Polynomial Normal Forms with Exponentially Small Remainder for Analytic Vector Fields

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Abstract

A key tool in the study of the dynamics of vector fields near an equilibrium point is the theory of normal forms, invented by Poincaré, which gives simple forms to which a vector field can be reduced close to the equilibrium. In the class of formal vector valued vector fields the problem can be easily solved, whereas in the class of analytic vector fields divergence of the power series giving the normalizing transformation generally occurs. Nevertheless the study of the dynamics in a neighborhood of the origin, can very often be carried out via a normalization up to finite order. This paper is devoted to the problem of optimal truncation of normal forms for analytic vector fields in $\mathbb{R}^m$. More precisely we prove that for any vector field in $\mathbb{R}^m$ admitting the origin as a fixed point with a semi-simple linearization, the order of the normal form can be optimized so that the remainder is exponentially small.We also give several examples of non semi-simple linearization for which this result is still true.

Key words: Analytic vector fields, Normal forms, exponentially small remaiders

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1 Introduction

1.1 Position of the problem

A key tool in the study of the dynamics of vector fields near an equilibrium point is the theory of normal forms, invented by Poincaré, which gives simple forms to which a vector field can be reduced close to the equilibrium [1],[3]. In the class of formal vector valued vector fields the problem can be easily solved [1], whereas in the class of analytic vector fields divergence of the power series giving the normalizing transformation generally occurs [3], [21],[22]. Nevertheless the study of the dynamics in a neighborhood of the origin, can very often be carried out via a normalization up to finite order (see for instance [4], [11], [15], [16], [19],[23]). Normal forms are not unique and various characterization exist in the literature [2],[5],[8],[13],[23]. In this paper we will consider the version given in [13]:

Theorem 1.1 (Unperturbed NF-Theorem) *Let V be a smooth (resp. analytic) vector field defined on a neighborhood of the origin in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$. Then, for any integer $p \geq 2$, there are polynomials $\mathcal{Q}_p, \mathcal{N}_p : \mathbb{R}^m \to \mathbb{R}^m$ (resp. $\mathbb{C}^m \to \mathbb{C}^m$) , of degree $\leq p$, satisfying*

$$\mathcal{Q}_p(0) = \mathcal{N}_p(0) = 0, D\mathcal{Q}_p(0) = D\mathcal{N}_p(0) = 0$$

such that under the near identity change of variable $X = Y + \mathcal{Q}_p(Y)$, the vector field

$$\frac{dX}{dt} = V(X) \tag{1}$$

becomes

$$\frac{dY}{dt} = LY + \mathcal{N}_p(Y) + \mathcal{R}_p(Y) \tag{2}$$

where $DV(0) = L$, where the remainder $\mathcal{R}_p$ is a smooth (resp. analytic) function satisfying $\mathcal{R}_p(X) = \mathcal{O}(\|X\|^{p+1})$ and where the normal form polynomial $\mathcal{N}_p$ of degree p satisfies

$$\mathcal{N}_p(\mathrm{e}^{tL^*}Y) = \mathrm{e}^{tL^*}\mathcal{N}_p(Y)$$

for all $Y \in \mathbb{R}^m$ (resp. in $\mathbb{C}^m$) and $t \in \mathbb{R}$ or equivalently

$$D\mathcal{N}_p(Y)L^*Y = L^*\mathcal{N}_p(Y)$$

where L^* is the adjoint of L. Moreover, if T is a unitary linear map which commutes with V, then for every Y,

$$\mathcal{Q}_p(TY) = T\mathcal{Q}_p(Y) \qquad \mathcal{N}_p(TY) = T\mathcal{N}_p(Y).$$

Similarly, if V is reversible with respect to some linear unitary symmetry S ($S = S^{-1} = S^*$), i.e. if V anticommutes with this symmetry, then for every Y,

$$\mathcal{Q}_p(SY) = S\mathcal{Q}_p(Y) \qquad \mathcal{N}_p(SY) = -S\mathcal{N}_p(Y).$$

This version of the Normal Form Theorem up to any finite order has the following advantages : its proof is elementary and the characterization given is global in terms of a unique commutation property. Moreover it uses a simple hermitian structure of the space of homogeneous polynomials of given degree.

Since a usual way to study the dynamics of vector fields close to an equilibrium is to see the full vector field as a perturbation of its normal form $L + \mathcal{N}_p$ by higher order terms, it happens to be of great interest to obtain sharp upper bounds of the remainders $\mathcal{R}_p$. A similar theory of resonant normal forms was developed for Hamiltonians systems written in action-angle coordinates (see for instance [6], [9], [20]). A sticking result obtained by Nekhoroshev [17], [18], in order to study the stability of the action variables over exponentially large interval of time, is that up to an optimal choice of the order p of the normal form , the remainder can be made exponentially small. For more details of such Normal Form Theorems with exponentially small remainder for Hamiltonian systems written in action angle variables, we refer to the work of Pöschel in [20]. A similar result of exponential smallness of the remainder was also obtained by Giorgilli and Posilicano in [10] for a *reversible system* with a linear part composed of harmonic oscillators.

So a natural question is to determine for which class of analytic vector fields, such results of normalization up to exponentially small remainder can be obtained?

Since in the previously mentioned works dealing with particular hamiltonian or reversible systems, the normalization procedure is based on diagonalizable homological operators, a first natural class to consider, is the class $\mathcal{C}_{\mathrm{s}}$ of analytic vector fields, fixing the origin, and such that their linearization at the origin is *semi-simple* (i.e. is diagonalizable). This is indeed the largest class for which the homological operators involved in the normalization procedure are diagonalizable (see Lemma 2.5-(a)). More precisely, we prove in this paper that such results of normalization up to exponentially small remainder can be obtained for any analytic vector fields in $\mathcal{C}_{\mathrm{s}}$ provided that the spectrum of the linearization at the origin satisfies some "nonresonance assumptions" which enable to control the small divisor effects.

The question of the validity of such results for analytic vector fields with *non semi-simple linearization* is far more intricate : we give two examples of non semi-simple linearizations for which the result is still true. However, the question remains totally open for other non semi simple linearizations. We perform some estimates which suggest that the results should not be true in general for non semi-simple linearizations, but theses estimates are not sufficient to build a counter example (see Remark 2.9).

1.2 Statement of the results

To state our results we need to specify some "nonresonance assumptions" which enable to control the small divisor effects. In many problems, one uses one of the two following classical "nonresonance assumptions" : for a subset $\mathcal{Z}$ of $\mathbb{Z}^m$, for $K \in \mathbb{N}$, and for $\gamma > 0$, a vector $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^m$, is said to be $\boldsymbol{\gamma}, K$*-nonresonant modulo* $\mathcal{Z}$ if for every $k \in \mathbb{Z}^m$ with $|k| \leq K$,

$$| \langle \lambda, k \rangle | \geq \gamma \qquad \text{when} \qquad k \notin \mathcal{Z}. \tag{3}$$

Similarly, for $\gamma > 0, \tau > m-1$, λ is said to be $\boldsymbol{\gamma}, \boldsymbol{\tau}$*-Diophantine modulo* $\mathcal{Z}$ if for every $k \in \mathbb{Z}^m$,

$$| \langle \lambda, k \rangle | \geq \frac{\gamma}{|k|^\tau} \qquad \text{when} \qquad k \notin \mathcal{Z}, \tag{4}$$

where for $k = (k_1, \cdots, k_m) \in \mathbb{Z}^m$, $|k| := |k_1| + \cdots + |k_m|$. However, in the problem of normal forms, the small divisors appear as eigenvalues of the homological operator giving the normal forms by induction (see Subsection 2.1 and Lemma 2.5). To control these small divisors let us introduce two slightly different definitions :

Definition 1.2 *Let us define* $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{C}^m$*,* $K \in \mathbb{N}$*,* $\gamma > 0$ *and* $\tau > m-1$*.*

(a) *The vector* λ *is said to be* $\boldsymbol{\gamma}, K$**-homologically without small divisors** *if for every* $\alpha \in \mathbb{N}^m$ *with* $2 \leq |\alpha| \leq K$*, and every* $j \in \mathbb{N}$*,* $1 \leq j \leq m$*,*

$$| \langle \lambda, \alpha \rangle - \lambda_j | \geq \gamma \qquad \textit{when} \qquad \langle \lambda, \alpha \rangle - \lambda_j \neq 0.$$

(b) *The vector* λ *is said to be* $\boldsymbol{\gamma}, \boldsymbol{\tau}$**-homologically Diophantine** *if for every* $\alpha \in \mathbb{N}^m$*,* $|\alpha| \geq 2$*,*

$$| \langle \lambda, \alpha \rangle - \lambda_j | \geq \frac{\gamma}{|\alpha|^\tau} \qquad \textit{when} \qquad \langle \lambda, \alpha \rangle - \lambda_j \neq 0.$$

(c) *For a linear operator* L *in* $\mathbb{R}^m$*, let us denote by* $\lambda_1, \cdots, \lambda_m$ *its eigenvalues and* $\lambda_L := (\lambda_1, \cdots, \lambda_m)$*. Then* L *is said to be* $\boldsymbol{\gamma}, K$**-homologically without small divisors** *(resp.* $\boldsymbol{\gamma}, \boldsymbol{\tau}$**-homologically Diophantine***) if* λ_L *is so.*

Remark 1.3 Observe that in the above definitions, the components of α are nonnegative integers whereas in (3) and (4), k lies in $\mathbb{Z}^m$.

In what follows we use Arnold's notations [1] for denoting matrices under complex Jordan normal forms : $\boldsymbol{\lambda}^{\mathbf{2}}$ denotes the 2×2 complex Jordan block corresponding to $\lambda \in \mathbb{C}$ whereas $\boldsymbol{\lambda}.\boldsymbol{\lambda}$ represents 2×2 complex diagonal matrix $\operatorname{diag}(\lambda, \lambda)$, i.e.

$$\boldsymbol{\lambda}^{\mathbf{2}} := \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \text{whereas} \qquad \boldsymbol{\lambda}.\boldsymbol{\lambda} := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

A matrix under complex Jordan normal form is then denoted by the products of the name of its Jordan blocs. Moreover since for real matrices the Jordan blocks corresponding to non zero matrices occur by pairs $\boldsymbol{\lambda}^r$ and $\overline{\boldsymbol{\lambda}}^r$ we shorten their name as follows : for $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$, $\mathbf{0^2}.\boldsymbol{\lambda}_{\mathbf{1}}^{r_1}.\boldsymbol{\lambda}_{\mathbf{2}}^{r_2}.\overline{\boldsymbol{\lambda}}_{\mathbf{1}}^{r_1}.\overline{\boldsymbol{\lambda}}_{\mathbf{2}}^{r_2}$ is simply denoted by $\mathbf{0^2}.\boldsymbol{\lambda}_{\mathbf{1}}^{r_1}.\boldsymbol{\lambda}_{\mathbf{2}}^{r_2}|_{\mathbb{C}}$. Moreover, when one works with vector fields in $\mathbb{R}^m$, one may want to remain in $\mathbb{R}^m$ and thus to use real Jordan normal forms for the linearization of the vector field. So, for $\mu \in \mathbb{R}$ and $\lambda = x + \mathrm{i}y \in \mathbb{C} \setminus \mathbb{R}$, we denote by $\boldsymbol{\mu}^{\mathbf{2}}\boldsymbol{\lambda}^{\mathbf{2}}|_{\mathbb{R}}$ the real Jordan matrix

$$\begin{pmatrix} \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} x & -y \\ y & x \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \end{pmatrix}.$$

Finally, we equip $\mathbb{R}^m$ and $\mathbb{C}^m$ with the canonical inner product and norm, i.e. for $X = (X_1, \cdots, X_m) \in \mathbb{C}^m$, $\quad \|X\|^2 := \langle X, X \rangle = \sum_{j=1}^{m} \overline{X_j} X_j$. We are now ready to state our main result:

Theorem 1.4 (NF-Theorem with exponentially small remainder)
Let V be an analytic vector field in a neighborhood of 0 in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$, i.e.

$$V(X) = LX + \sum_{k \geq 2} V_k[X^{(k)}] \tag{5}$$

where L is a linear operator in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) and where V_k is bounded k-linear symmetric and

$$\|V_k[X_1, \cdots, X_k]\| \leq c \frac{\|X_1\| \cdots \|X_k\|}{\rho^k} \tag{6}$$

with $c, \rho > 0$ independent of k.

(a) *If L is semi-simple and under real (resp. complex) Jordan normal form, then*

(i) *if L is γ, τ-homologically Diophantine, then for every $\delta > 0$ such that $p_{\text{opt}} \geq 2$, the remainder $\mathcal{R}_p$, given by the Normal Form Theorem 1.1 for $p = p_{\text{opt}}$, satisfies*

$$\sup_{\|Y\| \leq \delta} \|\mathcal{R}_{p_{\text{opt}}}(Y)\| \leq M_\tau \delta^2 \mathrm{e}^{-\frac{w}{\delta^b}} \tag{7}$$

with

$$b = \frac{1}{1+\tau}, \quad p_{\text{opt}} = \left[\frac{1}{\mathrm{e}(C\delta)^b}\right], \quad w = \frac{1}{\mathrm{e}C^b}$$

and

$$M_\tau = \frac{10}{9} c C^2 \left\{ \left(\mathfrak{m} \sqrt{\frac{27}{8\mathrm{e}}} \right)^{1+\tau} + (2\mathrm{e})^{2+2\tau} \right\}$$

where

$$C = \frac{\sqrt{m}}{\rho^2} \left\{ \left(\frac{5}{2} m + 2 \right) ac + 3\rho \right\}, \qquad \mathfrak{m} = \sup_{p \in \mathbb{N}} \frac{\mathrm{e}^2\, p!}{p^{p+\frac{1}{2}} \mathrm{e}^{-p}}, \qquad a = \gamma^{-1}.$$

(ii) *if L is γ, K-homologically without small divisors, then for every $\delta > 0$ such that $K \geq p_{\text{opt}} \geq 2$ then the remainder $\mathcal{R}_p$ given by the Normal Form Theorem 1.1 for $p = p_{\text{opt}}$ satisfies (7) with $\tau = 0$, i.e. $b = 1$.*

(b) *For $L = \mathbf{0}^2 \underbrace{\mathbf{0} \cdots \mathbf{0}}_{q \text{ times}}$ and $L = \mathbf{0}^3 \underbrace{\mathbf{0} \cdots \mathbf{0}}_{q \text{ times}}$, estimate (7) still holds with $\tau = 0$, i.e $b = 1$, and with $a = 1$.*

The proof of this theorem is given in section 2.

Remark 1.5 Stirling's formula ensures that $\mathfrak{m}$ is finite.

Remark 1.6 Theorem 1.1 gives a polynomial upper bound of the remainder $\mathcal{R}_p$ of the form $\sup_{\|Y\| \leq \delta} \|\mathcal{R}_p(Y)\| \leq C(p)\delta^{p+1}$ whereas the above theorem ensures that with an optimal choice of p we have $\sup_{\|Y\| \leq \delta} \|\mathcal{R}_{p_{\text{opt}}}(Y)\| \leq M\delta^2 \mathrm{e}^{-\frac{w}{\delta^b}}$. The proof heavily relies on a precise control of the divergence of $C(p)$ with p.

Remark 1.7 A semi simple matrix under complex Jordan normal form is simply a diagonal matrix whereas a real semi simple matrix under real Jordan normal form is the direct sum of a diagonal matrix with 2×2 blocks of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ with $x, y \in \mathbb{R}$.

Remark 1.8 The characterization of the normal form and the exponentially small estimates are invariant under unitary changes of coordinates. Indeed, if we perform in (2) a unitary change of coordinates $Y = Q\widetilde{Y}$ where Q is a unitary linear operator ($Q^* = Q^{-1}$), then it becomes

$$\frac{d\widetilde{Y}}{dt} = \widetilde{L}\widetilde{Y} + \widetilde{\mathcal{N}}_p(\widetilde{Y}) + \widetilde{\mathcal{R}}_p(\widetilde{Y})$$

with $\widetilde{L} = Q^{-1}LQ$, $\widetilde{\mathcal{N}}_p(\widetilde{Y}) = Q^{-1}\mathcal{N}_p(Q\widetilde{Y})$, $\widetilde{\mathcal{R}}_p(Y) = Q^{-1}\mathcal{R}_p(QY)$, where $\widetilde{\mathcal{N}}_p$ satisfies the same normal form criteria as $\mathcal{N}_p$, i.e. $\widetilde{\mathcal{N}}(\mathrm{e}^{t\widetilde{L}^*}\widetilde{Y}) = \mathrm{e}^{t\widetilde{L}^*}\widetilde{\mathcal{N}}(\widetilde{Y})$ and where $\widetilde{\mathcal{R}}_p$ admits the same exponentially small upper bound as $\mathcal{R}_p$ given by (7).

However, when Q is not unitary then $\widetilde{\mathcal{N}}_p$ satisfies a slightly different normal form criteria given by

$$\widetilde{\mathcal{N}}(\mathrm{e}^{t\overset{\vee}{L}}\widetilde{Y}) = \mathrm{e}^{t\overset{\vee}{L}}\widetilde{\mathcal{N}}(\widetilde{Y})$$

where $\overset{\vee}{L} = Q^{-1}L^*Q$ which is not equal to $\widetilde{L}^*$ when Q is not unitary. In this case, $\widetilde{\mathcal{R}}_{p_{\mathrm{opt}}}$ also admits a slightly different upper bound given by

$$\sup_{\|\widetilde{Y}\|\leq\widetilde{\delta}} \|\widetilde{\mathcal{R}}_{p_{\mathrm{opt}}}(Y)\| \leq M_\tau |||Q^{-1}|||\ |||Q|||^2\ \widetilde{\delta}^2\ \mathrm{e}^{-\frac{w}{|||Q|||^b\ \widetilde{\delta}^b}}$$

where $|||Q||| = \sup_{\|Y\|=1} \|Q(Y)\|$.

The above remark enables to state a corollary without assuming that L is under real or complex Jordan normal form

Corollary 1.9 *Let V be an analytic vector field in a neighborhood of 0 in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$, i.e. satisfying (5) and (6). Denote $L = DV(0)$ and let Q be an invertible matrix such that $J = QLQ^{-1}$ is under real (resp. complex) Jordan normal form.*

Then, there are polynomials $\mathsf{Q}_{\mathsf{p}_{\mathrm{opt}}}, \mathsf{N}_{\mathsf{p}_{\mathrm{opt}}} : \mathbb{R}^m \to \mathbb{R}^m$ (resp. $\mathbb{C}^m \to \mathbb{C}^m$), of degree $\leq p_{\mathrm{opt}}$, satisfying $\mathsf{Q}_{\mathsf{p}_{\mathrm{opt}}}(0) = \mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(0) = 0, D\mathsf{Q}_{\mathsf{p}_{\mathrm{opt}}}(0) = D\mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(0) = 0$ such that under the near identity change of variable $X = Y + \mathsf{Q}_{\mathsf{p}_{\mathrm{opt}}}(Y)$, the vector field (1) becomes

$$\frac{dY}{dt} = LY + \mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(Y) + \mathsf{R}_{\mathsf{p}_{\mathrm{opt}}}(Y)$$

where the remainder $\mathsf{R}_{\mathsf{p}_{\mathrm{opt}}} = \mathcal{O}(\|Y\|^{\mathsf{p}_{\mathrm{opt}}+1})$ is analytic and where $\mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}$ satisfies the normal form criteria

$$\mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(\mathrm{e}^{t\overset{\vee}{L}}Y) = \mathrm{e}^{t\overset{\vee}{L}}\mathsf{N}_{\mathsf{p}_{\mathrm{opt}}}(Y) \qquad \textit{with } \overset{\vee}{L} = Q^{-1}J^*Q$$

for all $Y \in \mathbb{R}^m$ (resp. in $\mathbb{C}^m$) and $t \in \mathbb{R}$. Moreover,

(a) *if L is semi-simple and γ,τ-homologically Diophantine, then for every $\delta > 0$ such that $\mathsf{p}_{\rm opt} \geq 2$, the remainder $\mathsf{R}_{\mathsf{p}_{\rm opt}}$ satisfies*

$$\sup_{\|Y\|\leq\delta} \|\mathsf{R}_{\mathsf{p}_{\rm opt}}(Y)\| \leq \mathsf{M}_\tau \delta^2 \mathrm{e}^{-\frac{\mathsf{w}}{\delta^b}} \tag{8}$$

with

$$b = \frac{1}{1+\tau}, \quad \mathsf{p}_{\rm opt} = \left[\frac{1}{\mathrm{e}(\mathsf{C}\delta)^b}\right], \quad \mathsf{w} = \frac{1}{\mathrm{e}\mathsf{C}^b}$$

and

$$\mathsf{M}_\tau = \frac{10}{9} c |||Q|||\, |||Q^{-1}|||\ \mathsf{C}^2 \left\{ \left(\mathfrak{m}\sqrt{\frac{27}{8\mathrm{e}}}\right)^{1+\tau} + (2\mathrm{e})^{2+2\tau} \right\}$$

where

$$\mathsf{C} = \frac{\sqrt{m}}{\rho^2} \ \left\{ \left(\frac{5}{2}m + 2\right) ac|||Q|||^2\, |||Q^{-1}|||^2 + 3\rho |||Q|||\, |||Q^{-1}||| \right\},$$

and $\mathfrak{m} = \sup_{p\in\mathbb{N}} \frac{\mathrm{e}^2\ p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$, $a = \gamma^{-1}$*;*

(b) *if L is semi-simple and γ, K-homologically without small divisors, then for every $\delta > 0$ such that $K \geq \mathsf{p}_{\rm opt} \geq 2$ then the remainder $\mathsf{R}_{\mathsf{p}_{\rm opt}}$ satisfies (8) with $\tau = 0$, i.e. $b = 1$;*

(c) *for $J = \mathbf{0}^2 \underbrace{\mathbf{0}\cdots\mathbf{0}}_{q\ times}$ and $J = \mathbf{0}^3 \underbrace{\mathbf{0}\cdots\mathbf{0}}_{q\ times}$, estimate (8) still holds with $\tau = 0$, i.e $b = 1$, and with $a = 1$.*

Proof. Starting with (1), perform a first change of coordinates $X = Q^{-1}\widetilde{X}$ to obtain a vector field $\widetilde{V}$ such that $D\widetilde{V}(0) = J$ is under Jordan normal form, then apply Theorem 1.4, i.e perform a second change of coordinates $\widetilde{X} = \widetilde{\mathcal{Q}}_{\mathsf{p}_{\rm opt}}(\widetilde{Y})$ and finally perform a last change of coordinates $\widetilde{Y} = QY$ to get the desired result. $\square$

The previous corollary readily enables to state a second one which holds for perturbed vector fields

$$\frac{du}{dt} = V(u,\mu), \quad u \in \mathbb{R}^m, \quad \mu \in \mathbb{R}^s \tag{9}$$

by setting $U = (u,\mu)$, $\mathcal{V} = (V, 0)$ and observing that (9) is equivalent to

$$\frac{dU}{dt} = \mathcal{V}(U).$$

Theorem 1.10 *Let $V : \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^m$ be an analytic family of vector fields defined in a neighborhood of* 0 *in $\mathbb{C}^m \times \mathbb{C}^s$ such that $V(0,\mu) = 0$, i.e.*

$$V(X,\mu) = L_0 X + \sum_{\substack{n+\ell\geq 2\\ k\geq 1}} V_{k,\ell}[X^{(k)}, \mu^{(\ell)}] \tag{10}$$

where $L_0 = D_uV(0,0)$ *is a linear operator in* $\mathbb{R}^m$ *(resp. in* $\mathbb{C}^m$*) and where* $V_{k,\ell}$ *is bounded* $k+\ell$*-linear symmetric and*

$$\|V_{k,\ell}[X_1,\cdots,X_k,\mu_1,\cdots,\mu_\ell]\| \le c\frac{\|X_1\|\cdots\|X_n\|}{\rho^k}\ \frac{\|\mu_1\|\cdots\|\mu_\ell\|}{\rho^\ell} \tag{11}$$

with $c, \rho > 0$ *independent of* n *and* ℓ*.*

Let Q *be an invertible matrix such that* $J = QL_0Q^{-1}$ *is under real (resp. complex) Jordan normal form.*

Then, there are polynomials $\mathsf{Q}_{\mathsf{p}_{\rm opt}}, \mathsf{N}_{\mathsf{p}_{\rm opt}} : \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^m$ *(resp.* $\mathbb{C}^m \times \mathbb{C}^s \to \mathbb{C}^m$*) , of degree* $\le p_{\rm opt}$*, satisfying* $\mathsf{Q}_{\mathsf{p}_{\rm opt}}(0,0) = \mathsf{N}_{\mathsf{p}_{\rm opt}}(0,0) = 0, D\mathsf{Q}_{\mathsf{p}_{\rm opt}}(0,0) = D\mathsf{N}_{\mathsf{p}_{\rm opt}}(0,0) = 0$ *such that under the near identity change of variable* $X = Y + \mathsf{Q}_{\mathsf{p}_{\rm opt}}(Y)$*, the vector field (9) becomes*

$$\frac{dY}{dt} = L_0Y + \mathsf{N}_{\mathsf{p}_{\rm opt}}(Y,\mu) + \mathsf{R}_{\mathsf{p}_{\rm opt}}(Y,\mu)$$

where the remainder $\mathsf{R}_{\mathsf{p}_{\rm opt}} = \mathcal{O}\Big((\|Y\|+\|\mu\|)^{\mathsf{p}_{\rm opt}+1}\Big)$ *is analytic and where* $\mathsf{N}_{\mathsf{p}_{\rm opt}}$ *satisfies the normal form criteria*

$$\mathsf{N}_{\mathsf{p}_{\rm opt}}(\mathrm{e}^{t\overset{\vee}{L}_0}Y,\mu) = \mathrm{e}^{t\overset{\vee}{L}_0}\mathsf{N}_{\mathsf{p}_{\rm opt}}(Y,\mu) \qquad \textit{with } \overset{\vee}{L}_0 = Q^{-1}J^*Q$$

for all $Y \in \mathbb{R}^m$ *(resp. in* $\mathbb{C}^m$*) and* $t \in \mathbb{R}$*. Moreover,*

(a) *if* L_0 *is semi-simple and* γ,τ*-homologically Diophantine, then for every* $\delta > 0$ *such that* $\mathsf{p}_{\rm opt} \ge 2$*, the remainder* $\mathsf{R}_{\mathsf{p}_{\rm opt}}$ *satisfies*

$$\sup_{\|Y\|+\|\mu\|\le\delta} \|\mathsf{R}_{\mathsf{p}_{\rm opt}}(Y,\mu)\| \le \mathsf{M}_\tau\delta^2\mathrm{e}^{-\frac{\mathsf{w}}{\delta^b}} \tag{12}$$

with

$$b = \frac{1}{1+\tau}, \quad \mathsf{p}_{\rm opt} = \left[\frac{1}{\mathrm{e}(\mathsf{C}\delta)^b}\right], \quad \mathsf{w} = \frac{1}{\mathrm{e}\mathsf{C}^b}$$

and

$$\mathsf{M}_\tau = \frac{10}{9}c|||Q|||\ |||Q^{-1}|||\ \mathsf{C}^2\left\{\left(\mathfrak{m}\sqrt{\frac{27}{8\mathrm{e}}}\right)^{1+\tau} + (2\mathrm{e})^{2+2\tau}\right\}$$

where

$$\mathsf{C} = \frac{\sqrt{m}}{\rho^2}\ \left\{\left(\frac{5}{2}m+2\right)ac|||Q|||^2\ |||Q^{-1}|||^2 + 3\rho|||Q|||\ |||Q^{-1}|||\right\},$$

and $\mathfrak{m} = \sup_{p\in\mathbb{N}} \frac{\mathrm{e}^2\ p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$*,* $a = \gamma^{-1}$*;*

(b) *if* L_0 *is semi-simple and* γ, K*-homologically without small divisors, then for every* $\delta > 0$ *such that* $K \ge \mathsf{p}_{\rm opt} \ge 2$ *then the remainder* $\mathsf{R}_{\mathsf{p}_{\rm opt}}$ *satisfies (12) with* $\tau = 0$*, i.e.* $b = 1$*;*

(c) *for* $J = \mathbf{0}^2 \underbrace{\mathbf{0}\cdots\mathbf{0}}_{q\ times}$ *and* $J = \mathbf{0}^3 \underbrace{\mathbf{0}\cdots\mathbf{0}}_{q\ times}$, *estimate (12) still holds with* $\tau = 0$, *i.e* $b = 1$, *and with* $a = 1$.

The proof of this theorem is given in section 3.

Remark 1.11 In the non semi-simple case, we get the exponential smallness of the remainder only in two special nilpotent cases. For the other cases the problem remains open. Nevertheless, in [14], we study analytic reversible families of vector fields $V(X,\mu)$ in $\mathbb{R}^4$ admitting a $0^2 i\omega$ resonance (i.e. $D_X V(X,0) = \mathbf{0^2 i\boldsymbol{\omega}}|_{\mathbb{R}}$) and we show how the above theorem can be used to get an exponentially small upper bound of the remainder of the form

$$\sup_{Y\in B(\mu)} \|\mathsf{R}_{\mathsf{p}_{\rm opt}}(Y,\mu)\| \leq \mathsf{M}_\tau \mathrm{e}^{-\frac{\mathsf{w}}{\sqrt{|\mu|}}} \tag{13}$$

where $B(\mu)$ is some appropriate neighborhood of the origin of size of order $|\mu|$. We then deduce from (13), the existence of homoclinic connections to exponentially small periodic orbits.

2 Exponential estimates for unperturbed vector fields

This section is devoted to the proof of Theorem 1.4. We first recall in few words the proof of Theorem 1.1.

2.1 Normalization and Homological equations

Let V be an analytic vector field in a neighborhood of 0 in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$, i.e. a vector field satisfying (5) and (6). Let $\mathcal{H}$ be the space of the polynomial $\Phi : \mathbb{R}^m \mapsto \mathbb{R}^m$ (resp. $\mathbb{C}^m \mapsto \mathbb{C}^m$) and let $\mathcal{H}_k$ be the space of the homogeneous ones of degree k. We are interested in polynomial changes of variables, of the form $X = Y + \mathcal{Q}_p(Y)$ with

$$\mathcal{Q}_p(Y) = \sum_{2\leq k\leq p} \Phi_k(Y), \quad \Phi_k \in \mathcal{H}_k$$

such that by the change of variable, equation (1) becomes of the form (2) with

$$\mathcal{N}_p(Y) = \sum_{2\leq k\leq p} N_k(Y), \qquad N_k \in \mathcal{H}_k,$$

where $\mathcal{N}_p$ is as simple as possible. A basic identification of powers of Y leads to

$$\begin{aligned} &\{\mathrm{Id} + \sum_{2\le k\le p} D\Phi_k(Y)\}\{LY + \sum_{2\le k\le p} N_k(Y) + \mathcal{R}_p(Y)\} \\ &= L\{\sum_{1\le k\le p} \Phi_k(Y)\} + \sum_{q\ge 2} V_q\left[\{\sum_{1\le k\le p} \Phi_k(Y)\}^{(q)}\right]. \end{aligned} \tag{14}$$

where $\Phi_1(Y) = Y$. This leads to the following hierarchy of homological equations in $\mathcal{H}_n$ for $2 \le n \le p$,

$$\mathcal{A}_{_L}\Phi_n + N_n = F_n, \tag{15}$$

with

$$F_n = -\sum_{2\le k\le n-1} D\Phi_k.N_{n-k+1} + \sum_{2\le q\le n} \sum_{p_1+\cdots+p_q=n} V_q[\Phi_{p_1}, \cdots, \Phi_{p_q}], \tag{16}$$

where $\mathcal{A}_{_L}$ is the homological operator given by

$$(\mathcal{A}_{_L}\Phi)(Y) = D\Phi(Y).LY - L\Phi(Y).$$

In (16), by convention, the sums corresponding to an empty index set are equal to 0. Observe that $\mathcal{A}_{_L}$ induces on each $\mathcal{H}_n$ a linear endomorphism denoted by $\mathcal{A}_{_L}|_{\mathcal{H}_n} : \mathcal{H}_n \to \mathcal{H}_n$. Generally $\mathcal{A}_{_L}|_{\mathcal{H}_n}$ is not invertible. So when F_n lies in the range $\mathrm{ran}(\mathcal{A}_{_\mathrm{L}}|_{\mathcal{H}_\mathrm{n}})$ of $\mathcal{A}_{_L}|_{\mathcal{H}_n}$, one can take $N_n = 0$ and for Φ_n any preimage of F_n. When F_n does not lie in $\mathrm{ran}(\mathcal{A}_{_\mathrm{L}}|_{\mathcal{H}_\mathrm{n}})$, one has to choose N_n in an appropriate supplementary space of $\mathrm{ran}(\mathcal{A}_{_\mathrm{L}}|_{\mathcal{H}_\mathrm{n}})$ so that $F_n - N_n$ belongs to $\mathrm{ran}(\mathcal{A}_{_\mathrm{L}}|_{\mathcal{H}_\mathrm{n}})$.

The key idea of the proof of Theorem 1.1 contained in [13] is to introduce an appropriate inner product on $\mathcal{H}$ such that the adjoint $\mathcal{A}_{_L}^*$ of $\mathcal{A}_{_L}$ is given by $\mathcal{A}_{_{L^*}}$. Hence,

$$\mathcal{H}_n = \ker \mathcal{A}_{_L}|_{\mathcal{H}_n} \overset{\perp}{\oplus} \mathrm{ran}\mathcal{A}_{_{\mathrm{L}^*}}|_{\mathcal{H}_\mathrm{n}}, \qquad \mathcal{H}_\mathrm{n} = \mathrm{ran}\mathcal{A}_{_\mathrm{L}}|_{\mathcal{H}_\mathrm{n}} \overset{\perp}{\oplus} \ker \mathcal{A}_{_{\mathrm{L}^*}}|_{\mathcal{H}_\mathrm{n}}.$$

Then for solving (15), we use the orthogonal projection π_n on $\ker \mathcal{A}_{_{L^*}}|_{\mathcal{H}_n}$ for obtaining N_n and the pseudo-inverse $\widetilde{\mathcal{A}_{_L}}|_{\mathcal{H}_n}^{-1}$of $\mathcal{A}_{_L}|_{\mathcal{H}_n}$ defined in $(\ker \mathcal{A}_{_{L^*}})^\perp = \mathrm{ran}\mathcal{A}_{_\mathrm{L}}|_{\mathcal{H}_\mathrm{n}}$ taking values in $(\ker \mathcal{A}_{_L}|_{\mathcal{H}_n})^\perp$ for Φ_n, i.e.

$$N_n = \pi_n(F_n) \qquad \text{and} \qquad \Phi_n = \widetilde{\mathcal{A}_{_L}}|_{\mathcal{H}_n}^{-1}((\mathrm{Id} - \pi_n)(F_n)). \tag{17}$$

This completes the proof of theorem 1.1 and ensures that N_n belongs to $\ker \mathcal{A}_{_{L^*}}|_{\mathcal{H}_n}$ and thus that $\mathcal{N}_p$ lies in $\ker \mathcal{A}_{_{L^*}} := \{\mathcal{N}/D\mathcal{N}(Y)L^*Y - L^*\mathcal{N}(Y) = 0\}$.

To conclude this subsection, the appropriate inner product in $\mathcal{H}$ introduced in [13] is given by

$$\langle\Phi|\Phi'\rangle_{_\mathcal{H}} = \sum_{j=1}^{m} \left\langle \Phi_j|\Phi'_j \right\rangle$$

with $\Phi = (\Phi_1, \cdots, \Phi_m)$, $\Phi' = (\Phi'_1, \cdots, \Phi'_m)$, where for any pair of polynomial $P, P' : \mathbb{R}^m \to \mathbb{R}$ (resp. $\mathbb{C}^m \to \mathbb{C}$) ,

$$\langle P|P'\rangle = \overline{P}(\partial_Y)P'(Y)|_{Y=0}.$$

where by definition $\overline{P}(Y) := \overline{P\left(\overline{Y}\right)}$. E.g, for all positive integers $\alpha_1, \cdots, \alpha_m$, $\beta_1, \cdots, \beta_m$

$$\left\langle Y_1^{\alpha_1}.\cdots.Y_m^{\alpha_m}|Y_1^{\beta_1}.\cdots.Y_m^{\beta_m}\right\rangle = \alpha_1!.\cdots.\alpha_m!\ \delta_{\alpha_1,\beta_1}.\cdots.\delta_{\alpha_m,\beta_m}$$

where $\delta_{\alpha,\beta} = 1$ if $\alpha = \beta$, and 0 otherwise. It what follows we norm $\mathcal{H}_n$ with the associated euclidian norm $|\Phi|_2 := \sqrt{\langle\Phi|\Phi\rangle_{\mathcal{H}}}$

2.2 Exponential upper bounds for the remainder: main results

2.2.1 Main result.

We want to give an estimate on $\mathcal{R}_p(Y)$ depending on p and on the size of the ball where Y lies. Given the size of this ball, the aim is to optimize the degree p of the normal form, and show that $\mathcal{R}_p(Y)$ can be made exponentially small with respect to δ. For unperturbed vector fields, all follows from the following proposition which ensures that the exponentially estimates of the remainder follows from the estimates of the growth with respect to k of the euclidian norm of the pseudo inverse of $\mathcal{A}_L|_{\mathcal{H}_k}$.

Remark 2.1 A priori the pseudo inverse of the homological operator $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ is only defined from $(\ker \mathcal{A}_{L^*})^\perp = \mathrm{ran}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_{\mathrm{k}}}$ onto $(\ker \mathcal{A}_L|_{\mathcal{H}_k})^\perp$. From now on, we extend it on the whole space $\mathcal{H}_k$ as follows

$$\mathcal{A}_L|_{\mathcal{H}_k}\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\Phi = \Phi \quad \text{for } \Phi \in (\ker \mathcal{A}_{L^*})^\perp, \qquad \widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\Phi = 0, \quad \text{for } \Phi \in \ker \mathcal{A}_{L^*}.$$

Proposition 2.2 (Exponential estimates of the remainder) *Let V be an analytic vector field in a neighborhood of* 0 *in $\mathbb{R}^m$ (resp. in $\mathbb{C}^m$) such that $V(0) = 0$, i.e. a vector field satisfying (5) and (6). Denote*

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 = \sup_{|\Phi|_2=1} \left|\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}.\Phi\right|_2.$$

Then, if there exits $K \geq 2$, $a \geq 0$ and $\tau \geq 0$ such that $a_k(L) \leq ak^\tau$ for every k with $2 \leq k \leq K \leq +\infty$, then for every $\delta > 0$ such that $K \geq p_{\mathrm{opt}} \geq 2$ the remainder $\mathcal{R}_p$ given by the Normal Form Theorem 1.1 for $p = p_{\mathrm{opt}}$ satisfies

$$\sup_{\|Y\|\leq\delta} \|\mathcal{R}_{p_{\mathrm{opt}}}(Y)\| \leq M\ \delta^2 \mathrm{e}^{-\frac{w}{\delta^b}}$$

with

$$b=\frac{1}{1+\tau},\ p_{\mathrm{opt}}=\left[\frac{1}{\mathrm{e}(C\delta)^b}\right],\ w=\frac{1}{\mathrm{e}C^b},\ M=\tfrac{10}{9}cC^2\left\{\left(\mathfrak{m}\sqrt{\tfrac{27}{8\mathrm{e}}}\right)^{1+\tau}+(2\mathrm{e})^{2+2\tau}\right\}$$

where $C=\frac{\sqrt{m}}{\rho^2}\left\{\left(\frac{5}{2}m+2\right)ac+3\rho\right\}$, $\mathfrak{m}=\sup_{p\in\mathbb{N}}\frac{\mathrm{e}^2\ p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$ *and where for a real number* x*, we denote by* $[x]$ *its integer part.*

Remark 2.3 Stirling's formula ensures that $\mathfrak{m}$ is finite.

The proof of this proposition is performed in two main steps. We first prove that roughly speaking, $\mathcal{R}_p$ admits an upper bound of the form

$$\sup_{\|Y\|\le\delta}\|\mathcal{R}_p(Y)\|\le M(C\delta)^{p+1}(p!)^{1+\tau}.$$

where M depends on τ but not on δ nor p. Then we optimize p (see Lemma 2.19), so that $(C\delta)^{p+1}(p!)^{\tau+1}$ is exponentially small for $p=p_{\mathrm{opt}}$. In fact, the upper bound for $\mathcal{R}_p$ is a little bit more complicated (see Lemma 2.17) and we obtain it only for $(C\delta)^{\frac{1}{1+\tau}}p\le\mathrm{e}^{-1}$, which is just enough to obtain the desired exponentially small upper bound of the remainder. The detailed proof of this proposition is postponed to subsection 2.3.

Remark 2.4 The euclidian norms $a_k(L)$ of the homological operator are invariant under unitary changes of coordinates. Indeed, if Q is a unitary linear operator, let us denote $L'=Q^{-1}LQ$ and $a_k(L')=|||\widetilde{\mathcal{A}_{L'}}|_{\mathcal{H}_k}^{-1}|||_2$. Then, since $\mathcal{A}_{L'}|_{\mathcal{H}_k}=\mathcal{T}_Q\mathcal{A}_L|_{\mathcal{H}_k}\mathcal{T}_Q^{-1}$ where $(\mathcal{T}_Q\Phi)(Y)=Q^{-1}\Phi(QY)$ and since $\mathcal{T}_Q$ is unitary when Q is unitary (see Appendix A.3), we get that $a_k(L')=a_k(L)$ for every $k\ge 1$.

2.2.2 The semi-simple case: proof of Theorem 1.4-(a)

Theorem 1.4-(a) directly follows from proposition 2.2 and from the following lemma

Lemma 2.5 *Let* L *be a linear operator in* $\mathbb{R}^m$ *or* $\mathbb{C}^m$*.*

(a) *Denote by* $\sigma(L):=\{\lambda_1,\cdots,\lambda_m\}$ *the spectrum of* L*. Then, for every* $k\ge 2$ *the spectrum* $\sigma(\mathcal{A}_L|_{\mathcal{H}_k})$ *of* $\mathcal{A}_L|_{\mathcal{H}_k}$ *is given by*

$$\sigma(\mathcal{A}_L|_{\mathcal{H}_k}):=\{\Lambda_{j,\alpha}:=\langle\lambda_L,\alpha\rangle-\lambda_j,\quad 1\le j\le m,\ \alpha\in\mathbb{N}^m,|\alpha|=k\}.\quad(18)$$

Moreover, $\mathcal{A}_L|_{\mathcal{H}_k}$ *is semi simple if an only if* L *is so.*

(b) *If L is semi-simple and is under real or complex Jordan normal form, then for every $k \geq 2$,*

$$a_k(L) := |||\widetilde{\mathcal{A}_{_L}}|_{\mathcal{H}_k}^{-1}|||_2 \leq \max_{\substack{1\leq j\leq m, |\alpha|=k \\ \Lambda_{j,\alpha}\neq 0}} |\Lambda_{j,\alpha}|^{-1}.$$

Remark 2.6 When L is semi simple, under Jordan normal form, and γ, K-homologically without small divisors, the above lemma ensures that $a_k(L) \leq \gamma^{-1}$ for $2 \leq k \leq K$ and if L is γ,τ-homologically Diophantine, then $a_k(L) \leq \gamma^{-1}\ k^{\tau}$ for $k \geq 2$.

Proof of Lemma 2.5. (a): Although this result is classical (see [7]), we give its short proof for self-containedness of the paper. Let Q be an invertible matrix such that $J = Q^{-1}LQ$ is under complex Jordan normal form and observe that $\mathcal{A}_{_L}|_{\mathcal{H}_k} = \mathcal{T}_{_Q}^{-1}\mathcal{A}_{_J}|_{\mathcal{H}_k}\mathcal{T}_{_Q}$ where $(\mathcal{T}_{_Q}\Phi)(Y) = Q^{-1}\Phi(QY)$. Hence the spectrum of $\mathcal{A}_{_L}|_{\mathcal{H}_k}$ is equal to the spectrum of $\mathcal{A}_{_J}|_{\mathcal{H}_k}$. Let $\{c_j\}_{1\leq j\leq m}$ be the canonical basis of $\mathbb{C}^m$. Then, since J is under Jordan normal form, we have $Jc_j = \lambda_j c_j + \delta_{j-1}c_{j-1}$ with $\delta_0 = 0$ and where $\delta_{j-1} = 0$ if $\lambda_j \neq \lambda_{j-1}$ and $\delta_{j-1} = 0$ or 1 otherwise. Let $\{P_{j,\alpha}\}_{1\leq j\leq m, \alpha\in\mathbb{N}^m, |\alpha|=k}$ be the basis of $\mathcal{H}_k$ given by

$$P_{j,\alpha}(Y) := Y_1^{\alpha_1}.\cdots.Y_m^{\alpha_m}c_j$$

we order this basis with the lexicographical order, i.e. $P_{j,\alpha} < P_{\ell,\beta}$ if the first non zero integer $\ell - j, \beta_1 - \alpha_1.\cdots.\beta_m - \alpha_m$ is positive. Within this order, $\mathcal{A}_{_J}$ is upper triangular and

$$\mathcal{A}_{_J}P_{j,\alpha} = (\langle\lambda_L, \alpha\rangle - \lambda_j)P_{j,\alpha} + \sum_{\ell=1}^{m}\alpha_\ell\delta_\ell P_{j,\alpha-\sigma_\ell+\sigma_{\ell+1}} - \delta_{j-1}P_{j-1,\alpha} \tag{19}$$

with $\sigma_\ell = (0,\cdots,0,1,\cdots,0)$ where the coefficient 1 is at the ℓ-th position. Hence the spectrum of $\mathcal{A}_{_J}|_{\mathcal{H}_k}$ and thus the spectrum of $\mathcal{A}_{_L}|_{\mathcal{H}_k}$ is given by (18). Formula (19) also ensures that $\mathcal{A}_{_J}$ is semi simple if and only if J is so.

(b) : We proceed in two steps.

Step 1. First assume that L is semi-simple and is under complex Jordan normal form i.e. assume that $L = J$ is diagonal. Then $\delta_j = 0$ for $1 \leq j \leq m$. Thus, by (19), $\mathcal{A}_{_L}|_{\mathcal{H}_k}$ is also semi simple and $\{P_{j,\alpha}\}_{1\leq j\leq m,\alpha\in\mathbb{N}^m,|\alpha|=k}$ is a basis of eigenvectors of $\mathcal{A}_{_L}|_{\mathcal{H}_k}$. For $\Phi \in \mathcal{H}_k$, let us denote

$$\Phi = \widehat{\Phi} + \overset{\vee}{\Phi}, \quad \overset{\vee}{\Phi} = \pi_k\Phi \in \ker(\mathcal{A}_{_{L^*}}|_{\mathcal{H}_k}), \quad \widehat{\Phi} = \sum_{\substack{1\leq j\leq m, |\alpha|=k \\ \Lambda_{j,\alpha}\neq 0}} \widehat{\Phi}_{j,\alpha}P_{j,\alpha} \in \mathrm{ran}(\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_{\mathrm{k}}}),$$

and $\mathsf{M} = \max\limits_{\substack{1\le j\le m, |\alpha|=k \\ \Lambda_{j,\alpha}\neq 0}} |\Lambda_{j,\alpha}|^{-1}$. Then since $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\check{\Phi} = 0$ and $\langle P_{j,\alpha}|P_{\ell,\beta}\rangle_{\mathcal{H}} = 0$ for $(j,\alpha) \neq (\ell,\beta)$ we have

$$
\begin{aligned}
\left|\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\Phi\right|_2^2 &= \sum_{\substack{1\le j\le m, |\alpha|=k \\ \Lambda_{j,\alpha}\neq 0}} |\Lambda_{j,\alpha}|^{-2}\ |\widehat{\Phi}_{j,\alpha}|^2\ |P_{j,\alpha}|_2^2\,, \\
&\le \mathsf{M}^2 \sum_{\substack{1\le j\le m, |\alpha|=k \\ \Lambda_{j,\alpha}\neq 0}} |\widehat{\Phi}_{j,\alpha}|^2\ |P_{j,\alpha}|_2^2\,, \\
&= \mathsf{M}^2 \left|\widehat{\Phi}\right|_2^2.
\end{aligned}
$$

Finally, since $\langle\check{\Phi}|\widehat{\Phi}\rangle_{\mathcal{H}} = 0$,

$$
\left|\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\Phi\right|_2 \le \mathsf{M}\,|\Phi|_2\,. \tag{20}
$$

Step 2. If L is real semi simple and is under real Jordan normal form then it is conjugated to its complex Jordan normal form by a unitary matrix since

$$
\begin{pmatrix} x+\mathrm{i}y & 0 \\ 0 & x-\mathrm{i}y \end{pmatrix} = Q^{-1}\begin{pmatrix} x & -y \\ y & x \end{pmatrix} Q, \qquad \text{with} \qquad Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\mathrm{i}\sqrt{2}} & \frac{-1}{\mathrm{i}\sqrt{2}} \end{pmatrix}.
$$

Then, remark 2.4 and the previous step ensures that (20) still holds when L is real, semi simple and under real Jordan normal form. $\square$

2.2.3 The non semi-simple case: proof of Theorem 1.4-(b)

Theorem 1.4-(b) directly follows from proposition 2.2 and from the following lemma

Lemma 2.7 *For* $L = \mathbf{0}^2\underbrace{\mathbf{0}\cdots\mathbf{0}}_{q\ times}$, $L = \mathbf{0}^3\underbrace{\mathbf{0}\cdots\mathbf{0}}_{q\ times}$, $a_k(L)$ *satisfies*

$$
a_k(L) \le a, \qquad \textit{for every } k \ge 1
$$

with $a = 1$.

The detailed proof of this lemma is postponed to subsection 2.4. (see Lemmas 2.22 and 2.24). For non semi-simple operators L the direct computation of the norm of $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ is in general quite intricate. So, in subsection 2.4, the computation of $a_k(L)$ for $L = \mathbf{0}^j\underbrace{\mathbf{0}\cdots\mathbf{0}}_{q\text{ times}}$, $j = 2,3$ is performed via the following lemma which gives this norm in terms of the spectrum of the self adjoint operator $(\mathcal{A}_L|_{\mathcal{H}_k})^*\mathcal{A}_L|_{\mathcal{H}_k} = \mathcal{A}_{L^*}|_{\mathcal{H}_k}\mathcal{A}_L|_{\mathcal{H}_k}$ which appears easier to handle.

Lemma 2.8 *For every linear operator L in $\mathbb{R}^m$ or $\mathbb{C}^m$ and every $k \geq 1$, let us denote by $\Sigma_k(L) \subset \mathbb{R}^+$ the spectrum of the positive self adjoint operator $(\mathcal{A}_L|_{\mathcal{H}_k})^* \mathcal{A}_L|_{\mathcal{H}_k} = \mathcal{A}_{L^*}|_{\mathcal{H}_k} \mathcal{A}_L|_{\mathcal{H}_k}$. Then,*

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 = \left(\min_{\sigma \in \Sigma_k(L) \backslash \{0\}} |\sigma| \right)^{-\frac{1}{2}}.$$

Proof. Observe that

$$a_k = \sup_{\Phi \in \mathcal{H}_k \backslash \{0\}} \frac{\left| \widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1} \Phi \right|_2}{|\Phi|_2} = \sup_{\Psi \in (\ker \mathcal{A}_L|_{\mathcal{H}_k})^\perp} \frac{|\Psi|_2}{\left| \mathcal{A}_L|_{\mathcal{H}_k} \Psi \right|_2} = \left(\inf_{\Psi \in (\ker \mathcal{A}_L|_{\mathcal{H}_k})^\perp} \frac{\left\langle \mathcal{A}_{L^*}|_{\mathcal{H}_k} \mathcal{A}_L|_{\mathcal{H}_k} \Psi | \Psi \right\rangle}{|\Psi|_2^2} \right)^{-\frac{1}{2}}.$$

Then, since $\ker \mathcal{A}_L|_{\mathcal{H}_k} = \ker \mathcal{A}_{L^*}|_{\mathcal{H}_k} \mathcal{A}_L|_{\mathcal{H}_k}$ and since $\mathcal{A}_{L^*}|_{\mathcal{H}_k} \mathcal{A}_L|_{\mathcal{H}_k}$ is a positive self adjoint operator, we get $a_k(L) := \left(\min_{\sigma \in \Sigma_k(L) \backslash \{0\}} |\sigma| \right)^{-\frac{1}{2}}$.

□

Remark 2.9 For $L = \mathbf{0^2 i\omega}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$, the above strategy leads to an estimate of $a_k(L)$ of the form

$$a_k(L) \leq a(Ck)^{k-1}$$

which is far too large to get an exponential estimate of the remainder. So, at the present time, we do not know whether an estimate of the form (7) is still true for $L = \mathbf{0^2 i\omega}|_{\mathbb{R}\,\mathrm{or}\,\mathbb{C}}$.

2.3 Exponentially small estimates of the remainder for polynomially bounded pseudo inverse of the homological operator.

This subsection is devoted to the proof of proposition 2.2. To fix the notations we make the proof for vector fields in $\mathbb{R}^m$. The proof is the same for $\mathbb{C}^m$. So, let V be an analytic vector field in a neighborhood of 0 in $\mathbb{R}^m$ such that $V(0) = 0$, i.e. a vector field satisfying (5) and (6). We assume that the pseudo inverse of the homological operator is polynomially bounded on $\mathcal{H}_k$ for $2 \leq k \leq K \leq +\infty$, i.e we assume that there exists $a > 0$ and $\tau \geq 0$ such that

$$a_k = \left| \widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1} . \Phi \right|_2 \leq a k^\tau \qquad \text{for } 2 \leq k \leq K.$$

Our aim is to find an exponential upper bound of the remainder $\mathcal{R}_p(Y)$ for Y in a ball of radius δ. Since the remainder $\mathcal{R}_p(Y)$ is given by equation (14), for estimating it, we successively compute upper bounds for $\Phi_n(Y)$, $N_n(Y)$,

$\sum\limits_{2\leq k\leq p} D\Phi_k(Y)$, $\sum\limits_{1\leq k\leq p} \Phi_k(Y)$ and finally for $\mathcal{R}_p(Y)$. For the polynomials N_n and Φ_n the natural norm to finally compute an upper bound of $\sup\limits_{\|Y\|\leq\delta} \|\mathcal{R}_p(Y)\|$ is the "sup-norm" defined for any $\Phi \in \mathcal{H}_n$ by

$$|\Phi|_{0,n} = \sup_{Y\in\mathbb{C}^m} \frac{\|\Phi(Y)\|}{\|Y\|^n}.$$

However, N_n and Φ_n are the solutions of the Homological Equation (15) given by (17), i.e. defined via the orthogonal projector π_n which has nice properties for the euclidian norm and not for the sup norm. These two norm can be compared has follows :

Lemma 2.10 (Comparison of the euclidian and the sup norm)

For every $\Phi \in \mathcal{H}_k$,

$$|\Phi|_{0,k} \leq \frac{1}{\sqrt{k!}} |\Phi|_2 \leq \sqrt{C_{k+m-1}^{m-1}} \; |\Phi|_{0,k} \leq \sqrt{m}\; k^{\frac{m-1}{2}} \; |\Phi|_{0,k}\,.$$

where $C_n^r = \dfrac{n!}{r!(n-r)!}$.

The proof of this Lemma is given in Appendix A (see Lemmas A.3 and A.5). Moreover if we normalize the euclidian norm on $\mathcal{H}_n$ by defining

$$|\Phi|_{2,n} := \frac{1}{\sqrt{n!}} |\Phi|_2\,, \qquad \text{for every } \Phi \in \mathcal{H}_n,$$

then the normalized euclidian norm has very nice properties with respect to multiplication and derivation :

Lemma 2.11 (Multiplicativity of the normalized euclidian norm)

(a) *Let q and $\{p_\ell\}_{1\leq\ell\leq q}$ be positive integers and let $R_q \in \mathcal{L}_q(\mathbb{R}^m)$ be q-linear. Then for every $\Phi_{p_\ell} \in \mathcal{H}_{p_\ell}$, $1 \leq \ell \leq q$, the polynomial $R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}]$ lies in $\mathcal{H}_n$ with $n = p_1 + \cdots + p_q$ and*

$$\left|R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}]\right|_{2,n} \leq |||R_q|||_{\mathcal{L}_q(\mathbb{R}^m)} |\Phi_{p_1}|_{2,p_1} \cdots \left|\Phi_{p_q}\right|_{2,p_q}.$$

(b) *Let $k > 0$ and $p \geq 0$ be two integers and let Φ_k, N_p lie respectively in $\mathcal{H}_k$ and $\mathcal{H}_p$. Then $D\Phi_k.N_p$ lies in $\mathcal{H}_n$ with $n = k-1+p$ and*

$$|D\Phi_k.N_p|_{2,n} \leq \sqrt{k^2 + (m-1)k} \; |\Phi_k|_{2,k} |N_p|_{2,p}$$

This Lemma is also proved in Appendix A (see lemmas A.8 and A.9).

Hence to compute by induction upper bounds of Φ_n, N_n defined via π_n, we use the normalized euclidian norms

$$\nu_n = |N_n|_{2,n}\,, \quad \text{for } n \geq 2,$$
$$\phi_n = |\Phi_n|_{2,n}\,, \quad \text{for } n \geq 1,$$

with the convention $\Phi_1(Y) = Y$ and thus $\phi_1 = |Y|_{2,1} = \sqrt{m}$. Lemma 2.10 ensures that the same upper bounds will also hold for the sup norms of N_n, Φ_n. Since π_n is orthogonal, we deduce from (17) that

$$\nu_n = |N_n|_{2,n} = |\pi_n(F_n)|_{2,n} = \tfrac{1}{\sqrt{n!}}\,|\pi_n(F_n)|_2 \leq \tfrac{1}{\sqrt{n!}}\,|F_n|_2 = |F_n|_{2,n}$$

and similarly

$$\phi_n \leq |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_n}^{-1}|||_2 \;\; |F_n|_{2,n} \leq an^{\tau}\,|F_n|_{2,n}\,.$$

Hence using the multipicativity and the derivation properties of the normalized euclidian norms, we get that

$$\nu_n \leq \sum_{2\leq k\leq n-1} \Big(k^2 + (m-1)k\Big)^{\frac{1}{2}}\,\phi_k \nu_{n-k+1} + \sum_{2\leq q\leq n}\;\sum_{p_1+\cdots+p_q=n} c\,\frac{\phi_{p_1}\cdots\phi_{p_q}}{\rho^q}, \qquad (21)$$

$$\phi_n \leq an^{\tau} \sum_{2\leq k\leq n-1} \Big(k^2+(m-1)k\Big)^{\frac{1}{2}}\,\phi_k \nu_{n-k+1} + an^{\tau} \sum_{2\leq q\leq n}\;\sum_{p_1+\cdots+p_q=n} c\,\frac{\phi_{p_1}\cdots\phi_{p_q}}{\rho^q} \qquad (22)$$

for $2 \leq n \leq K$ with the convention $\phi_1 = |\Phi_1|_{2,1} = |Y|_{2,1} = \sqrt{m}$. Hence using that $(k^2 + (m-1)k)^{\frac{1}{2}} \leq \sqrt{m}k$, we check by induction that

Lemma 2.12 *Let $\{\beta_n\}_{n\geq 1}$ be the sequence defined by induction*

$$\beta_n = m \sum_{2\leq k\leq n-1} k\beta_k\beta_{n-k+1} + \sum_{2\leq q\leq n}\;\sum_{p_1+\cdots+p_q=n} \left(\frac{\rho}{ac}\right)^{q-2} \beta_{p_1}\cdots\beta_{p_q}, \quad n \geq 2, \qquad (23)$$
$$\beta_1 = 1.$$

Then we have the estimates

$$\nu_n \leq \frac{\sqrt{m}}{a}\left(\frac{ac\sqrt{m}}{\rho^2}\right)^{n-1} \Big((n-1)!\Big)^{\tau}\beta_n, \quad \textit{for } 2 \leq n \leq K, \qquad (24)$$

$$\phi_n \leq \sqrt{m}\,\left(\frac{ac\sqrt{m}}{\rho^2}\right)^{n-1} (n!)^{\tau}\beta_n, \quad \textit{for } 1 \leq n \leq K. \qquad (25)$$

Proof. We proceed by induction. For $n = 1$, the above inequality is true since $\phi_1 = \sqrt{m}$. For $n = 2$, equation (23) ensures that $\beta_2 = 1$ and (21), (22) ensure that $\nu_2 \leq cm\rho^{-2}$ and $\phi_2 \leq acm2^{\tau}\rho^{-2}$, and thus (25), (24) are true for $n = 2$. Assume now that (24), (25) holds for $k < n$ with $n \geq 3$. Then (21) ensures that

$$\nu_n \leq \frac{\sqrt{m}}{a}\left(\frac{ac\sqrt{m}}{\rho^2}\right)^{n-1}\Big((n-1)!\Big)^{\tau}\left(m\sum_{2\leq k\leq n-1} k\beta_k\beta_{n-k+1}\,(D'_{n,k})^{\tau}\right.$$
$$\left.+\sum_{2\leq q\leq n}\sum_{p_1+\cdots+p_q=n}\left(\frac{\rho}{ac}\right)^{q-2}\beta_{p_1}\cdots\beta_{p_q}(D_{n,p_1,\cdots,p_q})^{\tau}\right)$$

where

$$D'_{n,k}=\frac{k!(n-k)!}{(n-1)!}\qquad\text{and}\qquad D_{n,p_1,\cdots,p_q}=\frac{p_1!\cdots p_q!}{(n-1)!}.$$

It remains to prove that $D'_{n,k}\leq 1$ for $2\leq k\leq n-1$ and that $D_{n,p_1,\cdots,p_q}\leq 1$ for $2\leq q\leq n,\ p_1+\cdots+p_q=n,\ p_j\geq 1$, to ensures that (24) holds for n and similarly that (25) holds also for n.

Denoting $C_n^k=\frac{n!}{k!(n-k)!}$ and observing that $C_n^k\geq n$ for $1\leq k\leq n-1$, we get

$$D'_{n,k}=\frac{n}{C_n^k}\leq 1.$$

Finally to prove that $D_{n,p_1,\cdots,p_q}\leq 1$ we proceed by induction on q. For $q=2$, we have

$$D_{n,p_1,p_2}=\frac{p_1!(n-p_1)!}{(n-1)!}=D'_{n,p_1}\leq 1$$

since $1\leq p_1\leq n-1$. Assume now that $D_{n,p_1,\cdots,p_q}\leq 1$ for $q\geq 2$ and every $n\geq q$, then

$$\begin{aligned}D_{n,p_1,\cdots,p_{q+1}}&=D_{p_1+\cdots+p_q,p_1,\cdots,p_q}\,(p_1+\cdots+p_q-1)!\,\frac{p_{q+1}!}{(n-1)!},\\&=D_{p_1+\cdots+p_q,p_1,\cdots,p_q}\,\frac{1}{C_{n-1}^{p_{q+1}}}\leq 1,\end{aligned}$$

since for every $r\in\mathbb{N}$ and j with $0\leq j\leq r$, we have $C_r^j\geq 1$. This completes the proof of Lemma 2.12. $\square$

The study of the sequence $\{\beta_n\}_{n\geq 1}$ enables to obtain Gevrey estimates for ϕ_n,ν_n.

Lemma 2.13 *In choosing $\alpha_1=1$ and*

$$\alpha_n=\Theta^{n-2}(n-2)!,\qquad \textit{for } n\geq 2,$$

and Θ large enough such that

$$ac\Theta>\rho, \tag{26}$$

and

$$\frac{\frac{5}{2}m+2}{\Theta}+\frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}}\leq 1, \tag{27}$$

then β_n in (23) satisfies $\beta_n \leq \alpha_n$ for $n \geq 1$ and thus

$$\phi_n \leq \frac{acm}{\rho^2}\left(\frac{ac\sqrt{m}\ \Theta}{\rho^2}\right)^{n-2} (n!)^\tau\ (n-2)!, \quad \textit{for } 2 \leq n \leq K, \ \textit{and } \phi_1 = \sqrt{m},$$
$$\nu_n \leq \frac{cm}{\rho^2}\left(\frac{ac\sqrt{m}\ \Theta}{\rho^2}\right)^{n-2} \Big((n-1)!\Big)^\tau\ (n-2)!, \qquad \textit{for } 2 \leq n \leq K.$$

Proof. We proceed by induction. We have $\beta_1 = 1 = \alpha_1 \leq \alpha_1$ and $\beta_2 = 1 = \alpha_2 \leq \alpha_2$. Assume now that $\beta_k \leq \alpha_k$ for $k < n$ and $n \geq 3$.

Step 1. Splitting of the bounds. Then by induction hypothesis,

$$\beta_n \leq \Delta_n^1 + \Delta_n^2 \tag{28}$$

with

$$\Delta_n^1 = m \sum_{2\leq k\leq n-1} k\ \alpha_k\ \alpha_{n-k+1} + \sum_{1\leq k\leq n-1} \alpha_k\ \alpha_{n-k},$$
$$\Delta_n^2 = \sum_{3\leq q\leq n}\ \sum_{p_1+\ldots+p_q=n} \left(\frac{\rho}{ac}\right)^{q-2} \alpha_{p_1}\ldots\ldots\alpha_{p_q}.$$

Step 2. Two auxiliary sums for $\boldsymbol{\Delta_n^1}$.

Step 2.1 Upper bound for $\boldsymbol{S_n}$. Let us we define

$$S_n = \sum_{2\leq k\leq n-1} \frac{k(k-2)!(n-k-1)!}{(n-2)!}.$$

Explicit computations show that $S_3 = 2,\ S_4 = S_5 = \frac{5}{2}$. Hence, $S_n \leq 5/2$ for $3 \leq n \leq 5$. To prove that it also holds for $n \geq 5$, observe that for $n \geq 5$,

$$\begin{aligned} S_{n+1} - S_n &= \sum_{2\leq k\leq n} \frac{k(k-2)!(n-k)!}{(n-1)!} - \sum_{2\leq k\leq n-1} \frac{k(k-2)!(n-k-1)!}{(n-2)!}, \\ &= \sum_{3\leq k\leq n-2} k(k-2)!\left(\frac{(n-k)!}{(n-1)!} - \frac{(n-k-1)!}{(n-2)!}\right) + \frac{n+2}{n-1} + \frac{1}{n-2} - \frac{n+1}{n-2}, \\ &= \sum_{3\leq k\leq n-2} k(k-2)!\left(\frac{1}{(n-1)\ldots(n-k+1)} - \frac{1}{(n-2)\ldots(n-k)}\right) + \frac{n-4}{(n-1)(n-2)}, \\ &= -\sum_{3\leq k\leq n-2} \frac{k!}{(n-1)\ldots(n-k)} + \frac{n-4}{(n-1)(n-2)}, \\ &= -\sum_{3\leq k\leq n-3} \frac{k!}{(n-1)\ldots(n-k)} - \frac{(n-2)!}{(n-1)!} + \frac{n-4}{(n-1)(n-2)}, \\ &= -\sum_{3\leq k\leq n-3} \frac{k!}{(n-1)\ldots(n-k)} - \frac{2}{(n-1)(n-2)} \leq 0 \end{aligned}$$

where, here again, by convention, the sums corresponding to an empty index set are equal to 0.

Hence $S_{n+1} \leq S_n$ for $n \geq 5$, and we can conclude

$$S_n \leq 5/2, \text{ for } n \geq 3. \tag{29}$$

Step 2.2 Upper bound for P_n. We now define

$$P_n = \sum_{2 \leq k \leq n-2} \frac{(k-2)!(n-k-2)!}{(n-2)!}.$$

Observe that $P_4 = \frac{1}{2}$ and that for $n \geq 4$,

$$\begin{aligned}
P_{n+1} - P_n &= \sum_{2 \leq k \leq n-1} \frac{(k-2)!(n-k-1)!}{(n-1)!} - \sum_{2 \leq k \leq n-2} \frac{k(k-2)!(n-k-2)!}{(n-2)!}, \\
&= \sum_{2 \leq k \leq n-2} (k-2)! \left(\frac{1}{(n-1)...(n-k)} - \frac{1}{(n-2)...(n-k-1)} \right) + \frac{1}{(n-1)(n-2)}, \\
&= - \sum_{2 \leq k \leq n-2} \frac{k(k-2)!}{(n-1)...(n-k-1)} + \frac{1}{(n-1)(n-2)}, \\
&= - \sum_{2 \leq k \leq n-3} \frac{k(k-2)!}{(n-1)...(n-k-1)} - \frac{(n-2)(n-4)!}{(n-1)!} + \frac{1}{(n-1)(n-2)}, \\
&= - \sum_{2 \leq k \leq n-3} \frac{k(k-2)!}{(n-1)...(n-k-1)} - \frac{1}{(n-1)(n-2)(n-3)} \leq 0.
\end{aligned}$$

Hence, $P_{n+1} \leq P_n,$ for $n \geq 4$ and we can conclude

$$P_n \leq 1/2, \quad \text{for } n \geq 4. \tag{30}$$

Step 3. Upper bound for $\boldsymbol{\Delta^1_n}$. It results from (29) and (30) that for $\Theta \geq 1$,

$$\Delta^1_n \leq \frac{\frac{5}{2}m + 2}{\Theta} \alpha_n, \quad n \geq 3, \tag{31}$$

where the proof of this inequality is direct for $n = 3$.

Step 4. Auxiliary sums for $\boldsymbol{\Delta^2_n}$. Now, we define for $n \geq q \geq 2$:

$$\Pi_{q,n} = \sum_{p_1 + ... + p_q = n} \alpha_{p_1} .... \alpha_{p_q},$$

then we already have

$$\Pi_{n,n} = 1 \leq \frac{1}{\Theta^{n-2}}\alpha_n, \quad n \geq 3,$$
$$\Pi_{2,2} = 1,$$
$$\Pi_{2,n} \leq \frac{2}{\Theta}\alpha_n, \quad n \geq 3,$$

where the last inequality comes easily from the inequality for P_n. For estimating $\Pi_{q,n}$ with $n \geq q+1$, we proceed as follows

$$\Pi_{q,n} = \sum_{1\leq k\leq n-q+1} \alpha_k \Pi_{q-1,n-k} = \Pi_{q-1,n-1} + \alpha_{n-q+1} + \sum_{2\leq k\leq n-q} \alpha_k \Pi_{q-1,n-k}$$

and prove by induction that

$$\Pi_{q,n} \leq \frac{2}{\Theta^{q-1}}\alpha_n, \quad n \geq q+1 \geq 3.$$

Finally, gathering all our results, we get

$$\Pi_{q,n} \leq \frac{2}{\Theta^{q-2}}\alpha_n, \quad n \geq q \geq 3, \tag{32}$$

Step 5. Upper bound for $\boldsymbol{\Delta_n^2}$. We deduce from (32) that

$$\Delta_2^n = \sum_{3\leq q\leq n} \left(\frac{\rho}{ac}\right)^{q-2} \Pi_{q,n} \leq \sum_{3\leq q\leq n} 2\left(\frac{\rho}{ac\Theta}\right)^{q-2} \alpha_n \leq \alpha_n \left\{\frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}}\right\}, \tag{33}$$

provided that $\frac{\rho}{ac\Theta} < 1$.

Step 6. Upper bound for $\boldsymbol{\beta_n}$. Hence, (31) and (33) ensure that

$$\beta_n \leq \left\{\frac{\frac{5}{2}m+2}{\Theta} + \frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}}\right\}\alpha_n \leq \alpha_n$$

provided that $\dfrac{\rho}{ac\Theta} < 1$ and $(\frac{5}{2}m+2)\dfrac{1}{\Theta} + \dfrac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}} \leq 1.$

□

In all what follows we choose

$$\Theta = \frac{5}{2}m + 2 + \frac{3\rho}{ac} \tag{34}$$

which ensures that (26) and (27) are simultaneously satisfied since with this choice

$$\frac{\rho}{ac\Theta} < \frac{1}{3}, \quad \text{and} \quad (\frac{5}{2}m+2)\,\frac{1}{\Theta} + \frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}} \le \frac{(\frac{5}{2}m+2)\,\frac{1}{\Theta}}{1-\frac{\rho}{ac\Theta}} + \frac{2\frac{\rho}{ac\Theta}}{1-\frac{\rho}{ac\Theta}} = 1.$$

We can now compute an upper bound for the change of coordinates and for its differential.

Lemma 2.14 *For every $\delta > 0$ and every p, $2 \le p \le K$ satisfying*

$$\delta p^{1+\tau} \le \frac{\rho^2}{2ac\sqrt{m}\;\Theta}. \tag{35}$$

we have

$$\|\sum_{1\le k\le p} \Phi_k(Y)\| \qquad \le \frac{10}{9}\sqrt{m}\;\delta, \tag{36}$$

$$|||\sum_{2\le k\le p} D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)} \le 2/5. \tag{37}$$

for every $Y \in \mathbb{R}^m$ with $||Y|| \le \delta$.

Remark 2.15 Observe that the size δ of the ball where Y lies and the degree p of the normal form, i.e. the degree of the polynomial change of variable are now mutually constrained by (35).

Proof. We proceed in three steps.

Step 1. Upper bound for $\|\sum_{1\le k\le p} \boldsymbol{\Phi_k(Y)}\|$. Lemmas 2.10, 2.13 ensure that

$$\begin{aligned}
\|\sum_{1\le k\le p} \Phi_k(Y)\| &\le \sum_{1\le k\le p} |\Phi_k|_{0,k}\,\|Y\|^k,\\
&\le \sum_{1\le k\le p} |\Phi_k|_{2,k}\,\|Y\|^k,\\
&\le \sum_{1\le k\le p} \phi_k\delta^k,\\
&\le \delta\sqrt{m}\; + \sum_{2\le k\le p} \frac{acm\;\delta^2}{\rho^2}\left(\frac{ac\sqrt{m}\Theta\;\delta}{\rho^2}\right)^{k-2} (k!)^\tau\;(k-2)!\\
&\le \delta\sqrt{m}\left\{1+\frac{1}{\Theta}\sum_{2\le k\le p}\left(\frac{1}{2p^{1+\tau}}\right)^{k-1}(k!)^\tau\;(k-2)!\right\},\\
&\le \delta\sqrt{m}\left\{1+\frac{1}{\Theta\;p}\sum_{2\le k\le p}\left(\frac{1}{2}\right)^{k-1}\right\},
\end{aligned}$$

since for $2 \leq k \leq p$,

$$\frac{(k-2)!}{p^{k-1}} \leq \frac{1}{p}, \quad \text{and} \quad \frac{k!}{p^{k-1}} = \frac{2}{p}.\cdots.\frac{k}{p} \leq 1. \tag{38}$$

Hence,

$$\| \sum_{1 \leq k \leq p} \Phi_k(Y) \| \leq \delta \sqrt{m} \left\{ 1 + \frac{1}{p\Theta} \right\} \leq \frac{10}{9} \sqrt{m}\, \delta,$$

since $\Theta \geq \frac{5}{2} m + 2 \geq \frac{9}{2}$ and $p \geq 2$.

Step 2. Upper bound for $|||D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)}$. For $Y, Z \in \mathbb{R}^m$ seeing Z as an homogeneous polynomial of degree 0, Lemmas 2.10, 2.11 ensure that

$$\begin{aligned}
\frac{\|D\Phi_k(Y).Z\|}{\|Y\|^{k-1}} &\leq |D\Phi_k(Y).Z|_{0,k}\,, \\
&\leq |D\Phi_k(Y).Z|_{2,k}\,, \\
&\leq \sqrt{k^2 + (m-1)k}\ \ |\Phi_k|_{2,k}\, |Z|_{2,0}\,, \\
&= \sqrt{k^2 + (m-1)k}\ \phi_k\ \|Z\|.
\end{aligned}$$

Hence using that $\sqrt{k^2 + (m-1)k} \leq \sqrt{m}k$ we obtain

$$|||D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)} \leq \sqrt{m} k \phi_k \|Y\|^{k-1}.$$

Step 3. Upper bound for $|||\sum_{2 \leq k \leq p} D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)}$. Lemma 2.13, the previous step and estimate (38) ensure that for $\|Y\| \leq \delta$, with δ, p satisfying (35) we have

$$\begin{aligned}
||| \sum_{2 \leq k \leq p} D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)} &\leq \frac{m}{\Theta} \sum_{2 \leq k \leq p} \left(\frac{ac\sqrt{m}\Theta\delta}{\rho^2} \right)^{k-1} k\ (k!)^\tau\ (k-2)!, \\
&\leq \frac{m}{\Theta} \sum_{2 \leq k \leq p} \left(\frac{1}{2p^{1+\tau}} \right)^{k-1} k\ (k!)^\tau\ (k-2)!, \\
&\leq \frac{m}{\Theta} \sum_{2 \leq k \leq p} \left(\frac{1}{2} \right)^{k-1}, \\
&\leq \frac{m}{\Theta}, \\
&\leq \frac{2}{5},
\end{aligned}$$

since $\Theta \geq \frac{5}{2} m$. $\square$

We have now enough material to compute an upper bound of the remainder.We first prove

Lemma 2.16 *For every $\delta > 0$, every p, $2 \leq p \leq K$ satisfying (35) and for every $Y \in \mathbb{R}^m$ with $\|Y\| \leq \delta$, we have*

$$\|\mathcal{R}_p(Y)\| \leq \frac{5}{3}\left(\Delta_p^1 + \Delta_p^2 + \Delta_p^3\right) \tag{39}$$

where

$$\begin{aligned}
\Delta_p^1 &= \sum_{\substack{2\leq k\leq p\\ p+1\leq n\leq p+k-1}} \sqrt{m}\; k\phi_k\nu_{n-k+1}\delta^n,\\
\Delta_p^2 &= \sum_{\substack{2\leq q\leq p\\ p+1\leq n=p_1+\cdots+p_q,\\ 1\leq p_j\leq p}} \frac{c\delta^n}{\rho^q}\phi_{p_1}\cdots\phi_{p_q},\\
\Delta_p^3 &= \sum_{p+1\leq q}\frac{c}{\rho^q}(\frac{10}{9}\sqrt{m}\;\delta)^q.
\end{aligned} \tag{40}$$

Proof. The remainder $\mathcal{R}_p(Y)$ is given by equation (14) where it gathers all the terms of order larger than p. To bound it, we proceed in several steps.

Step 1. Explicit formula for the remainder $\mathcal{R}_p$. Identifying the powers of Y in (14), we get that the remainder $\mathcal{R}_p$ is explicitely given by

$$\mathfrak{L}_p\;\mathcal{R}_p(Y) = \mathfrak{N}_p^1 + \mathfrak{N}_p^2 + \mathfrak{N}_p^3 \tag{41}$$

with

$$\begin{aligned}
\mathfrak{L}_p \quad &= \mathrm{Id} + \sum_{2\leq k\leq p} D\Phi_k(Y),\\
\mathfrak{N}_p^1(Y) &= \sum_{\substack{2\leq k\leq p, 2\leq k'\leq p\\ k-1+k'\geq p+1}} D\Phi_k(Y).N_{k'}(Y),\\
\mathfrak{N}_p^2(Y) &= \sum_{\substack{2\leq q\leq p,\; 1\leq p_j\leq p\\ p_1+\cdots+p_q\geq p+1}} V_q\left[\Phi_{p_1}(Y),\cdots,\Phi_{p_q}(Y)\right],\\
\mathfrak{N}_p^3(Y) &= \sum_{q\geq p+1} V_q\left[\{\sum_{1\leq k\leq p}\Phi_k(Y)\}^{(q)}\right].
\end{aligned}$$

Step 2. Upper bound for $|||\mathfrak{L}_p^{-1}|||_{\mathcal{L}(\mathbb{R}^m)}$. Since lemma 2.14 ensures that

$$|||\sum_{2\leq k\leq p} D\Phi_k(Y)|||_{\mathcal{L}(\mathbb{R}^m)} \leq 2/5 < 1$$

we get that $\mathfrak{L}_p$ *is invertible and for every* $\delta > 0$*, every* $Y \in \mathbb{R}^m$ *with* $||Y|| \leq \delta$ *and every* p*,* $2 \leq p \leq K$ *satisfying* (35),

$$|||\mathfrak{L}_p^{-1}|||_{\mathcal{L}(\mathbb{R}^m)} \leq \frac{1}{1-\frac{2}{5}} = \frac{5}{3}. \tag{42}$$

Step 3. Upper bound for $\|\mathfrak{N}_p^1(Y)\|$. Setting $n = k-1+k'$ in the sum defining $\mathfrak{N}_p^1(Y)$ we obtain

$$\mathfrak{N}_p^1(Y) = \sum_{\substack{2\leq k\leq p,\\ p+1\leq n\leq p+k-1}} D\Phi_k(Y).N_{n-k+1}(Y).$$

Thus, using lemmas 2.10, 2.11 we get

$$\begin{aligned}
\|\mathfrak{N}_p^1(Y)\| &\leq \sum_{\substack{2\leq k\leq p,\\ p+1\leq n\leq p+k-1}} \|D\Phi_k(Y).N_{n-k+1}(Y)\|,\\
&\leq \sum_{\substack{2\leq k\leq p,\\ p+1\leq n\leq p+k-1}} \left|D\Phi_k.N_{n-k+1}\right|_{0,n} \|Y\|^n,\\
&\leq \sum_{\substack{2\leq k\leq p,\\ p+1\leq n\leq p+k-1}} \left|D\Phi_k.N_{n-k+1}\right|_{2,n} \delta^n,\\
&\leq \sum_{\substack{2\leq k\leq p,\\ p+1\leq n\leq p+k-1}} \sqrt{k^2+(m-1)k}\,\left|\Phi_k\right|_{2,k} . \left|N_{n-k+1}\right|_{2,n-k+1} \delta^n.
\end{aligned}$$

Hence, for every $\delta > 0$*, every* $Y \in \mathbb{R}^m$ *with* $||Y|| \leq \delta$ *and every* p*,* $2 \leq p \leq K$ *satisfying* (35),

$$\|\mathfrak{N}_p^1(Y)\| \leq \sum_{\substack{2\leq k\leq p\\ p+1\leq n\leq p+k-1}} \sqrt{m}\; k\phi_k\nu_{n-k+1}\delta^n. \tag{43}$$

Step 4. Upper bound for $\|\mathfrak{N}_p^2(Y)\|$. Here again, using lemmas 2.10, 2.11 we obtain

$$\begin{aligned}
\|\mathfrak{N}_p^2(Y)\| &\leq \sum_{\substack{2\leq q\leq p,\ 1\leq p_j\leq p\\ p_1+\cdots+p_q\geq p+1}} \|V_q\left[\Phi_{p_1}(Y),\cdots,\Phi_{p_q}(Y)\right]\|,\\
&\leq \sum_{\substack{2\leq q\leq p,\ 1\leq p_j\leq p\\ p_1+\cdots+p_q=n,\ n\geq p+1}} \left|V_q\left[\Phi_{p_1},\cdots,\Phi_{p_q}\right]\right|_{0,n} \|Y\|^n,\\
&\leq \sum_{\substack{2\leq q\leq p,\ 1\leq p_j\leq p\\ p_1+\cdots+p_q=n,\ n\geq p+1}} \left|V_q\left[\Phi_{p_1},\cdots,\Phi_{p_q}\right]\right|_{2,n} \delta^n,\\
&\leq \sum_{\substack{2\leq q\leq p,\ 1\leq p_j\leq p\\ p_1+\cdots+p_q=n,\ n\geq p+1}} \frac{c}{\rho^q}\left|\Phi_{p_1}\right|_{2,n}.\cdots.\left|\Phi_{p_q}\right|_{2,n} \delta^n.
\end{aligned}$$

So, in conclusion, for every $\delta > 0$, every $Y \in \mathbb{R}^m$ with $||Y|| \leq \delta$ and every p, $2 \leq p \leq K$ satisfying (35),

$$\|\mathfrak{N}_p^2(Y)\| \leq \sum_{\substack{2\leq q\leq p \\ p+1\leq n=p_1+\cdots+p_q, \\ 1\leq p_j\leq p}} \frac{c\delta^n}{\rho^q}\phi_{p_1}\cdots\phi_{p_q}. \tag{44}$$

Step 5. Upper bound for $\|\mathfrak{N}_p^3(Y)\|$. First, observe that

$$\|\mathfrak{N}_p^3(Y)\| \leq \sum_{q\geq p+1} \|V_q\big[\{\sum_{1\leq k\leq p} \Phi_k(Y)\}^{(q)}\big]\| \leq \sum_{q\geq p+1} \frac{c}{\rho^q}\|\sum_{1\leq k\leq p} \Phi_k(Y)\|^q.$$

Then, using lemma 2.14, we get that *for every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying* (35),

$$\|\mathfrak{N}_p^3(Y)\| \leq \sum_{p+1\leq q} \frac{c}{\rho^q}(\frac{10}{9}\sqrt{m}\ \delta)^q. \tag{45}$$

Finally, gathering (41), (43), (44), (45), we get the desired upper bound for $\|\mathcal{R}_p(Y)\|$.

$\square$

Lemma 2.17 *For every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying*

$$\delta p^{1+\tau} \leq \frac{\rho^2}{e^{1+\tau}ac\sqrt{m}\ \Theta}. \tag{46}$$

we have

$$\|\mathcal{R}_p(Y)\| \leq \frac{10}{9}c\left((C\delta)^{p+1}(p!)^{1+\tau} + \frac{1}{p^{2+2\tau}}\ \left(\frac{1}{e^{1+\tau}}\right)^{p+1}\right)$$

for every $Y \in \mathbb{R}^m$ with $||Y|| \leq \delta$ where

$$C = \frac{ac\sqrt{m}\Theta}{\rho^2} = \frac{\sqrt{m}}{\rho^2}\left\{\left(\frac{5}{2}m + 2\right)ac + 3\rho\right\}.$$

Remark 2.18 Observe that the constraint (46) imposed on δ and p is slightly stronger than the one (35) imposed in Lemma 2.14 since $\frac{1}{e^{1+\tau}} \leq \frac{1}{2}$. The constraint (46) has been chosen to get the optimal exponential decay rate for the upper bound of $\mathcal{R}_p$ obtained by an optimal choice of $p_{\text{opt}} = \left[\frac{1}{\mathrm{e}(C\delta)^{\frac{1}{1+\tau}}}\right]$, i.e $\delta(p_{\text{opt}})^{1+\tau} \approx \frac{1}{\mathrm{e}^{1+\tau}C}$ (for details see below lemmas 2.19 and 2.21).

Proof. Lemma 2.16, we get that for every $\delta > 0$, every p, $2 \le p \le K$ satisfying (35) and for every $Y \in \mathbb{R}^m$ with $\|Y\| \le \delta$,

$$\frac{3}{5}\|\mathcal{R}_p(Y)\| \le \Delta_p^1 + \Delta_p^2 + \Delta_p^3$$

where $\Delta_1, \Delta_2, \Delta_3$ are given by (40). The sums Δ_p^1 and Δ_p^3 can be optimally bounded with constraint (35) whereas for Δ_p^2 we use the stronger constraint (46).

Step 1. Upper bound for $\boldsymbol{\Delta_p^1}$. Defining $C = \frac{ac\sqrt{m}\ \Theta}{\rho^2}$ and using lemma 2.13 we get

$$\begin{aligned}
\Delta_p^1 &\le m^{\frac{5}{2}} \frac{ac^2}{\rho^4} \sum_{\substack{2\le k\le p \\ p+1\le n\le p+k-1}} \left(\frac{ac\sqrt{m}\ \Theta}{\rho^2}\right)^{n-3} \delta^n\ k(k!)^\tau (k-2)!\ \Big((n-k)!\Big)^\tau\ (n-k-1)!, \\
&\le \frac{m\rho^2}{a^2c\ \Theta^3} \sum_{2\le k\le p} k(k!)^\tau (k-2)! (C\delta)^{p+1} \sum_{p+1\le n\le p+k-1} (C\delta)^{n-p-1} (n-k-1)!\ \Big((n-k)!\Big)^\tau, \\
&\le \frac{m\rho^2}{a^2c\ \Theta^3} \sum_{2\le k\le p} k(k!)^\tau (k-2)!\ (C\delta)^{p+1} \sum_{p+1\le n\le p+k-1} \left(\tfrac{1}{2}\right)^{n-p-1} \tfrac{(n-k-1)!}{p^{n-p-1}} \left(\tfrac{(n-k)!}{p^{n-p-1}}\right)^\tau,
\end{aligned}$$

since $C\delta \le \frac{1}{(ep)^{1+\tau}} \le \frac{1}{2\ p^{1+\tau}}$ (here we do not need the strongest constraint). Then, observe that for $p+1 \le n \le p+k-1$,

$$\frac{(n-k-1)!}{(p-2)^{n-p-1}} \le (p-k)! \qquad \text{and} \qquad \frac{(n-k)!}{p^{n-p-1}} \le (p-k+1)!.$$

Thus, we obtain

$$\begin{aligned}
\Delta_p^1 &\le \frac{m\rho^2}{a^2c\ \Theta^3} \sum_{2\le k\le p} k(k!)^\tau (k-2)!\ \ (C\delta)^{p+1}\ 2(p-k)!\Big((p-k+1)!\Big)^\tau, \\
&\le \frac{2m\rho^2}{a^2c\ \Theta^3}\ (C\delta)^{p+1}(p!)^{1+\tau} \sum_{2\le k\le p} \frac{1}{C_p^k(k-1)} \left(\frac{p+1}{C_{p+1}^k}\right)^\tau, \\
&\le \frac{2m\rho^2}{a^2c\ \Theta^3}\ (C\delta)^{p+1}(p!)^{1+\tau} \sum_{2\le k\le p} \frac{1}{C_p^k(k-1)}, \\
&\le \frac{2m\rho^2}{a^2c\ \Theta^3}\ (C\delta)^{p+1}(p!)^{1+\tau} \sum_{2\le k\le p} \frac{1}{p-1}.
\end{aligned}$$

Hence, for every $\delta > 0$ and every p, $2 \le p \le K$ satisfying (35),

$$\Delta_p^1 \le \frac{2m\rho^2}{a^2c\ \Theta^3}\ (C\delta)^{p+1}(p!)^{1+\tau} \tag{47}$$

Step 2. Upper bound for Δ_p^2. Observing that $\alpha_n \leq (n-2)!\,\Theta^{n-1}$ for any $n \geq 1$ where $(-1)! = 0! = 1$ and using Lemma 2.13 we get

$$\begin{aligned}
\Delta_p^2 &= \sum_{2\leq q\leq p} \sum_{n\geq p+1} \sum_{\substack{p_1+\cdots+p_q=n\\ 1\leq p_j\leq p}} \tfrac{c(\sqrt{m})^q}{\rho^q} \left(\tfrac{ac\sqrt{m}}{\rho^2}\right)^{n-q} \delta^n \Big((p_1!)^\tau \alpha_{p_1}\Big)\cdots\Big((p_q!)^\tau \alpha_{p_q}\Big),\\
&\leq \sum_{2\leq q\leq p} \sum_{n\geq p+1} \sum_{\substack{p_1+\cdots+p_q=n\\ 1\leq p_j\leq p}} \tfrac{c(\sqrt{m})^q}{\rho^q} \left(\tfrac{ac\sqrt{m}\Theta}{\rho^2}\right)^{n-q} \delta^n\, (p_1!)^\tau (p_1-2)!\cdots(p_q!)^\tau(p_q-2)!\\
&\leq \sum_{2\leq q\leq p} \sum_{n\geq p+1} \sum_{\substack{p_1+\cdots+p_q=n\\ 1\leq p_j\leq p}} c\left(\tfrac{ac\sqrt{m}\delta\Theta}{\rho^2}\right)^n \left(\tfrac{\rho}{ac\Theta}\right)^q (p_1!)^\tau (p_1-2)!\cdots(p_q!)^\tau(p_q-2)!,\\
&\leq c \sum_{2\leq q\leq p} r^q \sum_{n\geq p+1} \sum_{\substack{p_1+\cdots+p_q=n\\ 1\leq p_j\leq p}} (C\delta)^n\, (p_1!)^\tau (p_1-2)!\cdots(p_q!)^\tau(p_q-2)!,
\end{aligned}$$

since $C = \frac{ac\sqrt{m}\ \Theta}{\rho^2}$ and where $r := \frac{\rho}{ac\Theta} \leq \frac{1}{3}$ with our choice of Θ given by (34). Moreover, for $\delta > 0$ and $p \geq 2$ satisfying (46) (here we use the stronger constraint), i.e. for $C\delta \leq \frac{1}{(ep)^{1+\tau}}$, we obtain

$$\begin{aligned}
\Delta_p^2 &\leq c \sum_{2\leq q\leq p} r^q \sum_{n\geq p+1} \sum_{\substack{p_1+\cdots+p_q=n\\ 1\leq p_j\leq p}} \left(\tfrac{1}{ep}\right)^{n(1+\tau)} (p_1!)^\tau (p_1-2)!\cdots(p_q!)^\tau(p_q-2)!,\\
&\leq c\left(\tfrac{1}{e^{1+\tau}}\right)^{p+1} \sum_{2\leq q\leq p} r^q \sum_{n\geq p+1} \sum_{\substack{p_1+\cdots+p_q=n\\ 1\leq p_j\leq p}} \left(\tfrac{1}{p}\right)^{n(1+\tau)} (p_1!)^\tau (p_1-2)!\cdots(p_q!)^\tau(p_q-2)!.
\end{aligned}$$

Then, recalling that $n \leq pq$, we get that

$$\begin{aligned}
\Delta_p^2 &\leq c\left(\frac{1}{e^{1+\tau}}\right)^{p+1} \sum_{2\leq q\leq p} r^q \left(\sum_{j=1}^{p} \left(\frac{1}{p^{1+\tau}}\right)^j (j!)^\tau (j-2)!\right)^q,\\
&\leq c\left(\frac{1}{e^{1+\tau}}\right)^{p+1} \sum_{2\leq q\leq p} r^q \left(\frac{1}{p^{1+\tau}} + \frac{p-1}{p^{\tau+2}}\right)^q,\\
&\leq c\left(\frac{1}{e^{1+\tau}}\right)^{p+1} \sum_{2\leq q\leq p} \left(\frac{2r}{p^{1+\tau}}\right)^q,\\
&\leq c\left(\frac{1}{e^{1+\tau}}\right)^{p+1} \frac{4r^2}{p^{2+2\tau}}\ \frac{1}{1-\frac{2r}{p^{1+\tau}}},
\end{aligned}$$

since $\frac{2}{p^{1+\tau}} \leq 1$. *Hence, for every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying* (46),

$$\Delta_p^2 \leq 4c\left(\frac{\rho}{ac\Theta}\right)^2 \frac{1}{1-\frac{\rho}{ac\Theta}}\frac{1}{p^{2+2\tau}} \left(\frac{1}{e^{1+\tau}}\right)^{p+1}. \tag{48}$$

Step 3. Upper bound for Δ_p^3. Observing that with our choice of Θ given by (34), for every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying (35), we obtain

$$\frac{\sqrt{m}\ \delta}{\rho} \leq \frac{\rho}{2ac\Theta p^{1+\tau}} \leq \frac{1}{12}$$

and thus,

$$\Delta_p^3 = c \sum_{p+1 \leq q} \left(\frac{10}{9} \frac{\sqrt{m}\ \delta}{\rho} \right)^q \leq c \left(\frac{10}{9} \frac{\sqrt{m}\ \delta}{\rho} \right)^{p+1} \sum_{q \geq 0} \left(\frac{5}{54} \right)^q .$$

Hence, for every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying (35),

$$\Delta_p^3 \leq \frac{54}{49} c \left(\frac{10}{9} \frac{\sqrt{m}\ \delta}{\rho} \right)^{p+1} . \tag{49}$$

Step 4. Upper bound for $\|\mathcal{R}_p(Y)\|$. Gathering the upper bounds for Δ_p^1, Δ_p^2, Δ_p^3 given by (47), (48), (49), that with our choice of Θ given by (34),

$$\frac{\rho}{ac\Theta} \leq \frac{1}{3}, \qquad \frac{m}{\Theta} \leq \frac{2}{5}$$

we obtain that for every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying (46)

$$\begin{aligned} \|\mathcal{R}_p(Y)\| &\leq \tfrac{5}{3} \left(\Delta_p^1 + \Delta_p^2 + \Delta_p^3 \right), \\ &\leq \tfrac{4c}{27}\ (C\delta)^{p+1} (p!)^{1+\tau} + \tfrac{10c}{9} \frac{1}{p^{2+2\tau}} \ \left(\frac{1}{e^{1+\tau}} \right)^{p+1} + \tfrac{90c}{49} \left(\tfrac{10}{9} \frac{\sqrt{m}\ \delta}{\rho} \right)^{p+1}, \\ &\leq \left(\tfrac{4}{27} + \tfrac{90}{49} \left(\tfrac{10}{27} \right)^3 \right)\ c\ (C\delta)^{p+1} (p!)^{1+\tau} + \tfrac{10c}{9} \frac{1}{p^{2+2\tau}} \ \left(\frac{1}{e^{1+\tau}} \right)^{p+1} \end{aligned}$$

since with our choice of Θ given by (34),

$$\frac{10\sqrt{m}}{9\rho} = \frac{10}{9} \frac{\rho}{ac\Theta} C \leq \frac{10}{27} C.$$

Hence, since $\left(\frac{4}{27} + \frac{90}{49} \left(\frac{10}{27} \right)^3 \right) \leq \frac{10}{9}$, for every $\delta > 0$ and every p, $2 \leq p \leq K$ satisfying (46) we have

$$\|\mathcal{R}_p(Y)\| \leq \frac{10}{9} c \left((C\delta)^{p+1} (p!)^{1+\tau} + \frac{1}{p^{2+2\tau}} \ \left(\frac{1}{e^{1+\tau}} \right)^{p+1} \right)$$

for every $Y \in \mathbb{R}^m$ with $||Y|| \leq \delta$. $\square$

The upper bound of the norm of the remainder $\|\mathcal{R}_p(Y)\|$ contains two terms. The second one, $\frac{1}{p^{2+2\tau}} \ \left(\frac{1}{e^{1+\tau}} \right)^{p+1}$ tends to 0 as p tends to infinity whereas the first one $(C\delta)^{p+1}(p!)^{1+\tau}$ tends to infinity. The key idea is to choose an optimal p for which $(C\delta)^{p+1}(p!)^{1+\tau} = \left((C\delta)^{\frac{p+1}{1+\tau}}p!\right)^{1+\tau}$ is minimal and prove that this minimal value is exponentially small with respect to δ. This results from the following lemma :

Lemma 2.19 *Choose $\varepsilon > 0$ and let us define $f_\varepsilon(p) := \varepsilon^{p+1}p!$ for $p \in \mathbb{N}$. Moreover, for $x \in \mathbb{R}$, denote by $[x]$ its integer part.*

Then, for $p_{\rm opt} := \left[\frac{1}{\varepsilon e}\right]$, $f_\varepsilon(p_{\rm opt})$ is exponentially small with respect to ε. Indeed,

$$f_\varepsilon\left(\left[\frac{1}{\varepsilon \mathrm{e}}\right]\right) \leq \mathfrak{m}\sqrt{\frac{\varepsilon}{\mathrm{e}}}\ \mathrm{e}^{-\frac{2}{\varepsilon \mathrm{e}}}$$

where $\mathfrak{m} = \sup\limits_{p\in\mathbb{N}} \dfrac{\mathrm{e}^2\ p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$.

Remark 2.20 Stirling's formula ensure that $\mathfrak{m}$ is finite.

Proof.

$$\begin{aligned} f_\varepsilon\left(\left[\frac{1}{\varepsilon \mathrm{e}}\right]\right) &\leq \frac{\mathfrak{m}\varepsilon}{\mathrm{e}^2}\exp\left\{\left(\left[\frac{1}{\varepsilon \mathrm{e}}\right]+\frac{1}{2}\right)\ln\left[\frac{1}{\varepsilon \mathrm{e}}\right] + \left[\frac{1}{\varepsilon \mathrm{e}}\right]\ln\frac{\varepsilon}{\mathrm{e}}\right\}, \\ &\leq \frac{\mathfrak{m}\varepsilon}{\mathrm{e}^2}\exp\left\{\left(\left[\frac{1}{\varepsilon \mathrm{e}}\right]+\frac{1}{2}\right)\ln\frac{1}{\varepsilon \mathrm{e}} + \left[\frac{1}{\varepsilon \mathrm{e}}\right]\ln\frac{\varepsilon}{\mathrm{e}}\right\}, \\ &= \frac{\mathfrak{m}\varepsilon}{\sqrt{\varepsilon \mathrm{e}}}\exp\left\{-2\left(\left[\frac{1}{\varepsilon \mathrm{e}}\right]+1\right)\right\} \leq \mathfrak{m}\sqrt{\frac{\varepsilon}{\mathrm{e}}}\ \mathrm{e}^{-\frac{2}{\varepsilon \mathrm{e}}}. \end{aligned}$$

□

Using this lemma we finally obtain the desired exponentially small upper bound for $\mathcal{R}_p(Y)$.

Lemma 2.21 *If there exits $K \geq 2$, $a > 0$ and $\tau \geq 0$ such that $a_k := \||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\||_2 \leq ak^\tau$ for every k with $2 \leq k \leq K$, then for every $\delta > 0$ such that $K \geq p_{\rm opt} \geq 2$, the remainder $\mathcal{R}_p$ given by the Normal Form Theorem 1.1 for $p = p_{\rm opt}$ satisfies*

$$\sup_{\|Y\|\leq\delta} \|R_{p_{\rm opt}}(Y)\| \leq M\delta^2\exp\left(-\tfrac{w}{\delta^b}\right)$$

with

$$b = \frac{1}{1+\tau},\ \ p_{\rm opt} = \left[\frac{1}{\mathrm{e}(C\delta)^b}\right],\ \ w = \frac{1}{\mathrm{e}C^b},\ \ M = \tfrac{10}{9}cC^2\left\{\left(\mathfrak{m}\sqrt{\tfrac{27}{8\mathrm{e}}}\right)^{1+\tau} + (2\mathrm{e})^{2+2\tau}\right\}$$

where $C = \frac{\sqrt{m}}{\rho^2}\left\{\left(\frac{5}{2}m+2\right)ac+3\rho\right\}$ *and* $\mathfrak{m} = \sup\limits_{p\in\mathbb{N}} \dfrac{\mathrm{e}^2\ p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$.

Proof. Let $\delta > 0$ be such that $p_{\mathrm{opt}} = \left[\frac{1}{\mathrm{e}(C\delta)^b}\right]$ satisfies $K \geq p_{\mathrm{opt}} \geq 2$. Observe that condition (46) reads $\delta^b p \leq \frac{1}{\mathrm{e}C^b}$ and thus that p_{opt} satisfies it. Then since,

$$p_{\mathrm{opt}} + 1 \geq \frac{1}{\mathrm{e}(C\delta)^b} \geq p_{\mathrm{opt}} \geq 2 \qquad \text{and} \qquad \frac{1}{p_{\mathrm{opt}}} \leq 2\ \mathrm{e}(C\delta)^b$$

lemmas 2.17 and 2.19 with $\varepsilon = (C\delta)^b$ ensure that

$$\begin{aligned}
\sup_{\|Y\|\leq\delta} \|R_{p_{\mathrm{opt}}}(Y)\| &\leq \tfrac{10}{9} c \left\{ \left(\mathfrak{m}\sqrt{\tfrac{(C\delta)^b}{\mathrm{e}}}\mathrm{e}^{-\frac{2}{\mathrm{e}(C\delta)^b}}\right)^{1+\tau} + \left(2\mathrm{e}(C\delta)^b\right)^{2+2\tau}\ \mathrm{e}^{-\frac{1+\tau}{\mathrm{e}(C\delta)^b}} \right\}, \\
&\leq \tfrac{10}{9} c (\mathrm{e}^{1+\tau} C\delta)^2\ \mathrm{e}^{-\frac{1+\tau}{\mathrm{e}(C\delta)^b}} \left\{ \left(\tfrac{\mathfrak{m}}{\mathrm{e}}(\mathrm{e}(C\delta)^b)^{-\frac{3}{2}}\mathrm{e}^{-\frac{1}{\mathrm{e}(C\delta)^b}}\right)^{1+\tau} + 4^{1+\tau} \right\}, \\
&\leq \tfrac{10}{9} c (\mathrm{e}^{1+\tau} C\delta)^2\ \mathrm{e}^{-\frac{1+\tau}{\mathrm{e}(C\delta)^b}} \left\{ \left(\tfrac{\mathfrak{m}}{\mathrm{e}}\sqrt{\tfrac{27}{8}}\mathrm{e}^{-\frac{3}{2}}\right)^{1+\tau} + 4^{1+\tau} \right\}, \\
&= \tfrac{10}{9} c C^2 \left\{ \left(\mathfrak{m}\sqrt{\tfrac{27}{8\mathrm{e}}}\right)^{1+\tau} + (2\mathrm{e})^{2+2\tau} \right\}\ \delta^2\ \mathrm{e}^{-\frac{1+\tau}{\mathrm{e}(C\delta)^b}},
\end{aligned}$$

since $x^{\frac{3}{2}}\mathrm{e}^{-x} \leq \sqrt{\frac{27}{8}}\mathrm{e}^{-\frac{3}{2}}$ for any $x \geq 0$. □

2.4 Computations of the norm of the pseudo inverse of the homological operator for non semi simple-operators.

This subsection is devoted to the computation of the norm of $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ for two examples of non semi simple operator L. We begin with the $\mathbf{0^2}$ singularity. In both cases, the computations of the norm of the pseudo inverse of the homological operator are performed via lemma 2.8. Hence, in all this subsection we denote by $\Sigma_k(L) \subset \mathbb{R}^+$ the spectrum of the positive self adjoint operator $(\mathcal{A}_L|_{\mathcal{H}_k})^*\mathcal{A}_L|_{\mathcal{H}_k} = \mathcal{A}_{L^*}|_{\mathcal{H}_k}\mathcal{A}_L|_{\mathcal{H}_k}$.

Lemma 2.22 (Norm of the pseudo inverse $\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}$ for $L = \mathbf{0^2}$)

For $L = \mathbf{0^2}$ and for every $k \geq 2$, we have $\min\limits_{\lambda\in\Sigma_k(L)\setminus\{0\}}\{\lambda\} \geq 1$ and thus

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq 1.$$

Proof. We are in dimension 2, with $Y = (x, y)$ and $L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We intend to give a lower bound of the non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in the subspace $\mathcal{H}_k$ of homogeneous polynomials of degree k. We recall that

$$\mathcal{A}_L\Phi(Y) = D\Phi(Y)LY - L\Phi(Y).$$

Thus, denoting $\Phi = (\phi_1, \phi_2)$ in $\mathcal{H}_k$, we have

$$\mathcal{A}_{_L}\Phi = (y\frac{\partial \phi_1}{\partial x} - \phi_2, y\frac{\partial \phi_2}{\partial x}) \qquad \text{and} \qquad \ker \mathcal{A}_{_L} = \operatorname{span}\{(y^k, 0), (xy^{k-1}, y^k)\}.$$

Now we look for the eigenvalues λ ($\lambda \geq 0$) of $\mathcal{A}_{_{L^*}}\mathcal{A}_{_L}$ in the subspace $\mathcal{H}_k$. They are given by

$$\begin{aligned} xy\frac{\partial^2 \phi_1}{\partial x \partial y} + x\frac{\partial \phi_1}{\partial x} - x\frac{\partial \phi_2}{\partial y} \qquad &= \lambda \phi_1, \\ xy\frac{\partial^2 \phi_2}{\partial x \partial y} + x\frac{\partial \phi_2}{\partial x} - y\frac{\partial \phi_1}{\partial x} + \phi_2 &= \lambda \phi_2. \end{aligned} \tag{50}$$

We check that

i) $\Phi = (0, x^k)$ gives $\lambda = k+1$;

ii) $\Phi = (y^k, 0)$ gives $\lambda = 0$;

iii) $\Phi = (x^\alpha y^\beta, x^{\alpha-1}y^{\beta+1})$ gives $\lambda = (\alpha-1)(\beta+1)$ with $\alpha+\beta = k$, $\alpha = 1, ...k$;

iv) $\Phi = ((\beta+1)x^\alpha y^\beta, -\alpha x^{\alpha-1}y^{\beta+1})$ gives $\lambda = \alpha(\beta+2)$ with $\alpha+\beta = k$, $\alpha = 1, ...k$.

These are the $2(k+1)$ eigenvalues of the operator $\mathcal{A}^*\mathcal{A}$ in the subspace $\mathcal{H}_k$, corresponding to a family of orthogonal eigenvectors. It is clear that $\min\limits_{\lambda \in \Sigma_k(L)\backslash\{0\}} \{\lambda\} \geq 1$ and thus,

$$a_k := |||\widetilde{\mathcal{A}_{_L}}|_{\mathcal{H}_k}^{-1}|||_2 \leq 1.$$

$\square$

Lemma 2.23 (Norm of the pseudo inverse $\widetilde{\mathcal{A}_{_L}}|_{\mathcal{H}_k}^{-1}$ for $L = \mathbf{0}^2.\underbrace{\mathbf{0}\cdots\mathbf{0}}_{q\ \mathbf{times}}$)

For $L = \mathbf{0^2}.\underbrace{\mathbf{0}\cdots\mathbf{0}}_{q\ times}$ *and for every* $k \geq 2$, *we have* $\min\limits_{\lambda \in \Sigma_k(L)\backslash\{0\}} \{\lambda\} \geq 1$ *and thus*

$$a_k(L) := |||\widetilde{\mathcal{A}_{_L}}|_{\mathcal{H}_k}^{-1}|||_2 \leq 1.$$

Proof. We are in dimension $2+q$, with $Y = (x, y, \widetilde{x}_1, \cdots, \widetilde{x}_q)$, $\Phi = (\phi_1, \phi_2, \widetilde{\phi}_1, \cdots, \widetilde{\phi}_q)$ and

$$L = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}.$$

Here again, we intend to give a lower bound of the non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in the subspace $\mathcal{H}_k$. We have

$$\mathcal{A}_L\Phi = (y\frac{\partial\phi_1}{\partial x} - \phi_2, y\frac{\partial\phi_2}{\partial x}, y\frac{\partial\widetilde{\phi}_1}{\partial x}, \cdots, y\frac{\partial\widetilde{\phi}_q}{\partial x}),$$
$$\mathcal{A}_{L^*}\Phi = (x\frac{\partial\phi_1}{\partial y}, x\frac{\partial\phi_2}{\partial y} - \phi_1, x\frac{\partial\widetilde{\phi}_1}{\partial y}, \cdots, x\frac{\partial\widetilde{\phi}_q}{\partial y}).$$

Hence, for $\delta = (\delta_1, \cdots, \delta_q) \in \mathbb{N}^q$ and $1 \leq \ell \leq q$, the spaces

$$\mathcal{H}^{\delta}_{k,12} = \mathcal{H}_k \cap \Big\{\Phi/\Phi(Y) = \widetilde{x}_1^{\delta_1}\cdots\widetilde{x}_q^{\delta_q}(\phi_1(x,y), \phi_2(x,y), 0, \cdots, 0), \quad \phi_1, \phi_2 \text{ polynomials}\Big\},$$
$$\widetilde{\mathcal{H}}^{\alpha,\beta,\delta}_{k,\ell} = \mathcal{H}_k \cap \Big\{\Phi/\Phi(Y) = x^\alpha y^\beta \widetilde{x}_1^{\delta_1}\cdots\widetilde{x}_q^{\delta_q}(0, 0, \cdots, 0, \widetilde{\phi}_\ell, 0, \cdots, 0),\ \widetilde{\phi}_\ell \in \mathbb{R}\Big\}$$

are stable under $\mathcal{A}_{L^*}\mathcal{A}_L$. Then, since $\mathcal{H}_k = \bigoplus_{|\delta|\leq k} \mathcal{H}^{\delta}_{k,12} \oplus \bigoplus_{\substack{1\leq\ell\leq q\\ \alpha+\beta+|\delta|=k}} \widetilde{\mathcal{H}}^{\alpha,\beta,\delta}_{k,\ell}$, we have

$$\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_\mathrm{k}}) = \bigcup_{|\delta|\leq \mathrm{k}} \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}^{\delta}_{\mathrm{k},12}}) \cup \bigcup_{\substack{1\leq\ell\leq \mathrm{q}\\ \alpha+\beta+|\delta|=\mathrm{k}}} \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\widetilde{\mathcal{H}}^{\alpha,\beta,\delta}_{\mathrm{k},\ell}}).$$

On one hand, in $\mathcal{H}^{\delta}_{k,12}$ the spectral equation $\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \lambda\Phi$ reads (50). *So the proof of Lemma 2.22 ensures that the spectrum of* $\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{H}^{\delta}_{k,12}}$ *is composed of non-negative integers.*

On the other hand, in $\widetilde{\mathcal{H}}^{\alpha,\beta,\delta}_{k,\ell}$ the spectral equation $\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \lambda\Phi$ reads

$$\alpha(\beta+1)\ \widetilde{\phi}_\ell = \lambda\widetilde{\phi}_\ell.$$

Hence, $\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\widetilde{\mathcal{H}}^{\alpha,\beta,\delta}_{\mathrm{k},\ell}}) = \{\alpha(\beta+1)\}$ *and the spectrum of* $\mathcal{A}_{L^*}\mathcal{A}_L\mathcal{H}_k$ *is composed of non-negative integers.* So, $\min_{\lambda\in\Sigma_k(L)\setminus\{0\}}\{\lambda\} \geq 1$ and thus,

$$a_k := |||\widetilde{\mathcal{A}_L}|^{-1}_{\mathcal{H}_k}|||_2 \leq 1.$$

$\square$

Lemma 2.24 (Norm of the pseudo inverse $\widetilde{\mathcal{A}_L}|^{-1}_{\mathcal{H}_k}$ for $L = 0^3$)

For $L = \mathbf{0^3}$ *and for every* $k \geq 2$*, we have* $\min_{\lambda\in\Sigma_k(L)\setminus\{0\}}\{\lambda\} \geq 1$ *and thus*

$$a_k(L) := |||\widetilde{\mathcal{A}_L}|^{-1}_{\mathcal{H}_k}|||_2 \leq 1.$$

Proof. We are in dimension 3, with $Y = (x, y, z)$, $\Phi = (\phi_1, \phi_2, \phi_3)$ and

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here again, we intend to give a lower bound of the non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$. This is performed in several steps.

Step 1. Splitting of the operators. We define differential operators $\mathcal{D}$ and $\mathcal{D}^*$ by

$$\mathcal{D} = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}, \qquad \mathcal{D}^* = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \tag{51}$$

then

$$\mathcal{A}_L\Phi = \mathcal{D}\Phi - L\Phi, \qquad \mathcal{A}_{L^*}\Psi = \mathcal{D}^*\Psi - L^*\Psi$$

and

$$\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \begin{pmatrix} \mathcal{D}^*(\mathcal{D}\phi_1 - \phi_2) \\ \mathcal{D}^*(\mathcal{D}\phi_2 - \phi_3) - \mathcal{D}\phi_1 + \phi_2 \\ \mathcal{D}^*\mathcal{D}\phi_3 - \mathcal{D}\phi_2 + \phi_3 \end{pmatrix}.$$

Moreover, we check that $\ker \mathcal{A}_L$ is spanned by

$$\begin{pmatrix} z^\alpha(xz - \frac{y^2}{2})^\beta \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} yz^\alpha(xz - \frac{y^2}{2})^\beta \\ z^{\alpha+1}(xz - \frac{y^2}{2})^\beta \\ 0 \end{pmatrix}, \begin{pmatrix} xz^\alpha(xz - \frac{y^2}{2})^\beta \\ yz^\alpha(xz - \frac{y^2}{2})^\beta \\ z^{\alpha+1}(xz - \frac{y^2}{2})^\beta \end{pmatrix}.$$

In what follows we use the properties

$$\begin{gathered} \mathcal{D}x = y, \;\; \mathcal{D}y = z, \;\; \mathcal{D}z = 0, \; \mathcal{D}(xz - \frac{y^2}{2}) = 0, \\ \mathcal{D}^*x = 0, \;\; \mathcal{D}^*y = x, \;\; \mathcal{D}^*z = y, \; \mathcal{D}^*(xz - \frac{y^2}{2}) = 0. \end{gathered}$$

Step 2. Splitting of $\mathcal{H}_k$. Using the basis of monomials, for α, β, γ non negative integers

$$\phi_{\alpha,\beta,\gamma} = x^\alpha z^\beta (xz - \frac{y^2}{2})^\gamma, \qquad \text{and} \qquad \psi_{\alpha,\beta,\gamma} = x^\alpha y z^\beta (xz - \frac{y^2}{2})^\gamma. \tag{52}$$

we split $\mathcal{H}_k$ into the direct sum

$$\mathcal{H}_k = \mathcal{H}'_k \oplus \mathcal{H}''_k$$

where

$$\mathcal{H}'_k = \left\{ \Phi = (\phi_1, \phi_2, \phi_3) / \phi_1, \phi_3 \in \operatorname*{span}_{\alpha+\beta+2\gamma=k} \{\phi_{\alpha,\beta,\gamma}\}, \ \phi_2 \in \operatorname*{span}_{\alpha+\beta+2\gamma+1=k} \{\psi_{\alpha,\beta,\gamma}\} \right\},$$
$$\mathcal{H}''_k = \left\{ \Phi = (\phi_1, \phi_2, \phi_3) / \phi_1, \phi_3 \in \operatorname*{span}_{\alpha+\beta+2\gamma+1=k} \{\psi_{\alpha,\beta,\gamma}\}, \ \phi_2 \in \operatorname*{span}_{\alpha+\beta+2\gamma=k} \{\phi_{\alpha,\beta,\gamma}\} \right\}.$$

Then, using the identities

$$\begin{aligned}
\mathcal{D}\phi_{\alpha,\beta,\gamma} &= \alpha\psi_{\alpha-1,\beta,\gamma}, \\
\mathcal{D}\psi_{\alpha,\beta,\gamma} &= (1+2\alpha)\phi_{\alpha,\beta+1,\gamma} - 2\alpha\phi_{\alpha-1,\beta,\gamma+1} \\
\mathcal{D}^*\phi_{\alpha,\beta,\gamma} &= \beta\psi_{\alpha,\beta-1,\gamma}, \\
\mathcal{D}^*\psi_{\alpha,\beta,\gamma} &= (1+2\beta)\phi_{\alpha+1,\beta,\gamma} - 2\beta\phi_{\alpha,\beta-1,\gamma+1}, \\
\mathcal{D}^*\mathcal{D}\phi_{\alpha,\beta,\gamma} &= \alpha(1+2\beta)\phi_{\alpha,\beta,\gamma} - 2\alpha\beta\phi_{\alpha-1,\beta-1,\gamma+1}, && (53) \\
\mathcal{D}^*\mathcal{D}\psi_{\alpha,\beta,\gamma} &= (2\alpha+1)(\beta+1)\psi_{\alpha,\beta,\gamma} - 2\alpha\beta\psi_{\alpha-1,\beta-1,\gamma+1}, && (54)
\end{aligned}$$

we observe that $\mathcal{H}'_k$ and $\mathcal{H}''_k$ are both invariant under $\mathcal{A}_{L^*}\mathcal{A}_L$. *Hence, the spectrum of the operator $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$ is the union of its spectrum when restricted to $\mathcal{H}'_k$ and to $\mathcal{H}''_k$.*

Step 3. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}'_k$. We also split $\mathcal{H}'_k$ into subspaces invariant under $\mathcal{A}_{L^*}\mathcal{A}_L$.

Step 3.1. Splitting of $\mathcal{H}'_k$. First observe that for $\alpha+\beta+2\gamma = k$, the subspace $\mathcal{E}'_{\alpha,\beta,\gamma}$ of $\mathcal{H}'_k$ gathering the polynomials Φ of the form

$$\begin{aligned}
\phi_1 &= \sum_p a_p \phi_{\alpha-p,\beta-p,\gamma+p}, \\
\phi_2 &= \sum_p b_p \psi_{\alpha-p-1,\beta-p,\gamma+p}, \\
\phi_3 &= \sum_p c_p \phi_{\alpha-p-1,\beta-p+1,\gamma+p}
\end{aligned}$$

where

for $\alpha \le \beta$, $\quad 0 \le p \le \alpha$, $\ b_\alpha = c_\alpha = 0$,

for $\beta+1 \le \alpha$, $0 \le p \le \beta+1$, $\ a_{\beta+1} = b_{\beta+1} = 0 \quad$ and $c_{\beta+1} = 0$ if $\alpha = \beta+1$.

is invariant under the operator $\mathcal{A}_{L^*}\mathcal{A}_L$. Indeed, we have

$$
\begin{aligned}
\mathcal{D}\phi_1-\phi_2 &= \sum\{(\alpha-p)a_p-b_p\}\psi_{\alpha-p-1,\beta-p,\gamma+p},\\
\mathcal{D}\phi_2-\phi_3 &= \sum\{(2\alpha-2p-1)b_p-c_p\}\phi_{\alpha-p-1,\beta-p+1,\gamma+p}\\
&\qquad\qquad -2(\alpha-p-1)b_p\phi_{\alpha-p-2,\beta-p,\gamma+p+1},\\
\mathcal{D}^*(\mathcal{D}\phi_1-\phi_2) &= \sum(2\beta-2p+1)\{(\alpha-p)a_p-b_p\}\phi_{\alpha-p,\beta-p,\gamma+p}\\
&\qquad\qquad -2(\beta-p)\{(\alpha-p)a_p-b_p\}\phi_{\alpha-p-1,\beta-p-1,\gamma+p+1},\\
\mathcal{D}^*(\mathcal{D}\phi_2-\phi_3) &= \sum(\beta-p+1)\{(2\alpha-2p-1)b_p-c_p\}\psi_{\alpha-p-1,\beta-p,\gamma+p}\\
&\qquad\qquad -2(\beta-p)(\alpha-p-1)b_p\psi_{\alpha-p-2,\beta-p-1,\gamma+p+1},\\
\mathcal{D}^*\mathcal{D}\phi_3 &= \sum(\alpha-p-1)(2\beta-2p+3)c_p\phi_{\alpha-p-1,\beta-p+1,\gamma+p}\\
&\qquad\qquad -2(\alpha-p-1)(\beta-p+1)c_p\phi_{\alpha-p-2,\beta-p,\gamma+p+1}.
\end{aligned}
$$

Moreover, $\Phi'_k=(0,0,\phi_{k,0,0})$ is an eigenvector of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$ belonging to the eigenvalue $\lambda=k+1$.

Then, since $\Phi=(\phi_{\alpha,\beta,\gamma},0,0)$, $\Phi=(0,\psi_{\alpha-1,\beta,\gamma},0)$, $\Phi=(0,0,\phi_{\alpha-1,\beta+1,\gamma})$ and $\Phi=(0,0,\phi_{\alpha-2,\beta,\gamma+1})$ belong to $\mathcal{E}'_{\alpha,\beta,\gamma}$ respectively for $\alpha\geq 0$, $\alpha\geq 1$, $\alpha\geq 1$ and $\alpha\geq 2$, we have the splitting of $\mathcal{H}'_k$ into the non direct sum

$$\mathcal{H}'_k=\mathbb{C}\Phi'_k+\sum_{\alpha+\beta+2\gamma=k}\mathcal{E}'_{\alpha,\beta,\gamma}.$$

Hence, the spectrum $\operatorname{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}'_{\mathrm{k}}})$ *of the operator* $\mathcal{A}_{L^*}\mathcal{A}_L$ *in* $\mathcal{H}'_k$ *is given by the union with possibly many overlaps*

$$\operatorname{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}'_{\mathrm{k}}})=\{\mathrm{k}+1\}\cup\bigcup_{\alpha+\beta+2\gamma=\mathrm{k}}\operatorname{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{E}'_{\alpha,\beta,\gamma}}).$$

Step 3.2. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}'_{\alpha,\beta,\gamma}$. The spectral equation $\mathcal{A}_{L^*}\mathcal{A}_L\Phi=\lambda\Phi$, for $\Phi\in\mathcal{E}'_{\alpha,\beta,\gamma}$ can be written as a hierarchy of systems of equation $(55)_p$ where for $p=0$ we have

$$
\begin{aligned}
&(2\beta+1)(\alpha a_0-b_0) &&=\lambda a_0,\\
&(\beta+1)\{(2\alpha-1)b_0-c_0\}+b_0-\alpha a_0 &&=\lambda b_0,\\
&(\alpha-1)(2\beta+3)c_0+c_0-(2\alpha-1)b_0 &&=\lambda c_0
\end{aligned}
\tag{$55)_0$}
$$

and for $1 \le p \le \min\{\alpha, \beta+1\}$,

$$\begin{aligned}
\lambda a_p &= (2\beta - 2p + 1)\{(\alpha - p)a_p - b_p\} \\
&\quad -2(\beta - p + 1)\{(\alpha - p + 1)a_{p-1} - b_{p-1}\}, \\
\lambda b_p &= (\beta - p + 1)\{(2\alpha - 2p - 1)b_p - c_p\} - (\alpha - p)a_p + b_p \\
&\quad -2(\beta - p + 1)(\alpha - p)b_{p-1}, \\
\lambda c_p &= (\alpha - p - 1)(2\beta - 2p + 3)c_p - (2\alpha - 2p - 1)b_p + c_p \\
&\quad -2(\alpha - p)(\beta - p + 2)c_{p-1} + 2(\alpha - p)b_{p-1}.
\end{aligned} \tag{55$_p$}$$

In particular, when $\alpha \le \beta$ the last system of the hierarchy is obtained for $p = \alpha$ ($b_\alpha = c_\alpha = 0$) and it reads

$$\begin{aligned}
\lambda a_\alpha &= -2(\beta - \alpha + 1)(a_{\alpha-1} - b_{\alpha-1}), \\
0 &= 0,
\end{aligned} \tag{55$_\alpha$}$$

while for $\beta \le \alpha - 1$ the last system is obtained for $p = \beta + 1$ ($a_{\beta+1} = b_{\beta+1} = 0$, and $c_{\beta+1} = 0$ if $\alpha = \beta + 1$) and it reads

$$\begin{aligned}
\lambda c_{\beta+1} &= (\alpha - \beta - 1)c_{\beta+1} - 2(\alpha - \beta - 1)\{c_\beta - b_\beta\}, \\
0 &= 0.
\end{aligned} \tag{55$_{\beta+1}$}$$

The system with $p = 0$ gives the eigenvalues:

$$\begin{aligned}
&\lambda_1 = (\alpha-1)(2\beta+1), && a_0 = b_0 = c_0 = 1, \\
&\lambda_2 = \alpha(2\beta+3), && a_0 = (\beta+1)(2\beta+1),\ b_0 = -2\alpha(\beta+1),\ c_0 = \alpha(2\alpha-1), \\
&\lambda_3 = (2\alpha-1)(\beta+1), && a_0 = -(2\beta+1),\ b_0 = \alpha - \beta - 1,\ c_0 = 2\alpha - 1.
\end{aligned}$$

We check that for for $\alpha = 0$ or 1, we recover known eigenvectors belonging to the 0 eigenvalue, all other eigenvalues are positive integers.

For proving that they indeed give eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ it is needed to check that for $1 \le p < \min\{\alpha, \beta+1\}$ the determinant Δ_p does not cancel for $\lambda = \lambda_1$ or λ_2 or λ_3 where

$$\Delta_p = \begin{vmatrix} (2\beta' + 1)\alpha' - \lambda & -(2\beta' + 1) & 0 \\ -\alpha' & (\beta' + 1)(2\alpha' - 1) + 1 - \lambda & -(\beta' + 1) \\ 0 & -2\alpha' + 1 & (\alpha' - 1)(2\beta' + 3) + 1 - \lambda \end{vmatrix}$$

with $\alpha' = \alpha - p$, $\beta' = \beta - p$. It results that

$$\Delta_p = (\lambda_1' - \lambda)(\lambda_2' - \lambda)(\lambda_3' - \lambda)$$

with

$$\begin{aligned}
\lambda_1' &= (\alpha' - 1)(2\beta' + 1) = \lambda_1 - p(2\alpha + 2\beta - 2p - 1), \\
\lambda_2' &= \alpha'(2\beta' + 3) = \lambda_2 - p(2\alpha + 2\beta - 2p + 3), \\
\lambda_3' &= (2\alpha' - 1)(\beta' + 1) = \lambda_3 - p(2\alpha + 2\beta - 2p + 1).
\end{aligned}$$

It is then easy to see (using the fact that $1 \leq p \leq \min\{\alpha - 1, \beta\}$) that the only case when $\Delta_p(\lambda_j) = 0$ is when $p = 1$ and $\lambda_2' = \lambda_1$:

$$\lambda_2' - \lambda_1 = (1 - p)(2\alpha + 2\beta - 2p + 1).$$

The case $p = 1$, $\lambda = \lambda_1 = (\alpha - 1)(2\beta + 1)$ leads to

$$\begin{aligned}
-2(\alpha - 1)a_1 - (2\beta - 1)b_1 &= 2\beta(\alpha - 1), \\
-(\alpha - 1)a_1 - (\alpha + \beta - 2)b_1 - \beta c_1 &= 2\beta(\alpha - 1), \\
-(2\alpha - 3)b_1 - 2\beta c_1 &= 2\beta(\alpha - 1)
\end{aligned}$$

where the compatibility condition is satisfied, hence giving a one parameter family of eigenvectors.

Finally, it remains to study the cases when the limiting equations cannot be solved, i.e. the two cases

i) when $\alpha \leq \beta$, $\lambda = 0$ (i.e. $\alpha = 0$, or 1), $p = \alpha$; the case $\alpha = 0$, $p = 0$, $\lambda = 0$ gives a known eigenvector, while $\alpha = p = 1$, $\lambda = 0$ gives $a_0 = b_0 = c_0 = 1$ and the equation for a_1 gives $0.a_1 = -2(a_0 - b_0) = 0$, hence the compatibility condition is satisfied.

ii) When $\beta \leq \alpha - 2$, $\lambda = \alpha - \beta - 1$, $p = \beta + 1$. The only possibility is $\lambda_1 = \alpha - \beta - 1$ which happens if $\beta = 0$. Then $p = 1$, and we need to solve $c_1 = c_1 - 2(c_0 - b_0)$ where $a_0 = b_0 = c_0 = 1$. Hence the compatibility condition is satisfied. This ends the study in the first invariant subspace.

In conclusion, all the eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}'_{\alpha,\beta,\gamma}$ and thus in $\mathcal{H}'_k$ are non negative integers.

Step 4. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}''_k$. We also split $\mathcal{H}''_k$ into subspaces invariant by $\mathcal{A}_{L^*}\mathcal{A}_L$.

Step 4.1. Splitting of $\mathcal{H}''_k$. For $\alpha+\beta+2\gamma+1=k$, let us denote $\mathcal{E}''_{\alpha,\beta,\gamma}$ the subspace of $\mathcal{H}''_k$ gathering the polynomials Φ of the form

$$\begin{aligned}\phi_1 &= \sum_p a_p \psi_{\alpha-p,\beta-p,\gamma+p},\\ \phi_2 &= \sum_p b_p \phi_{\alpha-p,\beta-p+1,\gamma+p},\\ \phi_3 &= \sum_p c_p \psi_{\alpha-p-1,\beta-p+1,\gamma+p}\end{aligned}$$

where

$$\begin{aligned}&\text{for } \alpha \le \beta, && 0 \le p \le \alpha,\quad c_\alpha = 0\\ &\text{for } \beta \le \alpha-1, && 0 \le p \le \beta+1,\quad a_{\beta+1}=0, \text{ and } c_{\beta+1}=0 \text{ if } \alpha=\beta+1.\end{aligned}$$

The following identities

$$\begin{aligned}\mathcal{D}\phi_1 - \phi_2 &= \sum \{(2\alpha-2p+1)a_p - b_p\}\phi_{\alpha-p,\beta-p+1,\gamma+p}\\ &\qquad\qquad - 2(\alpha-p)a_p\phi_{\alpha-p-1,\beta-p,\gamma+p+1},\\ \mathcal{D}\phi_2 - \phi_3 &= \sum\{(\alpha-p)b_p - c_p\}\psi_{\alpha-p-1,\beta-p+1,\gamma+p},\\ \mathcal{D}^*(\mathcal{D}\phi_1-\phi_2) &= \sum(\beta-p+1)\{(2\alpha-2p+1)a_p - b_p\}\psi_{\alpha-p,\beta-p,\gamma+p}\\ &\qquad\qquad - 2(\alpha-p)(\beta-p)a_p\psi_{\alpha-p-1,\beta-p-1,\gamma+p+1},\\ \mathcal{D}^*(\mathcal{D}\phi_2-\phi_3) &= \sum(2\beta-2p+3)\{(\alpha-p)b_p - c_p\}\phi_{\alpha-p,\beta-p+1,\gamma+p}\\ &\qquad\qquad - 2(\beta-p+1)\{(\alpha-p)b_p-c_p\}\phi_{\alpha-p-1,\beta-p,\gamma+p+1},\\ \mathcal{D}^*\mathcal{D}\phi_3 &= \sum(2\alpha-2p-1)(\beta-p+2)c_p\psi_{\alpha-p-1,\beta-p+1,\gamma+p}\\ &\qquad\qquad - 2(\alpha-p-1)(\beta-p+1)c_p\psi_{\alpha-p-2,\beta-p,\gamma+p+1}\end{aligned}$$

ensure that subspace $\mathcal{E}''_{\alpha,\beta,\gamma}$ is invariant under $\mathcal{A}_{L^*}\mathcal{A}_L$.

Moreover, the two dimensional subspace $\mathcal{P}''_k = \operatorname{span}\{\Phi''_k, \Psi''_k\}$ where $\Phi''_k = (0, \phi_{k,0,0}, 0)$ and $\Psi''_k = (0,0,\psi_{k-1,0,0})$ is stable by $\mathcal{A}_{L^*}\mathcal{A}_L$ since

$$\mathcal{A}_{L^*}\mathcal{A}_L\Phi''_k = (k+1)\Phi''_k - k\Psi''_k, \qquad \text{and} \qquad \mathcal{A}_{L^*}\mathcal{A}_L\Psi''_k = -\Phi''_k + 2k\Psi''_k.$$

Then, since $\Phi = (\psi_{\alpha,\beta,\gamma},0,0)$, $\Phi = (0,\phi_{\alpha,\beta+1,\gamma},0)$, $\Phi = (0,\phi_{\alpha-1,\beta,\gamma+1},0)$, $\Phi = (0,0,\psi_{\alpha-1,\beta+1,\gamma})$ and $\Phi = (0,0,\psi_{\alpha-2,\beta,\gamma+1})$ belong to $\mathcal{E}''_{\alpha,\beta,\gamma}$ respectively for $\alpha \ge 0$, $\alpha \ge 0$, $\alpha \ge 1$, $\alpha \ge 1$ and $\alpha \ge 2$, we have the splitting of $\mathcal{H}''_k$ into the non direct sum

$$\mathcal{H}''_k = \mathcal{P}''_k + \sum_{\alpha+\beta+2\gamma=k} \mathcal{E}''_{\alpha,\beta,\gamma} \qquad \text{with } \mathcal{P}''_k = \operatorname{span}\{\Phi''_k, \Psi''_k\}.$$

Hence, the spectrum $\mathrm{spec}(\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{H}''_k})$ *of the operator* $\mathcal{A}_{L^*}\mathcal{A}_L$ *in* $\mathcal{H}''_k$ *is given by the union with possibly many overlaps*

$$\mathrm{spec}(\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{H}''_k}) = \mathrm{spec}(\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{P}''_k}) \cup \bigcup_{\alpha+\beta+2\gamma=k} \mathrm{spec}(\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{E}''_{\alpha,\beta,\gamma}}).$$

Step 4.2. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{P}''_k$. In the basis $\{\Phi''_k, \Psi''_k\}$ the matrix of $\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{P}''_k}$ reads

$$\begin{pmatrix} k+1 & -k \\ -1 & 2k \end{pmatrix}.$$

Hence, the spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ *in* $\mathcal{P}''_k$ *is given by*

$$\mathrm{spec}(\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{P}''_k}) = \{2k+1, k\}.$$

Step 4.3. Spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}''_{\alpha,\beta,\gamma}$. The spectral equation $\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \lambda\Phi$, for $\Phi \in \mathcal{E}''_{\alpha,\beta,\gamma}$ can be written as a hierarchy of systems of equation $(56)_p$ where for $p = 0$ we have

$$\begin{aligned} &(\beta+1)\{(2\alpha+1)a_0 - b_0\} &&= \lambda a_0, \\ &(2\beta+3)(\alpha b_0 - c_0) + b_0 - (2\alpha+1)a_0 &&= \lambda b_0, \\ &(2\alpha-1)(\beta+2)c_0 + c_0 - \alpha b_0 &&= \lambda c_0 \end{aligned} \tag{56_0}$$

for $1 \le p \le \min\{\alpha, \beta+1\}$

$$\begin{aligned} \lambda a_p &= (\beta-p+1)\{(2\alpha-2p+1)a_p - b_p\} - 2(\alpha-p+1)(\beta-p+1)a_{p-1}, \\ \lambda b_p &= (2\beta-2p+3)\{(\alpha-p)b_p - c_p\} - (2\alpha-2p+1)a_p + b_p + \\ &\quad -2(\beta-p+2)\{(\alpha-p+1)b_{p-1} - c_{p-1}\} + 2(\alpha-p+1)a_{p-1}, \\ \lambda c_p &= (2\alpha-2p-1)(\beta-p+2)c_p - (\alpha-p)b_p + c_p + \\ &\quad -2(\alpha-p)(\beta-p+2)c_{p-1}. \end{aligned} \tag{56_p}$$

In particular, when $\alpha \le \beta$ the last system of the hierarchy is reached for $p = \alpha$ ($c_\alpha = 0$) and it reads

$$\begin{aligned} \lambda a_\alpha &= (\beta-\alpha+1)(a_\alpha - b_\alpha) - 2(\beta-\alpha+1)a_{\alpha-1}, \\ \lambda b_\alpha &= -a_\alpha + b_\alpha - 2(\beta-\alpha+2)(b_{\alpha-1} - c_{\alpha-1}) + 2a_{\alpha-1}, \\ 0 &= 0. \end{aligned}$$

This last system enables to compute $a_\alpha,\ b_\alpha$ if $\lambda \neq 0$ and $\lambda \neq \beta - \alpha + 2$.

When $\beta \leq \alpha - 1$, the last system of the hierarchy is reached for $p = \beta + 1$ ($a_{\beta+1} = 0$ and $c_{\beta+1} = 0$ if $\beta = \alpha - 1$) and it reads

$$\begin{aligned}
0 &= 0,\\
\lambda b_{\beta+1} &= (\alpha-\beta)b_{\beta+1} - c_{\beta+1} + 2\{(\alpha-\beta)(a_\beta - b_\beta) + c_\beta\},\\
\lambda c_{\beta+1} &= (\alpha-\beta-1)\{2c_{\beta+1} - b_{\beta+1} - 2c_\beta\}.
\end{aligned}$$

This last system enables to compute $b_\alpha,\ c_\alpha$ if $\lambda \neq 1$, when $\beta = \alpha - 1$ and if $\lambda \neq \alpha - \beta - 1$ and $\lambda \neq 2\alpha - 2\beta - 1$ when $\beta \leq \alpha - 2$.

The system for $p = 0$ gives the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ where

$$\begin{aligned}
&\lambda_1 = \alpha(2\beta+3) && a_0 = \beta+1,\ b_0 = \beta-\alpha+1,\ c_0 = -\alpha,\\
&\lambda_2 = (\beta+1)(2\alpha-1) && a_0 = 1,\ b_0 = 2,\ c_0 = 1,\\
&\lambda_3 = (\beta+2)(2\alpha+1) && a_0 = (\beta+1)(2\beta+3), b_0 = -(2\alpha+1)(2\beta+3), c_0 = \alpha(2\alpha+1).
\end{aligned}$$

Notice that $\lambda_1 = 0$ for $\alpha = 0$, which corresponds to a already known eigenvector in the kernel of $\mathcal{A}_L$. The coefficients a_p, b_p, c_p can be computed by induction provided that for $\lambda = \lambda_1$ or λ_2 or λ_3 the determinant

$$\Delta_p(\lambda) = (\lambda_1' - \lambda)(\lambda_2' - \lambda)(\lambda_3' - \lambda)$$

does not cancel, where

$$\begin{aligned}
\lambda_1' &= \lambda_1 - p(2\alpha + 2\beta - 2p + 3),\\
\lambda_2' &= \lambda_2 - p(2\alpha + 2\beta - 2p + 1),\\
\lambda_3' &= \lambda_3 - p(2\alpha + 2\beta - 2p + 5).
\end{aligned}$$

Using the fact that $1 \leq p \leq \min\{\alpha, \beta + 1\}$, we can see that the only problem comes when $\lambda_3' = \lambda_2$:

$$\lambda_3' - \lambda_2 = (1-p)(2\alpha + 2\beta + 3 - 2p)$$

which occurs when $p = 1$. This case $p = 1$, $\lambda = (\beta+1)(2\alpha-1)$ gives the system

$$\begin{aligned}
0 &= -(2\alpha-1)a_1 - \beta b_1 - 2\alpha\beta a_0,\\
0 &= -(2\alpha-1)a_1 + (1-\alpha-\beta)b_1 - (2\beta+1)c_1 - 2(\beta+1)(\alpha b_0 - c_0) + 2\alpha a_0,\\
0 &= (1-\alpha)b_1 - (2\beta+1)c_1 - 2(\alpha-1)(\beta+1)c_0
\end{aligned}$$

where the compatibility condition is satisfied with the values we found for a_0, b_0, c_0 $(a_0 - b_0 + c_0 = 0)$.

Finally, it then remains to study the last equation of the hierarchy:

i) when $\alpha \leq \beta$, $p = \alpha$, $\lambda = 0$ (i.e. $\alpha = p = 0$ leading to the know eigenvector in the kernel) or $\lambda = \beta - \alpha + 2$, i.e. $\lambda = \lambda_2$, $\alpha = 1 = p$ where the compatibility condition is satisfied due to $a_0 - b_0 + c_0 = 0$.

ii) When $\beta \leq \alpha - 1$, $p = \beta + 1$. Then for $\beta = \alpha - 1$, $\lambda_2 = 1$ (the bad case) for $\alpha = 1$ and this is again the case seen above. For $\beta \leq \alpha - 2$, the bad cases are when $\lambda_j = \alpha - \beta - 1$ or $2\alpha - 2\beta - 1$, i.e. $\lambda_2 = 2\alpha - 2\beta - 1$ for $\beta = 0$. We are again in the case $p = 1$ (notice that $a_1 = 0$) :

$$0 = (1-\alpha)b_1 - c_1 - 2(\alpha b_0 - c_0) + 2\alpha a_0,$$
$$0 = (1-\alpha)b_1 - c_1 - 2(\alpha - 1)c_0$$

which admits solutions since $a_0 - b_0 + c_0 = 0$.

In conclusion, all the eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{E}''_{\alpha,\beta,\gamma}$ and thus in $\mathcal{H}''_k$ are non negative integers. Gathering the results of step 3 and 4 we finally conclude that for every $k \geq 2$ all non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in $\mathcal{H}_k$ are positive integers. Hence, for every $k \geq 2$,

$$a_k := |||\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}|||_2 \leq 1.$$

□

Remark 2.25 For $L = \mathbf{0^4}$, the computation of eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ is more complicated and we could not find a lower estimate as in the 0^3 case. In particular, the kernel of this operator, which is also the kernel of $\mathcal{A}_L$ may be obtained as in the work [12], where it is observed for example that the polynomials invariant under $\mathcal{D}$ (same notation as for $\mathbf{0^3}$) are generated by 4 non independent polynomials of degree 1,2,3,4 with a non trivial relation between them. The same holds with $\mathcal{D}^*$. Moreover there are no common invariant polynomial under $\mathcal{D}$ and $\mathcal{D}^*$, contrary to the case $\mathbf{0^3}$. This does not allow to find a family of monomials giving a basis leading to a simple (triangular) matrix for the operator $\mathcal{A}_{L^*}\mathcal{A}_L$ in suitables subspaces (it seems necessary to obtain a not too complicated $\mathcal{D}^*\mathcal{D}$ operator applied to a suitable basis, for such a computation, as in the $\mathbf{0^3}$ case).

Lemma 2.26 (Norm of the pseudo inverse $\widetilde{\boldsymbol{\mathcal{A}}_L}|_{\mathcal{H}_k}^{-1}$ for $\boldsymbol{L} = \mathbf{0^3}.\underbrace{\mathbf{0\cdots 0}}_{q \text{ times}}$)

For $L = \mathbf{0^3}.\underbrace{\mathbf{0\cdots 0}}_{q\ times}$ *and for every* $k \geq 2$*, we have* $\min\limits_{\lambda\in\Sigma_k(L)\setminus\{0\}}\{\lambda\} \geq 1$ *and thus*

$$a_k(L) := \left\|\left|\widetilde{\mathcal{A}_L}|_{\mathcal{H}_k}^{-1}\right|\right\|_2 \leq 1.$$

Proof. We are in dimension $3+q$, with $Y = (x, y, z, \widetilde{x}_1, \cdots, \widetilde{x}_q)$, $\Phi = (\phi_1, \phi_2, \phi_3, \widetilde{\phi}_1, \cdots, \widetilde{\phi}_q)$ and

$$L = \begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}.$$

Here again, we intend to give a lower bound of the non zero eigenvalues of $\mathcal{A}_{L^*}\mathcal{A}_L$ in the subspace $\mathcal{H}_k$. We have

$$\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \begin{pmatrix} \mathcal{D}^*(\mathcal{D}\phi_1 - \phi_2) \\ \mathcal{D}^*(\mathcal{D}\phi_2 - \phi_3) - \mathcal{D}\phi_1 + \phi_2 \\ \mathcal{D}^*\mathcal{D}\phi_3 - \mathcal{D}\phi_2 + \phi_3 \\ \mathcal{D}^*\mathcal{D}\widetilde{\phi}_1 \\ \vdots \\ \mathcal{D}^*\mathcal{D}\widetilde{\phi}_q \end{pmatrix}.$$

Hence, for $\delta = (\delta_1, \cdots, \delta_q) \in \mathbb{N}^q$ and $1 \leq \ell \leq q$, the spaces

$$\begin{aligned} \mathcal{H}_{k,123}^{\delta} = \mathcal{H}_k \cap \Big\{ & \Phi/\Phi(Y) = \widetilde{x}_1^{\delta_1}\cdots\widetilde{x}_q^{\delta_q}(\widehat{\phi}(x,y,z), 0, \cdots, 0), \\ & \qquad\qquad\qquad\qquad \widehat{\phi} : \mathbb{R}^3 \to \mathbb{R}^3 \text{ polynomial}\Big\}, \\ \widetilde{\mathcal{H}}_{k,\ell}^{\delta} = \mathcal{H}_k \cap \Big\{ & \Phi/\Phi(Y) = \widetilde{x}_1^{\delta_1}\cdots\widetilde{x}_q^{\delta_q}(0, 0, \cdots, 0, \widetilde{\phi}_\ell(x,y,z), 0, \cdots, 0), \\ & \qquad\qquad\qquad\qquad \widetilde{\phi}_\ell : \mathbb{R}^3 \to \mathbb{R} \text{ polynomial}\Big\} \end{aligned}$$

are stable under $\mathcal{A}_{L^*}\mathcal{A}_{L}$. Then, since $\mathcal{H}_k = \bigoplus_{|\delta|\le k} \mathcal{H}^\delta_{k,123} \oplus \bigoplus_{\substack{1\le\ell\le q\\|\delta|\le k}} \widetilde{\mathcal{H}}^\delta_{k,\ell}$, we have

$$\mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}_\mathrm{k}}) = \bigcup_{|\delta|\le \mathrm{k}} \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\mathcal{H}^\delta_{\mathrm{k},123}}) \cup \bigcup_{\substack{1\le\ell\le \mathrm{q}\\|\delta|\le \mathrm{k}}} \mathrm{spec}(\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\widetilde{\mathcal{H}}^\delta_{\mathrm{k},\ell}}).$$

On one hand, in $\mathcal{H}^\delta_{k,123}$ the spectral equation $\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \lambda\Phi$ reads $\mathcal{A}^*_{\mathbf{0^3}}\mathcal{A}_{\mathbf{0^3}}\widehat{\phi} = \lambda\widehat{\phi}$. *So the proof of Lemma 2.24 ensures that the spectrum of* $\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{H}^\delta_{k,123}}$ *is composed of non-negative integers.*

On the other hand, in $\widetilde{\mathcal{H}}^\delta_{k,\ell}$ the spectral equation $\mathcal{A}_{L^*}\mathcal{A}_L\Phi = \lambda\Phi$ reads

$$\mathcal{D}^*\mathcal{D}\widetilde{\phi}_\ell = \lambda\widetilde{\phi}_\ell.$$

Let us decompose $\widetilde{\mathcal{H}}^\delta_{k,\ell}$

$$\widetilde{\mathcal{H}}^\delta_{k,\ell} = \widetilde{\mathcal{H}}^{\delta\,\prime}_{k,\ell} \oplus \widetilde{\mathcal{H}}^{\delta\,\prime\prime}_{k,\ell}$$

with

$$\widetilde{\mathcal{H}}^{\delta\,\prime}_{k,\ell} = \Big\{\Phi \in \widetilde{\mathcal{H}}^\delta_{k,\ell} / \widetilde{\phi}_\ell \in \operatorname*{span}_{\alpha+\beta+2\gamma=k-|\delta|} \{\phi_{\alpha,\beta,\gamma}\}\Big\},$$
$$\widetilde{\mathcal{H}}^{\delta\,\prime\prime}_{k,\ell} = \Big\{\Phi \in \widetilde{\mathcal{H}}^\delta_{k,\ell} / \widetilde{\phi}_\ell \in \operatorname*{span}_{\alpha+\beta+2\gamma+1=k-|\delta|} \{\psi_{\alpha,\beta,\gamma}\}\Big\}$$

where $\phi_{\alpha,\beta,\gamma}$, $\psi_{\alpha,\beta,\gamma}$ are defined in (52). Formulas (53)-(54) ensures that $\widetilde{\mathcal{H}}^{\delta\,\prime}_{k,\ell}$ and $\widetilde{\mathcal{H}}^{\delta\,\prime\prime}_{k,\ell}$ are both stable under $\mathcal{D}^*\mathcal{D}$. Moreover, ordering the basis $\phi_{\alpha,\beta,\gamma}$ (resp. $\psi_{\alpha,\beta,\gamma}$) by lexicographical order for (α,β,γ), formulas (53)-(54) also ensures that the matrix of $\mathcal{D}^*\mathcal{D}|_{\widetilde{\mathcal{H}}^{\delta\,\prime}_{k,\ell}}$ (resp. of $\mathcal{D}^*\mathcal{D}|_{\widetilde{\mathcal{H}}^{\delta\,\prime\prime}_{k,\ell}}$) in this basis is upper triangular with diagonal coefficient given by $\alpha(1+2\beta)$ (resp. by $(2\alpha+1)(\beta+1)$). Thus

$$\mathrm{spec}\mathcal{A}_{\mathrm{L}^*}\mathcal{A}_{\mathrm{L}}|_{\widetilde{\mathcal{H}}^\delta_{\mathrm{k},\ell}} = \bigcup_{\alpha+\beta+2\gamma=\mathrm{k}-|\delta|} \{\alpha(1+2\beta)\} \cup \bigcup_{\alpha+\beta+2\gamma+1=\mathrm{k}-|\delta|} \{(2\alpha+1)(\beta+1)\}.$$

Hence the spectrum of $\mathcal{A}_{L^*}\mathcal{A}_L|_{\mathcal{H}_k}$ *is composed of non-negative integers.* So, $\min_{\lambda\in\Sigma_k(L)\setminus\{0\}}\{\lambda\} \ge 1$ and thus,

$$a_k := |||\widetilde{\mathcal{A}_L}|^{-1}_{\mathcal{H}_k}|||_2 \le 1.$$

$\square$

3 Exponential estimates for perturbed vector fields

This section is devoted to the proof of Theorem 1.10. So, let $V : \mathbb{R}^m \times \mathbb{R}^s \to \mathbb{R}^m$ be an analytic family of vector fields in a neighborhood of 0 in $\mathbb{C}^m \times \mathbb{C}^s$

$$\frac{du}{dt} = V(u,\mu) \tag{57}$$

admitting the origin as a fixed point, i.e. satisfying $V(0,\mu) = 0$, (10) and (11). To deduce Theorem 1.10 from Theorem 1.4 which deals with the non perturbed case, we set $U = (u,\mu)$, $\mathcal{V}(U) = (V(u,\mu),0)$ and we observe that (57) is equivalent to

$$\frac{dU}{dt} = \mathcal{V}(U). \tag{58}$$

Let us denote by $\mathcal{L} = D_U\mathcal{V}(0)$ the linear operator corresponding to the $(m+s)\times(m+s)$ square matrix

$$\mathcal{L} = \begin{pmatrix} L_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proposition 2.2 ensures that when $a_k(\mathcal{L}) := |||\widetilde{\mathcal{A}_{\mathcal{L}}}|_{\mathcal{H}_k}^{-1}|||_2 \leq ak^\tau$, an optimal choice of the order $p = p_{\mathrm{opt}}$ of the change of coordinates $U = \mathcal{Y} + \mathcal{Q}_{p_{\mathrm{opt}}}(\mathcal{Y})$ leads to a normal form equation

$$\frac{d\mathcal{Y}}{dt} = \mathcal{L}\mathcal{Y} + \mathcal{N}_{p_{\mathrm{opt}}}(\mathcal{Y}) + \mathcal{R}_{p_{\mathrm{opt}}}(\mathcal{Y})$$

where $\mathcal{R}_p$ given by the Normal Form Theorem 1.1 for $p = p_{\mathrm{opt}}$ satisfies

$$\sup_{\|\mathcal{Y}\|\leq\delta} \|\mathcal{R}_{p_{\mathrm{opt}}}(\mathcal{Y})\| \leq M\ \delta^2 \mathrm{e}^{-\frac{w}{\delta^b}} \tag{59}$$

with

$$b = \frac{1}{1+\tau},\ p_{\mathrm{opt}} = \left[\frac{1}{\mathrm{e}(C\delta)^b}\right],\ w = \frac{1}{\mathrm{e}C^b},\ M = \tfrac{10}{9}cC^2\left\{\left(\mathfrak{m}\sqrt{\tfrac{27}{8\mathrm{e}}}\right)^{1+\tau} + (2\mathrm{e})^{2+2\tau}\right\}$$

where $C = \frac{\sqrt{m}}{\rho^2}\left\{\left(\frac{5}{2}m+2\right)ac + 3\rho\right\}$, $\mathfrak{m} = \sup_{p\in\mathbb{N}} \dfrac{\mathrm{e}^2\ p!}{p^{p+\frac{1}{2}}\mathrm{e}^{-p}}$ and where for a real number x, we denote by $[x]$ its integer part. Here, the situation is particular, since $\mathcal{V}(U) = (V(U),0)$. So, let us decompose $\mathcal{H}_k$ as follows

$$\mathcal{H}_k = \mathcal{H}_{k,m} \oplus \mathcal{H}_{k,s}$$

where $\mathcal{H}_{k,m}$ (resp. $\mathcal{H}_{k,s}$) is the space of the homogeneous polynomial $\mathcal{Q}$ of degree k from $\mathbb{R}^m \times \mathbb{R}^s$ to $\mathbb{R}^m \times \mathbb{R}^s$ such that $\mathfrak{p}_s(\mathcal{Q}) = 0$ (resp. $\mathfrak{p}_m(\mathcal{Q}) = 0$)

where $\mathfrak{p}_s$ (resp. $\mathfrak{p}_m$) is the canonical projection from $\mathbb{R}^m \times \mathbb{R}^s$ onto $\mathbb{R}^s$ (resp. onto $\mathbb{R}^m$). Thus, since $V_{n,\ell} \in \mathcal{H}_{n+\ell,m}$ for every n, ℓ and since $\mathcal{H}_{k,m}$ and $\mathcal{H}_{k,s}$ are both stable under $\mathcal{A}_{\mathcal{L}}$, we can choose $\mathcal{Q}_p$ and $\mathcal{N}_p$ of the form

$$\mathcal{Q}_p(\mathcal{Y}) = (\mathsf{Q}_p(\mathcal{Y}), 0) = \sum_{k=2}^{p} \Phi_k(\mathcal{Y}), \qquad \mathcal{N}_p(\mathcal{Y}) = (\mathsf{N}_p(\mathcal{Y}), 0) = \sum_{k=2}^{p} N_k(\mathcal{Y})$$

where $\Phi_k,\ N_k$ lie in $\mathcal{H}_{k,m}$. With this choice

$$\mathcal{Y} = (Y, \mu),\ \text{with } Y \in \mathbb{R}^m,\ \mu \in \mathbb{R}^s \quad \text{and} \quad \mathfrak{p}_s(\mathcal{R}_p(\mathcal{Y})) = 0.$$

Moreover with this choice, the homological equation (15) which is the center of this analysis reads

$$\begin{cases} \mathcal{A}_{\mathcal{L}} \Phi_k = N_k + F_k & \text{in } \mathcal{H}_{k,m}, \\ \quad 0 \quad = 0 & \text{in } \mathcal{H}_{k,s}. \end{cases}$$

So with this particular form of the homological equation, we only need to have

$$a_{k,m}(L_0) := \left|\left|\left| \widetilde{\mathcal{A}_{\mathcal{L}}}|_{\mathcal{H}_{k,m}}^{-1} \right|\right|\right|_2 \leq a k^{\tau}$$

to get the exponential estimate (59) given by Proposition 2.2. Then, Theorem 1.10 follows directly from the following lemma which gives $a_{k,m}(L_0)$ when either L_0 is semi simple or $L_0 = \mathbf{0^2}.\underbrace{\mathbf{0}.\cdots.\mathbf{0}}_{q \text{ times}}$ or $L_0 = \mathbf{0^3}.\underbrace{\mathbf{0}.\cdots.\mathbf{0}}_{q \text{ times}}$.

Lemma 3.1

(a) *Let L_0 be a semi simple matrix under real or complex Jordan normal form. Then,*

$$a_{k,m}(L_0) \leq \max_{\substack{1 \leq j \leq m, |\alpha| \leq k \\ \Lambda_{j,\alpha} \neq 0}} |\Lambda_{j,\alpha}|^{-1},$$

where $\Lambda_{j,\alpha} = \langle \lambda_{L_0}, \alpha \rangle - \lambda_j$ where $\lambda_{L_0} = \{\lambda_1, \cdots, \lambda_m\}$ is the spectrum of L_0.

(b) *For $L_0 = \mathbf{0}^2.\underbrace{\mathbf{0}.\cdots.\mathbf{0}}_{q \text{ times}}$ and $L_0 = \mathbf{0}^3.\underbrace{\mathbf{0}.\cdots.\mathbf{0}}_{q \text{ times}}$, $a_{k,m}(L_0) \leq 1$.*

Remark 3.2 When L_0 is semi-simple, under real or complex Jordan normal form and γ, K-homologically non-resonant we deduce from this lemma that

$$a_{k,m}(L_0) \leq \gamma^{-1}$$

and when L_0 is semi-simple, under real or complex Jordan normal form and γ, τ-homologically diophantine we get that

$$a_{k,m}(L_0) \leq \gamma^{-1} \max_{\substack{1 \leq j \leq m, |\alpha| \leq k \\ \Lambda_{j,\alpha} \neq 0}} |\alpha|^{\tau} = \gamma^{-1} k^{\tau}.$$

Proof of Lemma 3.1. **(a):** Let L_0 be a semi simple matrix under real or complex Jordan normal form.

Step 1. We first assume that L_0 is under complex Jordan normal form, i.e. that L_0 is diagonal. Then, $\mathcal{L}$ is also diagonal and we deduce from (19) that the spectrum of $\mathcal{A}_{\mathcal{L}}|_{\mathcal{H}_{k,m}}$ is given by

$$\begin{aligned}\operatorname{spec}(\mathcal{A}_{\mathcal{L}}|_{\mathcal{H}_{\mathrm{k,m}}}) &= \Big\{ \langle \lambda_{L_0}, \alpha_m\rangle + \langle 0, \alpha_s\rangle - \lambda_j,\ 1 \le j \le m,\ |\alpha_m| + |\alpha_s| = k, \\ &\qquad\qquad \alpha_m \in \mathbb{N}^m, \alpha_s \in \mathbb{N}^s \Big\}, \\ &= \Big\{ \langle \lambda_{L_0}, \alpha_m\rangle - \lambda_j,\ 1 \le j \le m,\ |\alpha_m| \le k, \alpha_m \in \mathbb{N}^m \Big\}.\end{aligned}$$

Hence, as in the proof of Lemma 2.5, we deduce from the above formula that

$$a_{k,m}(L_0) \le \max_{\substack{1\le j\le m, |\alpha|\le k \\ \Lambda_{j,\alpha}\neq 0}} |\Lambda_{j,\alpha}|^{-1}. \tag{60}$$

Step 2. When L_0 is semi-simple and under real Jordan normal form, but not diagonal, then it is conjugated to a complex diagonal matrix J_0 via a unitary map Q. So, $\mathcal{L}$ is conjugated to the complex $(m+s) \times (m+s)$ diagonal matrix $J_0 \oplus 0$ by the unitary map $Q \oplus I_s$. Hence, Remark 2.4 and step 1 ensure that $a_{k,m}(L_0)$ still satisfies (60) in this case.

(b): For $L_0 = \mathbf{0}^j.\underbrace{\mathbf{0}.\cdots.\mathbf{0}}_{q \text{ times}}$ with j=2,3, Lemmas 2.23, 2.26 ensure that

$$a_{k,m}(L_0) := |||\widetilde{\mathcal{A}_{\mathcal{L}}}|^{-1}_{\mathcal{H}_{k,m}}|||_2 \le |||\widetilde{\mathcal{A}_{\mathcal{L}}}|^{-1}_{\mathcal{H}_k}|||_2 = a_k\Big(\mathbf{0}^j.\underbrace{\mathbf{0}.\cdots.\mathbf{0}}_{q+s \text{ times}}\Big) \le 1.$$

□

A Properties of the normalized euclidian norm

A.1 Comparison of the euclidian and the sup norm

We begin with two technical lemmas which are used several times

Lemma A.1 *Let k, m be two positive integers and $\{u_j\}_{1\le j\le m}$ be m complex numbers. Then*

$$\frac{(u_1 + \cdots + u_m)^k}{k!} = \sum_{|\alpha|=k} \frac{u_1^{\alpha_1}}{\alpha_1!} \cdots \frac{u_m^{\alpha_m}}{\alpha_m!}.$$

Proof. We proceed by induction. For $m = 1$ this is trivial and for $m = 2$ this is true because of the binomial formula. Assume now that it is true for $m \geq 2$, then

$$\begin{aligned}
\sum_{|\alpha|=k} \frac{u_1^{\alpha_1}}{\alpha_1!} \cdots \frac{u_{m+1}^{\alpha_{m+1}}}{\alpha_{m+1}!} &= \sum_{\alpha_{m+1}=0}^{k} \frac{u_{m+1}^{\alpha_{m+1}}}{\alpha_{m+1}!} \sum_{\alpha_1+\cdots+\alpha_m=k-\alpha_{m+1}} \frac{u_1^{\alpha_1}}{\alpha_1!} \cdots \frac{u_m^{\alpha_m}}{\alpha_m!}, \\
&= \sum_{\alpha_{m+1}=0}^{k} \frac{u_{m+1}^{\alpha_{m+1}}}{\alpha_{m+1}!} \, \frac{(u_1+\cdots+u_m)^{k-\alpha_{m+1}}}{(k-\alpha_{m+1})!}, \\
&= \frac{(u_1+\cdots+u_{m+1})^k}{k!}.
\end{aligned}$$

$\square$

Lemma A.2 *Let k, m be two positive integers and*

$$\begin{aligned}
\mathcal{E}^1_{k,m} &= \{\beta = (\beta_1, \cdots, \beta_m) \in \mathbb{N}^m, \beta_j \geq 1, |\beta| = k\}, \\
\mathcal{E}^0_{k,m} &= \{\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}^m, \alpha_j \geq 0, |\alpha| = k\}.
\end{aligned}$$

Then, the cardinals $d^j_{k,m}$ of $\mathcal{E}^j_{k,m}$, $j = 0, 1$, are given by

$$d^1_{k,m} = C^{m-1}_{k-1}, \qquad d^0_{k,m} = C^{m-1}_{k+m-1}.$$

where $C^r_n = \dfrac{n!}{r!(n-r)!}$.

Proof. The cardinal of $d^1_{k,m}$ is equal to the number of ways for placing $(m-1)$ distinct separators among $k-1$ possible locations, the order of the separators being meaningless. For instance, the cardinal of $d^1_{k,3}$, is equal to the number of ways for placing 2 distinct separators among $k-1$ possible locations, the order of the separators being meaningless.

$$\overbrace{[\underbrace{\cdot\,|\,\cdot\,|\,\cdot}_{\alpha_1}|\underbrace{\cdot\,|\,\cdot}_{\alpha_2}|\underbrace{\cdot\,|\,\cdot\,|\,\cdot\,|\,\cdot}_{\alpha_3}]}^{k}$$

Hence, $d^1_{k,3} = C^2_{k-1}$ and more generally, $d^1_{k,m} = C^{m-1}_{k-1}$.

Finally, the map $\mathcal{E}^0_{k,m} \to \mathcal{E}^1_{k+m,m} : (\alpha_1, \cdots, \alpha_m) \mapsto (\beta_1 := \alpha_1 + 1, \cdots, \beta_m := \alpha_m + 1)$ is one to one. Hence

$$d^0_{k,m} = d^1_{m+k} = C^{m-1}_{m+k-1}.$$

$\square$

Lemma A.3 *For every* $\Phi \in \mathcal{H}_k$, $\quad |\Phi|_{0,k} \leq |\Phi|_{2,k} = \frac{1}{\sqrt{k!}} |\Phi|_2$.

Proof. For $\Phi \in \mathcal{H}_k$ with $\Phi = \sum\limits_{\substack{1\leq j\leq m\\|\alpha|=n}} \Phi_{j,\alpha} Y_1^{\alpha_1} \cdots Y_m^{\alpha_m}\, c_j$ where $\{c_j\}_{1\leq j\leq m}$ is the canonical basis of $\mathbb{R}^m$ we have

$$|\Phi|_{2,k} = \frac{1}{\sqrt{k!}} \sqrt{\sum_{\substack{1\leq j\leq m\\|\alpha|=k}} |\Phi_{j,\alpha}|^2 \alpha_1!.\cdots.\alpha_m!}$$

and

$$\begin{aligned}\frac{\|\Phi(Y)\|^2}{\|Y\|^{2k}} &= \sum_{j=1}^m \left| \sum_{|\alpha|=k} \Phi_{j,\alpha} \frac{Y_1^{\alpha_1}}{\|Y\|^{\alpha_1}} \cdots \frac{Y_m^{\alpha_m}}{\|Y\|^{\alpha_m}} \right|^2 \\ &\leq \sum_{j=1}^m \left(\sum_{|\alpha|=k} |\Phi_{j,\alpha}|^2\ \alpha_1! \cdots \alpha_m! \right) \left(\sum_{|\alpha|=k} \frac{Y_1^{2\alpha_1}}{\alpha_1! \|Y\|^{2\alpha_1}} \cdots \frac{Y_m^{2\alpha_m}}{\alpha_m! \|Y\|^{2\alpha_m}} \right)\end{aligned}$$

by the Cauchy Schwarz formula. Then using Lemma A.1 we get

$$\sum_{|\alpha|=k} \frac{Y_1^{2\alpha_1}}{\alpha_1! \|Y\|^{2\alpha_1}} \cdots \frac{Y_m^{2\alpha_m}}{\alpha_m! \|Y\|^{2\alpha_m}} = \frac{1}{k!} \left(\frac{Y_1^2}{\|Y\|^2} + \cdots + \frac{Y_m^2}{\|Y\|^2} \right)^k = \frac{1}{k!}.$$

Hence,

$$|\Phi|_{0,k} = \sup_{Y\in\mathbb{C}^k\setminus\{0\}} \frac{\|\Phi(Y)\|}{\|Y\|^k} \leq \sqrt{\frac{1}{k!} \sum_{j=1}^m \left(\sum_{|\alpha|=k} |\Phi_{j,\alpha}|^2\ \alpha_1! \cdots \alpha_m! \right)} = \frac{|\Phi|_2}{\sqrt{k!}} = |\phi|_{2,k}.$$

$\square$

We now prove a Parseval like formula :

Lemma A.4 *For every* $\Phi \in \mathcal{H}_k$,

$$|\Phi|_2^2 = \frac{1}{(2\pi)^m} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_m \int_0^{+\infty} dr_1 \cdots \int_0^{+\infty} dr_m \|\Phi(\sqrt{r_1}\mathrm{e}^{i\theta_1}, \cdots, \sqrt{r_m}\mathrm{e}^{i\theta_m})\|^2 \mathrm{e}^{-r_1} \cdots \mathrm{e}^{-r_m}.$$

Proof. We have

$$\|\Phi(\sqrt{r_1}\mathrm{e}^{i\theta_1}, \cdots, \sqrt{r_m}\mathrm{e}^{i\theta_m})\|^2 = \\ \sum_{j=1}^m \sum_{\substack{|\alpha|=k\\|\beta|=k}} \Phi_{j,\alpha} \overline{\Phi_{j,\beta}}\ r_1^{\frac{\alpha_1+\beta_1}{2}} \cdots r_m^{\frac{\alpha_m+\beta_m}{2}} \mathrm{e}^{\mathrm{i}\theta_1(\alpha_1-\beta_1)} \cdots \mathrm{e}^{\mathrm{i}\theta_m(\alpha_m-\beta_m)}.$$

Hence,

$$\frac{1}{(2\pi)^m}\int_0^{2\pi}\!d\theta_1\cdots\int_0^{2\pi}\!d\theta_m\int_0^{+\infty}\!dr_1\cdots\int_0^{+\infty}\!dr_m\|\Phi(\sqrt{r_1}\mathrm{e}^{i\theta_1},\cdots,\sqrt{r_m}\mathrm{e}^{i\theta_m})\|^2\ \mathrm{e}^{-r_1}\cdots\mathrm{e}^{-r_m},$$
$$=\sum_{j=1}^m\sum_{|\alpha|=k}|\Phi_{j,\alpha}|^2\int_0^{+\infty}\!dr_1\cdots\int_0^{+\infty}\!dr_m\ r_1^{\alpha_1}\cdots r_m^{\alpha_m}\ \mathrm{e}^{-r_1}\cdots\mathrm{e}^{-r_m},$$
$$=\sum_{j=1}^m\sum_{|\alpha|=k}|\Phi_{j,\alpha}|^2\alpha_1!\cdots\alpha_m! = |\Phi|_2^2.$$

□

Finally, we ready to prove the opposite comparison of the two norms in $\mathcal{H}_k$.

Lemma A.5 *For every $\Phi\in\mathcal{H}_k$,* $\quad |\Phi|_{2,k}\leq\sqrt{C_{k+m-1}^{m-1}}\ |\Phi|_{0,k}$.

Proof. Using the Lemmas A.1, A.4 we get

$$\begin{aligned}
|\Phi|_{2,k}^2 &\leq |\Phi|_{0,k}^2\int_0^{+\infty}\!dr_1\cdots\int_0^{+\infty}\!dr_m\ \frac{(r_1+\cdots+r_m)^k}{k!}\ \mathrm{e}^{-r_1}\cdots\mathrm{e}^{-r_m},\\
&= |\Phi|_{0,k}^2\int_0^{+\infty}\!dr_1\cdots\int_0^{+\infty}\!dr_m\ \sum_{|\alpha|=k}\frac{r_1^{\alpha_1}}{\alpha_1!}\cdots\frac{r_m^{\alpha_m}}{\alpha_m!}\ \mathrm{e}^{-r_1}\cdots\mathrm{e}^{-r_m},\\
&= |\Phi|_{0,k}^2\sum_{|\alpha|=k}1,\\
&= |\Phi|_{0,k}^2\ C_{m+k-1}^{m-1}.
\end{aligned}$$

□

A.2 Multiplicativity of the normalized Euclidian Norm

To handle the computations, we need in this subsection more compact notations. For $Y=(Y_1,\cdots,Y_m)\in\mathbb{C}^m$ and $\alpha=(\alpha_1,\cdots,\alpha_m)\in\mathbb{N}^m$ let us denote

$$\alpha! = \alpha_1!\cdots\alpha_m! \qquad\text{and}\qquad Y^\alpha = Y_1^{\alpha_1}\cdots Y_m^{\alpha_m}.$$

With these notations, for $\Phi\in\mathcal{H}_n$ with $\Phi(Y)=\sum_{|\alpha|=n}\Phi_\alpha Y^\alpha$ where $\Phi_\alpha\in\mathbb{R}^m$, we have

$$|\Phi|_{2,n} = \frac{1}{\sqrt{n!}}\sqrt{\sum_{|\alpha|=n}\|\Phi_\alpha\|^2\ \alpha!}.$$

We start with two technical lemmas which are used several times.

Lemma A.6 *For $\alpha\in\mathbb{N}^m$ and $n\in\mathbb{N}$ let us denote*

$$\mathfrak{B}_n^\alpha = \frac{n!}{\alpha!}$$

Then for every positive integers q and $\{p_\ell\}_{1\le\ell\le q}$ and every $\gamma\in\mathbb{N}^m$ with $|\gamma| = p_1+\cdots+p_q$, we have

$$\mathfrak{B}^{\gamma}_{p_1+\cdots+p_q} = \sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell \\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} \mathfrak{B}^{\alpha^{(1)}}_{p_1}\cdots\mathfrak{B}^{\alpha^{(q)}}_{p_q}.$$

Proof. Using Lemma A.1 we get that for every $u=(u_1,\cdots,u_m)\in\mathbb{C}^m$,

$$\begin{aligned}(u_1+\cdots+u_m)^{p_1+\cdots+p_q} &= \sum_{|\gamma|=p_1+\cdots+p_q} \mathfrak{B}^{\gamma}_{p_1+\cdots+p_q}\ u^{\gamma}, \\ &= (u_1+\cdots+u_m)^{p_1}\cdots(u_1+\cdots+u_m)^{p_q}, \\ &= \sum_{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell} \mathfrak{B}^{\alpha^{(1)}}_{p_1}\cdots\mathfrak{B}^{\alpha^{(q)}}_{p_q}\ u^{\alpha^{(1)}+\cdots+\alpha^{(q)}}.\end{aligned}$$

Identifying the powers of u we get the desired result. □

Lemma A.7 *Let $k>0, p\ge 0$ be two integers. Then for every $\gamma\in\mathbb{N}^m$ with $|\gamma|=n$ with $n:=k-1+p$*

$$(k^2+(m-1)k)\ \mathfrak{B}^{\gamma}_n = \sum_{j=1}^{m}\sum_{\substack{|\alpha|=k,|\beta|=p \\ \alpha-\sigma_j+\beta=\gamma}} (\alpha_j)^2\ \mathfrak{B}^{\alpha}_k\ \mathfrak{B}^{\beta}_p.$$

where $\sigma_j=(0,\cdots,0,1,0,\cdots,0)\in\mathbb{N}^m$ with the coefficient 1 placed at the j-th position.

Proof. Observe that for every $u=(u_1,\cdots,u_m)\in\mathbb{C}^m$,

$$\begin{aligned}(u_1+\cdots+u_m)^p \sum_{j=1}^{m}\left(u_j\frac{\partial^2}{\partial u_j^2}+\frac{\partial}{\partial u_j}\right)&\left((u_1+\cdots+u_m)^k\right) \\ &= (k^2+(m-1)k)(u_1+\cdots+u_m)^n.\end{aligned}$$

Hence, since $\left(u_j\frac{\partial^2}{\partial u_j^2}+\frac{\partial}{\partial u_j}\right)u^{\alpha} = (\alpha^j)^2 u^{\alpha-\sigma_j}$, we get

$$(k^2+(m-1)k)\sum_{|\gamma|=n}\mathfrak{B}^{\gamma}_q\ u^{\gamma} = \sum_{j=1}^{m}\sum_{\substack{|\alpha|=k, \\ |\beta|=p}} (\alpha_j)^2\ \mathfrak{B}^{\alpha}_k\ \mathfrak{B}^{\beta}_p\ u^{\alpha+\beta-\sigma_j}.$$

Identifying the powers of u we immediately get the desired result. □

We are now ready to prove the multiplicativity of the normalized euclidian norm in $\mathcal{H}_n$.

Lemma A.8 *Let q and $\{p_\ell\}_{1\le\ell\le q}$ be positive integers and let $R_q \in \mathcal{L}_q(\mathbb{R}^m)$ be q-linear. Then for every $\Phi_{p_\ell} \in \mathcal{H}_{p_\ell}$, $1 \le \ell \le q$, the polynomial $R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}]$ lies in $\mathcal{H}_n$ with $n = p_1 + \cdots + p_q$ and*

$$\Big|R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}]\Big|_{2,n} \le |||R_q|||_{\mathcal{L}_q(\mathbb{R}^m)} \left|\Phi_{p_1}\right|_{2,p_1} \cdots \left|\Phi_{p_q}\right|_{2,p_q}.$$

Proof. For $1 \le \ell \le q$, let us denote

$$\Phi_{p_\ell}(Y) = \sum_{|\alpha|=p_\ell} \Phi_\alpha^{(p_\ell)}\, Y^\alpha.$$

Since R_q is q-linear we get

$$\begin{aligned} R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}] &= \sum_{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell} Y^{\alpha^{(1)}+\cdots+\alpha^{(q)}}\ R_q[\Phi_{\alpha^{(1)}}^{(p_1)}, \cdots, \Phi_{\alpha^{(q)}}^{(p_q)}], \\ &= \sum_{|\gamma|=n} Y^\gamma \sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell \\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} R_q[\Phi_{\alpha^{(1)}}^{(p_1)}, \cdots, \Phi_{\alpha^{(q)}}^{(p_q)}]. \end{aligned}$$

Hence,

$$\begin{aligned} \Big|R_q[\Phi_{p_1}, \cdots, \Phi_{p_q}]\Big|_{2,n}^2 &= \frac{1}{n!} \sum_{|\gamma|=n} \gamma! \Big\| \sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell \\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} R_q[\Phi_{\alpha^{(1)}}^{(p_1)}, \cdots, \Phi_{\alpha^{(q)}}^{(p_q)}] \Big\|^2, \\ &\le \sum_{|\gamma|=n} \frac{1}{\mathfrak{B}_n^\gamma} \Big(\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell \\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} |||R_q|||_{\mathcal{L}_q(\mathbb{R}^m)} \|\Phi_{\alpha^{(1)}}^{(p_1)}\| \cdots \|\Phi_{\alpha^{(q)}}^{(p_q)}\| \Big)^2, \\ &\le |||R_q|||_{\mathcal{L}_q(\mathbb{R}^m)}^2 \sum_{|\gamma|=n} \Bigg[\frac{1}{\mathfrak{B}_n^\gamma} \Big(\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell \\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} (\alpha^{(1)}!\|\Phi_{\alpha^{(1)}}^{(p_1)}\|^2) \cdots (\alpha^{(q)}!\|\Phi_{\alpha^{(q)}}^{(p_q)}\|^2) \Big) \\ &\qquad \times \Big(\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell \\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} \frac{1}{\alpha^{(1)}!} \cdots \frac{1}{\alpha^{(q)}!} \Big) \Bigg], \end{aligned}$$

by the Cauchy-Schwarz formula. Then since Lemma A.6 ensures that

$$\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell \\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} \frac{1}{\alpha^{(1)}!} \cdots \frac{1}{\alpha^{(q)}!} = \frac{1}{p_1! \cdots p_q!} \sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m, |\alpha^{(\ell)}|=p_\ell \\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}} \mathfrak{B}_{p_1}^{\alpha^{(1)}} \cdots \mathfrak{B}_{p_q}^{\alpha^{(q)}} = \frac{1}{p_1! \cdots p_q!} \mathfrak{B}_n^\gamma,$$

we obtain

$$
\begin{aligned}
\Big|R_q[\Phi_{p_1},\cdots,\Phi_{p_q}]\Big|_{2,n}^2 &\leq \frac{|||R_q|||^2_{\mathcal{L}_q(\mathbb{R}^m)}}{p_1!\cdots p_q!}\sum_{|\gamma|=n}\sum_{\substack{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell\\ \alpha^{(1)}+\cdots+\alpha^{(q)}=\gamma}}(\alpha^{(1)}!\|\Phi^{(p_1)}_{\alpha^{(1)}}\|^2)\cdots(\alpha^{(q)}!\|\Phi^{(p_q)}_{\alpha^{(q)}}\|^2),\\
&= \frac{|||R_q|||^2_{\mathcal{L}_q(\mathbb{R}^m)}}{p_1!\cdots p_q!}\sum_{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell}(\alpha^{(1)}!\|\Phi^{(p_1)}_{\alpha^{(1)}}\|^2)\cdots(\alpha^{(q)}!\|\Phi^{(p_q)}_{\alpha^{(q)}}\|^2),\\
&= |||R_q|||^2_{\mathcal{L}_q(\mathbb{R}^m)}\prod_{\ell=1}^q\Big(\frac{1}{p_\ell!}\sum_{\alpha^{(\ell)}\in\mathbb{N}^m,|\alpha^{(\ell)}|=p_\ell}\alpha^{(\ell)}!\|\Phi^{(p_\ell)}_{\alpha^{(\ell)}}\|^2\Big),\\
&= |||R_q|||^2_{\mathcal{L}_q(\mathbb{R}^m)}\,|\Phi_{p_1}|^2_{2,p_1}\cdots\Big|\Phi_{p_q}\Big|^2_{2,p_q}.
\end{aligned}
$$

□

Lemma A.9 *Let $k>0$, $p\geq 0$ be two integers and let Φ_k, N_p lie respectively in $\mathcal{H}_k$ and $\mathcal{H}_p$. Then $D\Phi_k.N_p$ lies in $\mathcal{H}_n$ with $n=k-1+p$ and*

$$
|D\Phi_k.N_p|_{2,n}\leq\sqrt{k^2+(m-1)k}\;\;|\Phi_p|_{2,k}\,|N_p|_{2,p}\,.
$$

Proof. Let us denote

$$
\Phi_k(Y)=\sum_{|\alpha|=k}Y^\alpha\Phi_\alpha,\qquad N_p(Y)=\sum_{|\beta|=p}Y^\beta N_\beta
$$

where $\Phi_\alpha, N_\beta\in\mathbb{C}^m$, and $N_\beta=(N_{\beta,1},\cdots,N_{\beta,m})$. Then,

$$
D\Phi_k.N_p=\sum_{j=1}^m\sum_{\substack{|\alpha|=k\\|\beta|=p}}\alpha_j\;Y^{\alpha-\sigma_j+\beta}N_{\beta,j}\Phi_\alpha=\sum_{|\gamma|=n}Y^\gamma\sum_{j=1}^m\sum_{\substack{|\alpha|=k,|\beta|=p\\ \alpha-\sigma_j+\beta=\gamma}}\alpha_jN_{\beta,j}\Phi_\alpha.
$$

where $\sigma_j=(0,\cdots,0,1,0,\cdots,0)$ with the coefficient 1 placed at the j-th position. Hence,

$$
\begin{aligned}
|D\Phi_k.N_p|^2_{2,n} &\leq \sum_{|\gamma|=n}\frac{1}{\mathfrak{B}^\gamma_n}\Big(\sum_{j=1}^m\sum_{\substack{|\alpha|=k,|\beta|=p\\ \alpha-\sigma_j+\beta=\gamma}}\alpha_j|N_{\beta,j}|\|\Phi_\alpha\|\Big)^2\\
&\leq \sum_{|\gamma|=n}\frac{1}{\mathfrak{B}^\gamma_n}\Bigg[\Big(\sum_{j=1}^m\sum_{\substack{|\alpha|=k,|\beta|=p\\ \alpha-\sigma_j+\beta=\gamma}}\alpha!\beta!|N_{\beta,j}|^2\|\Phi_\alpha\|^2\Big),\\
&\qquad\qquad\times\Big(\sum_{j=1}^m\sum_{\substack{|\alpha|=k,|\beta|=p\\ \alpha-\sigma_j+\beta=\gamma}}(\alpha_j)^2\frac{1}{\alpha!}\frac{1}{\beta!}\Big)\Bigg]
\end{aligned}
$$

by the Cauchy-Schwarz formula. Then, since Lemma A.7 ensures that

$$\sum_{j=1}^{m}\sum_{\substack{|\alpha|=k,|\beta|=p\\ \alpha-\sigma_j+\beta=\gamma}}\alpha_j^2\,\frac{1}{\alpha!}\frac{1}{\beta!}=\frac{1}{k!p!}\sum_{j=1}^{m}\sum_{\substack{|\alpha|=k,|\beta|=p\\ \alpha-\sigma_j+\beta=\gamma}}\alpha_j^2\,\mathfrak{B}_k^\alpha\mathfrak{B}_p^\beta=\frac{1}{k!p!}(k^2+(m-1)k)\mathfrak{B}_n^\gamma,$$

we finally obtain

$$\begin{aligned}
|D\Phi_k.N_p|_{2,n}^2 &\leq \frac{1}{k!p!}(k^2+(m-1)k)\sum_{|\gamma|=n}\sum_{j=1}^{m}\sum_{\substack{|\alpha|=k,|\beta|=p\\ \alpha-\sigma_j+\beta=\gamma}}\alpha!\beta!|N_{\beta,j}|^2\|\Phi_\alpha\|^2,\\
&=\frac{1}{k!p!}(k^2+(m-1)k)\sum_{\substack{|\alpha|=k,\\ |\beta|=p}}\sum_{j=1}^{m}\alpha!\beta!\|\Phi_\alpha\|^2|N_{\beta,j}|^2,\\
&=\frac{1}{k!p!}(k^2+(m-1)k)\sum_{\substack{|\alpha|=k,\\ |\beta|=p}}\alpha!\beta!\|\Phi_\alpha\|^2\|N_\beta\|^2,\\
&=(k^2+(m-1)k)\ \ |\Phi_k|_{2,k}^2\,|N_p|_{2,p}^2\,.
\end{aligned}$$

□

A.3 Invariance of the euclidian norm under unitary linear change of coordinates

Lemma A.10 *Let Q be a unitary linear map in $\mathbb{R}^m$ or $\mathbb{C}^m$ and denote $\mathcal{T}_Q : \mathcal{H}\to\mathcal{H}, \Phi\mapsto Q^{-1}\circ\Phi\circ Q$. Then $\mathcal{T}_Q$ is a unitary linear operator in $\mathcal{H}$, i.e. for every $\Phi\in\mathcal{H}$,*

$$\left|\mathcal{T}_Q\Phi\right|_2=|\Phi|_2\,.$$

Proof. Using lemma A.4 we get that

$$\left|\mathcal{T}_Q\Phi\right|_2^2=$$
$$\frac{1}{(2\pi)^m}\int_0^{2\pi}\!d\theta_1\cdots\int_0^{2\pi}\!d\theta_m\int_0^{+\infty}\!dr_1\cdots\int_0^{+\infty}\!dr_m\ \|\Phi(Q(\sqrt{r_1}\mathrm{e}^{i\theta_1},\cdots,\sqrt{r_m}\mathrm{e}^{i\theta_m}))\|^2\mathrm{e}^{-r_1}\cdots\mathrm{e}^{-r_m}.$$

Then performing the change of coordinates

$$(r_1,\cdots,r_m,\theta_1,\cdots,\theta_m)\mapsto(r_1',\cdots,r_m',\theta_1',\cdots,\theta_m')$$

with

$$(\sqrt{r_1'}\mathrm{e}^{i\theta_1'},\cdots,\sqrt{r_m'}\mathrm{e}^{i\theta_m'})=Q(\sqrt{r_1}\mathrm{e}^{i\theta_1},\cdots,\sqrt{r_m}\mathrm{e}^{i\theta_m})$$

the Jacobian of which is equal to 1 and observing that

$$r_1'+\cdots+r_m'=\|Q(\sqrt{r_1}\mathrm{e}^{i\theta_1},\cdots,\sqrt{r_m}\mathrm{e}^{i\theta_m})\|^2=r_1+\cdots+r_m$$

we get the desired result. $\square$

References

[1] Arnold V.I. *Geometrical methods in the theory of ordinary differential equations.* Springer-Verlag, New York-Berlin, (1983).

[2] Belitskii G. Normal forms relative to a filtering action of group. *Trans. Mosc. Math. Soc.*, **40**, 1-39 (1981).

[3] Brjuno A. D. Analytic normal forms of differential equations. I, II. *Trans. Moscow Math. Soc.* **25**, 131-288 (1971) and **26** 199-239 (1972).

[4] Chossat P., Iooss G. The Couette -Taylor problem. *Appl. Math. Sci.* **102** Springer Verlag (1994).

[5] Cushman R., Sanders J. Splitting algorithm for nilpotent normal forms. *J. Dynamics and Stability of systems*, **2**, 4, 235-246 (1988).

[6] Delshams A. & Gutierrez P. Effective stability and KAM theory. *J. Differential Equations* **128**, no.2, 415–490 (1996).

[7] Duistermaat I.J. Bifurcations of periodic solutions near equilibrium points of hamiltonian systems. 57-105 Springer Lecture Notes in Mathematics bf 1057 (Bifurcation theory and applications, Montecatini 1983 Salvadori ed.).

[8] Elphick C., Tirapegui E., Brachet M.E., Coullet P., Iooss G. A simple global characterization for normal forms of singular vector fields. *Physica D* **29** p. 95–127 (1987).

[9] Fassò F. Lie series method for vector fields and Hamiltonian perturbation theory. *J. Appl. Math. Phys.* **41** 843-864 (1990).

[10] Giorgilli A. & Posilicano A. Estimates for normal forms of differential equation near an equilibrium point. *ZAMP*, vol. 39, 713-732 (1988).

[11] Guckenheimer J., Holmes P. Nonlinear oscillations, dynamical systems and bifurcations of vector fields. *Applied Mathematical Sciences* vol.**42** Berlin, Heidelberg, New York: Springer-Verlag (1983)

[12] G.Iooss, A codimension two bifurcation for reversible vector fields. *Fields Inst. Com.* **4**, 201-217 (1995).

[13] Iooss G., Adelmeyer M. *Topics in bifurcation theory and applications.* Advanced Series in Non Linear Dynamics **3** World Scientific, (1992).

[14] Iooss G., Lombardi E. Normal forms with exponentially small remainder : application to homoclinic connections for the $0^{2+}\mathrm{i}\omega$ resonance. *Submitted to CRAS*

[15] Iooss G., Pérouème M.C. Perturbed homoclinic solutions in 1:1 resonance vector fields. *Journal of Differential Equations.* Vol 102 No.1, (1993).

[16] Lanford 0. Bifurcations of periodic solutions into invariant tori. Springer Lecture Notes in Math. **322**, 159-192.

[17] Nekhoroshev, N.N. An exponential estimate of the time of stability of nearly integrable Hamiltonian systems, I. Usp. Mat. Nauk **32** 5-66 (1997); Russ. Math. Surv. **32**, 1-65 (1977).

[18] Nekhoroshev, N.N. An exponential estimate of the time of stability of nearly integrable Hamiltonian systems, II. Tr. Semin. Petrovsk. **5.**, 5-50 (1979); In:Oleineik, O.A.(ed.) Topics in Modern Mathematics, Petrovskii Semin., no. 5. New York: Consultant Bureau (1985).

[19] Lombardi E. Orbits homoclinic to exponentially small periodic orbits for a class of reversible systems. Application to water waves. *Arch. Rational Mech. Anal.* **137**, 227-304, (1997).

[20] Pöschel J. Nekoroshev estimates for quasi convex hamiltonian systems. *Math. Z.* **213**, 187-216, (1993) .

[21] Stolovitch L. Classification analytique de champs de vecteurs 1-résonnants de $(C^n, 0)$. *Asymptotic Anal.* **12** , no. 2, 91–143 (1996).

[22] Stolovitch L. Sur un théorème de Dulac. *Ann. Inst. Fourier (Grenoble)* **44** , no. 5, 1397–1433. (1994)

[23] Vanderbauwhede A. Center manifolds, normal forms and elementary bifurcations. *Dynamics Reported* **Vol. 2**, 89-169 (1992).