Approximate invariant manifolds up to exponentially small terms

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Abstract

This paper is devoted to analytic vector fields near an equilibrium for which the linearized system is split in two invariant subspaces E_0 (dim m_0), E_1 (dim m_1). Under light diophantine conditions on the linear part, we prove that there is a polynomial change of coordinate in E_1 allowing to eliminate, in the E_1 component of the vector field, all terms depending only on the coordinate $u_0 \in E_0$, up to an exponentially small remainder. This main result enables to prove the existence of analytic center manifolds up to exponentially small terms and extends to infinite dimensional vector fields. In the elliptic case, our results also proves, with very light assumptions on the linear part in E_1, that for initial data very close to a certain analytic manifold, the solution stays very close to this manifold for a very long time, which means that the modes in E_1 stay very small.

Keywords: analytic vector fields; normal forms; exponentially small remainders; center manifolds

AMS: 34M45; 34G20

1 Introduction

Let us consider an analytic vector field in the neighborhood of an equilibrium which we take at the origin. A natural idea is to try to uncouple a subset of coordinates from the other ones, by using a change of variables. This is used in particular since Poincaré and Dulac, and this is one of the main tool in the search of invariant manifolds of vector fields. Eliminating most of components of the vector field, expecting to only keep the relevant ones for the dynamics, is precisely the idea of center manifold reduction, which is widely used in many physical systems, to simplify the study of the dynamics. However this reduction is only valid when we want to eliminate the hyperbolic part of the vector field and it has the defect to kill the analyticity after the reduction process. For systems fully elliptic near the origin, it may be expected to use a change of variables to uncouple all oscillatory modes. If this were possible, and if the initial data does not excite some modes, these ones would not be awaken for all times. Unfortunately, this is not possible in general, even though for hamiltonian systems, with suitable non resonant eigenvalues of the linearized system, it is nearly the case (Arnold diffusion between invariant tori corresponding to the "normal form" system with uncoupled modes).

In the present work, we consider systems for which the linearized system is split in two invariant subspaces E_0 (dim m_0), E_1 (dim m_1). With light assumptions on the linear part, our main result is that there is a polynomial change of coordinate in E_1 allowing to eliminate, in the E_1 component of the vector field, all terms depending only on the coordinate $u_0 \in E_0$, up to an exponentially small remainder (see Theorem 1). The proof of this theorem is based on a Gevrey estimate of the divergence of the remainder, which can be exponentially small by an optimal choice of the degree of the polynomial change of coordinates.

Gevrey estimates of the divergence of remainders, to get exponentially small upper bounds after an optimal choice of the order, were already used in the theory of normal forms for Hamiltonian systems in action-angle coordinates [2], [3], [14] following the pioneering work of Nekhoroshev [11, 12]. A similar result of exponential smallness of the remainder was also obtained by Giorgilli and Posilicano in [4] for a *reversible system* with a linear part composed of harmonic oscillators. For an extension of the result of normal forms with an exponentially remainder to any analytic vector fields with semi simple linear part see [7].

Direct normalization up to exponentially small terms is not available for vector fields studied in this paper since $\mathbf{L}_1$ is not assumed to be diagonalizable. However we can eliminate from the E_1 component of the vector field all terms depending only on the coordinate $u_0 \in E_0$, up to an exponentially small remainder.

A first application of this result is when the linear part in E_1 is hyperbolic, while the linear part in E_0 has all its eigenvalues on the imaginary axis. It is well known that the center manifold reduction applies for small bounded solutions [8], which then lie on a manifold of same dimension as E_0. It is also well known that this manifold is in general not analytic [13], [20], [1], [16]. Our result allows to obtain a center manifold which is the graph of a function sum of a polynomial of degree $p = O(\delta^{-b})$ and an exponentially small function of order $O(e^{-c/\delta^b})$ where δ is the size of the ball where we study the solutions, and c and b are positive numbers (see Theorem 5). It results in particular that the loss of analyticity is located in exponentially small terms. This result extends in infinite dimensional cases, then applicable in particular for a large class of PDE's. So combining, this result on center manifolds with the normal form theorem with exponentially small remainder [7] for the E_0 component ($\mathbf{L}_0$ is diagonalizable), we can transform (1) into a new system with a "simplified" analytic leading part, perturbed by exponentially small terms. Such a transformation can be very useful when dealing with exponentially small phenomena (see [10]).

Another application, important in particular for engineering systems, is when the two linear subsystems in E_0 and E_1 have their eigenvalues on the imaginary axis. In particular, this situation happens for non linear vibrations of structures. Our result gives a sort of justification of a popular elimination process made in a formal way (see for example [9], [15], [17]), which allows to roughly state that for a class of initial data which do not excite in some sense the high frequencies (corresponding to E_1), then these ones are not awaken for all times....Our results prove, with very light diophantine assumptions (4) on the linear part in E_1, that for initial data very close to a certain analytic manifold, the solution stays very close to this manifold for a very long time, which means that the modes in E_1 stay very small (see theorem 8). This type of result is related to Arnold's diffusion for Hamiltonian systems (see a related result in [5]), while it should be noticed that we do not assume our system to be Hamiltonian, our assumptions on the eigenvalues being much lighter that usually done on such systems. In particular the linear part in E_1 is not assumed to be diagonalizable. Finally, notice the particular case studied in the same spirit by Groves and Schneider [6], for which E_0 is 2-dimensional and corresponds to a double eigenvalue in 0, while E_1 corresponds to eigenvalues all imaginary. In this example there is no need of the diophantine condition (4), but E_1 is infinite dimensional and the result we obtain here needs to be adapted. In [18], Touzé and Amabili consider the damped case with an external periodic forcing. They assume that high frequency modes lie at a growing distance from the imaginary axis. Our method might be used in such a case, to rigorously prove that the high frequency modes do not awake as t goes to infinity, provided certain non resonance condition between the forcing frequency and natural frequencies are realized, and provided the initial data is taken on a certain manifold in the spirit of Theorem 8.

2 Main results

We gather in this section the main theorems proved in this paper. Our main theorem is the following

SplittingThm **Theorem 1** *Consider the following system in* $\mathbb{R}^m$ *(resp.* $\mathbb{C}^m$*)*

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u), \tag{1}$$

basicSyst

where $u(t) \in \mathbb{R}^m$ *(resp.* $\mathbb{C}^m$*),* $\mathbf{L}$ *is a linear operator, and* $\mathbf{R}$ *is analytic in a neighborhood of the origin, such that*

$$\mathbf{R}(u) = \sum_{2 \leq k} \mathbf{R}_k[u^{(k)}], \tag{2}$$

ExpR

where $\mathbf{R}_k$ *is a* k *- linear symmetric map on* $(\mathbb{R}^m)^k$ *(resp.* $(\mathbb{C}^m)^k$*) satisfying*

$$||\mathbf{R}_k[u_1, u_2, \cdots, u_k]|| \leq \frac{c}{\rho^k}||u_1|| \cdots ||u_k||, \tag{3}$$

AnalyticR

for a certain radius of convergence $\rho > 0$ *(here* $[u^{(k)}]$ *means the* k *- uple of vectors* $[u, u, \cdots, u]$*). Assume that the linear operator* $\mathbf{L}$ *is the direct sum of two linear operators* $\mathbf{L}_0$ *on* E_0 *(dim* m_0*), and* $\mathbf{L}_1$ *on* E_1 *(dim* m_1*), such that* $\mathbf{L}_0$ *is diagonalizable with eigenvalues* $\lambda_1^{(0)}, \cdots, \lambda_{m_0}^{(0)}$ *and that there exist constants* $\gamma > 0, \tau \geq 0$ *such that*

$$|\langle \alpha, \lambda^{(0)} \rangle - \lambda_j^{(1)}| \geq \frac{\gamma}{|\alpha|^\tau} \tag{4}$$

diophCond

holds for any $\alpha \in \mathbb{N}^{m_0} \backslash \{0\}$*, and any eigenvalue* $\lambda_j^{(1)}$ *of* $\mathbf{L}_1$*.*

Then there exists a polynomial $\mathbf{\Phi} : E_0 \to E_1$ *of degree* $p = O(\delta^{-b})$ *such that the change of variables in* E_1

$$u_1 = v_1 + \mathbf{\Phi}(u_0) \tag{5}$$

Change Var

transforms the system (1) into the following system in $E_0 \times E_1$ *:*

$$\begin{aligned} \frac{du_0}{dt} &= \mathbf{L}_0 u_0 + \mathbf{R}^{(0)}(u_0, v_1), \\ \frac{dv_1}{dt} &= \mathbf{L}_1 v_1 + \mathbf{R}^{(1)}(u_0, v_1) + \boldsymbol{\rho}(u_0), \end{aligned} \tag{6}$$

newSyst

in which $\mathbf{R}^{(0)}, \mathbf{R}^{(1)}$, $\boldsymbol{\rho}$ *are analytic in their arguments, and where*

$$\mathbf{R}^{(0)}(u_0, u_1) = \mathbf{P}_0 \mathbf{R}(u_0 + v_1 + \mathbf{\Phi}(u_0)),$$

$\mathbf{P}_0$ *being the projection on* E_0 *which commutes with* $\mathbf{L}$, *and*

$$\mathbf{R}^{(1)}(u_0, v_1) = O(||v_1||(||u_0|| + ||v_1||)), \tag{7}$$

EqRun

$$\sup_{||u_0|| \leq \delta} ||\boldsymbol{\rho}(u_0)|| \leq M e^{-\frac{w}{\delta^b}}, \tag{8}$$

Eqrho

with $M, w > 0$ *depending only on* τ, m_0, c, ρ, $\mathbf{L}_1$ *and*

$$b = \frac{1}{1 + \nu\tau}$$

where ν *is the maximal index (size of Jordan blocks) of eigenvalues of* $\mathbf{L}_1$.

Remark 2 *Notice that the constants* M *and* w *do not depend on the dimension* m_1 *of the subspace* E_1 *if* $\mathbf{L}_1$ *is a priori in Jordan form. This allows to consider systems with large (even infinite) dimensions.*

Remark 3 *Since all the norms are equivalent on* $\mathbb{R}^m$ *(resp.* $\mathbb{C}^m$*), (7),(8) remains true for any norm on* $\mathbb{R}^m$ *(resp.* $\mathbb{C}^m$*). A change of norm simply change the values of* M *and* w*. So, estimates (7),(8) remain true under linear change of coordinates up to a change of values of* M *and* w*. Hence without loss of generality we can assume that the complexified space of* E_0 *and* E_1*, still denoted by* E_0 *and* E_1 *read respectively* $E_0 = \mathbb{C}^{m_0} \times \underbrace{(0, \cdots, 0)}_{m_1 \text{ times}}$ *and* $E_1 = \underbrace{(0, \cdots, 0)}_{m_0 \text{ times}} \times \mathbb{C}^{m_1}$ *and that in the canonical basis of* $\mathbb{C}^m$*,* L_0 *is diagonal and* L_1 *is under Jordan normal form.*

We deduce from the above theorem a corollary which deals with vector fields depending on parameters.

perturbed vector field

Corollary 4 *Consider the following system in* $\mathbb{R}^m$ *(resp.* $\mathbb{C}^m$*)*

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u, \mu), \tag{9}$$

perturbed syst

where $u(t) \in \mathbb{R}^m$ *(resp.* $\mathbb{C}^m$*),* $\mathbf{L}$ *is a linear operator, and* $\mathbf{R}$ *is analytic in a neighborhood of the origin in* $\mathbb{R}^m \times \mathbb{R}^q$ *(resp.* $\mathbb{C}^m \times \mathbb{R}^q$*) and such that*

$$\mathbf{R}(0, \mu) = 0, \ D_u\mathbf{R}(0, 0) = 0. \tag{10}$$

0staysSolu

Assuming the same hypothesis on $\mathbf{L}$ *as in Theorem 1, and that* 0 *is not eigenvalue of* $\mathbf{L}_1$*, then there exists a polynomial* $\mathbf{\Phi} : E_0 \times \mathbb{R}^q \to E_1$ *of degree* $p = O(\delta^{-b})$ *such that the change of variables in* E_1

$$u_1 = v_1 + \mathbf{\Phi}(u_0, \mu)$$

transforms the system (1) into the following system in $E_0 \times E_1$:

$$\begin{aligned} \frac{du_0}{dt} &= \mathbf{L}_0 u_0 + \mathbf{R}^{(0)}(u_0, v_1, \mu), \\ \frac{dv_1}{dt} &= \mathbf{L}_1 v_1 + \mathbf{R}^{(1)}(u_0, v_1, \mu) + \boldsymbol{\rho}(u_0, \mu), \end{aligned}$$

in which $\mathbf{R}^{(0)}, \mathbf{R}^{(1)}$, $\boldsymbol{\rho}$ *are analytic in their arguments, and where*

$$\mathbf{R}^{(0)}(u_0, u_1, \mu) = \mathbf{P}_0 \mathbf{R}(u_0 + v_1 + \boldsymbol{\Phi}(u_0, \mu), \mu),$$

$\mathbf{P}_0$ *being the projection on* E_0 *which commutes with* $\mathbf{L}$, *and*

$$\mathbf{R}^{(1)}(u_0, v_1, \mu) = O(||v_1||(||u_0|| + ||v_1|| + ||\mu||)),$$

$$\sup_{||u_0|| + ||\mu|| \leq \delta} ||\boldsymbol{\rho}(u_0, \mu)|| \leq M e^{-\frac{w}{\delta^b}},$$

with $M, w > 0$ *depending only on* τ, m_0, c, ρ, $\mathbf{L}_1$ *and* b *is as in Theorem 1.*

Another application of theorem 1, is the existence of analytic center manifolds up to exponentially small term. More precisely, consider the case when the spectrum of $\mathbf{L}_0 \subset i\mathbb{R}$, and $\mathbf{L}_1$ is hyperbolic, i.e. the eigenvalues of $\mathbf{L}_1$ lie at a distance $\gamma > 0$ from the imaginary axis. Then in finite dimension we have the following

centermanifoldRm **Theorem 5 (Center manifold analytic up to exp. small terms)** *Consider the analytic system (1) in* $\mathbb{R}^m$ *and assume that* $\mathbf{L}_0$ *is diagonalizable with all its eigenvalues on the imaginary axis, and assume that* $\mathbf{L}_1$ *has its eigenvalues at least at a distance* $\gamma > 0$ *from the imaginary axis.*

Then for any $k \geq 2$, *there exists a polynomial* $\boldsymbol{\Phi} : E_0 \to E_1$ *of degree* $O(1/\delta)$, *with* $\boldsymbol{\Phi}(0) = 0$, $D\boldsymbol{\Phi}(0) = 0$, *a neighborhood* $\mathcal{O}$ *of* 0 *in* $\mathbb{R}^m$, *and a map* $\boldsymbol{\Psi} \in \mathcal{C}^k(E_0, E_1)$ *which is* $O(e^{-\frac{C}{\delta}})$ *for* $||u_0||_{E_0} \leq \delta$ *and a certain constant* $C > 0$, *such that the manifold*

$$\mathcal{M}_0 = \{u_0 + \boldsymbol{\Phi}(u_0) + \boldsymbol{\Psi}(u_0) \; ; \; u_0 \in E_0\} \tag{11}$$

has the following properties.

(a) $\mathcal{M}_0$ *is locally invariant, i.e., if* u *is a solution of (1) satisfying* $u(0) \in \mathcal{M}_0 \cap \mathcal{O}$ *and* $u(t) \in \mathcal{O}$ *for all* $t \in [0, T]$, *then* $u(t) \in \mathcal{M}_0$ *for all* $t \in [0, T]$.

(b) $\mathcal{M}_0$ *contains the set of bounded solutions of (1) staying in* $\mathcal{O}$ *for all* $t \in \mathbb{R}$*, i.e., if* u *is a solution of (1) satisfying* $u(t) \in \mathcal{O}$ *for all* $t \in \mathbb{R}$*, then* $u(0) \in \mathcal{M}_0$*.*

Remark 6 *The interest of Theorem 5 is that it implies that the reduced system on the center manifold is analytic, up to exponentially small terms. This property is clearly still true after the polynomial new change of variables which put the reduced system under normal form (the usual one). In considering the analytic part of the reduced vector field, this normal form may be derived up to an optimal degree, as made in [7], since* $\mathbf{L}_0$ *is diagonalizable. This may be helpful when dealing with exponentially small phenomena associated with the original system (1).*

Remark 7 *This theorem is also true in the infinite dimensional case (see Theorem 20 in subsection 4.2)*

A last application of theorem 1, important in particular for engineering systems, is when the two linear subsystems in E_0 and E_1 have both their eigenvalues on the imaginary axis. More precisely in section 5, we prove

EllipticThm

Theorem 8 *(Elliptic vector fields) Assume that assumptions of Theorem 1 hold, and in addition that* $\mathbf{L}_1$ *has only imaginary eigenvalues. Then for any small initial data* $u(0)$ *chosen on the manifold* $\mathcal{M}'_0 = \{u = u_0 + \mathbf{\Phi}(u_0); u_0 \in E_0\}$ *the solution* $u(t)$ *stays at a distance* $O(e^{-\frac{C}{\delta^b}})$ *to* $\mathcal{M}'_0$ *for* $t \in [0,T]$*, with* $T = O(\delta^{-(b+1/\nu)})$*, where* $b = (1+\nu\tau)^{-1}$ *and* ν *is the maximal index of eigenvalues of* $\mathbf{L}_1$*.*

Remark 9 *We observe (see 39) that in going up to exponentially small terms in Theorem 1, we win the exponential smallness of* $||v_1(t)||$ *for a long range of time, without more precise assumption on* $\mathbf{L}_1$*. If we assume more specific properties of the system, we may have a longer range of time for the validity of this exponential smallness. First, if* $\mathbf{L}_1$ *is diagonalizable this range of time is* $O(\delta^{-[1+(1+\tau)^{-1}]})$*.*

Remark 10 *Let assume in addition that (1) is a* reversible system *such that* $\mathbf{L}$ *has only pairs of simple imaginary eigenvalues, satisfying the* γ*,* τ*-homologically diophantine assumption defined in [7]: for every* $\alpha \in \mathbb{N}^m$*,* $|\alpha| \geq 2$

$$|\langle \alpha, \lambda \rangle - \lambda_j| \geq \frac{\gamma}{|\alpha|^\tau} \text{ when } \langle \alpha, \lambda \rangle - \lambda_j \neq 0,$$

and $\langle \alpha, \lambda \rangle - \lambda_j = 0$ only for the trivial cases $2\lambda_j + \overline{\lambda_j} - \lambda_j = 0$ (non resonance assumption). In such a case, we can use the normal form theorem of [7] which gives a normal form up to an exponentially small term, which improves the final form of Theorem 1 since the coupling between the subsystems in E_0 and in E_1 only appears in exponentially small terms.Taking $v_1(0) = 0$, it is easy to show that $||v_1(t)||$ stays exponentially small now for an exponentially long time (analogue to Arnold diffusion).

3 Proof of the main theorem

We first deduce corollary 4 from theorem 1 and then we prove this theorem.

Proof of Corollary 4. Let us define

$$\widetilde{u} = (u, \mu) \in \mathbb{R}^m \times \mathbb{R}^q,$$

then the system reads

$$\frac{d\widetilde{u}}{dt} = \widetilde{\mathbf{L}}\widetilde{u} + \widetilde{\mathbf{R}}(\widetilde{u}), \tag{12}$$ extendSyst

with

$$\widetilde{\mathbf{L}}\widetilde{u} = (\mathbf{L}u, 0), \; \widetilde{\mathbf{R}}(\widetilde{u}) = (\mathbf{R}(u, \mu), 0).$$

Then, it is clear that the system (12) satisfies all assumptions of Theorem 1. In particular, the operator $\widetilde{\mathbf{L}}$ is the direct sum of $\widetilde{\mathbf{L}}_0$ and $\widetilde{\mathbf{L}}_1$ defined by

$$\begin{aligned} \widetilde{\mathbf{L}}_0\widetilde{u}_0 &= (\mathbf{L}_0 u_0, 0), \text{ for } \widetilde{u}_0 \in \widetilde{E_0} = E_0 \times \mathbb{R}^q, \\ \widetilde{\mathbf{L}}_1\widetilde{u}_1 &= (\mathbf{L}_1 u_1, 0), \text{ for } \widetilde{u}_1 \in \widetilde{E_1} = E_1 \times \{0\}, \end{aligned}$$

and the eigenvalues of $\widetilde{\mathbf{L}}_1$ are those of $\mathbf{L}_1$, while the eigenvalues of $\widetilde{\mathbf{L}}_0$ are those of $\mathbf{L}_0$ with 0 still semi-simple, having an additional q - dimensional eigenspace: $(0, \mu), \mu \in \mathbb{R}^q$ and the diophantine condition (4) is still satisfied. Hence the Corollary is proved.

Proof of Theorem 1. In the proof below we use several algebraic properties which were proved in [7]. Performing the change of coordinates $u = u_0 + u_1 + \Phi(u_0)$, we check that (1) is equivalent to (6) close to the origin if and only if

$$\begin{aligned} \boldsymbol{P_0 R}(u_0 + v_1 + \boldsymbol{\phi}(u_0)) &= \boldsymbol{R}^{(0)}(u_0, v_1), \\ D\boldsymbol{\Phi}(u_0).\boldsymbol{L}_0 u_0 - \boldsymbol{L}_1 \boldsymbol{\Phi}(u_0) &= -D\boldsymbol{\Phi}(u_0).\boldsymbol{R}^{(0)}(u_0, v_1) - \boldsymbol{\rho}(u_0) \\ &\quad + \boldsymbol{P_1 R}\big(u_0 + u_1 + \boldsymbol{\Phi}(u_0)\big) - \boldsymbol{R}^{(1)}(u_0, v_1). \end{aligned}$$

Then Setting $v_1 = 0$ and using (7), we obtain the following basic identity

$$D\mathbf{\Phi}(u_0)\mathbf{L}_0 u_0 - \mathbf{L}_1\mathbf{\Phi}(u_0) = -D\mathbf{\Phi}(u_0)\mathbf{P}_0\mathbf{R}(u_0 + \mathbf{\Phi}(u_0)) \\ +\mathbf{P}_1\mathbf{R}(u_0 + \mathbf{\Phi}(u_0)) - \boldsymbol{\rho}(u_0). \tag{13}$$

basicIdent

Let decompose the polynomial $\mathbf{\Phi}$ into a sum of homogeneous polynomials of increasing degrees

$$\mathbf{\Phi}(u_0) = \sum_{2\leq k\leq p} \mathbf{\Phi}_k[u_0^{(k)}]$$

with k - linear symmetric maps $\mathbf{\Phi}_k:(E_0)^k \to E_1$. For convenience we denote by $\mathbf{\Phi}_1(u_0) \equiv u_0$ which takes its values in E_0 (contrary to $\mathbf{\Phi}_k$ for $k \geq 2$, which takes its values in E_1). Then we have for $2 \leq n \leq p$

$$D\mathbf{\Phi}_n[u_0^{(n)}]\mathbf{L}_0 u_0 - \mathbf{L}_1\mathbf{\Phi}_n[u_0^{(n)}] = \mathbf{F}_n[u_0^{(n)}], \tag{14}$$

homologicEqu

with

$$\begin{aligned}\mathbf{F}_n[u_0^{(n)}] &= \sum_{\substack{2\leq q\leq n\\ k_1+\cdots+k_q=n,\ \ k_j\geq 1}} \mathbf{P}_1\mathbf{R}_q[\mathbf{\Phi}_{k_1},\cdots,\mathbf{\Phi}_{k_q}] + \\ &- \sum_{\substack{2\leq \ell\leq n-1,\ 2\leq q\leq n-\ell+1\\ k_1+\cdots+k_q=n-\ell+1,\ \ k_j\geq 1}} D\mathbf{\Phi}_\ell[u_0^{(\ell)}]\mathbf{P}_0\mathbf{R}_q[\mathbf{\Phi}_{k_1},\cdots,\mathbf{\Phi}_{k_q}].\end{aligned}$$

Equation (14) is of the form

$$\mathcal{A}\mathbf{\Phi}_n = \mathbf{F}_n$$

with the homological operator $\mathcal{A}$ defined on the vector space of polynomials $\mathbf{\Phi} : E_0 \to E_1$, by

$$\mathcal{A}\mathbf{\Phi} = D\mathbf{\Phi}(u_0)\mathbf{L}_0 u_0 - \mathbf{L}_1\mathbf{\Phi}(u_0). \tag{15}$$

homologic

We then need to introduce the scalar product in the space $\mathcal{H}$ of polynomials of a variable in E_0, taking values in $\mathbb{C}^m$ (which could be in the complexified space of the subspace E_1 or E_0) as done in [7].

Given two polynomials $\mathbf{\Phi}$ and $\mathbf{\Phi}'$ we define their scalar product by

$$\langle \mathbf{\Phi}, \mathbf{\Phi}' \rangle_{\mathcal{H}} := \sum_{1\leq j\leq n} \langle \mathbf{\Phi}_j, \mathbf{\Phi}'_j \rangle$$

with $\mathbf{\Phi} = (\mathbf{\Phi}_1, \cdots, \mathbf{\Phi}_n)$, $\mathbf{\Phi}' = (\mathbf{\Phi}'_1, \cdots, \mathbf{\Phi}'_n)$, and where for a pair of polynomials $P, Q : E_0 \to \mathbb{C}$,

$$\langle P, Q \rangle = \overline{P}(\partial_X)Q(X)|_{X=0},$$

where by definition

$$\overline{P}(X) = \overline{P(\overline{X})}.$$

Then the associated euclidian norm is defined by

$$|\mathbf{\Phi}|_2 := \sqrt{\langle \mathbf{\Phi}, \mathbf{\Phi} \rangle_{\mathcal{H}}}.$$

It is clear that for any $n \geq 1$, the linear operator $\mathcal{A}$ leaves invariant the subspace $\mathcal{H}_n$ of homogeneous polynomials of degree n, and we have the following Lemma proved in Appendix:

Eric's Lemma

Lemma 11 *The operator $\mathcal{A}$ is invertible in the subspace $\mathcal{H}_n$ and there exists a constant a, depending only on γ and $\mathbf{L}_1$, such that*

$$|||\mathcal{A}|_{\mathcal{H}_n}^{-1}|||_2 := \sup_{|\mathbf{\Phi}|_2=1} |\mathcal{A}|_{\mathcal{H}_n}^{-1}\mathbf{\Phi}| \leq a n^{\tau'},$$

where $\tau' = \nu\tau$, and ν is the maximal index of the eigenvalues of $\mathbf{L}_1$.

This lemma is proved in Appendix A.

Remark 12 *If $\mathbf{L}_1$ is in Jordan form, the constant a depends only on γ and ν. If $\mathbf{L}_1$ is diagonal then $\tau' = \tau$ and $a = 1/\gamma$.*

Moreover, defining the norm

$$\phi_n := |\mathbf{\Phi}|_{2,n} := \frac{1}{\sqrt{n!}} |\mathbf{\Phi}|_2, \text{ for } \mathbf{\Phi} \in \mathcal{H}_n,$$

we have the following lemma, proved in [7] (see Lemmas 2.10, 2.11):

Lemma 13

(i) For $k_1 + ... + k_q = n$

$$|\mathbf{R}_q[\mathbf{\Phi}_{k_1}, \cdots, \mathbf{\Phi}_{k_q}]|_{2,n} \leq \frac{c}{\rho^q} \phi_{k_1} \cdots \phi_{k_q},$$

(ii) for $2 \leq \ell \leq p$, $\ell + k = n + 1$, and any $\mathbf{N}_k \in \mathcal{H}_k$

$$|D\mathbf{\Phi}_\ell \cdot \mathbf{N}_k|_{2,n} \leq \sqrt{\ell^2 + (m_0 - 1)\ell}\ \phi_\ell |\mathbf{N}_k|_{2,k} \leq \ell\sqrt{m_0}\ \phi_\ell |\mathbf{N}_k|_{2,k}.$$

Then, the proof of Theorem 1 is performed in several steps giving respectively estimates of ϕ_n, $\|\sum \mathbf{\Phi}(u_0)\|$, and ρ_0 gathered in the following lemmas:

Lemphin **Lemma 14** *There exists $K > 0$ depending only on c, c_{01}, ρ, m_0, a such that for every n with $1 \leq n \leq p$,*

$$\phi_n \leq \sqrt{m_0}\ K^{n-1}(n!)^{1+\tau'}. \tag{16}$$ estimPhi_na

where $c_{01} := max\Big(|||\boldsymbol{P}_0|||, |||\boldsymbol{P}_1|||\Big)$.

LemSigmaphiopt **Lemma 15** *Let us choose p such that*

$$p = p_{\text{opt}} := \left[\frac{1}{(2\delta K)^b}\right],\ b = \frac{1}{1+\tau'}, \tag{17}$$ lowerBounda

where $[\cdot]$ denotes the integer part of a number. Then for $||u_0|| \leq \delta$ we have

$$\left\| \sum_{1\leq k\leq p_{\text{opt}}} \mathbf{\Phi}_k(u_0) \right\| \leq 2\delta\sqrt{m_0}.$$

rhoopt **Lemma 16** *The remainder $\boldsymbol{\rho}$ satisfies*

$$\boldsymbol{\rho}(u_0) = \mathfrak{R}_1(u_0) + \mathfrak{R}_2(u_0) + \mathfrak{R}_3(u_0) + \mathfrak{R}_4(u_0),$$

with

$$\begin{aligned}
\mathfrak{R}_1(u_0) &= \sum_{p+1\leq q} \mathbf{P}_1\mathbf{R}_q\left[\left(\sum_{1\leq k\leq p}\mathbf{\Phi}_k(u_0)\right)^{(q)}\right],\\
\mathfrak{R}_2(u_0) &= -\sum_{2\leq \ell\leq p,\ p+1\leq q} D\mathbf{\Phi}_\ell[u_0^{(\ell)}]\mathbf{P}_0\mathbf{R}_q\left[\left(\sum_{1\leq k\leq p}\mathbf{\Phi}_k(u_0)\right)^{(q)}\right],
\end{aligned}$$

$$\mathfrak{R}_3(u_0) = \sum_{\substack{2\leq q\leq p,\ 1\leq k_j\leq p\\ k_1+\cdots+k_q\geq p+1}} \mathbf{P}_1\mathbf{R}_q[\mathbf{\Phi}_{k_1}(u_0), \cdots, \mathbf{\Phi}_{k_q}(u_0)],$$

$$\mathfrak{R}_4(u_0) = -\sum_{\substack{2\leq \ell\leq p,\ q\leq p,\ 1\leq k_j\leq p\\ k_1+\ldots k_q\geq p-l+2}} D\mathbf{\Phi}_\ell[u_0^{(\ell)}]\mathbf{P}_0\mathbf{R}_q[\mathbf{\Phi}_{k_1}(u_0), \cdots, \mathbf{\Phi}_{k_q}(u_0)],$$

and for $p = p_{\text{opt}}$, it satisfies

$$\sup_{||u_0||\leq\delta} ||\boldsymbol{\rho}(u_0)|| \leq Me^{-\frac{w}{\delta^b}}, \tag{18}$$ Eqrhoa

with $M, w > 0$ depending only on $\tau, m_0, c, \rho,\ \mathbf{L}_1$.

Proof of lemma 14. *Estimate of* ϕ_n. We obtain from (14)

$$\phi_n \le acc_{01} n^{\tau'} \left\{ \sum_{\substack{2\le q\le n \\ k_1+\cdots+k_q=n,\ k_j\ge 1}} \frac{1}{\rho^q}\phi_{k_1}\cdots\phi_{k_q} \right. \\ \left. + \sum_{\substack{2\le \ell\le n-1,\ 2\le q\le n-\ell+1 \\ k_1+\cdots+k_q=n-\ell+1,\ k_j\ge 1}} \frac{\ell\sqrt{m_0}}{\rho^q}\phi_\ell\phi_{k_1}\cdots\phi_{k_q} \right\}. \tag{19}$$ firstEstimate

Then, notice that by construction

$$\phi_1 = \sqrt{m_0}.$$

For suppressing the factor $n^{\tau'}$ in the inequality (19), we introduce the following sequence α_n defined by

$$\alpha_1 = 1 \quad \text{and} \quad \phi_n = \sqrt{m_0} K_1^{n-1} (n!)^{\tau'} \alpha_n, \qquad \text{for} \quad n \ge 1$$

where K_1 will be chosen later. Using the following inequalities proved in [7]-lemma 2.12,

$$\frac{k_1! \cdots + k_q!}{(n-1)!} \le 1 \qquad \text{for } 2 \le q \le n,\ k_1 + \cdots + k_q = n,$$

and

$$\frac{\ell! k_1! \cdots k_q!}{(n-1)!} = \frac{\ell! k_1! \cdots k_q!}{(n-\ell)!}\ \frac{(n-\ell)!}{(n-1)!} \le 1, \quad \text{for} \begin{cases} 2 \le \ell \le n-1, \\ 2 \le q \le n-\ell+1, \\ k_1 + \cdots + k_q = n-\ell+1, \end{cases}$$

$$\alpha_n \le \frac{acc_{01} K_1}{\sqrt{m_0}} \left\{ \sum_{\substack{2\le q\le n \\ k_1+\cdots+k_q=n,\ k_j\ge 1}} \left(\frac{\sqrt{m_0}}{K_1\rho}\right)^q \alpha_{k_1}\cdots\alpha_{k_q} + \right. \\ \left. \sum_{\substack{2\le \ell\le n-1,\ 2\le q\le n-\ell+1 \\ k_1+\cdots+k_q=n-\ell+1,\ k_j\ge 1}} m_0 \left(\frac{\sqrt{m_0}}{K_1\rho}\right)^q \ell\alpha_\ell\alpha_{k_1}\cdots\alpha_{k_q} \right\},$$

and by choosing

$$K_1 \ge \frac{acc_{01} m_0^{3/2}}{\rho^2}, \tag{20}$$ cond1

we finally get

$$\begin{aligned}\alpha_n \quad \leq \quad & \sum_{\substack{2\leq q\leq n \\ k_1+\cdots+k_q=n,\ k_j\geq 1}} \left(\frac{\sqrt{m_0}}{K_1\rho}\right)^{q-2} \alpha_{k_1}\cdots\alpha_{k_q} + \\ & + \sum_{\substack{2\leq \ell\leq n-1,\ 2\leq q\leq n-\ell+1 \\ k_1+\cdots+k_q=n-\ell+1,\ k_j\geq 1}} \left(\frac{\sqrt{m_0}}{K_1\rho}\right)^{q-2} \ell\alpha_\ell\alpha_{k_1}\cdots\alpha_{k_q}.\end{aligned} \tag{21}$$

inegAlpha_n

Now, the idea is to use the majorizing sequence β_n defined by

$$\beta_1 = 1,\ \ \beta_n = \Theta^{n-2}(n-2)! \text{ for } n \geq 2,$$

the number Θ being chosen later, large enough. It is clear that

$$\begin{aligned}\alpha_1 &= 1 \leq \beta_1, \\ \alpha_2 &\leq \alpha_1^2 = 1 \leq \beta_2.\end{aligned}$$

Assuming that $\alpha_k \leq \beta_k$ for $1 \leq k \leq n-1$, we intend to prove that $\alpha_n \leq \beta_n$. Indeed, by replacing α_k by β_k, $1 \leq k \leq n-1$, in the right hand side of (21) we find that

$$\alpha_n \leq \sum_{2\leq q\leq n} \left(\frac{\sqrt{m_0}}{K_1\rho}\right)^{q-2} \Pi_{q,n} + \sum_{2\leq \ell\leq n-1,\ 2\leq q\leq n-\ell+1} \left(\frac{\sqrt{m_0}}{K_1\rho}\right)^{q-2} \ell\beta_\ell \Pi_{q,n-\ell+1},$$

with, for $2 \leq q \leq n$

$$\Pi_{q,n} := \sum_{k_1+\cdots+k_q=n,\ \ k_j\geq 1} \beta_{k_1}\cdots\beta_{k_q}.$$

It is shown in [7]-lemma 2.13 that

$$\begin{aligned}&\Pi_{2,n} \leq \tfrac{2}{\Theta}\beta_n, && \text{for } n \geq 3, \\ &\Pi_{q,n} \leq \tfrac{2}{\Theta^{q-2}}\beta_n, && \text{for } 3 \leq q \leq n.\end{aligned}$$

Hence

$$\begin{aligned}\alpha_n \quad \leq \quad & \left(\frac{2}{\Theta} + \sum_{3\leq q\leq n} 2\left(\frac{\sqrt{m_0}}{\Theta K_1\rho}\right)^{q-2}\right)\beta_n + \\ & + \frac{2}{\Theta}\sum_{2\leq \ell\leq n-1} \ell\beta_\ell\beta_{n-\ell+1} + \sum_{2\leq \ell\leq n-1,\ 3\leq q\leq n-\ell+1} 2\left(\frac{\sqrt{m_0}}{\Theta K_1\rho}\right)^{q-2} \ell\beta_\ell\beta_{n-\ell+1}.\end{aligned}$$

We choose now Θ and K_1 such that

$$\frac{1}{\Theta}+\frac{\sqrt{m_0}}{\Theta K_1\rho-\sqrt{m_0}}\le\frac{1}{4}, \qquad (22) \quad \boxed{\texttt{cond2}}$$

then

$$\alpha_n\le\frac{1}{2}\left(\beta_n+\sum_{2\le\ell\le n-1}\ell\beta_\ell\beta_{n-\ell+1}\right).$$

Since it is shown in [7] (p.22) that

$$\sum_{2\le\ell\le n-1}\frac{\ell(\ell-2)!(n-\ell-1)!}{(n-2)!}\le\frac{5}{2}\text{ for }n\ge 3,$$

we then obtain

$$\alpha_n\le\frac{1}{2}\left(1+\frac{5}{2\Theta}\right)\beta_n.$$

Hence, it suffices to take

$$\Theta\ge\frac{5}{2} \qquad (23) \quad \boxed{\texttt{cond3}}$$

for having $\alpha_n\le\beta_n$, which finally proves that

$$\phi_n\le\sqrt{m_0}K_1(\Theta K_1)^{n-2}(n!)^{\tau'}(n-2)!,\quad n\ge 2,$$

provided that conditions (20), (22), (23) on Θ and K_1 are satisfied. We can take for example

$$\Theta=8,\ K=8K_1=\max\left\{\frac{9\sqrt{m_0}}{\rho},\frac{8acc_{01}m_0^{3/2}}{\rho^2}\right\}. \qquad (24) \quad \boxed{\texttt{Kdef}}$$

The first conclusion is that there exists $K>0$ depending only on c, c_{01}, ρ, m_0, a such that

$$\phi_n\le\sqrt{m_0}K^{n-1}(n!)^{1+\tau'}\text{ for }1\le n\le p. \qquad (25) \quad \boxed{\texttt{estimPhi\_n}}$$

■

Proof of Lemma 15. Estimate of $\sum_{n=1}^{p}\mathbf{\Phi}_n$.

First we have for $||u_0|| \leq \delta$ and from Lemma 2.10 of [7]

$$\begin{aligned}
\left\| \sum_{1\leq n\leq p} \mathbf{\Phi}_n(u_0) \right\| &\leq \sum_{1\leq n\leq p} \phi_n \delta^n \\
&\leq \sum_{1\leq n\leq p} \frac{\sqrt{m_0}}{K}(\delta K)^n (n!)^{1+\tau'} \\
&\leq \delta\sqrt{m_0} \sum_{1\leq n\leq p} (\delta K p^{1+\tau'})^{n-1}.
\end{aligned}$$

Let us choose p such that

$$p = \left[\frac{1}{(2\delta K)^b}\right], \; b = \frac{1}{1+\tau'}, \tag{26}$$ lowerBound

where $[\cdot]$ denotes the integer part of a number, then

$$\left\| \sum_{1\leq n\leq p} \mathbf{\Phi}_n(u_0) \right\| \leq 2\delta\sqrt{m_0},$$

and for $\delta < \rho/(2\sqrt{m_0})$ we have $||u_0 + \mathbf{\Phi}(u_0)|| < \rho$. ■

Proof of Lemma 16. *Estimate of the remainder* $\boldsymbol{\rho}(u_0)$.

We estimate each term $\mathfrak{R}_k$ separately.

Step 1. First we estimate $\mathfrak{R}_1(u_0)$. We have for every $\delta < \rho/(4\sqrt{m_0})$, and p satisfying (26)

$$\begin{aligned}
||\mathfrak{R}_1(u_0)|| \leq \sum_{q\geq p+1} cc_{01} \left(\tfrac{2\delta\sqrt{m_0}}{\rho}\right)^q &\leq cc_{01} \left(\tfrac{2\delta\sqrt{m_0}}{\rho}\right)^p \\
&\leq 2cc_{01} \left(\tfrac{1}{2}\right)^{p+1} \\
&\leq 2cc_{01} \left(\tfrac{1}{2}\right)^{\frac{1}{(2\delta K)^b}} \\
&\leq 2cc_{01} e^{-\frac{\ln 2}{(2\delta K)^b}}.
\end{aligned} \tag{27}$$ estimR_1

Step 2. For estimating $\mathfrak{R}_2(u_0)$ we have for $\delta < \rho/(4\sqrt{m_0})$

$$\begin{aligned}
||\mathfrak{R}_2(u_0)|| &\leq cc_{01} \sum_{2\leq \ell\leq p,\; q\geq p+1} \ell\phi_\ell \sqrt{m_0}\delta^{\ell-1} \left(\frac{2\delta\sqrt{m_0}}{\rho}\right)^q \\
&\leq cc_{01} \sum_{2\leq \ell\leq p} \ell\phi_\ell \sqrt{m_0}\delta^{\ell-1} \left(\frac{2\delta\sqrt{m_0}}{\rho}\right)^p.
\end{aligned}$$

Now, for p satisfying (26)

$$\begin{aligned}\sum_{2\le\ell\le p}\ell\phi_\ell\delta^{\ell-1} &\le \sum_{2\le n\le p}\sqrt{m_0}(K\delta)^{n-1}n(n!)^{1+\tau'}\\ &\le \sum_{2\le n\le p}\sqrt{m_0}p(K\delta p^{1+\tau'})^{n-1}\\ &\le \sqrt{m_0}2K\delta p^{2+\tau'}\le\sqrt{m_0}p\le\frac{\sqrt{m_0}}{(2\delta K)^b}.\end{aligned}$$

Hence, for $\delta<\delta_1=\min\{\rho/(4\sqrt{m_0}),\frac{1}{2K(2e)^{1+\tau'}}\}$ an using that that for $x\ge 2$, $\ln x\le x^{\frac{\ln 2}{2}}$, we get that

$$\begin{aligned}||\mathfrak{R}_2(u_0)|| &\le \frac{cc_{01}m_0}{(2\delta K)^b}\left(\frac{2\delta\sqrt{m_0}}{\rho}\right)^p\\ &\le \frac{2cc_{01}m_0}{(2\delta K)^b}\left(\frac{1}{2}\right)^{p+1}\\ &\le \frac{2cc_{01}m_0}{(2K\delta)^b}e^{-\frac{\ln 2}{(2\delta K)^b}}\\ &\le 2cc_{01}m_0e^{-\frac{\ln 2}{2(2\delta K)^b}}. \qquad (28) \quad \boxed{\texttt{estimR\_2}}\end{aligned}$$

Step 3. We now estimate $\mathfrak{R}_3(u_0)$:

$$||\mathfrak{R}_3(u_0)||\le cc_{01}\sum_{\substack{2\le q\le p,\ 1\le k_j\le p\\ p+1\le k_1+\cdots+k_q=n\le qp}}\left(\frac{\sqrt{m_0}}{K\rho}\right)^q(\delta K)^n(k_1!)^{1+\tau'}...(k_q!)^{1+\tau'}$$

and from (24) we have $\frac{\sqrt{m_0}}{K\rho}=r\le 1/9$ and from (26) we have $K\delta\le\frac{1}{2p^{1+\tau'}}$. Hence,

$$\begin{aligned}||\mathfrak{R}_3(u_0)|| &\le cc_{01}\sum_{\substack{2\le q\le p,\ 1\le k_j\le p\\ p+1\le k_1+\cdots+k_q=n\le qp}} r^q\frac{1}{2^n}(\frac{k_1!}{p^{k_1}})^{1+\tau'}...(\frac{k_q!}{p^{k_q}})^{1+\tau'}\\ &\le \frac{cc_{01}}{2^{p+1}}\sum_{2\le q\le p}r^q\left(\sum_{1\le j\le p}\left(\frac{j!}{p^j}\right)^{1+\tau'}\right)^q.\end{aligned}$$

Moreover, we have

$$\sum_{1\le j\le p}\left(\frac{j!}{p^j}\right)^{1+\tau'}\le\frac{1}{p^{1+\tau'}}+\sum_{2\le j\le p}\frac{1}{p^{1+\tau'}}=\frac{1}{p^{\tau'}},$$

hence, since $\frac{r}{p^{\tau'}} \leq r \leq 1/9$

$$\begin{aligned} ||\mathfrak{R}_3(u_0)|| &\leq \frac{cc_{01}}{2^{p+1}} \sum_{2\leq q\leq p} \left(\frac{r}{p^{\tau'}}\right)^q \\ &\leq \frac{cc_{01}}{72\cdot 2^{p+1}} \leq \frac{cc_{01}}{72} e^{-\frac{\ln 2}{(2\delta K)^b}}. \end{aligned} \tag{29}$$ $\boxed{\texttt{estimR\_3}}$

Step 4. Finally, for the estimate of $\mathfrak{R}_4(u_0)$ we have by the same way

$$\begin{aligned} ||\mathfrak{R}_4(u_0)|| &\leq cc_{01}\frac{m_0 p^{1+\tau'}}{2^{p+1}} \sum_{\substack{2\leq \ell\leq p,\ q\leq p,\ 1\leq k_j\leq p \\ \ell+k_1+\cdots+k_q=n+1\geq p+2}} r^q \ell (\frac{\ell!}{p^\ell})^{1+\tau'} (\frac{k_1!}{p^{k_1}})^{1+\tau'} ... (\frac{k_q!}{p^{k_q}})^{1+\tau'} \\ &\leq cc_{01}\frac{m_0 p^{1+\tau'}}{2^{p+1}} \sum_{2\leq q\leq p,\ 2\leq \ell\leq p} r^q \ell (\frac{\ell!}{p^\ell})^{1+\tau'} \left(\sum_{1\leq j\leq p}\left(\frac{j!}{p^j}\right)^{1+\tau'}\right)^q \\ &\leq cc_{01}\frac{m_0 p^{1+\tau'}}{2^{p+1}} \sum_{2\leq q\leq p} \left(\frac{r}{p^{\tau'}}\right)^q \frac{p}{p^{\tau'}} \\ &\leq \frac{cc_{01}}{72}\frac{m_0 p^{2(1-\tau')}}{2^{p+1}} \\ &\leq \frac{cc_{01}}{72} m_0 p^{2(1-\tau')} e^{-\frac{\ln 2}{(2\delta K)^b}} \\ &\leq \frac{cc_{01} m_0}{72} e^{-\frac{\ln 2}{2(2\delta K)^b}}, \end{aligned} \tag{30}$$ $\boxed{\texttt{estimR\_4}}$

provided that $\delta \leq \delta_0 = \min\{\delta_1, \delta_2\}$ where δ_2 is small enough, such that

$$4(1-\tau')\ln(2K\delta_2)^{-b} \leq (2K\delta_2)^{-b}\ln 2,$$

this condition being empty for $\tau' \geq 1$. ■

Now collecting estimates (27), (28), (29), (30), proves Theorem 1.

4 Analytic center manifolds up to Exponentially small terms

4.1 Finite dimensional case. Proof of theorem 5

This subsection is entirely devoted to the proof of Theorem 5 which ensures the existence of analytic center manifolds up to Exponentially small terms.

We notice that the diophantine condition (4) is automatically satisfied, since

$$\langle \alpha, \lambda^{(0)} \rangle \in i\mathbb{R},$$

and

$$|\langle \alpha, \lambda^{(0)} \rangle - \lambda_j^{(1)}| \geq \gamma$$

for all $\alpha \in \mathbb{N}^{m_0} \backslash \{0\}$ and all eigenvalues $\lambda_j^{(1)}$ of $\mathbf{L}_1$. Hence Theorem 1 applies directly, ensuring that there exists a polynomial $\mathbf{\Phi} : E_0 \to E_1$ such that the change of variable in E_1

$$u_1 = v_1 + \mathbf{\Phi}(u_0)$$

transforms the system (1) into the following system in $E_0 \times E_1$

$$\frac{d\widetilde{u}}{dt} = \mathbf{F}(\widetilde{u}) + \widetilde{\boldsymbol{\rho}}(\widetilde{u}), \tag{31}$$ Equtilde

where $\widetilde{u} = (u_0, v_1) \in E_0 \times E_1$ and

$$\mathbf{F}(\widetilde{u}) = \begin{pmatrix} \mathbf{L}_0 u_0 + \mathbf{R}^{(0)}(u_0, v_1) \\ \mathbf{L}_1 v_1 + \mathbf{R}^{(1)}(u_0, v_1) \end{pmatrix}, \ \widetilde{\boldsymbol{\rho}}(\widetilde{u}) = \begin{pmatrix} 0 \\ \boldsymbol{\rho}(u_0) \end{pmatrix}$$

with

$$\sup_{||u_0|| \leq \delta} ||\boldsymbol{\rho}(u_0)|| \leq M e^{-\frac{w}{\delta}}.$$

For $\widetilde{\boldsymbol{\rho}} \equiv 0$, the truncated system

$$\frac{d\widetilde{u}}{dt} = \mathbf{F}(\widetilde{u}), \tag{32}$$ TruncatedSyst

admits the invariant manifold

$$\widetilde{\mathcal{M}}_0' = \{\widetilde{u} = (u_0, v_1) \in E_0 \times E_1 / v_1 = 0\}$$

which appears to be an analytic center manifold (see [8], or [19] and references therein) . In original coordinates this manifold reads

$$\mathcal{M}_0' = \{u \in \mathbb{R}^m / u = u_0 + \mathbf{\Phi}(u_0)\}$$

which is analytic since $\mathbf{\Phi}$ is polynomial.

Our aim is now to prove that for the full system (1), i.e. when $\widetilde{\boldsymbol{\rho}} \not\equiv 0$, this manifold is close to any center manifold up to an exponentially small term. For that purpose we see the full system in new coordinates (31) as a perturbation of the truncated system (32) by the exponentially small term $\widetilde{\boldsymbol{\rho}}(\widetilde{u})$.

We introduce three scalar parameters $(C, \varepsilon, \nu) \in [0,1]^3$ and consider the analytic vector field

$$\frac{d\widehat{v}}{dt} = \mathbf{V}(\widehat{v}, \varepsilon, C, \nu) := \mathbf{L}\widehat{v} + \frac{1}{\varepsilon}\mathbf{R}(\varepsilon\widehat{v}) + \frac{\nu e^{\frac{C}{\varepsilon}}}{\varepsilon}\widetilde{\boldsymbol{\rho}}(\varepsilon\widehat{v}). \tag{33}$$

RescaledSyst

For $\nu = 0$, (33) admits an analytic center manifold $\widetilde{\mathcal{M}'_0}$ obtained from $\mathcal{M}'_0$ by the scaling $\widetilde{u} = \varepsilon\widehat{v}$.

Since for every $(C, \varepsilon, \nu) \in [0,1]^3$ and every $\widehat{v} \in E_0 \times E_1$ satisfying $\|\widehat{v}\| \leq \delta_0 = \frac{w}{2}$,

$$\nu\frac{e^{\frac{C}{\varepsilon}}}{\varepsilon}||\widetilde{\boldsymbol{\rho}}(\varepsilon\widehat{v})\| \leq \frac{\nu}{\varepsilon}e^{\frac{C-w/\delta_0}{\varepsilon}} \leq \frac{\nu}{\varepsilon}e^{-\frac{1}{\varepsilon}} \leq \mathfrak{m}\nu < \infty$$

where $\mathfrak{m} := \sup\limits_{x\geq 0}(xe^{-x})$, we know (see [19]) that there is a family of center manifolds $\mathcal{M}_{\varepsilon,\nu}$ for $||\widehat{v}|| + \varepsilon + |\nu| + C \leq r$ with $r < 1$ which holds for

$$0 \leq C \leq \frac{r}{2} \quad \text{and} \quad ||\widehat{v}|| + \varepsilon + |\nu| \leq \frac{r}{2}.$$

Since we can choose $C_* < \min\left(\frac{r}{2}, \frac{4w}{r}\right)$ such that for every $\nu \in [0, r/4]$ and every $\widehat{v} \in E_0 \times E_1$ and $\varepsilon \in]0,1]$ satisfying $\|\widehat{v}\| + \varepsilon \leq \frac{r}{4}$,

$$\frac{e^{\frac{C_*}{\varepsilon}}}{\varepsilon}||\widetilde{\boldsymbol{\rho}}(\varepsilon\widehat{v})\| \leq \frac{1}{\varepsilon}e^{\frac{C_*-4w/r}{\varepsilon}} \leq \frac{\mathfrak{m}}{4w/r - C_*} < \infty$$

the value $\nu = e^{-\frac{C_*}{\varepsilon}}$ is eligible for a center manifold which corresponds to the original system rescaled. The regularity results on center manifolds allow to claim that the graph satisfies

$$u = u_0 + \boldsymbol{\Phi}(u_0) + \boldsymbol{\Psi}(u_0),$$

with constants M and C' such that

$$||\boldsymbol{\Psi}(u_0)|| \leq Me^{-\frac{C'}{\delta}}, \text{ for } ||u_0|| \leq \delta.$$

Notice that we loose analyticity only in the term $\boldsymbol{\Psi}$ which is exponentially small.

SubsecInfinitedim

4.2 Infinite dimensional case

The above result extends to the infinite dimensional case in the following way which needs an adapted assumption to replace Lemma 11. Indeed, still in $\mathbb{R}^m$, and assuming that $\mathbf{L}_0$ is diagonal in E_0 where the norm is such that $e^{L_0 t}$ is an isometry, we can solve the homological equation (14) in $\mathcal{H}_n$ in setting

$$\mathbf{v}(t) = \Phi_n[(e^{\mathbf{L}_0 t}u_0)^{(n)}] \in \mathcal{H}_n,$$

then

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{L}_1\mathbf{v}(t) + \mathbf{F}_n[(e^{\mathbf{L}_0 t}u_0)^{(n)}], \tag{34}$$

diffEquPhi_n

and it is easy to see that the unique solution which is allowed to possibly grow as $e^{\eta|t|}$ as $t \to \pm\infty$, with $\eta \in [0, \gamma[$, is given by

$$\mathbf{v}(t) = \int_{-\infty}^{t} e^{\mathbf{L}_1^-(t-s)}\mathbf{P}_-\mathbf{F}_n[(e^{\mathbf{L}_0 s}u_0)^{(n)}]ds - \int_{t}^{\infty} e^{\mathbf{L}_1^+(t-s)}\mathbf{P}_+\mathbf{F}_n[(e^{\mathbf{L}_0 s}u_0)^{(n)}]ds, \tag{35}$$

GreenKernel

where the linear operators $\mathbf{P}_\pm$ are the projections commuting with $\mathbf{L}_1$, corresponding to the separation of its spectrum into eigenvalues with positive or negative real parts, and $\mathbf{L}_1^\pm = \mathbf{P}_\pm\mathbf{L}_1$. Moreover $\mathbf{v}(t)$ is smooth and bounded for $t \in \mathbb{R}$, and $t = 0$ gives

$$\Phi_n[(u_0)^{(n)}] = \int_{-\infty}^{0} e^{-\mathbf{L}_1^- s}\mathbf{P}_-\mathbf{F}_n[(e^{\mathbf{L}_0 s}u_0)^{(n)}]ds - \int_{0}^{\infty} e^{-\mathbf{L}_1^+ s}\mathbf{P}_+\mathbf{F}_n[(e^{\mathbf{L}_0 s}u_0)^{(n)}]ds, \tag{36}$$

Phi_n

and there is a constant a depending only on the bounds of $e^{-\mathbf{L}_1^- s}$ for $s < 0$ and of $e^{-\mathbf{L}_1^+ s}$ for $s > 0$ such that

$$\phi_n \leq a|\mathbf{F}_n|_{2,n}.$$

Formula (35) which is valid in the finite dimensional space E_1 leads to a basic assumption for the center manifold theorem as formulated in [21], which is verified in many cases of physical interest (see examples in [21]).

Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be (real or complex) Banach spaces such that

$$\mathcal{Y} \hookrightarrow \mathcal{Z} \hookrightarrow \mathcal{X},$$

with continuous embeddings. We consider a differential equation in $\mathcal{X}$, of the form

$$\frac{du}{dt} = \mathbf{L}u + \mathbf{R}(u), \tag{37}$$

systdim

in which we assume that the following holds.

h:1 **Hypothesis 17** *We assume that* $\mathbf{L}$ *and* $\mathbf{R}$ *in (37) have the following properties:*

(a) $\mathbf{L} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$;

(b) *There exists* $\rho > 0$ *such that* $\mathbf{R} : \mathcal{Y} \to \mathcal{Z}$ *is analytic in the ball* $||u||_{\mathcal{Y}} \leq \rho$ *and satisfies (2) and (3).*

Besides the Hypothesis 17, we make two further assumptions on the linear operator $\mathbf{L}$, which are essential for the center manifold theorem.

h:2 **Hypothesis 18 (Spectral decomposition)** *Consider the spectrum* σ *of* $\mathbf{L}$*, and write*

$$\sigma = \sigma_+ \cup \sigma_0 \cup \sigma_-$$

in which

$$\sigma_+ = \{\lambda \in \sigma \; ; \; \mathrm{Re}\lambda > 0\}, \quad \sigma_0 = \{\lambda \in \sigma \; ; \; \mathrm{Re}\lambda = 0\}, \quad \sigma_- = \{\lambda \in \sigma \; ; \; \mathrm{Re}\lambda < 0\}.$$

We assume that

(a) *there exists a positive constant* $\gamma > 0$ such that

$$\inf_{\lambda \in \sigma_+} (\mathrm{Re}\lambda) > \gamma, \quad \sup_{\lambda \in \sigma_-} (\mathrm{Re}\lambda) < -\gamma;$$

(b) *the set* σ_0 *consists of a finite number of eigenvalues with finite algebraic multiplicities and geometric multiplicity one.*

This decomposition of the spectrum allows to define a projection $\mathbf{P}_0 \in \mathcal{L}(\mathcal{X}, E_0)$ on the finite-dimensional invariant "central" space E_0, which commutes with $\mathbf{L}$. The complementary projection $\mathbf{P}_h = \mathbb{I} - \mathbf{P}_0$ is also a projection commuting with $\mathbf{L}$, bounded in $\mathcal{X}_h = \mathbf{P}_h \mathcal{X}$ as well as in $\mathcal{Y}_h = \mathbf{P}_h \mathcal{Y}$ and $\mathcal{Z}_h = \mathbf{P}_h \mathcal{Z}$. The restriction of $\mathbf{L}$ to $\mathcal{Y}_h$ is denoted by $\mathbf{L}_h$.

h:3 **Hypothesis 19 (Linear equation)** *For any* $\eta \in [0, \gamma]$ *and any*

$$f \in \mathcal{C}_\eta(\mathbb{R}, \mathcal{Z}_h) = \left\{ v \in \mathcal{C}^0(\mathbb{R}, \mathcal{Z}_h); ||v||_{\mathcal{C}_\eta} = \sup_{t \in \mathbb{R}} \left(e^{-\eta|t|} ||v(t)||_{\mathcal{Z}_h} \right) < \infty \right\},$$

the linear problem

$$\frac{du_h}{dt} = \mathbf{L}_h u_h + f(t),$$

has a unique solution $u_h = \mathbf{K}_h f \in \mathcal{C}_\eta(\mathbb{R}, \mathcal{Y}_h)$*. Furthermore, the linear map* $\mathbf{K}_h$ *belongs to* $\mathcal{L}(\mathcal{C}_\eta(\mathbb{R}, \mathcal{Z}_h), \mathcal{C}_\eta(\mathbb{R}, \mathcal{Y}_h))$*, and there exists a continuous map* $C : [0, \gamma] \to \mathbb{R}$ *such that*

$$||\mathbf{K}_h||_{\mathcal{L}(\mathcal{C}_\eta(\mathbb{R}, \mathcal{Z}_h), \mathcal{C}_\eta(\mathbb{R}, \mathcal{Y}_h))} \leq C(\eta).$$

Then, we have the following theorem, which extends Theorem 5 to infinite dimensional cases:

center manifold **Theorem 20 (Center manifold analytic up to exp. small term)**
Assume that the Hypotheses 17, 18, and 19 hold. Then for any $k > 2$, *there exists a polynomial* $\mathbf{\Phi} : E_0 \to E_h$ *of degree* $O(1/\delta)$, *with* $\mathbf{\Phi}(0) = 0$, $D\mathbf{\Phi}(0) = 0$, *a neighborhood* $\mathcal{O}$ *of* 0 *in* $\mathcal{Y}$, *and a map* $\mathbf{\Psi} \in \mathcal{C}^k(E_0, \mathcal{Y}_h)$ *which is* $O(e^{-\frac{C}{\delta}})$ *for* $||u_0||_{E_0} \leq \delta$ *and a certain constant* $C > 0$, *such that the manifold*

$$\mathcal{M}_0 = \{u_0 + \mathbf{\Phi}(u_0) + \mathbf{\Psi}(u_0)\ ;\ u_0 \in E_0\} \subset \mathcal{Y} \tag{38}$$ e:m01

has the following properties.

(a) $\mathcal{M}_0$ *is locally invariant, i.e., if* u *is a solution of (37) satisfying* $u(0) \in \mathcal{M}_0 \cap \mathcal{O}$ *and* $u(t) \in \mathcal{O}$ *for all* $t \in [0, T]$, *then* $u(t) \in \mathcal{M}_0$ *for all* $t \in [0, T]$.

(b) $\mathcal{M}_0$ *contains the set of bounded solutions of (37) staying in* $\mathcal{O}$ *for all* $t \in \mathbb{R}$, *i.e., if* u *is a solution of (37) satisfying* $u(t) \in \mathcal{O}$ *for all* $t \in \mathbb{R}$, *then* $u(0) \in \mathcal{M}_0$.

Proof. We use the result proved in [21], complemented by the proof of Theorem 5, for which we need to use Hypothesis 19 to solve the homological equation (14), as in (36), by

$$\mathbf{\Phi}_n[u_0^{(n)}] = \mathbf{K}_h \mathbf{F}_n[\big(e^{\mathbf{L}_0 \cdot} u_0\big)^{(n)}]|_{t=0},$$

and to obtain the basic estimate

$$\phi_n \leq a|\mathbf{F}_n|_{2,n}.$$

■

5 Case of Elliptic vector fields

SecElliptic

Consider now the system (1) in $\mathbb{R}^n$ when both spectra of $\mathbf{L}_0$ and $\mathbf{L}_1$ lie on the imaginary axis. This is the natural situation for nonlinear vibrating systems, typically with a large number of coupled nonlinear oscillators. This section is devoted to the proof of theorem 8.

Theorem 1 applies and it results that the manifold $\mathcal{M}_0'$ defined by

$$u = u_0 + \mathbf{\Phi}(u_0),$$

which has the dimension of E_0 and is tangent to E_0 in 0, is "nearly" invariant. More precisely, assume that the initial condition at $t = 0$ is such that $v_1|_{t=0} = 0$, i.e. $u|_{t=0} \in \mathcal{M}'_0$. Then consider the second component of the vector field (6). If the remainder $\boldsymbol{\rho}(u_0)$ would be identically 0, the manifold $\mathcal{M}'_0$ would be an invariant manifold, since $v_1(t) = 0$ would be the unique solution of the initial value problem. Now assume $v_1(0) = 0$ and that $u_0(t)$ satisfies for $t \in [0, T]$

$$||u_0(t)|| \leq \delta.$$

Then (6) and the estimate for $\mathbf{R}^{(1)}$ gives as soon as $||v_1(t)|| \leq \delta$ for $t \in [0, T]$

$$||v_1(t)|| \leq c\delta \int_0^t ||e^{\mathbf{L}_1(t-s)}|| ||v_1(s)|| ds + Mte^{-\frac{w}{\delta^b}}.$$

For any $\xi > 0$, there exists $C = \beta(\nu)\xi^{-(\nu-1)}$ where ν is the maximal index of eigenvalues of $\mathbf{L}_1$, such that for any $t \in \mathbb{R}$

$$||e^{\mathbf{L}_1 t}|| \leq Ce^{\xi|t|},$$

then by Gronwall Lemma we get

$$||v_1(t)|| \leq Me^{-\frac{w}{\delta^b}} \left\{ t + \frac{cC\delta}{(cC\delta + \xi)^2} e^{(cC\delta+\xi)t} \right\},$$

and in choosing $\xi = (c\beta\delta)^{1/\nu}$

$$||v_1(t)|| \leq Me^{-\frac{w}{\delta^b}} \left\{ t + \frac{1}{4(c\beta\delta)^{1/\nu}} e^{2t(c\beta\delta)^{1/\nu}} \right\} \tag{39}$$

first time estimate

which shows that $||v_1(t)||$ stays smaller than $M_1 e^{-\frac{w}{2\delta^b}}$ for $t = O(\delta^{-[b+1/\nu]})$. This means that the trajectory stays exponentially close to the manifold $\mathcal{M}'_0$ for a very long time of order $O(\delta^{-[b+1/\nu]})$ and it achieves the proof of theorem 8.

A Norm of the inverse of the homological operator

AppendixA

LemAcal

Lemma 21 *Let* $\mathbf{L}$ *be a linear operator in* $\mathbb{C}^m$ *and assume that the linear operator* $\mathbf{L}$ *is the direct sum of two linear operators* $\mathbf{L}_0$ *on* E_0 *(dim* m_0*), and* $\mathbf{L}_1$ *on* E_1 *(dim* m_1*), such that* $\mathbf{L}_0$ *is diagonalizable with eigenvalues* $\lambda_1^{(0)}, \cdots, \lambda_{m_0}^{(0)}$ *and that there exist constants* $0 < \gamma \leq 1, \tau \geq 0$ *such that* $\Lambda_{\alpha,j} := \langle \alpha, \lambda^{(0)} \rangle - \lambda_j^{(1)}$ *satisfies*

$$|\Lambda_{\alpha,j}| \geq \frac{\gamma}{|\alpha|^\tau} \tag{40}$$

App_diophCond

for any $\alpha \in \mathbb{N}^{m_0}\backslash\{0\}$*, and any eigenvalue* $\lambda_j^{(1)}$ *of* $\mathbf{L}_1$.

Let $(e_k)_{1\leq k\leq m}$ *be the canonical basis of* $\mathbb{C}^m$*. We assume that* $(e_k)_{1\leq k\leq m_0}$ *is a basis of eigenvectors of* $\boldsymbol{L}_0$:

$$\boldsymbol{L}_0 e_k = \lambda_j^{(0)} e_k.$$

Moreover we also assume that $f_j = e_{m_0+j}$ *with* $1 \leq j \leq m_1$*, is a basis of generalized eigenvectors in which* $\boldsymbol{L}_1$ *is under Jordan complex normal form, i.e.*

$$\boldsymbol{L}_1 f_j = \lambda_j^{(1)} f_j + \delta_{j-1} f_{j-1}$$

where $\delta_0 = 0$ *and where* $\delta_j = 0$ *if* $\lambda_j^{(1)} \neq \lambda_{j-1}^{(1)}$ *and* $\delta_j = 0$ *or* 1 *otherwise.*

Let $\mathcal{H}$ *be the set of all polynomials from* E_0 *to* E_1 *and let* $\mathcal{H}_n$ *be the subset of homogeneous polynomials of degree* n*. Finally let us denote by* $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ *the homological operator defined by*

$$(\mathcal{A}\boldsymbol{\Phi})(u_0) = D\boldsymbol{\Phi}(u_0)\boldsymbol{L}_0 u_0 - \boldsymbol{L}_1\boldsymbol{\Phi}(u_0).$$

Then,

LemAcala (a) $\mathcal{A}$ *maps* $\mathcal{H}_n$ *into* $\mathcal{H}_n$ *and the spectrum of its restriction to* $\mathcal{H}_n$*,* $\mathcal{A}|_{\mathcal{H}_n}$*, is given by*

$$\sigma(\mathcal{A}|_{\mathcal{H}_n}) := \{\Lambda_{\alpha,j} = \langle \alpha, \lambda^{(0)}\rangle - \lambda_j^{(1)} /\ \alpha \in \mathbb{N}^{m_0},\ |\alpha| = n,\ 1 \leq j \leq m_1\}.$$

LemAcalb (b) $\mathcal{A}|_{\mathcal{H}_n}$ *is invertible in the subspace* $\mathcal{H}_n$ *and*

$$|||\mathcal{A}|_{\mathcal{H}_n}^{-1}|||_2 := \sup_{|\boldsymbol{\Phi}|_2=1} |\mathcal{A}|_{\mathcal{H}_n}^{-1}\boldsymbol{\Phi}|_2 \leq \nu\ \gamma^{-\nu}\ n^{\tau\nu}.$$

Proof of (a). Let us denote by $\boldsymbol{P}_{\alpha,j}$ with $\alpha \in \mathbb{N}^{m_0}$, $|\alpha| = n$ and $1 \leq j \leq m_1$ be the basis of $\mathcal{H}_n$ given by

$$\boldsymbol{P}_{\alpha,j}(u_0) = (u_{0,1})^{\alpha_1} \cdots (u_{0,m_0})^{\alpha_{m_0}} f_j$$

where $u_0 = \sum_{k=1}^{m_0} u_{0,k}\ e_k$. Then we check that

$$\mathcal{A}|_{\mathcal{H}_n}\boldsymbol{P}_{\alpha,j} = \Lambda_{\alpha,j}\ \boldsymbol{P}_{\alpha,j} - \delta_{j-1}\boldsymbol{P}_{\alpha,j-1}. \tag{41}$$

Let us order this basis by lexicographical order, i.e. $\boldsymbol{P}_{\alpha,j} < \boldsymbol{P}_{\beta,\ell}$ if the first non zero integer $\beta_1 - \alpha_1, \cdots,\ \beta_{m_0} - \alpha_{m_0}, \ell - j$ is positive. Within this order, the matrix $M_{\mathcal{A}|_{\mathcal{H}_n}}$ of $\mathcal{A}|_{\mathcal{H}_n}$ in the basis $\boldsymbol{P}_{\alpha,j}$ is upper triangular. More precisely, it is the direct sum of $m_1 \times m_1$ matrices M_α,

$$M_{\mathcal{A}|_{\mathcal{H}_n}} = \bigoplus_{\alpha \in \mathbb{N}^{m_0}, |\alpha|=m_0} M_\alpha, \qquad M_\alpha = \begin{pmatrix} \Lambda_{\alpha,1} & \delta_1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \delta_{m_1} \\ 0 & 0 & 0 & \Lambda_{\alpha,m_1} \end{pmatrix}. \tag{42}$$

EqMA

Hence the spectrum of $\mathcal{A}|_{\mathcal{H}_n}$ is given by

$$\sigma(\mathcal{A}|_{\mathcal{H}_n}) := \{\Lambda_{\alpha,j} = \langle \alpha, \lambda^{(0)} \rangle - \lambda_j^{(1)} / \ \alpha \in \mathbb{N}^{m_0},\ |\alpha| = n,\ 1 \leq j \leq m_1\}.$$

Proof of (b). Since by hypothesis, for every $\alpha \in \mathbb{N}^{m_0}$ and every $1 \leq j \leq m_1$, $|\Lambda_{\alpha,j}| \geq \frac{\gamma}{|\alpha|^\tau} > 0$, $\mathcal{A}|_{\mathcal{H}_n}$ is invertible and (42) ensures that

$$M_{\mathcal{A}|_{\mathcal{H}_n}^{-1}} = M_{\mathcal{A}|_{\mathcal{H}_n}}^{-1} = \bigoplus_{\alpha \in \mathbb{N}^{m_0}, |\alpha|=m_0} M_\alpha^{-1}.$$

Moreover M_α is block diagonal

$$M_\alpha = \bigoplus_{r=1}^{q} B_{j_r,p_r} \quad \text{with} \quad B_{j,p} = \begin{pmatrix} \Lambda_{\alpha,j} & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \Lambda_{\alpha,j+p} \end{pmatrix}$$

where $1 \leq j_r \leq m_1$ and $0 \leq p_r \leq \nu$ where ν is the maximal index of the eigenvalues of $\boldsymbol{L}_1$. For a polynomial $\boldsymbol{\Phi} \in \mathcal{H}_n$, we can write

$$\boldsymbol{\Phi} = \sum_{|\alpha|=n} \sum_{j=1}^{m_1} \Phi_{\alpha,j}\ \boldsymbol{P}_{\alpha,j} = \sum_{|\alpha|=n} \sum_{r=1}^{q} \sum_{j=j_r}^{j_r+p_r} \Phi_{\alpha,j}\ \boldsymbol{P}_{\alpha,j}$$

Then for $\boldsymbol{\Psi} \in \mathcal{H}_n$, $\mathcal{A}\boldsymbol{\Phi} = \boldsymbol{\Psi}$ if and only if, for every $\alpha \in \mathbb{N}^{m_0}$ with $|\alpha| = n$ and every $1 \leq r \leq q$

$$\begin{cases} \Phi_{\alpha,j_r+p_r} & = \Lambda_{\alpha,j_r+p_r}^{-1} \Psi_{\alpha,j_r+p_r} \\ \Phi_{\alpha,j_r+p_r-1} & = \Lambda_{\alpha,j_r+p_r-1}^{-1} \Psi_{\alpha,j_r+p_r-1} - \left(\Lambda_{\alpha,j_r+p_r} \Lambda_{\alpha,j_r+p_r-1}\right)^{-1} \Psi_{\alpha,j_r+p_r} \\ \quad\vdots & \\ \Phi_{\alpha,j_r} & = \Lambda_{\alpha,j_r}^{-1} \Psi_{\alpha,j_r+p_r-1} - \left(\Lambda_{\alpha,j_r} \Lambda_{\alpha,j_r+1}\right)^{-1} \Psi_{\alpha,j_r+1} \\ & \qquad + \cdots + (-1)^{p_r-1} \left(\Lambda_{\alpha,j_r} \cdots \Lambda_{\alpha,j_r+p_r}\right)^{-1} \Psi_{\alpha,j_r+p_r} \end{cases}$$

Then observe that for every $\alpha \in \mathbb{N}^{m_0}$ with $|\alpha| = n$,

$$\max_{\substack{1\le r\le q \\ j_r\le j\le \ell\le j_r+p_r}} (\Lambda_{\alpha,j}\cdots\Lambda_{\alpha,\ell})^{-1} \le \gamma^{-\nu}\ n^{\nu\tau}.$$

Thus, for every $\alpha \in \mathbb{N}^{m_0}$ with $|\alpha| = n$, every $1 \le r \le q$ and every $j_r \le j \le j_r + p_r$,

$$|\Phi_{\alpha,j}| \le \gamma^{-\nu} n^{\tau\nu}(|\Psi_{j_r}| + \cdots + |\Psi_{j_r+p_r}|.$$

Hence, since $\langle P_{\alpha,j}, P_{\beta,\ell}\rangle_{\mathcal{H}} = 0$ for $(j,\alpha) \neq (\ell,\beta)$ and since $|P_{j,\alpha}|_2 = |P_{\ell,\alpha}|_2 = \alpha!$, we have

$$\begin{aligned}
|\boldsymbol{\Phi}|_2^2 &= |\mathcal{A}^{-1}\boldsymbol{\Psi}|_2^2 \\
&= \sum_{|\alpha|=n}\sum_{r=1}^{q}\sum_{j=j_r}^{j_r+p_r} |\Phi_{\alpha,j}|^2\ |\boldsymbol{P}_{\alpha,j}|_2^2 \\
&\le (\gamma^{-\nu}\ n^{\nu\tau})^2 \sum_{|\alpha|=n}\sum_{r=1}^{q}\sum_{j=j_r}^{j_r+p_r}\left(\sum_{\ell=j_r}^{j_r+p_r}|\Psi_{\alpha,\ell}|\right)^2\ |\boldsymbol{P}_{\alpha,j}|_2^2 \\
&\le \nu(\gamma^{-\nu}\ n^{\nu\tau})^2 \sum_{|\alpha|=n}\sum_{r=1}^{q}\sum_{j=j_r}^{j_r+p_r}\sum_{\ell=j_r}^{j_r+p_r}|\Psi_{\alpha,\ell}|^2\ |\boldsymbol{P}_{\alpha,j}|_2^2 \\
&\le \nu^2(\gamma^{-\nu}\ n^{\nu\tau})^2 \sum_{|\alpha|=n}\sum_{r=1}^{q}\sum_{\ell=j_r}^{j_r+p_r}|\Psi_{\alpha,\ell}|^2\ |\boldsymbol{P}_{\alpha,\ell}|_2^2 \\
&= (\nu\gamma^{-\nu}\ n^{\nu\tau})^2|\boldsymbol{\Psi}|_2^2
\end{aligned}$$

Hence, $|\mathcal{A}^{-1}\boldsymbol{\Psi}|_2 \le \nu\gamma^{-\nu}\ n^{\nu\tau}|\boldsymbol{\Psi}|_2$. ∎

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