\tau)+D_X\Phi(\tau,X)][\tilde{L}_0X+N(\tau,X)]\right\}$$
$$=\mathscr{F}[u_0(\tau)+\tilde{Q}_0(\tau)X+\Phi(\tau,X)]. \tag{46}$$

Identification at order 1 in X leads to

$$\dot{u}_0(\tau)\Pi_1+\dot{\tilde{Q}}_0(\tau)+\tilde{Q}_0(\tau)\tilde{L}_0=\mathscr{L}(\tau)\,\tilde{Q}_0(\tau)$$

which is exactly (39). Now, at any order p, the identification gives

$$n_p(\tau,X)\,\dot{u}_0(\tau)+\frac{\partial\Phi_p}{\partial\tau}(\tau,X)+\tilde{Q}_0(\tau)\,N_p(\tau,X)+D_X\Phi_p(\tau,X)\cdot\tilde{L}_0X$$
$$=\mathscr{L}(\tau)\,\Phi_p(\tau,X)+\mathscr{R}_p(\tau,X), \tag{47}$$

where $\mathscr{R}_p$ only depends on $n_{p'}$, $N_{p'}$, $\Phi_{p'}$ with $p'\leqslant p-1$. For instance we have

$$\mathscr{R}_2(\tau,X)=\mathscr{F}_2[\tau,\tilde{Q}_0(\tau)X]-x_1[\dot{\tilde{Q}}_0(\tau)X+\tilde{Q}_0(\tau)\,\tilde{L}_0X].$$

Let us decompose Φ_p by using (38),

$$\Phi_p(\tau,X)=\Phi_p^{00}(\tau,X)\cdot\dot{u}_0(\tau)+\tilde{Q}_0(\tau)\,\Phi_p^0(\tau,X)+\Phi_p^-(\tau,X), \tag{48}$$

where $\Phi_p^{00}\in\mathbb{R}$, $\Phi_p^0\in\mathbb{R}^m$, $\Phi_p^-\in E_-(\tau)$. The same decomposition holds for $\mathscr{R}_p(\tau,x)$.

Property (39) allows one to write

$$\left[\frac{\partial}{\partial\tau}-\mathscr{L}(\tau)\right]\Phi_p(\tau,X)$$
$$=\frac{\partial\Phi_p^{00}}{\partial\tau}(\tau,X)\cdot\dot{u}_0(\tau)+\tilde{Q}_0(\tau)\frac{\partial\Phi_p^0}{\partial\tau}(\tau,X)-\tilde{Q}_0(\tau)\,\tilde{L}_0\Phi_p^0(\tau,X)$$
$$-\dot{u}_0(\tau)\cdot\Pi_1\Phi_p^0(\tau,X)+\left[\frac{\partial}{\partial\tau}-\mathscr{L}(\tau)\right]\Phi_p^-(\tau,X),$$

hence we recover exactly the system (27), (28) with $\tilde{L}_0$, $\tilde{Q}_0(\tau)$, instead of L_0, $Q_0(\tau)$, and instead of (26) we have

$$n_p(\tau,X)+\frac{\partial}{\partial\tau}\Phi_p^{00}(\tau,X)+D_X\Phi_p^{00}(\tau,X)\cdot\tilde{L}_0X-\Pi_1\Phi_p^0(\tau,X)=\mathscr{R}_p^{00}(\tau,X). \tag{49}$$

We observe that we first have to determine Φ_p^0 before computing Φ_p^{00} in the same way as before. Hence Theorem 2 is proved by the same arguments as Theorem 1.

5. Case When -1 Is a Floquet Multiplier

In this section we make the following assumptions:

(i) C.1. -1 *is an eigenvalue of* $S(T)$.

(ii) *Assumption* A.2 ($=$B.2) *holds.*

Let us denote by $E_0(t)$ and $E_-(t)$ respectively the T-periodic subspaces spanned by (i) the $m+1$ vector functions $\zeta_j(t)$ given in Lemmas 2 and 5 for all Jordan blocks belonging to eigenvalues of modulus 1 for $S(T)$ (-1 included) and (ii) the complementary subspace given in Lemma 3, belonging to eigenvalues of moduli less than 1 for $S(T)$. Let us denote by $P_0(t)$, $P_-(t)$ respectively the T-periodic projections on $E_0(t)$, $E_-(t)$ defined in Lemma 3.

Now, since we want in L_0, real Jordan blocks corresponding to the eigenvalue -1 for $S(T)$, we need Lemma 5 and we built a $2T$-periodic $\tilde{Q}_0(t)$ associated with $\tilde{L}_0$. We have in fact in $\tilde{E}_0(t)$

$$\tilde{Q}_0(t)X = \tilde{Q}_{00}(t)X_0 + Q_{01}(t)X_1, \tag{50}$$

where $X=(X_0, X_1)$, $X_0 \in \mathbb{R}^{m_0}$, $x_1 \in \mathbb{R}^{m_1}$, $m_0 + m_1 = m$, and $Q_{01}(t)X_1$ belongs to the space spanned by the eigenfunctions $\zeta_j(t)$ given in Lemma 5. Hence, we have

$$\tilde{Q}_0(t+T)X = \tilde{Q}_{00}(t)X_0 - Q_{01}(t)X_1,$$

i.e., $\tilde{Q}_{00}$ is T-periodic while Q_{01} is $2T$-periodic, and this allows one to define a symmetry S in $\mathbb{R}^m$,

$$X = (X_0, X_1) \mapsto SX = (X_0, -X_1),$$

such that

$$\tilde{Q}_0(t+T)X = \tilde{Q}_0(t)SX. \tag{51}$$

Here we use, for more generality, notations of Section 4. In the frame of Section 3 (i.e., if A.1 holds instead of B.1), we should write $Q_0(\tau)$ instead of $\tilde{Q}_0(\tau)$.

We can now establish:

THEOREM 3. *If we assume that* C.1 *holds instead of* A.3 *or* B.3, *then the results of Theorems* 1 *or* 2 *hold with the following modification*: Φ, n, *and* N *are* $2T$*-periodic in* τ *such that shifting* τ *by* T *is equivalent to acting* S *on* $X \in \mathbb{R}^m$ *for* Φ *and* n, *and*

$$N(\tau+T, SX) = SN(\tau, X) \qquad \text{in } \mathbb{R}^m. \tag{52}$$

Remark. We decompose Φ as in (25) or (48), then

$$\tilde{Q}_0(\tau+T)\,\Phi^0(\tau+T, X) = \tilde{Q}_0(\tau)\,\Phi^0(\tau, SX)$$

leads to

$$\Phi^0(\tau+T, SX) = S\Phi^0(\tau, X)$$

thanks to (51).

EXAMPLE 3. Let us consider the case where $S(T)$ restricted to the invariant subspace E_0 takes the following form in a suitable basis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}. \tag{53}$$

Then

$$\tilde{L}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and $X = X_1$, $SX = -X$. Finally n and N satisfy the same property as in Example 2, in addition to the symmetry property of Theorem 3. This leads to the following normal form (autonomous):

$$\begin{aligned} \frac{d\tau}{dt} &= 1 + \varphi_0(x_1^2), \\ \frac{dx_1}{d\tau} = x_2, \qquad \frac{dx_2}{d\tau} &= x_2\varphi_1(x_1^2) + x_1\varphi_2(x_1^2), \end{aligned} \tag{54}$$

where φ_0, φ_1, φ_2 are polynomials in their argument, at least of degree 1. It has to be noticed for the interpretation of the solutions that we have in fact

$$Z = u_0(\tau) + \tilde{Q}_0(\tau)X + \Phi(\tau, X)$$

with (here) $\tilde{Q}_0(\tau+T) = -\tilde{Q}_0(\tau)$, $\Phi(\tau+T, -X) = \Phi(\tau, X)$. For instance a nontrivial stationary solution in x_1, with $x_2 = 0$, leads to a periodic solution for (1) with period $T'=2T/1+\varphi_0(x_1^2)$, up to high order terms (not written in the normal form).

Proof of Theorem 3. We can make the same analysis as in the proofs of Theorems 1 or 2, depending on the assumption A.1 or B.1, but in a $2T$-periodic frame. We still write (21) or (40),

$$Z=u_0(\tau)+\tilde{Q}_0(\tau)X+\Phi(\tau, X), \tag{55}$$

where u_0 is T-periodic, but $\tilde{Q}_0$ and Φ are now $2T$-periodic in τ. We may observe that this does not cause much trouble since when σ is a Floquet exponent associated with a T-periodic vector function, σ is also a Floquet exponent associated with a $2T$-periodic vector function. In fact, if $e^{\sigma T}$ is an eigenvalue of $S(T)$, then $e^{\sigma(2T)}$ is an eigenvalue of $S(2T)=[S(T)]^2$. The only interest here in doing so is to consider $\sigma=0$ as a Floquet exponent for the case when -1 is an eigenvalue of $S(T)$. Now we have normal forms such as (16), (17) or (41), (42), but with $2T$-periodic in τ functions n and N.

Let us now observe that

$$\mathscr{R}_2(T+\tau, SX)=\mathscr{R}_2(\tau, X)$$

due to (51) and (in the case of Assumption B.1) to the fact that S commutes with $\tilde{L}_0$ and that $\Pi_1 SX=\Pi_1 X$ since the first component of X belongs to X_0 and not to X_1 (see (50)).

Let us assume that

$$\mathscr{R}_p(T+\tau, SX)=\mathscr{R}_p(\tau, X),$$

then, due to (51) and to the T-periodicity of $\dot{u}_0$, we have

$$\mathscr{R}_p^{00}(T+\tau, SX)=\mathscr{R}_p^{00}(\tau, X), \qquad S\mathscr{R}_p^{0}(T+\tau, SX)=\mathscr{R}_p^{0}(\tau, X). \tag{56}$$

Moreover we easily obtain, after formula (30),

$$\Phi_p^{-}(\tau+T, SX)=\Phi_p^{-}(\tau, X). \tag{57}$$

Now, due to (56) we observe that $e^{i\pi l}\Phi_{pl}^{00}(SX)$, $e^{i\pi l}n_{pl}(SX)$ satisfy the same equation (31) as $\Phi_{pl}^{00}(X)$, $n_{pl}(X)$, and that $e^{i\pi l}S\Phi_{pl}^{0}(SX)$, $e^{i\pi l}SN_{pl}(SX)$ satisfy the same equation (32) as $\Phi_{pl}^{0}(X)$, $N_{pl}(X)$. The aim is now to prove that we can choose Φ_{pl}^{00}, Φ_{pl}^{0} such that

$$\begin{aligned} e^{i\pi l}\Phi_{pl}^{00}(SX)&=\Phi_{pl}^{00}(X), \qquad & e^{i\pi l}n_{pl}(SX)&=n_{pl}(X),\\ e^{i\pi l}S\Phi_{pl}^{0}(SX)&=\Phi_{pl}^{0}(X), \qquad & e^{i\pi l}SN_{pl}(SX)&=N_{pl}(X). \end{aligned} \tag{58}$$

Once this is proved we have immediately

$$\Phi_p(\tau+T, SX)=\Phi_p(\tau, X), \qquad n_p(\tau+T, SX)=n_p(\tau, X),$$
$$N_p(\tau+T, SX)=SN_p(\tau, X), \tag{59}$$

and it is now clear that the recurrence assumption propagates at order $p+1$, and that Theorem 3 is then proved.

To prove the second line of (58) let us introduce a symmetry S_*^0 such that for any m-dimensional vector valued homogeneous polynomial Ψ of degree p in X we have

$$(S_*^0\Psi)(x)=e^{i\pi l}S\Psi(SX). \tag{60}$$

It is clear that $\mathscr{R}_{pl}^0$ is invariant under S_*^0 and that S_*^0 commutes with the homological operator defined in (32) and with its adjoint defined in (34). It results that the image of the homological operator and the kernel of its adjoint are invariant under S_*^0. As a consequence the orthogonal projection N_{pl} of $\mathscr{R}_{pl}^0$ on the kernel of the adjoint is then *invariant under* S_*^0. Now we can choose $\Phi_{pl}^0=S_*^0\Phi_{pl}^0$ and $(58)_2$ is proved. For $(58)_1$ the proof is the same with another symmetry S_*^{00} defined in an obvious way.

6. Applications and Additional Properties

6.1. *Perturbation by Parameters*

Let us consider a system like (1) but depending on a parameter $\mu\in\mathbb{R}^k$,

$$\frac{dZ}{dt}=\mathscr{F}(\mu, Z), \tag{61}$$

where we assume that for $\mu=0$ there is a periodic solution $t\mapsto u_0(t)$ the orbit of which is denoted by Γ. We assume that $\mathscr{F}$ is as regular as we wish in the neighborhood of $(0, \Gamma)$ in $\mathbb{R}^k\times\mathbb{R}^n$.

Then we may follow the same arguments as those developed in Elphick *et al.* [8] and prove Theorems 1, 2, 3 with the only difference being that μ enters into Φ, n, N and that Φ, n, N may have terms depending only on μ in their expansions in (μ, X), as well as terms linear in X with degree at least one in μ.

For the identification process in $[\mu^{(q)}, X^{(p)}]$ (degree q in μ, degree p in X) we follow the strategy explicited in Fig. 1.

Let us give the normal forms corresponding to the three examples presented above, now with parameters.

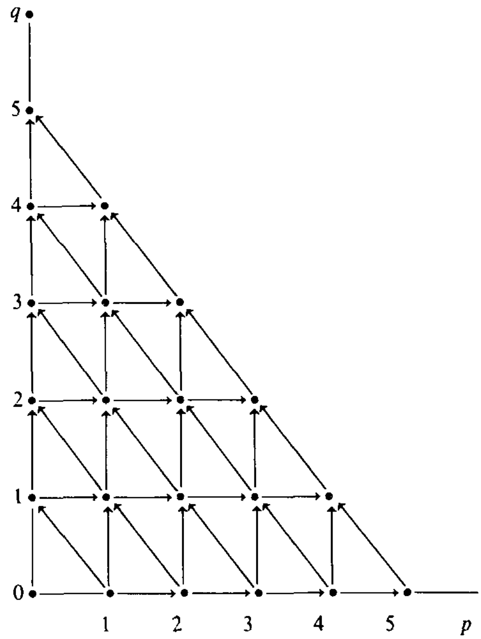

FIG. 1. For computing $\Phi_{qp}, n_{qp}, N_{qp}$ it is necessary to compute $\Phi_{q'p'}, n_{q'p'}, N_{q'p'}$ with $p'+q' \leqslant p+q-1$, $q' \leqslant q$ and $p'+q' = p+q$, $q' \leqslant q-1$.

EXAMPLE 1 (See (18)). We start with Eq. (20), where polynomials P, Q_0, Q_1 now depend on $\mu \in \mathbb{R}^k$, and with the only restriction being that there are no terms of degree 1 in $z, \bar{z}$ without μ into P (and Q_1 if $q=1$), and so terms of degree 0 in $z, \bar{z}$ without μ into Q_0 (and Q_1 if $q=2$).

In what follows we redefine some components of μ (equal at the first order to some linear combination of the k components of μ). After making the change of coordinates

$$z = e^{i\omega\tau}, \tag{62}$$

we then obtain, for the principal part of the unfolded field

$$\begin{aligned} \frac{d\tau}{dt} &= 1 + \nu_2 + \alpha\,|y|^2 + \beta\bar{y}^q + \bar{\beta}y^q + \cdots \in \mathbb{R}, \\ \frac{dy}{d\tau} &= \nu_1 y + ay\,|y|^2 + b\bar{y}^{q-1} + \cdots \in \mathbb{C}, \end{aligned} \tag{63}$$

where we specify $\nu_1 \in \mathbb{C}$, $\nu_2 \in \mathbb{R}$ components of μ, and where α, β, a, b depend on μ. Let us assume that we have a generic case, i.e., for $\mu = 0$,

$\alpha \neq 0$, $a \neq 0$, $\beta \neq 0$ if $q \geqslant 2$, $b \neq 0$ if $q \geqslant 3$. The relevant "small" parameters are $\nu_1 \in \mathbb{C}$ and $\nu_2 \in \mathbb{R}$ for $q \geqslant 3$; we add $b \in \mathbb{R}$ for $q = 2$, and $\beta \in \mathbb{C}$ if $q = 1$.

In fact ν_2 and β do not play any role into the qualitative behavior of trajectories, since they correspond to a change of time scale.

When $q \geqslant 3$ the main parameter is $\nu_1 \in \mathbb{C}$ where the imaginary part of ν_1 is the *detuning parameter*. This system was extensively studied by V. Arnold in [1, Chap. 6, Sect. 35]. For $q \geqslant 5$ this leads to the well-known Arnold tongues, which can be computed easily by looking for steady solutions in y of (63). These steady solutions give in fact subharmonic periodic solutions, which exist inside a resonant tongue, for the system (61).

When $q = 2$ or 1, the situation is much more complicated, because now b is close to 0 and plays the role of another component of the parameter μ. Then we refer to the work of J. M. Gambaudo [9] which contains the analysis of such vector fields.

EXAMPLE 2 (See (43)). We start with Eq. (45), where $\varphi_0, \varphi_1, \varphi_2$ now depend on $\mu \in \mathbb{R}^k$ and where there are no terms of degree 1 in x_1 without μ into φ_0 and φ_2, and no terms of degree 0 in x_1 without μ into φ_1. The unfolding of (45) in generic cases is well known [19, 2] and the study of the system in (x_1, x_2) is also classical [1, 10]. The principal part of our system may be written as

$$\begin{aligned} \frac{d\tau}{dt} &= 1 + x_1 + \mu_3, \\ \frac{dx_1}{d\tau} &= x_2, \\ \frac{dx_2}{d\tau} &= \mu_1 + \mu_2 x_2 + a x_1^2 + b x_1 x_2, \end{aligned} \tag{64}$$

where we suppressed a term in $\mu_2' x_1$ in $dx_2/d\tau$, by a small change of coordinates, using the generic assumption $a|_{\mu=0} \neq 0$.

EXAMPLE 3 (See (53)). Starting with Eq. (54) with $\varphi_0, \varphi_1, \varphi_2$ now depending on μ with no term of degree 0 in x_1^2 without μ, we obtain for the principal part of the unfolded system

$$\begin{aligned} \frac{d\tau}{dt} &= 1 + \nu_3 + \alpha x_1^2, \\ \frac{dx_1}{d\tau} &= x_2, \\ \frac{dx_2}{d\tau} &= \nu_1 x_1 + \nu_2 x_2 + a x_1^3 + b x_2 x_1^2. \end{aligned} \tag{65}$$

We may notice that the system in $(x_1, x_2))$ was extensively studied by J. Carr [3].

6.2. *Case of Additional Symmetries in the System*

6.2.1. *Case of a symmetry not broken by $u_0(t)$.* Let us assume that the system (1) satisfies for any Z

$$\mathscr{F}(\mathscr{C}Z) = \mathscr{C}\mathscr{F}(Z), \tag{66}$$

where $\mathscr{C}$ is a linear operator in $\mathbb{R}^n$ which represents some symmetry invariance of the system. Let us moreover assume that

$$\mathscr{C}u_0(t) = u_0(t), \tag{67}$$

i.e., $u_0(t)$ does not break this invariance. An easy consequence of (66), (67) is that the linear operators, defined in Section 2, $\mathscr{L}(t)$, $S(t, s)$, $S(T)$, $P(t)$ commute with $\mathscr{C}$. For $P(t)$ this results from the uniqueness of the solution of (11).

Let us now define $\mathscr{C}_0(\tau) \in \mathscr{L}(\mathbb{R}^m)$ as follows. Let us denote by $\tilde{Q}_0(\tau)$ as well the linear operator occurring in Theorem 2, as the linear operator $Q_0(\tau)$ occurring in Theorem 1. Now we define $\mathscr{C}_0(\tau)$ for any $X \in \mathbb{R}^m$ by

$$\tilde{Q}_0(\tau)\, \mathscr{C}_0(\tau) X = \mathscr{C} \tilde{Q}_0(\tau) X \qquad \text{in } \tilde{E}_0(\tau). \tag{68}$$

Then, by differentiating with respect to τ, we obtain

$$\begin{aligned} &\frac{d\mathscr{C}_0}{d\tau} = \tilde{L}_0 \mathscr{C}_0 - \mathscr{C}_0 \tilde{L}_0 && (\text{where } \tilde{L}_0 \equiv L_0 \text{ in the frame of Theorem 1}), \\ &\Pi_1 \mathscr{C}_0 = \Pi_1 && (\text{in the frame of Theorem 2}). \end{aligned} \tag{69}$$

Hence we have

$$\mathscr{C}_0(\tau) = e^{\tilde{L}_0 \tau} \mathscr{C}_0(0)\, e^{-\tilde{L}_0 \tau}, \tag{70}$$

and by definition $\mathscr{C}_0$ is T-periodic.

Now it is easy to check that (66), (67) and (68), (69) lead to

$$\mathscr{R}_2(\tau, \mathscr{C}_0(\tau) X) = \mathscr{C} \mathscr{R}_2(\tau, X)$$

in the frame of Theorem 1 as well as in the frame of Theorem 2.

Now assuming that the following identity holds,

$$\mathscr{R}_p(\tau, \mathscr{C}_0(\tau) X) = \mathscr{C} \mathscr{R}_p(\tau, X), \tag{71}$$

it is not hard to show that we can find Φ_p, n_p, N_p such that if $\mathscr{C}_0(\tau)$ is an isometry of $\mathbb{R}^m$, then

$$\begin{aligned}
&\Phi_p(\tau, \mathscr{C}_0(\tau)X) = \mathscr{C}\Phi_p(\tau, X),\\
&n_p(\tau, \mathscr{C}_0(\tau)X) = n_p(\tau, X),\\
&N_p(\tau, \mathscr{C}_0(\tau)X) = \mathscr{C}_0(\tau)\, N_p(\tau, X).
\end{aligned} \tag{72}$$

In fact for $\Phi_p^-(\tau, X)$ the property results from (30) and from the identity

$$\begin{aligned}
&\mathscr{C}S(\tau, s)\, P_-(s)\, \mathscr{R}_p[s, e^{\tilde{L}_0(s-\tau)}X]\\
&\quad = S(\tau, s)\, P_-(s)\, \mathscr{R}_p[s, \mathscr{C}_0(s)\, e^{\tilde{L}_0(s-\tau)}X]\\
&\quad = S(\tau, s)\, P_-(s)\, \mathscr{R}_p[s, e^{\tilde{L}_0(s-\tau)}\mathscr{C}_0(\tau)\, X].
\end{aligned}$$

Now, due to (69) we observe that $\Phi_p^{00}[\tau, \mathscr{C}_0(\tau)X]$, $n_p[\tau, \mathscr{C}_0(\tau)X]$ satisfy the same equation (26) as $\Phi_p^{00}(\tau, X)$, $n_p(\tau, X)$ ((49) if assumption B.1 holds). In the same way if $\mathscr{C}_0(\tau)$ is invertible we can easily see that $\mathscr{C}_0^{-1}(\tau)\, \Phi_p^0[\tau, \mathscr{C}_0(\tau)X]$, $\mathscr{C}_0^{-1}(\tau)\, N_p[\tau, \mathscr{C}_0(\tau)X]$ satisfy the same equation (27) as $\Phi_p^0(\tau, X)$, $N_p(\tau, X)$.

Let us define for any $\Psi(\tau, X)$ T-periodic in τ, homogeneous polynomial of degree p in X, taking values in $\mathbb{R}^m$:

$$(\mathscr{C}_{0*}^0 \Psi)(\tau, X) = \mathscr{C}_0^{-1}(\tau)\, \Psi[\tau, \mathscr{C}_0(\tau)X]. \tag{73}$$

It is clear that $\mathscr{R}_p^0$ is invariant under $\mathscr{C}_{0*}^0$ by hypothesis, and that $\mathscr{C}_{0*}^0$ commutes with the homological operator defined by (27):

$$\Psi \mapsto \mathscr{A}\Psi, \qquad \text{where}$$

$$(\mathscr{A}\Psi)(\tau, X) = \frac{\partial \Psi}{\partial \tau}(\tau, X) - \tilde{L}_0 \Psi(\tau, X) + D_X \Psi(\tau, X) \cdot \tilde{L}_0 X. \tag{74}$$

Now, if $\mathscr{C}_0(\tau)$ is *unitary*, then $\mathscr{C}_{0*}^0$ commutes also with the adjoint $\mathscr{A}^*$ of $\mathscr{A}$ defined by

$$(\mathscr{A}^*\Psi)(\tau, X) = -\frac{\partial \Psi}{\partial \tau}(\tau, X) - \tilde{L}_0^* \Psi(\tau, X) + D_x \Psi(\tau, X) \cdot \tilde{L}_0^* X. \tag{75}$$

To check (75) is easy using the fact that $\mathscr{C}_0^*(\tau) = \mathscr{C}_0^{-1}(\tau)$ satisfies

$$\frac{d\mathscr{C}_0^*}{d\tau} = \mathscr{C}_0^*(\tau)\, \tilde{L}_0^* - \tilde{L}_0^* \mathscr{C}_0^*(\tau).$$

In fact, if we introduce a scalar product taking account of T-periodicity in τ by defining

$$((\Phi \mid \Psi)) = \frac{1}{T} \int_0^T \langle \Phi(\tau, \cdot) \mid \Psi(\tau, \cdot) \rangle \, d\tau,$$

then it is a standard analysis to state that in a Hilbert space such that $L^2(\mathbb{T}^1)$ or any Sobolev space $H^m(\mathbb{T}^1)$ on τ, the linear operator $\mathscr{A}$ is unbounded, densely defined, with a closed range. Hence the Fredholm alternative applies, with the adjoint defined by (75). Finally, the orthogonal projection N_p of $\mathscr{R}_p^0$ on the kernel of $\mathscr{A}^*$ is then *invariant* under $\mathscr{C}_{0*}^0$, and we may use the same arguments, with a new operator $\mathscr{C}_{0*}^{00}$, to prove the corresponding result for n_p. In summing up all these results, we prove (72) and it is clear that the recurrence assumption (71) propagates at order $p+1$. Finally we prove the following:

THEOREM 4. *If the vector field is equivariant under a linear invertible operator $\mathscr{C}$, and if u_0 is pointwise invariant under $\mathscr{C}$, and if we enter into the assumptions of Theorems* 1 *or* 2, *then if $\mathscr{C}_0(\tau)$ defined by* (68) *is unitary on $\mathbb{R}^m$ we have in addition to the previous properties given in Theorem* 1 *or* 2

$$\begin{aligned} \Phi(\tau, \mathscr{C}_0(\tau) X) &= \mathscr{C} \Phi(\tau, X), \\ n(\tau, \mathscr{C}_0(\tau) X) &= n(\tau, X), \\ N(\tau, \mathscr{C}_0(\tau) X) &= \mathscr{C}_0(\tau) \, N(\tau, X). \end{aligned} \tag{76}$$

If the assumptions of Theorem 3 *hold, then $\mathscr{C}_0(\tau)$ is $2T$-periodic and* (76) *holds again.*

Remark 1. In fact (76) is valid *globally*, i.e., not only on any finite expansion; this general result on equivariance under $\mathscr{C}_0$ is the same as that in [18].

Remark 2. For using (76) it is essential to know how the linear operator $\mathscr{C}$ acts on the vector-functions $\zeta_j(t)$ defined in Section 2, which span $\tilde{E}_0(t)$. This gives the structure of $\mathscr{C}_0(t)$ in $\mathbb{R}^m$. It has to be noticed that if $\{\zeta_j(t); j = 1, ..., m\}$ is a basis for $\tilde{E}_0(t)$, then $\{\mathscr{C}\zeta_j(t); j = 1, ..., m\}$ is also a basis for $\tilde{E}_0(t)$, due to (9), and $\mathscr{C}$ respects the Jordan block structure of $S(T)$, eventually in exchanging blocks belonging to a same eigenvalue.

EXAMPLE 4. Let us consider the case where $S(T)$ restricted to E_0 takes the following form in a suitable basis $\{\dot{u}_0(0), \zeta_1\}$: $\left(\begin{smallmatrix}1 & 0 \\ 0 & 1\end{smallmatrix}\right)$. Let us assume that $\mathscr{F}$ commutes with a symmetry S,

$$S^2 = \mathrm{Id}, \qquad S \neq \mathrm{Id},$$

and assume that $Su_0(t) = u_0(t)$. Then we have

$$S\zeta_1 = \pm\zeta_1,$$

hence $S\zeta_1(t) = \pm\zeta_1(t)$, where $\zeta_1(t)$ is defined in Section 2. Finally here

$$\mathscr{C}_0(\tau)X = \pm X \qquad \text{in } \mathbb{R}.$$

In the case when $S\zeta_1 = -\zeta_1$, we obtain the following unfolded normal form:

$$\frac{d\tau}{dt} = 1 + \nu_2 + \alpha x^2 + \cdots,$$

$$\frac{dx}{dt} = \nu_1 x + ax^3 + \cdots,$$

which leads to a *pitchfork bifurcation of closed curves* when ν_1 crosses 0. Each bifurcated closed curve is mapped into the symmetric one by the acting S.

In the case when $S\zeta_1 = \zeta_1$, (76) gives nothing more on the normal form, and we recover a *saddle-node bifurcation* of pointwise symmetric closed curves.

EXAMPLE 5. Let us consider again Example 1 when $\mathscr{F}$ commutes with a representation R_φ of the symmetry group $SO(2)$. By definition, for any $\varphi_1, \varphi_2 \in \mathbb{R}$,

$$R_{\varphi_1+\varphi_2} = R_{\varphi_1} R_{\varphi_2}, \qquad R_0 = R_{2\pi} = \text{Id}.$$

Let us denote by $\{\dot{u}_0(0), \zeta_1, \bar{\zeta}_1\}$ the basis where the matrix (18) is written. Then, due to the commutation property, we have

$$R_\varphi \zeta_1 = e^{il\varphi}\zeta_1 \qquad \text{for some} \quad l \in \mathbb{Z},$$

hence, due to (9),

$$R_\varphi \zeta_1(t) = e^{il\varphi}\zeta_1(t).$$

Now, we note $X = (z, \bar{z})$, and it is clear that

$$\mathscr{C}_0(\tau, \varphi)X = (e^{il\varphi}z, e^{-il\varphi}\bar{z})$$

is independent of τ. The normal form (20) which satisfies (76) is now simpler,

$$\frac{d\tau}{dt} = 1 + P(|z|^2),$$

$$\frac{dz}{d\tau} = i\omega z + zQ_0(|z|^2),$$

and we see that the analysis of the unfolded vector field just leads to a simple Hopf bifurcation of an invariant T^2 torus with *no resonance tongue* such as for (63).

6.2.2. *Case of symmetry broken by $u_0(t)$—Rotating waves.* Let us just consider below the case when $\mathscr{F}$ commutes with a representation of $SO(2)$, denoted here by R_φ as in Example 5, and such that

$$R_\varphi u_0(t) = u_0(t + \varphi/\omega), \qquad \text{i.e., } u_0(t) = R_{\omega t} u_0(0). \tag{77}$$

If this is the case one usually says that we have a "rotating wave," and the form (15) or (40) may be much improved. In fact for any Z in the neighborhood of Γ we have, by the implicit function theorem, a unique couple $(\tau, \hat{Z})$ such that

$$Z = R_{\omega\tau}[u_0(0) + \hat{Z}], \tag{78}$$

where $\hat{Z} \in \tilde{E}_0 \oplus E_-$ (set $\tau = 0$ in the definitions of $\tilde{E}_0(\tau)$, $E_-(\tau)$, and $\tau \in \mathbb{R}/T\mathbb{Z}$). If we denote by L the generator of R_φ, then replacing (78) into (1) leads to

$$\frac{d\tau}{dt}\left[\omega L[u_0(0) + \hat{Z}] + \frac{d\hat{Z}}{d\tau}\right] = \mathscr{F}[u_0(0) + \hat{Z}], \tag{79}$$

where $\omega L u_0(0) = \mathscr{F}[u_0(0)]$. Projecting (79) on $Lu_0(0)$ and on $\tilde{E}_0 \oplus E_-$ we obtain an autonomous system

$$\begin{aligned} \frac{d\tau}{dt} &= 1 + g(\hat{Z}), \\ \frac{d\hat{Z}}{d\tau} &= F(\hat{Z}), \end{aligned} \tag{80}$$

and we reduce it to the normal form analysis of an autonomous vector field F near a singularity $\hat{Z} = 0$, in a space *one dimension less* than the original one. The case of a Hopf bifurcation from (80) is studied in [12] where the remnent symmetry of $u_0(0)$ is used to determine the structure of bifurcating solutions.

More precisely, we notice that the linear operator coccurring in $(80)_2$ is $D_Z\mathscr{F}[u_0(0)]-\omega L$, restricted to the $(n-1)$ dimensional subspace complementary to $Lu_0(0)$ and we can find a center manifold and a normal form of the vector field as

$$
\begin{aligned}
&Z = R_{\omega\tau}[u_0(0)+X+\Phi(X)],\\
&\frac{d\tau}{dt} = 1+n(X) \qquad (+x_1 \text{ if B.1 holds}),\\
&\frac{dX}{d\tau} = \tilde{L}_0 X + N(X),
\end{aligned} \tag{81}
$$

where $e^{\tilde{L}_0^*\tau}N(e^{-\tilde{L}_0^*\tau}x)=N(X)$ and $n(e^{-\tilde{L}_0^*\tau}x)=n(X)$ may be computed as in Sections 3 and 4, but without τ dependancy. Moreover we can observe that there is an integer p such that $R_{2\pi;p}u_0(t)=u_0(t)$, hence a remnent symmetry is added in $\mathbb{R}^m$ where a suitable representation of $R_{2\pi,p}$ acts on X, such that (n, N) is equivariant under this representation.

6.2.3. *Case of a group orbit of periodic solutions—Standing waves.* Another very common case is an $O(2)$-symmetry invariant system where time periodic solutions are either "traveling waves" or "standing waves." The first case fits into the frame treated in Section 6.2.2. The second case enters in the general frame of the occurrence of a group orbit of periodic solutions. This sitution was already studied by Chossat and Golubitsky [5] using a suitable Poincaré map. An analogous problem for a group orbit of fixed points for a vector field was also considered in [13].

Let us denote by R_φ as in Section 6.2.2 the $SO(2)$ representation and by S the symmetry such that

$$SR_\varphi = R_{-\varphi}S. \tag{82}$$

If the vector field $\mathscr{F}$ commutes with S and R_φ for any $\varphi\in\mathbb{R}$, the system is $O(2)$ invariant. A one parameter orbit of "standing waves" may be defined as

$$R_\psi u_0(t), \qquad \psi\in\mathbb{R}, \tag{83}$$

where u_0 is $T=2\pi/\omega$-periodic in t and

$$Su_0(t)=u_0(t). \tag{84}$$

Moreover, there is $p\in\mathbb{N}^*$ such that

$$R_{2\pi\cdot p}u_0(t)=u_0(t), \qquad R_{\pi\cdot p}u_0(t+\pi/\omega)=u_0(t). \tag{85}$$

The idea is now to factorize R_ψ like $R_{\omega\tau}$ in Section 6.2.2: we may look for a center manifold written as

$$Z = R_\psi [u_0(\tau) + \tilde{Q}_0(\tau) X + \Phi(\tau, X)]. \tag{86}$$

Now making the same analysis as in Section 3, 4, or 5, depending on assumptions, we find a normal form of the vector field, in a $(n-1)$ dimensional space, like (16) or (41) with an additional phase equation,

$$\frac{d\psi}{d\tau} = n_0(\tau, X), \tag{87}$$

where n_0 satisfies the same property (17) or (42) as n. Moreover we have the remnent symmetry due to the representations of S and $R_{\pi/p}\sigma_{\pi/\omega}$ (where σ_s is the shift by s of the time variable) on X in $\mathbb{R}^m$ as defined by (68).

Remark. In [4], the bifurcations occurring from "ribbons," which are standing waves with respect to the $O(2)$ action, may be treated as above. In fact it is simler since $u_0(\tau)$ is also a rotating wave due to the $SO(2)$ action (azimuthal variable), so amplitude equations become autonomous in such a case.

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