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A Codimension 2 Bifurcation for Reversible Vector Fields

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Abstract. For a family of reversible vector fields having a fixed point at the origin, we present the problem where, at criticality, the derivative at the origin has a multiple 0 eigenvalue with a 4×4 Jordan block. This is a codimension 2 singularity for reversible vector fields. This case happens in the water-wave problem for Bond number 1/3 and Froude number 1.

We study the persistence of all the known phenomena on the codimension one curves (in the parameter plane), especially concerning homoclinic orbits. One of these unfoldings is the 1:1 resonance Hopf bifurcation. The study strongly relies upon the knowledge of the reversible normal forms associated with the 4×4 Jordan block, and the unfolded situations, together with appropriate scalings.

1 Introduction

Let us consider a four-dimensional reversible vector field, having a fixed point at the origin, and such that 0 is a quadruple eigenvalue of the linear part, with a 4×4 Jordan block. This is a codimension 2 singularity for reversible vector fields, as we show in Section 2. For the unfolded vector field, we assume in what follows that 0 remains a fixed point. This is of course a restriction, but it leads to interesting cases, and simplifies the analysis. (It also implies the existence of another fixed point.) This situation can be found, for instance, in the water-wave problem, for Bond number 1/3 and Froude number 1, once the problem is reduced to a four-dimensional ordinary differential equation using a center manifold argument, where the evolution variable is the space variable (see Kirchgässner [1988]).

In the present work, we first give a normal form for the unfolded vector field. Then, we show how to pass to the limit for codimension one bifurcations which

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appear in the unfolding of our singularity. In particular, there is a curve in the parameter plane where a *reversible 1:1 resonance Hopf bifurcation* takes place, and similarly we have a curve for a *reversible Takens-Bogdanov singularity*, and another curve for the case when *a double zero eigenvalue exists with a pair of pure imaginary eigenvalues.* All the normal forms associated with such codimension one bifurcations are *integrable.* In particular, they all lead to the existence of homoclinic orbits. Many of these solutions for the truncated system are proved to persist for the full system (see Kirchgässner [1988], Iooss and Pérouème [1993], Iooss and Kirchgässner [1992]), but what was not known is what happens when we approach the codimension two point. Indeed, the coefficients of the previous codimension one normal forms become singular. We give in Section 4 the leading coefficients of such normal forms, and these coefficients give an idea of the scalings which are necessary to suppress the divergence as we approach the singularity. The effect of the scalings is to blow up the set of eigenvalues, for the purpose of entering into the frame of regular codimension one bifurcations. Applying the 'guessed' scalings to the codimension 2 normal form, we show that, in some horns of the parameter plane, tangent to the curves of codimension one bifurcations, all previously known results relative to such bifurcations stay valid.

2 Normal form

First we consider a vector field in $\mathbb{R}^4$ of the following form

$$\frac{dX}{dt} = L_0 X + R(X) \tag{2.1}$$

where L_0 is the linear operator given by the following matrix

$$L_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

and R is a regular (say C^k for k large enough) nonlinear function at least quadratic near the origin. We assume, in addition, that *the vector field (2.1) is reversible.* By this we mean that we have a linear symmetry S which leaves unchanged the eigenvector of L_0 and which anticommutes with the vector field

$$S^2 = Id, \quad SL_0 = -L_0 S, \quad SR(X) = -R(SX). \tag{2.3}$$

In particular, this means that the matrix of S reads as

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.4}$$

Remark 2.1 Let us notice here that the other possibility for S would be to change the eigenvector into its opposite, which would lead to the negative of the matrix given in (2.4); then the reversible normal form changes drastically, and will not be considered here.

Looking for a normal form for (2.1) means that we wish to find, near the origin, new coordinates for which the vector field looks simpler, up to a certain order (fixed in advance). This is a very useful tool, first introduced by Poincaré and Birkhoff, and developed in particular by Arnold [1983]. In a detailed and modern survey of normal forms, Cushman and Sanders [1990] give a normal form equivalent to ours (2.19), once restricted to reversible vector fields.[1] Instead of using their result directly, we prefer to present another self-contained quite elementary proof. The result we apply, first due to Belitskii [1981], was independently found by Elphick et al. [1987] and a detailed proof may be found, for instance, in Iooss and Adelmeyer [1992] Chapter 1.

Given a degree p, one can find a polynomial change of variable such that, up to order p, the function R is replaced by a function N satisfying the following linear partial differential equation:

$$DN(X)L_0^*X = L_0^*N(X) \tag{2.5}$$

which is equivalent to saying that N commutes with the group generated by the adjoint L_0^* of L_0. In addition, we keep the property that $SN = -NoS$, as in the original problem. Denoting by x_j and N_j, $j = 1,2,3,4$ the components of X and N in $\mathbb{R}^4$, the reversibility property gives

$$N_j(SX) = (-1)^j N_j(X) \tag{2.6}$$

and equation (2.5) reads as

$$\partial^* N_1 = 0, \quad \partial^* N_j = N_{j-1}, \quad for\ j = 2,3,4, \tag{2.7}$$

where the differential operator ∂^* is defined for any function f regular near 0 by:

$$\partial^* f = Df(X)L_0^*X = x_1\frac{\partial f}{\partial x_2} + x_2\frac{\partial f}{\partial x_3} + x_3\frac{\partial f}{\partial x_4}. \tag{2.8}$$

It is easy to find polynomial first integrals of the vector field L_0^*X, i.e. solutions f for $\partial^* f = 0$. The following are first integrals of degrees 1,2,3,4 respectively

$$\left.\begin{aligned} p_1 &= x_1, \\ p_2 &= x_2^2 - 2x_1x_3, \\ p_3 &= x_2^3 - 3x_1x_2x_3 + 3x_1^2x_4, \\ p_4 &= 3x_2^2x_3^2 - 6x_2^3x_4 - 8x_1x_3^3 + 18x_1x_2x_3x_4 - 9x_1^2x_4^2, \end{aligned}\right\} \tag{2.9}$$

where we notice the relationship

$$p_3^2 = p_2^3 - p_1^2p_4. \tag{2.10}$$

Lemma 2.2 *Any polynomial solution of $\partial^* f = 0$ is a polynomial P of variables p_1, p_2, p_3, p_4. As a consequence, it can be written as: $P_0(p_1,p_2,p_4)+p_3P_1(p_1,p_2,p_4)$, where P_0 and P_1 are both polynomials of their three arguments.*[2]

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[1] I thank Richard Cushman who made the comparison between the two normal forms.

[2] I thank A. Cerezo for this lemma. See also Cerezo [1988] for more details on similar problems.

Proof Let us choose the new variables p_1, p_2, p_3, x_2. The Jacobian of the transformation is $-6x_1^3$, and the inverse transformation is given by

$$x_1 = p_1,\ x_2 = x_2,\ x_3 = (2p_1)^{-1}[x_2^2 - p_2],\ x_4 = (3p_1^2)^{-1}(p_3 + \frac{1}{2}x_2^3 - \frac{3}{2}x_2 p_2).$$

It then results that any polynomial $P(x_1, x_2, x_3, x_4)$ may be written as a function $Q(p_1, p_2, p_3, x_2)$, where Q is a polynomial of p_2, p_3, x_2, with coefficients rational functions of p_1. Now, since $\partial^* P = 0$, we get $\partial Q/\partial x_2 = 0$, and then

$$P(x_1, x_2, x_3, x_4) = Q(p_1, p_2, p_3) = \sum_{\alpha,\beta,\gamma} a_{\alpha\beta\gamma} p_1^\alpha p_2^\beta p_3^\gamma$$

where the summation is *finite*, and taken over $\alpha \in \mathbb{Z}$ and $\beta, \gamma \in \mathbb{N}$. Let us now define the 'weight' of a monomial as follows: for x_1 we give the weight a, for x_2 we give $a+1$, for x_3 we give $a+2$, and for x_4 we give $a+3$. Then the differential operator ∂^* has the property to *conserve the degree of monomials and to decrease the weight by one*. It follows that one can consider *independently* monomials of different degrees and weights, and assume that all monomials in Q have the same degree and the same weight. Then, the monomials of Q have degree $\alpha + 2\beta + 3\gamma$ and weight $a(\alpha + 2\beta + 3\gamma) + 2\beta + 3\gamma$. This should be true for all values of a, hence we have α and $2\beta + 3\gamma$ which are the same for all our monomials, which may be written now as

$$Q(p_1, p_2, p_3) = p_1^\alpha \sum_{\beta,\gamma} b_{\beta\gamma} p_2^\beta p_3^\gamma$$

where α may be < 0. Now, if $\alpha < 0$, for $x_1 = 0$ we necessarily have

$$\sum_{\beta,\gamma} b_{\beta\gamma} x_2^{2\beta+3\gamma} = 0, \text{ hence} \sum_{\beta,\gamma} b_{\beta\gamma} = 0.$$

This leads to the property that

$$\sum_{\beta,\gamma} b_{\beta\gamma} p_2^\beta p_3^\gamma = \sum_{\beta,\gamma} b_{\beta\gamma} (p_2^3)^{\beta/3} (p_3^2)^{\gamma/2}$$

is homogeneous in (p_2^3, p_3^2) and equals zero if $p_2^3 - p_3^2 = 0$. Hence, $\sum_{\beta,\gamma} b_{\beta\gamma} p_2^\beta p_3^\gamma$ is divisible by $p_2^3 - p_3^2 = p_1^2 p_4$. This increases α by 2 and introduces p_4 into Q. It is then easy to iterate the reasoning until $\alpha \geq 0$. The conclusion of the lemma follows immediately from (2.10). □

Now, the reversibility operator S leaves invariant p_1, p_2, p_4 and changes p_3 into $-p_3$, hence from (2.6) and the above lemma we need to have

$$N_1(X) = p_3 P_1(p_1, p_2, p_4). \tag{2.11}$$

To make progress in constructing our normal form we need to introduce the following new polynomials q_j, r_j, s_j defined by

$$\partial^* q_j = p_j,\quad \partial^* r_j = q_j \quad \text{for } j = 1,2,3, \quad \text{and} \quad \partial^* s_j = r_j \quad \text{for } j = 1,3. \tag{2.12}$$

We have:

$$\begin{aligned} q_1 &= x_2, \\ q_2 &= -3x_1x_4 + x_2x_3, \\ q_3 &= 3x_1x_2x_4 - 2x_1x_3^2 + x_2^2x_3, \end{aligned} \tag{2.13}$$

$$\begin{aligned} r_1 &= x_3, \\ r_2 &= -3x_2x_4 + 2x_3^2, \\ r_3 &= -3x_1x_3x_4 + 3x_2^2x_4 - x_2x_3^2, \end{aligned} \tag{2.14}$$

$$\begin{aligned} s_1 &= x_4, \\ s_3 &= 3x_2x_3x_4 - (4/3)x_3^3 - 3x_1x_4^2. \end{aligned} \tag{2.15}$$

Notice that s_2, such that $\partial^* s_2 = r_2$, is not a polynomial, so we do not want to introduce it. The change of X into SX leaves unchanged q_3, r_1, r_2, s_3 and changes q_1, q_2, r_3, s_1 into their negative. Now equation (2.7) for N_2 may be solved by adding a particular solution q_3P_1 to the general solution of the equation $\partial^* f = 0$, so by the lemma, $N_2 = q_3P_1(p_1,p_2,p_4) + Q_0(p_1,p_2,p_4) + p_3Q_1(p_1,p_2,p_4)$ and we have $Q_1 \equiv 0$, because of (2.6). Now N_2 may be written as

$$N_2 = q_3P_1(p_1,p_2,p_4) + p_1P_2(p_1,p_2,p_4) + p_2Q(p_2,p_4) + R(p_4) \tag{2.16}$$

where P_2, Q, and R are polynomials in their arguments, defined by $p_2Q + R = Q_0(0,p_2,p_4)$, and $R(p_4) = Q_0(0,0,p_4)$. Now equation (2.7) for N_3 leads by the same arguments to

$$N_3 = r_3P_1(p_1,p_2,p_4) + q_1P_2(p_1,p_2,p_4) + q_2Q(p_2,p_4) + \frac{q_1}{p_1}R(p_4) + \varphi$$

where φ satisfies $\partial^*\varphi = 0$. Here the particular solution of (2.7) is not a polynomial, due to $(q_1/p_1)R(p_4)$. It is clear that $\psi = p_1\varphi$ has to be a polynomial and that $q_1R(p_4) + \psi$ cancels for $x_1 = 0$ (φ itself is not necessarily a polynomial). Then observing that $p_{4|x_1=0} = x_2^2y_4$, where $y_4 = 3x_3^2 - 6x_2x_4$, this last property may be written as

$$x_2R(x_2^2y_4) + \psi(0, x_2^2, x_2^3, x_2^2y_4) \equiv 0.$$

The fact that x_2 and $x_2^2y_4$ are independent variables and a simple examination of the monomials shows that we necessarily have $R \equiv 0$, therefore ψ divisible by p_1, i.e. φ is a polynomial. Finally by the reversibility property (2.6), N_3 reads

$$N_3 = r_3P_1(p_1,p_2,p_4) + q_1P_2(p_1,p_2,p_4) + q_2Q(p_2,p_4) + p_3P_3(p_1,p_2,p_4), \tag{2.17}$$

and we get directly, following the same lines

$$\begin{aligned} N_4 = s_3P_1(p_1,p_2,p_4) + r_1P_2(p_1,p_2,p_4) + r_2Q(p_2,p_4) + q_3P_3(p_1,p_2,p_4) \\ + P_4(p_1,p_2,p_4). \end{aligned} \tag{2.18}$$

Now, collecting the results (2.11), (2.16), (2.17), (2.18) we obtain the desired normal form for our reversible vector field

$$\frac{dX}{dt} = L_0X + P_4(x_1,p_2,p_4)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} + P_2(x_1,p_2,p_4)\begin{pmatrix}0\\x_1\\x_2\\x_3\end{pmatrix}$$
$$+Q(p_2,p_4)\begin{pmatrix}0\\p_2\\q_2\\r_2\end{pmatrix} + P_1(x_1,p_2,p_4)\begin{pmatrix}p_3\\q_3\\r_3\\s_3\end{pmatrix} + P_3(x_1,p_2,p_4)\begin{pmatrix}0\\0\\p_3\\q_3,\end{pmatrix}, \tag{2.19}$$

where $P_j, j = 1,2,3,4$ and Q are polynomials in their arguments, taking into account that the pure linear part is entirely contained into L_0X.

Assuming that 0 stays a fixed point of the unfolded vector fields, the principal part (linear and quadratic parts) of the right hand side of (2.19) is then generally given by

$$\begin{aligned} P_4(x_1,p_2,p_4) &= \mu_2x_1 + ax_1^2 + bp_2 + O(3) \\ P_2(x_1,p_2,p_4) &= \mu_1 + cx_1 + O(2) \\ Q(p_2,p_4) &= d + O(2) \end{aligned} \tag{2.20}$$

where μ_1 and μ_2 are the small unfolding parameters, and where $O(n)$ gives the order in X of unwritten terms. Other coefficients depend on parameters but in principle they are of order one when the parameter (μ_1,μ_2) is 0. We shall see in what follows, that the *most important nonlinear coefficient is a.*

Notice that quadratic terms of the system corresponding to (2.19) for the water-wave problem, not in this normal form, were given by Kirchgässner [1988]. This actual normal form for quadratic terms was recently completed by Menasce [1992].

3 Local study near fixed points of the normal form

The fixed points for the vector field (2.19) are given by the system of equations

$$x_2 + p_3P_1(x_1,p_2,p_4) = 0,$$

$$x_3 + x_1P_2(x_1,p_2,p_4) + p_2Q(p_2,p_4) + q_3P_1(x_1,p_2,p_4) = 0,$$

$$x_4 + x_2P_2(x_1,p_2,p_4) + q_2Q(p_2,p_4) + r_3P_1(x_1,p_2,p_4) + p_3P_3(x_1,p_2,p_4) = 0,$$

$$P_4(x_1,p_2,p_4) + x_3P_2(x_1,p_2,p_4) + r_2Q(p_2,p_4) + s_3P_1(x_1,p_2,p_4)$$
$$+q_3P_3(x_1,p_2,p_4) = 0,$$

for which the first 3 equations can be solved with respect to (x_2,x_3,x_4) using the implicit function theorem. Then there remains one equation for x_1 depending on parameters,

$$(\mu_2 - \mu_1^2)x_1 + [a + (2b-2c)\mu_1]x_1^2 + O(x_1^3) = 0, \tag{3.1}$$

which gives *two fixed points*: the first is the origin O, the second O', also invariant under S, is such that

$$\begin{aligned} x_1 &= a^{-1}(\mu_1^2 - \mu_2) + O(\mu_1^3, \mu_1\mu_2, \mu_2^2), \\ x_2 &= 0, \\ x_3 &= -a^{-1}\mu_1(\mu_1^2 - \mu_2) - ca^{-2}\mu_2^2 + O(\mu_1^4, \mu_1\mu_2^2, \mu_2\mu_1^2, \mu_2^3), \\ x_4 &= 0. \end{aligned} \tag{3.2}$$

The matrix of the derivative at the origin O is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \mu_1 & 0 & 1 & 0 \\ 0 & \mu_1 & 0 & 1 \\ \mu_2 & 0 & \mu_1 & 0 \end{pmatrix}, \tag{3.3}$$

and its eigenvalues λ are given by

$$\lambda^4 - 3\mu_1\lambda^2 + \mu_1^2 - \mu_2 = 0. \tag{3.4}$$

Notice that the linear operator (3.3) is reversible (anticommutes with S), so the set of eigenvalues is symmetric with respect to both axes in the complex plane. We give in Figure 1, the position of these four eigenvalues in the complex plane, as a function of (μ_1, μ_2).

The matrix of the derivative at the second fixed point O' is of the form:

$$\begin{pmatrix} 0 & 1+O(2) & 0 & O(2) \\ \begin{matrix}\mu_1 + 2cx_1 - 2dx_3 \\ +O(2)\end{matrix} & 0 & \begin{matrix}1 - 2dx_1 \\ +O(2)\end{matrix} & 0 \\ 0 & \mu_1 + cx_1 + dx_3 + O(2) & 0 & 1 - 3dx_1 + O(2) \\ \begin{matrix}\mu_2 + 2ax_1 \\ +(c-2b)x_3 + O(2)\end{matrix} & 0 & \begin{matrix}\mu_1 + (c-2b)x_1 \\ +4dx_3 + O(2)\end{matrix} & 0 \end{pmatrix} \tag{3.5}$$

where $O(2)$ means order 2 terms in x_j. The eigenvalues λ are solutions of the equation

$$\lambda^4 - [3\mu_1 + (2b - 4c)a^{-1}\mu_2 + \text{h.o.t.}]\lambda^2 - \mu_1^2 + \mu_2 + \text{h.o.t.} = 0. \tag{3.6}$$

We give in Figure 2 the position of these four eigenvalues in the complex plane, as a function of (μ_1, μ_2).

4 Computation of normal forms near curves $\Gamma_j (j = 0, 1, 2)$

Now we want to give the principal part of the normal forms near curves $\Gamma_j (j = 0, 1, 2)$ of the parameter plane (μ_1, μ_2), where codimension one bifurcations occur. We anticipate *singular coefficients* when approaching the origin of the parameter plane. This will give us an idea of the required scaling to be used on the normal form (2.19) for the codimension 2 problem.

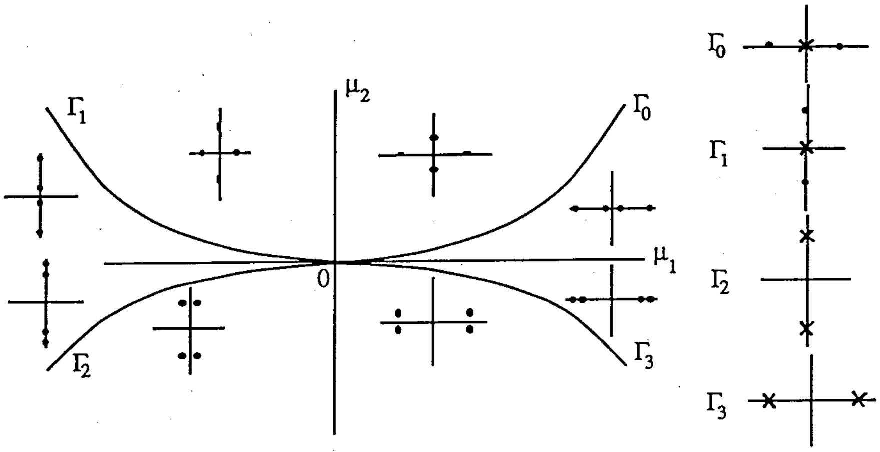

Figure 1 Eigenvalues of matrix (3.3)–linearization at 0. $\Gamma_0 : \mu_2 - \mu_1^2 = 0$, $\Gamma_1 : \mu_2 - \mu_1^2 = 0$, $\Gamma_2 : 4\mu_2 + 5\mu_1^2 = 0$, $\Gamma_3 : 4\mu_2 + 5\mu_1^2 = 0$.

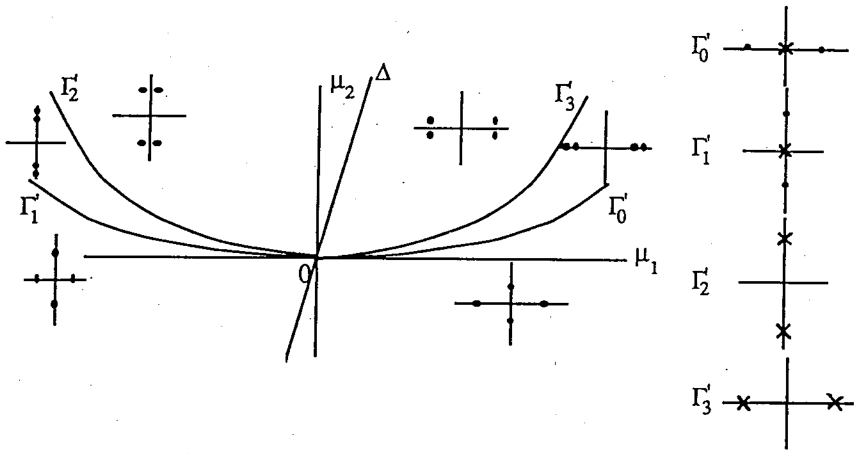

Figure 2 Eigenvalues of matrix (3.5)—linearization at $0'$. Γ'_0 and $\Gamma'_1 : \mu_2 - \mu_1^2 + \text{h.o.t.} = 0$, Γ'_2 and $\Gamma'_3 : 4\mu_2 - 13\mu_1^2 + \text{h.o.t.} = 0$, $\Delta : 3\mu_1 + (2b - 4c)a^{-1}\mu_2 + \text{h.o.t.} = 0$.

4.1 Normal form near Γ_1 Let us set $\mu_1 = \mu$ and $\mu_2 = \mu^2 + \epsilon$, $\mu < 0$. Then, in what follows we consider μ as fixed and ϵ is the bifurcation parameter. For $\epsilon = 0$, the eigenvalues are $\lambda_\pm = \pm i\sqrt{-3\mu}$ which are simple and 0 which is double. For $\epsilon > 0$, the double eigenvalue splits into two real eigenvalues $\pm\sqrt{-\epsilon/3\mu}$ + h.o.t., while for $\epsilon < 0$, they all are pure imaginary.

We need to introduce the eigenvectors $\xi_0, \zeta_0, \bar{\zeta}_0$ and the generalized eigenvector ξ_1

$$\xi_0 = \begin{pmatrix} 1 \\ 0 \\ -\mu \\ 0 \end{pmatrix} \quad \xi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2\mu \end{pmatrix} \quad \zeta_0 = \begin{pmatrix} 1 \\ \lambda_+ \\ 2\mu \\ \lambda_+\mu \end{pmatrix}, \tag{4.1}$$

which satisfy $S\xi_0 = \xi_0, S\xi_1 = -\xi_1, S\zeta_0 = \bar{\zeta}_0$.

Then the new coordinate system $(A, B, z, \bar{z})$ is defined by

$$X = A\xi_0 + B\xi_1 + z\zeta_0 + \bar{z}\bar{\zeta}_0 + \Phi(\epsilon, A, B, z, \bar{z}) \tag{4.2}$$

where the Taylor expansion of Φ may be computed in such a way that the system (2.19) takes the following normal form (see Iooss and Kirchgässner [1992] for the characterization and the computation of this normal form which is integrable, up to any order)

$$\begin{aligned} \tfrac{dA}{dt} &= B, \\ \tfrac{dB}{dt} &= (-3\mu)^{-1}\{\epsilon A + [a + 2\mu(b-c)]A^2 + \nu|z|^2\} + \ldots, \\ \tfrac{dz}{dt} &= i\sqrt{-3\mu}z[1 + (18\mu^2)^{-1}\epsilon + \delta A + \ldots], \end{aligned} \tag{4.3}$$

where $\nu = 2a + 2\mu(c - 7b) + 12\mu^2 d$, and $\delta = (9\mu^2)^{-1}[a + \mu(4c - b) - 12\mu^2 d]$.

We want to choose scales of A, B, z, t and ϵ, such that this system is no longer singular when μ goes to 0. The scales for A and ϵ are given by the z equation. The scale for t is also given by the last equation, in such a way that the eigenvalues do not meet at $\mu = 0$. Then the B scale is determined by keeping the form of the first equation, and it remains to check that it works in the B equation.

The following choice is then made

$$A = \mu^2\tilde{A}, \quad B = (-\mu)^{5/2}\tilde{B}, \quad z = \mu^2\tilde{z}, \quad t = (-\mu)^{-1/2}\tau, \quad \epsilon = \mu^2\tilde{\epsilon}. \tag{4.4}$$

Now, (4.1) and (4.2) show that we need to set, with $\mu_1 = \mu$ and $\mu_2 = \mu^2(1 + \tilde{\epsilon})$,

$$x_1 = \mu^2 y_1, \; x_2 = (-\mu)^{5/2} y_2, \; x_3 = -\mu^3 y_3, \; x_4 = (-\mu)^{7/2} y_4, \; t = (-\mu)^{-1/2}\tau. \tag{4.5}$$

4.2 Normal form near Γ_0 (reversible Takens-Bogdanov bifurcation) The computation is even simpler than in the previous case. We set again $\mu_1 = \mu$ and $\mu_2 = \mu^2 + \epsilon$, but now $\mu > 0$. The only relevant eigenvalue is 0 at $\epsilon = 0$. The two other eigenvalues are real $\lambda_\pm = \pm\sqrt{3\mu}$, and the corresponding modes are eliminated by a center manifold argument. So we have the vectors ξ_0 and ξ_1 as in (4.1), and a suitable Φ in

$$X = A\xi_0 + B\xi_1 + \Phi(\epsilon, A, B) \tag{4.6}$$

gives the two-dimensional set of amplitude equations (integrable up to any fixed order)

$$\begin{aligned} \tfrac{dA}{dt} &= B \\ \tfrac{dB}{dt} &= (-3\mu)^{-1}\{\epsilon A + [a + 2\mu(b-c)]A^2\} + \dots . \end{aligned} \tag{4.7}$$

Looking at this system is not sufficient to determine scales for the four-dimensional system (2.19). In fact, we also want to choose a scale in t such that the eigenvalues $\pm\sqrt{3\mu}$ become of order one (otherwise this introduces small denominators into the expression of the center manifold (4.6)). We then obtain the following choice of scales

$$A = \mu^2\tilde{A}, \quad B = \mu^{5/2}\tilde{B}, \quad z = \mu^2\tilde{z}, \quad t = \mu^{-1/2}\tau, \quad \epsilon = \mu^2\tilde{\epsilon}, \tag{4.8}$$

which leads to: $\mu_1 = \mu$ and $\mu_2 = \mu^2(1+\tilde{\epsilon})$,

$$x_1 = \mu^2 y_1, \; x_2 = \mu^{5/2} y_2, \; x_3 = \mu^3 y_3, \; x_4 = \mu^{7/2} y_4, \; t = \mu^{-1/2}\tau. \tag{4.9}$$

Remark 4.1 Indeed, the coefficients of the terms in $A^3, \epsilon A^2, \epsilon^2 A$ in the second equation of (4.7) have order μ^{-3} which agrees with (4.8). (See Menasce [1991] for the computation of higher order terms for the water-wave problem.) In principle, it would be possible for the special form of (2.19) to lead to more regular coefficients than expected, so that 'exploding' the real eigenvalues $\pm\sqrt{3\mu}$ would not be necessary.

4.3 Normal form near Γ_2 (reversible 1:1 resonance Hopf bifurcation) Let us set $\mu_1 = \mu$ and $\mu_2 = (-5/4)\mu^2 - \epsilon$, $\mu < 0$, and proceed as above. For $\epsilon = 0$, the eigenvalues are pure imaginary and double: $\lambda_\pm = \pm i\sqrt{(-3/2)\mu}$. For $\epsilon > 0$ they split and escape from the imaginary axis in two complex simple pairs symmetric with respect to both axes, while for $\epsilon < 0$, they split in two pairs of pure imaginary simple eigenvalues. Here we need to introduce the eigenvectors $\zeta_0, \bar{\zeta}_0$ and the generalized eigenvectors $\zeta_1, \bar{\zeta}_1$

$$\zeta_0 = \begin{pmatrix} 1 \\ \lambda_+ \\ \mu/2 \\ -\lambda_+\mu/2 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} 0 \\ 1 \\ 2\lambda_+ \\ 5\mu/2 \end{pmatrix}, \tag{4.10}$$

which satisfy

$$S\zeta_0 = \bar{\zeta}_0, \; S\zeta_1 = -\bar{\zeta}_1. \tag{4.11}$$

The new coordinates $(A, B, \bar{A}, \bar{B})$ are defined by

$$X = A\zeta_0 + B\zeta_1 + \bar{A}\bar{\zeta}_0 + \bar{B}\bar{\zeta}_1 + \Phi(\epsilon, A, B, \bar{A}, \bar{B}) \tag{4.12}$$

where Φ is such that the system (2.19) takes the following normal form (see for instance Iooss and Adelmeyer [1992] for the characterization and Iooss, Mielke and

Demay [1989] for the computation of such normal forms which appear to be *integrable up to any fixed order*)

$$\left.\begin{aligned}\tfrac{dA}{dt} &= i\sqrt{(-3/2)\mu}A + B - (6\mu\sqrt{-6\mu})^{-1}i\epsilon A + \dots\\ \tfrac{dB}{dt} &= i\sqrt{(-3/2)\mu}B - (6\mu\sqrt{-6\mu})^{-1}i\epsilon B\\ &\quad -(6\mu)^{-1}\epsilon A + (76a^2/243\mu^3)A\,|A|^2 + \dots\end{aligned}\right\}. \tag{4.13}$$

The choice of scales is then as follows: first the time scale has to place the double eigenvalue at distance $O(1)$ from O, then the scales for B/A and ϵ follow after a simple examination of the first equation, and then the scale of A by an examination of the second equation.

The following choice is then made

$$A = \mu^2\tilde{A},\ B = (-\mu)^{5/2}\tilde{B},\ t = (-\mu)^{-1/2}\tau,\ \epsilon = 5/4\mu^2\tilde{\epsilon}. \tag{4.14}$$

Now, (4.10) and (4.12) show that we need to set, with $\mu_1 = \mu$ and $\mu_2 = (-5/4)\mu^2(1+\tilde{\epsilon})$,

$$x_1 = \mu^2 y_1,\ x_2 = (-\mu)^{5/2}y_2,\ x_3 = -\mu^3 y_3,\ x_4 = (-\mu)^{7/2}y_4,\ t = (-\mu)^{-1/2}\tau, \tag{4.15}$$

which is identical to (4.5).

5 Rescaled codimension two normal forms: continuation results

Let us start with the scalings (4.5) (4.15), then the system (2.19) becomes

$$\begin{aligned}\tfrac{dY}{d\tau} &= L_0 Y + \mu^{-4}P_4(\mu^2 y_1, -\mu^5 p_2', -\mu^{11}p_4')\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\\ &-\mu^{-1}P_2(\mu^2 y_1, -\mu^5 p_2', -\mu^{11}p_4')\begin{pmatrix}0\\y_1\\y_2\\y_3\end{pmatrix} + \mu^2 Q(-\mu^5 p_2', -\mu^{11}p_4')\begin{pmatrix}0\\p_2'\\q_2'\\r_2'\end{pmatrix}\\ &-\mu^5 P_1(\mu^2 y_1, -\mu^5 p_2', -\mu^{11}p_4')\begin{pmatrix}p_3'\\q_3'\\r_3'\\s_3'\end{pmatrix} + \mu^4 P_3(\mu^2 y_1, -\mu^5 p_2', -\mu^{11}p_4')\begin{pmatrix}0\\0\\p_3'\\q_3'\end{pmatrix},\end{aligned}$$

whose principal part reads

$$\frac{dY}{d\tau} = \Lambda_\epsilon Y + [ay_1^2 - b\mu p_2']\begin{pmatrix}0\\0\\0\\1\end{pmatrix} - c\mu y_1\begin{pmatrix}0\\y_1\\y_2\\y_3\end{pmatrix} + \mu^2 d\begin{pmatrix}0\\p_2'\\q_2'\\r_2'\end{pmatrix} + \begin{pmatrix}O(\mu^5)\\O(\mu^3)\\O(\mu^3)\\O(\mu^2)\end{pmatrix}, \tag{5.1}$$

where

$$\Lambda_\epsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ \tilde{\mu}_2 & 0 & -1 & 0 \end{pmatrix},$$

and $\tilde{\mu}_2 = 1+\tilde{\epsilon}$ near Γ_1, and $\tilde{\mu}_2 = (-5/4)(1+\tilde{\epsilon})$ near Γ_2.

In the case where we are near Γ_0, the scaling (4.9) gives in an analogous way

$$\frac{dY}{d\tau} = \Lambda'_\epsilon Y + [ay_1^2 + b\mu p_2'] \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c\mu y_1 \begin{pmatrix} 0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} + \mu^2 d \begin{pmatrix} 0 \\ p_2' \\ q_2' \\ r_2' \end{pmatrix} + \begin{pmatrix} O(\mu^5) \\ O(\mu^3) \\ O(\mu^3) \\ O(\mu^2) \end{pmatrix}, \tag{5.2}$$

where

$$\Lambda'_\epsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \tilde{\mu}_2 & 0 & 1 & 0 \end{pmatrix}, \ \tilde{\mu}_2 = 1+\tilde{\epsilon}.$$

These rescaled normal forms (5.1),(5.2), are valid in the regions of the parameter plane (μ_1, μ_2) defined by $\mu_2 = \mu_1^2\tilde{\mu}_2$ with $\tilde{\epsilon}$ small with respect to 1. These are little 'horns' centered on curves $\Gamma_j, j = 0,1,2$. (See Figure 3.)

We observe that, as expected, the eigenvalues of Λ_ϵ are

i) for $\tilde{\mu}_2 = 1+\tilde{\epsilon}$: double 0 if $\tilde{\epsilon} = 0$, and $\pm i\sqrt{3}$

ii) for $\tilde{\mu}_2 = (-5/4)(1+\tilde{\epsilon})$: double $\pm i\sqrt{3/2}$ if $\tilde{\epsilon} = 0$

iii) of Λ'_ϵ for $\tilde{\epsilon} = 0$: double 0 and $\pm\sqrt{3}$.

The computation made above essentially shows that the coefficients of systems (5.1), (5.2) are regular functions of μ. Now, if we consider higher order terms, not written in the normal form (2.19), it should be clear that they are also regular functions of μ, since, to have that, it would be sufficient to consider the normal form up to quadratic terms.

Now the computation of the normal forms corresponding to the above three cases give coefficients which are *regular functions of the two parameters* $\tilde{\epsilon}$ *and* μ. The computations made in Section 4 show that the new normal forms are now:

i) for $\tilde{\mu}_2 = 1+\tilde{\epsilon}$, near $\Gamma_1(\mu < 0)$: (we still denote the amplitudes by A, B, z)

$$\begin{aligned} \tfrac{dA}{d\tau} &= B, \\ \tfrac{dB}{d\tau} &= (3)^{-1}\{\tilde{\epsilon}A + [a+2\mu(b-c)]A^2 + \nu|z|^2\} + \dots, \\ \tfrac{dz}{d\tau} &= i\sqrt{3}z[1+(18)^{-1}\tilde{\epsilon} + \delta' A + \dots], \end{aligned} \tag{5.3}$$

where $\nu = 2a + 2\mu(c-7b) + 12\mu^2 d$, and $\delta' = (9)^{-1}[a+\mu(4c-b) - 12\mu^2 d]$ and where unwritten terms are at least of order μ^2. (We could not guarantee this in Section 4.)

ii) for $\tilde{\mu}_2 = 1 + \tilde{\epsilon}$, near $\Gamma_0(\mu > 0)$: (we still denote the amplitudes by A, B)

$$\frac{dA}{d\tau} = B,$$
$$\frac{dB}{d\tau} = (-3)^{-1}\{\tilde{\epsilon}A + [a + 2\mu(b-c)]A^2\} + \dots, \tag{5.4}$$

where unwritten terms are at least of order μ^2. (We could not guarantee this in Section 4.)

iii) for $\tilde{\mu}_2 = (-5/4)(1+\tilde{\epsilon})$, near $\Gamma_2(\mu < 0)$: (we still denote the amplitudes by A and B)

$$\frac{dA}{d\tau} = i\sqrt{(3/2)}A + B + 5(24\sqrt{6})^{-1}i\tilde{\epsilon}A + \dots,$$
$$\frac{dB}{d\tau} = i\sqrt{(3/2)}B + 5(24\sqrt{6})^{-1}i\tilde{\epsilon}B + (5/24)\tilde{\epsilon}A \tag{5.5}$$
$$-(76a^2/243)A\,|A|^2 + \dots$$

where unwritten terms are of higher order. (We could not guarantee this in Section 4.)

The phase portraits of the normal forms of the reversible vector fields given by (5.3, 5.4, 5.5) are well known. However, the corresponding phase portraits of the full vector fields are only partially known. For instance, for the case (5.3) and (5.4), the detailed study is made in Iooss and Kirchgässner [1992].

In particular, we see on (5.3), i.e. near Γ_1, that $K = |z|^2$ is a first integral of the normal form, up to any order, and the trace of the phase portrait on the plane A, B is given in Figure 4. In such a case, for $K = 0$, there is a homoclinic orbit, starting and ending at the origin, for $\tilde{\epsilon} > 0$. It is known that this homoclinic orbit may not persist for the full vector field (only one stable and one unstable direction, in a four-dimensional space). There is a model equation, with the same linear part, for which the homoclinic orbit is proved not to persist (Eckhaus [1991], Hammersley and Mazzarino [1989], Pomeau, Ramani, and Grammaticos [1988]), and this leads to the fact that here it might not persist in a wide region of the parameter space. For $K \neq 0$, one has circles of solutions curves homoclinic to periodic orbits ($B = 0, A$ and $|z|$ fixed). It is shown by Iooss and Kirchgässner [1992] and Lombardi [1992] for very small periodic orbits, that in each of these families, the two reversible homoclinic orbits persist for the full vector field, and an analogous result holds for quasi-periodic solutions.

For (5.4), i.e. near Γ_0, the situation is simpler since it reduces to dimension 2. The phase portrait is the same as for $K = 0$ on Figure 4 except that we have to change $\tilde{\epsilon}$ into $-\tilde{\epsilon}$, hence for $\tilde{\epsilon} < 0$ such orbits homoclinic to 0 for the full original systems (2.1), exist in one side of the 'horn' centered on Γ_0, in the parameter plane.

In the case of the normal form (5.5), which corresponds to the reversible 1:1 resonance Hopf bifurcation, the detailed study is made in Iooss and Pérouème [1993]. We observe that the coefficient of $A\,|A|^2$ is negative, so we give in Figure 5 the projection of the phase portrait on the plane r_0, r_1 where $A = r_0e^{i\Psi_0}, B = r_1e^{i\Psi_1}$ and where we fix the first integral $K = A\bar{B} - B\bar{A}$ to be 0, i.e. we make $\Psi_0 = \Psi_1$ and let r_0 be ≥ 0 and r_1 take real values. It is proved in Iooss and Pérouème [1993] that the two reversible homoclinic orbits of each of these families persist for the full

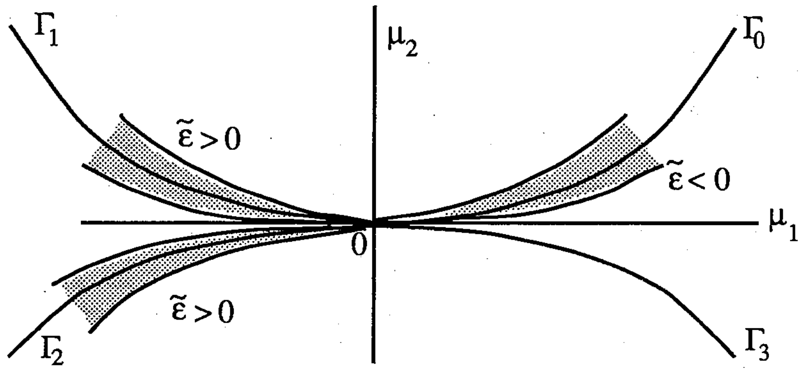

Figure 3 Three horns where the scaled normal forms are valid. We indicate for each normal form the sides where homoclinics to 0 exist.

vector fields. This implies that there are also two reversible orbits, homoclinic to 0, for the full system (2.1), in the side of the 'horn' centered on Γ_2 which is such that $\tilde{\epsilon} > 0$. These results hold provided the coefficient a is non-zero. This coefficient was recently computed in the water-wave problem, it is indeed non-zero and can be deduced from the computations made in Dias and Iooss [1993] where (4.13) is precisely given.

5.1 Simplified model and open problems The complete study of the phase portraits in dimension four is a wide open problem. However, we notice from the above computations, that the really important nonlinear coefficient is a. If we wildly truncate the problem, just keeping this nonlinear term in (2.19), we obtain an apparently simple 4th order differential equation for $x_1(t)$ denoted hereafter by $x(t)$:

$$x^{(4)} - 3\mu_1 x'' + (\mu_1^2 - \mu_2)x - ax^2 = 0. \tag{5.6}$$

This equation contains all the statements shown in the previous sections, concerning what we know in the horns in the (μ_1, μ_2) plane. Despite its apparent simplicity, and the fact that *one can integrate twice this equation* [set u(x)=$(x')^2(t)$ and integrate once], one should note a particular study relative to the neighbourhood of Γ_1, made by several authors on equations equivalent to (5.6) (see Eckhaus [1991], Amick and McLeod [1991], Hammersley and Mazzarino [1989], Pomeau, Ramani, and Grammaticos [1988]) showing that there is no reversible orbit homoclinic to 0, even though at any order the normal form of (5.6) possesses such orbits. Another interesting problem (work in progress) is what happens to the two homoclinics existing near Γ_2, when we escape from the corresponding little horn and go towards the horn where there exists only one homoclinic orbit near Γ_0. For instance, crossing of the curve Γ_3 shows a 'homoclinic multiplying' reversible bifurcation (see Champneys and Toland [1993]), among the new homoclinic orbits, one is clearly related with one of the two homoclinic orbits existing near Γ_2, while the other one is just the continuation of the basic one near Γ_0.

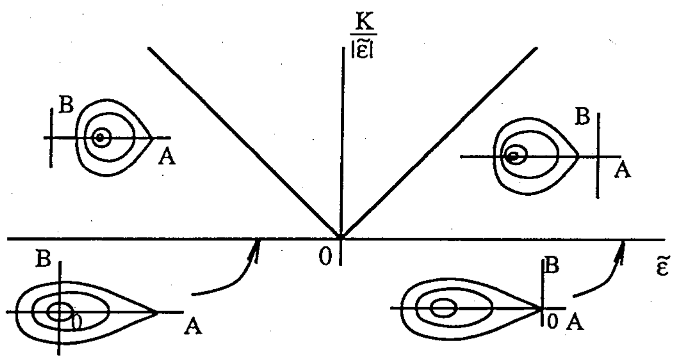

Figure 4 Phase portrait in the (A, B) plane for parameters in the horn near Γ_1. (See (5.3).)

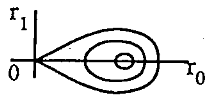

Figure 5 Phase portrait in the (r_0, r_1) plane for parameters in the horn near Γ_2, for $\tilde{\varepsilon} > 0$.

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