Travelling waves in a chain of coupled nonlinear oscillators

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Abstract

In a chain of nonlinear oscillators, linearly coupled to their nearest neighbors, all travelling waves of small amplitude are found as solutions of finite dimensional reversible dynamical systems. The coupling constant and the inverse wave speed form the parameter space. The groundstate consists of a one-parameter family of periodic waves. It is realized in a certain parameter region containing all cases of light coupling. Beyond the border of this region the complexity of wave-forms increases via a succession of bifurcations. In this paper we give an appropriate formulation of this problem, prove the basic facts about the reduction to finite dimensions, show the existence of the ground states and discuss the first bifurcation by determining a normal form for the reduced system. Finally we show the existence of *nanopterons*, which are localized waves with a noncancelling periodic tail at infinity whose amplitude is exponentially small in the bifurcation parameter.

1 Introduction

Consider the dynamics of a one-dimensional network of nonlinear oscillators, as described by the infinite system

$$\ddot{X}_n + V'(X_n) = \gamma(X_{n+1} - 2X_n + X_{n-1}), \qquad n \in \mathbb{Z}. \tag{1}$$

Here, $X_n(\widetilde{t})$, $\widetilde{t} \in \mathbb{R}$, gives the position of the nth particle, $V(X_n)$ its potential energy, V being a regular function independent of n, and the positive constant

γ measures the coupling between nearest neighbors, which is assumed to be linear. Furthermore, the function V satisfies $V'(0) = 0, V''(0) = 1$.

We shall construct solutions of (1) in the form of travelling waves. In fact, we shall develop a general method for classifying travelling waves of small amplitude via an infinite sequence of bifurcations. We shall discuss in detail the groundstate and the first of these bifurcations.

With the ansatz $X_n(\widetilde{t}) = \widetilde{x}(\widetilde{t} - n\tau)$, after scaling the time as $\widetilde{t} = \tau t$, and denoting $x(t) = \widetilde{x}(\tau t)$, system (1) is transformed to

$$\ddot{x}(t) + \tau^2 V'[x(t)] = \gamma\tau^2[x(t-1) - 2x(t) + x(t+1)] \tag{2}$$

which is a scalar "neutral" or "advance-delay" differential equation.

Equations of this type have been subject of various investigations on the dynamics of lattices. Friesecke and Wattis have shown in [6] the surprising fact that, in a unidimensional hamiltonian network, solitary waves exist, even if the coupling is nonlinear. They used a variational approach. How delicate this result really is, will appear also in the subsequent analysis. Further results along these lines were given by Smets and Willem [19].

Equation (2) has been investigated by MacKay and Aubry in [15] for the existence of time-periodic and localized-in-space standing waves, so-called breathers. Aubry then, while searching for "multibreathers", developed in [1] the technique of "phase torsion" to study the existence of travelling waves.

Rusticini also studied equations of the type considered here in [17], [18]. His motivation came from problems of optimal control. He proved a Hopf-bifurcation theorem by constructing 2d-center manifolds for periodic solutions via a Lyapunov-Schmidt argument. Some of his analysis is close to ours, like the ad hoc construction of C^0-semigroups on the positive and the negative spectral part – both being infinite dimensional –.

We should also mention the recent work of Mallet-Paret et al. in [3], [13], [14] on waves in higher dimensional lattices. There, the dynamics is restricted to discrete systems, but give a global picture of the solutions. The arguments rely on an advanced form of the Lyapunov-Schmidt method given by X.B. Lin (c.f. [14]).

With the method being developed here, we exploit two facts: first the ellipticity of (2) in its continuous parts, and the intrusion of hyperbolicity via the discrete terms. With increasing intensity of coupling, the effect of the latter will be more and more dominating, and the complexity of the solution behavior will explode. Nevertheless, one can perform the "continuous limit" for (1) and thus obtain travelling wave solutions of the following nonlinear wave equation

$$u_{\widetilde{t}\widetilde{t}} + V'(u) = K u_{\xi\xi} \tag{3}$$

for the function $u(\widetilde{t}, \xi)$. Its discretized form (1) is obtained with $X_n(\widetilde{t}) = u(\widetilde{t}, nh)$, and $K = \gamma h^2$, where h is the discretization step. Looking for solutions of (3) of the form of travelling waves

$$u(\widetilde{t}, \xi) = \widetilde{x}(\widetilde{t} - \xi/c) \tag{4}$$

leads to the discretized form (2) where $\tau = h/c$. Now, (4) implies for (3)

$$(1 - K/c^2)\frac{d^2\widetilde{x}}{d\widetilde{t}^2} + \widetilde{x} = g(\widetilde{x}), \tag{5}$$

where g is defined by $V'(x) = x - g(x)$, hence $g(x) = O(x^2)$. It is then clear that travelling waves, as solutions of (5), exist near 0, if and only if $K/c^2 < 1$, i.e. $\gamma\tau^2 < 1$. They form a *one parameter family* in the neighborhood of 0.

In the present work, we prove the existence of the corresponding travelling waves (if $\gamma\tau^2 < 1$) for the above discretized model (1), but we *also* prove the existence of infinitely many other types of travelling waves near 0, for values of (γ, τ) in regions such that $\gamma\tau^2 > 1$. This shows in particular how dangerous the belief might be that all nontrivial solutions of a discretized version of (5) survive the limit $h \to 0$.

The method we shall develop is based on previous work in [11], [16],[20], proving the reducibility of quasilinear elliptic systems in infinite cylindrical domains. Treating the system as evolutionary in the unbounded variable, one is able to show that, under quite general conditions, the original system, if restricted to a suitable neighborhood of 0, is equivalent to a flow on a *finite dimensional* manifold. Extending this idea to the problem under consideration in Section 2, we are able to prove the validity of a reduction of (2) to a system of ordinary differential equations whose dimension equals the dimension of the invariant subspace belonging to the central part of the spectrum of the linearization at 0, and which inherits the *"reversibility"* from the original equation (2). This is done in Sections 4 and 5. It should be emphasized that the extension of the previous results to the case considered here is by no means straightforward.

In the following sections we analyze the case of small coupling first, when no bifurcation occurs and all "small" travelling waves are periodic. Thereafter we treat the first bifurcation occuring at a critical value of the coupling constant γ (near 21 for $\tau = 1$). The difficulties of applying previous reduction results [20] will be apparent in that case and solved in a general way. Exploiting the reducibility to a finite system of ordinary differential equations, we apply normal form theory. The resulting system is integrable on this level of approximation and quite rich in its structure. In order to keep the scope of this paper limited, however, we suppress the instinct to describe all possible solutions as well as the proof of persistence for the full system - not just reduced to its normal form - of the solutions found. That would complete the analysis.

We rather construct some of the most interesting forms of waves, such as "nanopterons". These are roughly the superposition of a localized travelling (solitary) wave, whose principal part is given explicitly, and exponentially small (in the bifurcation parameter) periodic waves ("phonons"). The proof of their existence follows from the work of Lombardi in [12]. For other type of solutions, like periodic or quasi-periodic ones, see the methods developed in [8].

2 Extended formulation

Instead of treating (2) directly, we introduce a new variable $v \in [-1, 1]$ and functions $X(t, v) = x(t + v)$. The notation $U(t)(v) = (x(t), \xi(t), X(t, v))^T$ indicates our intention to construct U as a map from $\mathbb{R}$ into some function space living on the v-interval $[-1, 1]$. We use the notations $\xi(t) = \dot{x}(t)$, $\delta^1 X(t, v) = X(t, 1)$, and $\delta^{-1} X(t, v) = X(t, -1)$. Equation (2) can now be written as follows

$$\partial_t U = L_{\gamma,\tau} U + M_\tau(U), \tag{6}$$

where $L_{\gamma,\tau}$ is the linear, nonlocal operator

$$L_{\gamma,\tau} = \begin{pmatrix} 0 & 1 & 0 \\ -\tau^2(1+2\gamma) & 0 & \gamma\tau^2(\delta^1 + \delta^{-1}) \\ 0 & 0 & \partial_v \end{pmatrix},$$

and

$$M_\tau(U) = \tau^2(0,\, g(x),\, 0)^T,$$

where $g(x) = x - V'(x) = ax^2 + bx^3 + \ldots = 0(x^2)$ as $x \to 0$. Moreover, we require the boundary condition $X(t, 0) = x(t)$.

Observe that (6) is somewhat more general than the original equation (2), if we allow g to depend not only on x, but on ξ, X as well. In that case, the coupling could be a smooth nonlinear function as indicated in the introduction.

We introduce Banach-spaces $\mathbb{H}$ and $\mathbb{D}$ for $U(v) = (x, \xi, X(v))^T$

$$\begin{aligned} \mathbb{H} &= \mathbb{R}^2 \times \left(C^0[-1,1]\right) \\ \mathbb{D} &= \left\{U \in \mathbb{R}^2 \times (C^1[-1,1]) / X(0) = x\right\} \end{aligned} \tag{7}$$

with the usual maximum norms. The operator $L_{\gamma,\tau}$ then maps $\mathbb{D}$ into $\mathbb{H}$ continuously. The nonlinearity M_τ is supposed to satisfy $M_\tau \in C^k(\mathbb{D}, \mathbb{D})$, $k \geq 1$, and

$$\|M_\tau(U)\|_{\mathbb{D}} \leq c(\rho)\|U\|_{\mathbb{D}}^2 \tag{8}$$

for all $U \in \mathbb{D}$ with $\|U\|_{\mathbb{D}} \leq \rho$; ρ being an arbitrary positive constant. In our particular case $g \in C^2(\Omega)$ suffices for the validity of the assumption on M_τ; Ω denotes an open neighborhood of $0 \in \mathbb{R}$.

It is obvious that $L_{\gamma,\tau}$, acting in $\mathbb{H}$ with domain $\mathbb{D}$, has a compact resolvent in $\mathbb{H}$. Moreover, $L_{\gamma,\tau}$ and M_τ, both anticommute with the reflexion S in $\mathbb{H}$, given by

$$S(x, \xi, X)^T = (x, -\xi, X \circ s)^T, \tag{9}$$

where $X \circ s(v) = X(-v)$. Therefore, (6) is *reversible.*

Although (6) is illposed as an initial value problem, it is possible to construct, nevertheless, solutions bounded for all $t \in \mathbb{R}$. Using a proper extension of certain reduction methods for quasilinear elliptic systems (c.f. [11],[16],[20]) one is able to reduce (6) to a finite dimensional system of ordinary differential equations, which is reversible and has the property to contain all bounded solutions which are close to the trivial solution $U = 0$. The dimension of this reduced system will depend on the coupling parameter γ and on the delay-advance parameter τ. This dependence of the dimension as a function of (γ, τ) is detailed in the next section.

3 The Spectrum of $L_{\gamma,\tau}$

To determine the spectrum $\sum \equiv \sum L_{\gamma,\tau}$ of $L_{\gamma,\tau}$, the resolvent equation

$$(\lambda\mathbb{I} - L_{\gamma,\tau})U = F \tag{10}$$

has to be solved for any given $F = (f_0, f_1, F_2)^T \in \mathbb{H}$, with $\lambda \in \mathbb{C}$, and $U = (x, \xi, X)^T \in \mathbb{D}$. This is possible provided that $N(\lambda;\, \gamma, \tau) \neq 0$, where

$$N(\lambda;\, \gamma, \tau) = -\lambda^2 - \tau^2(1 + 2\gamma) + \gamma\tau^2(e^{\lambda} + e^{-\lambda}). \tag{11}$$

Indeed, we obtain

$$x = -[N(\lambda;\, \gamma, \tau)]^{-1}(\lambda f_0 + f_1 + \gamma\tau^2 \widetilde{f}_\lambda), \tag{12}$$

$$\xi = -[N(\lambda;\, \gamma, \tau)]^{-1}\{[\lambda^2 + N(\lambda;\, \gamma, \tau)]f_0 + \lambda f_1 + \gamma\tau^2\lambda\widetilde{f}_\lambda\}, \tag{13}$$

$$X(v) = e^{\lambda v}x - \int_0^v e^{\lambda(v-s)}F_2(s)ds, \tag{14}$$

with

$$\widetilde{f}_\lambda = \int_0^1 [-e^{\lambda(1-s)}F_2(s) + e^{-\lambda(1-s)}F_2(-s)]ds.$$

Since $N(\lambda;\, \gamma, \tau)$ is an entire function of λ for every $(\gamma, \tau) \in \mathbb{R}_+^2$, the spectrum $\sum L_{\gamma,\tau}$ consists of isolated eigenvalues λ. They are roots of $N(\lambda;\, \gamma, \tau)$, and thus have finite multiplicities.

Remark, that $L_{\gamma,\tau}$ is real and that $SL_{\gamma,\tau} + L_{\gamma,\tau}S = 0$ holds. $\sum L_{\gamma,\tau}$ is then invariant under $\lambda \mapsto \overline{\lambda}$ and $\lambda \mapsto -\lambda$. Thus, $\sum L_{\gamma,\tau}$ is invariant under reflexion on the real – and the imaginary axis in $\mathbb{C}$. Thus, we can restrict the following considerations to $\lambda = p + iq$ with nonnegative p and q.

The central part $\sum_0 \equiv \sum_0 L_{\gamma,\tau} = \sum L_{\gamma,\tau} \cap i\mathbb{R}$ of the spectrum is determined by $N(iq; \gamma, \tau) = 0, q \in \mathbb{R}$, i.e.

$$q^2 + 2\gamma\tau^2 \cos q - \tau^2(1 + 2\gamma) = 0. \tag{15}$$

For eigenvalues of higher multiplicity we have to solve in addition

$$q - \gamma\tau^2 \sin q = 0 \tag{16}$$

if the multiplicity is at least two. For triple eigenvalues

$$1 - \gamma\tau^2 \cos q = 0 \tag{17}$$

has to hold also. There are no eigenvalues of multiplicity greater than 3 in $\sum_0$.

In the parameter-space $(\gamma, \tau) \in \mathbb{R}^2_+$, the set DE, for which there are double eigenvalues on $\sum_0$, consists of a sequence of curves which we call "DEC". They are parametrized by $q = x \in \mathbb{R}^+$ as follows

$$d(\gamma,\, \tau)(x) : \tau^2 = x^2 - 2x \tan x/2, \quad \gamma = \tau^{-2} x/ \sin x. \tag{18}$$

For triple eigenvalues it follows in addition

$$x = \tan x. \tag{19}$$

Since $d(\gamma, \tau)(x)/dx$ vanishes on DEC if (19) holds, the triple eigenvalues appear as cusps on DEC. There are no eigenvalues of multiplicity higher that 3 on $\sum_0$.

In the following lemma we describe the character of $\sum_0$ on the bifurcation curves DEC. To conform with Figure 1, we restrict this description to $\sum_0^+$, i.e. those eigenvalues on $\sum_0$ having positive imaginary part. Due to reversibility the rest of $\sum_0$ is obtained by a simple reflexion on the real axis.

Lemma 1 *(i) For each $(\gamma, \tau) \in \mathbb{R}^2_+$, there exists $p_0 > 0$, such that all $\lambda \in \sum L_{\gamma,\,\tau} \setminus \sum_0$ satisfy $|\operatorname{Re} \lambda| \geq p_0$.*

(ii) Let $\lambda = p + iq \in \sum \setminus \sum_0$, then

$$|q| \leq \tau + 2\sqrt{\gamma\tau^2 + 4e^{-2}} \cosh(p/2) \tag{20}$$

holds.

(iii) Given any DEC. It contains exactly one cusp point, $(\gamma^*,\, \tau^*)$ say, corresponding to a triple eigenvalue iq^* of $\sum_0^+$. Moreover, $\sum_0 L_{\gamma^*,\,\tau^*} = \{\pm iq^*\}$. For all other $(\gamma,\, \tau)$ on that DEC, $\sum_0^+ L_{\gamma,\,\tau}$ contains either two or one double eigenvalue. The first case happens where two different DEC's intersect. - If $(\gamma,\, \tau)$ does not belong to any DEC, $\sum_0^+ L_{\gamma,\,\tau}$ consists of simple eigenvalues.

(iv) For each fixed $\tau \in (0, 2\pi]$, there exists a strictly increasing sequence $(\gamma_j^*(\tau))$, $0 < \gamma_1^* < \ldots$, such that $\sum_0^+ L_{\gamma_j^*,\tau}$ possesses a double eigenvalue $+iq_j^*$, which has largest modulus among all eigenvalues. All other eigenvalues in $\sum_0^+ L_{\gamma_j^*,\tau}$ are simple. If $\gamma \in (\gamma_j^*(\tau), \gamma_{j+1}^*(\tau)), \gamma_0^* \equiv 0, \sum_0^+ L_{\gamma,\tau}$ consists of $2j+1$ simple eigenvalues.

For larger values of τ, the situation $\sum_0^+ L_{\gamma,\tau}$ is described in Figure 1.

Proof: Ad (i) Let us denote $\lambda_n = p_n + iq_n$ the roots λ of $N(\lambda; \gamma, \tau) = 0$. Assuming $p_n \neq 0$, and $p_n \to 0$ as $n \to \infty$, we have

$$\begin{aligned} q_n^2 + 2\gamma\tau^2 \cos q_n - \tau^2(1 + 2\gamma) &= \varepsilon_n \to 0, \\ q_n - \gamma\tau^2 \sin q_n &= \varepsilon_n' \to 0. \end{aligned}$$

Hence, q_n is bounded, and we can extract a subsequence q_{k_n} converging towards q_*, and q_* satisfies

$$\begin{aligned} q_*^2 + 2\gamma\tau^2 \cos q_* - \tau^2(1 + 2\gamma) &= 0, \\ q_* - \gamma\tau^2 \sin q_* &= 0. \end{aligned}$$

This means that q_* is a (double) root of $N(iq;\,\gamma,\tau)$, contradicting the isolatedness of the roots. This completes the proof of (i).

Ad (ii). Denoting $\lambda = p + iq$ the roots λ of $N(\lambda;\,\gamma,\tau) = 0$, we have

$$\begin{aligned} p^2 - q^2 &= 2\gamma\tau^2 \cosh p \cos q - \tau^2(1+2\gamma), \\ pq &= \gamma\tau^2 \sinh p \sin q. \end{aligned}$$

It follows: $q^2 \le \tau^2 + p^2 + 4\gamma\tau^2\cosh^2 p/2 \le \tau^2 + 4(\gamma\tau^2 + 4e^{-2})\cosh^2 p/2$, and assertion (ii) is immediate.

Ad (iii). Since $\gamma \in \mathbb{R}^+$, the components of DE are defined by the inequalities $\sin x > 0, x - 2\tan x/2 > 0$. Hence, to the nth component belongs the interval $I_n = (2\pi n, x_n), n \in \mathbb{N}^+$, where x_n is defined by $2n\pi < x_n < (2n+1)\pi, x_n = 2\tan x_n/2$. We have $\tau(2n\pi) = 2n\pi$, $\tau(x_n) = 0$ and $\gamma(x) \to +\infty$ as $x \to 2n\pi^-$, or $x \to x_n^+$. There is a unique x_n^* in I_n satisfying (19). DEC is a smooth curve in $I_n \setminus \{x_n^*\}$. It is easy to check that $\gamma(x)$ resp. $\tau(x)$ decreases resp. increases on $(2n\pi, x_n^*)$ and reverses its type of growth on (x_n^*, x_n). The cusp points $[\gamma(x_n^*), \tau(x_n^*)]$ are the points of the parameter-space where $\sum_0$ contains triple eigenvalues. These are $\pm i x_n^*$. Moreover, the coordinates of the cusp points satisfy

$$\gamma = (1+\tau)\tau^{-2}, \cos\sqrt{2\tau+\tau^2} = (1+\tau)^{-1}, \sin\sqrt{2\tau+\tau^2} > 0. \tag{21}$$

For a given (γ,τ), the double eigenvalues iq are solutions of

$$[\tau^2(1+2\gamma) - q^2]^2 + 4[q^2 - \gamma^2\tau^4] = 0, \tag{22}$$

which is obtained after elimination of $\sin q$, and $\cos q$ in (15,16). This shows that we cannot have more than 2 double eigenvalues in $\sum_0^+ L_{\gamma,\tau}$. The case when q is a double root of (22) corresponds to (γ,τ) satisfying (21), i.e. $+iq$ is a triple eigenvalue of $L_{\gamma,\tau}$. In such a case, $\pm iq$ are the only pure imaginary eigenvalues of $L_{\gamma,\tau}$, since for fixed $\tau = \tau(x_n^*)$, we know by a continuity argument starting with $\gamma = 0$, that for $\gamma < \gamma(x_n^*)$, there is only one pair of simple pure imaginary eigenvalues in $\sum_0 L_{\gamma,\tau}$, or equivalently one positive simple solution q of (15). We conclude that for every $(\gamma,\tau) \in \mathbb{R}^2_+$, $\sum_0$ consists of simple eigenvalues, if (γ,τ) does not belong to DE. Otherwise $\sum_0^+$ contains exactly one double eigenvalue if (γ,τ) does not belong to the intersection of two components of DE, or is not a cusp point. Intersection points of two components of DEC give two pairs of double eigenvalues on $\sum_0$.

Ad (iv). Set $h_\tau(q) = (1-\cos q)(q^2-\tau^2)^{-1}$, where we assume $q > \tau$. One finds the set (γ,τ) of (18), in looking for q,γ,τ such that $h_\tau(q) = (2\gamma\tau^2)^{-1}, dh_\tau(q)/dq = 0$. It is easy to show, for $0 < \tau < 2\pi$, that h_τ has its minimum value 0 for $q = 2n\pi, n = 1, 2, ...$ and maxima for one value of q in every interval between these minima, the values of maxima decaying as q increases. Assertion (iv) of Lemma 1 then follows directly. Notice that for $\tau > 2\pi$ the function h_τ may have one minimum and one maximum before the first minimum of the form $2n\pi$. The case when h_τ has an horizontal inflexion point gives the cusp point. This completes the proof.

Remark that the spectrum of $L_{\gamma,\tau}$ is *not sectorial* (see part (ii) of the lemma). This implies, that we cannot use the traditional reduction tools based on estimates of the resolvent operator $(iq\mathbb{I}-L_{\gamma,\tau})^{-1}$ of order $1/|q|$ for $|q|$ large. Indeed, such an estimate implies the spectrum to be sectorial. Therefore, we have to solve subsequently the affine linear hyperbolic system (23) ad hoc.

4 Weak coupling and periodic waves

The bifurcations in system (6) will occur when the cardinality of $\sum_0 L_{\gamma,\tau}$ changes. Thus, the set $\{[\tau(x), \gamma(x)]\}$ described in Figure 1 is the critical set where bifurcations take place. Let Δ_0 denote the set of (γ, τ) where $\sum_0 L_{\gamma,\tau}$ contains only one pair of simple eigenvalues $\pm iq_1$. In this section the case $(\gamma,\, \tau) \in \Delta_0$ is treated. We separate (6) into a central and a hyperbolic part due to the separation $\sum = \sum_0 + \sum_h$ of $L_{\gamma,\tau}$. Then, we use the Reduction-Theorem 3 in [20] to justify the application of a center manifold argument. It will follow, that all small nontrivial solutions of (6) are periodic in this case.

Introduce the spaces $E^\alpha_j(Z)$ for $\alpha \in \mathbb{R}$, $j \in \mathbb{N}$, with norms $\|f\|_j$, and similarly the vector-valued version $\mathbb{E}^\alpha_j(Z)$, as follows

$$E^\alpha_j(Z) = \left\{ f \in C^j(\mathbb{R},\, Z) \Big/ \|f\|_j = \max_{0\le k\le j} \sup_{t\in\mathbb{R}} e^{-\alpha|t|} |D^k f(t)| < \infty \right\}.$$

For $\alpha > 0$, these Banach-spaces consist of functions, which may grow exponentially at infinity. Sometimes we need exponentially decaying functions, which will be denoted by $E^{-\alpha}(Z)$. If necessary, we use weights $\cosh(\alpha t)$ instead of $\exp(\alpha|t|)$.

The eigenprojection P_1, on the two-dimensional subspace spanned by eigenvectors belonging to $\pm iq_1$, is computed as the sum of the residues for the 3 components (12,13,14) of the solution of the resolvent equation (10). This leads to the following result

Lemma 2 *Assume $(\gamma, \tau) \in \Delta_0$, then $\sum_0 L_{\gamma,\tau} = \{\pm iq_1\}$, and the eigenprojection P_1 is defined in $\mathbb{H}$ by*

$$(P_1U)_0 = a_1(U)/N_1, \quad (P_1U)_1 = q_1 b_1(U)/N_1,$$
$$(P_1U)_2 = \frac{1}{N_1}[a_1(U)\cos q_1 v + b_1(U)\sin q_1 v],$$

where

$$
\begin{aligned}
U &= (x, \xi, X)^T \in \mathbb{H}, \\
a_1(U) &= q_1 x - \gamma\tau^2\sigma(U), \quad b_1(U) = \xi - \gamma\tau^2\rho(U), \\
\sigma(U) &= \int_0^1 \sin q_1(1-s)[X(s) + X(-s)]ds, \\
\rho(U) &= \int_0^1 \cos q_1(1-s)[X(s) - X(-s)]ds, \\
N_1 &= q_1 - \gamma\tau^2 \sin q_1.
\end{aligned}
$$

To prepare application of [20] we have to consider the affine linear system associated with (6) for the hyperbolic part. Set $Q_h = \mathbb{I} - P_1$, $U_h = Q_h U$, then the equation

$$
\partial_t U_h = L_{\gamma,\tau} U_h + Q_h F \tag{23}
$$

has to be solved for $U_h \in \mathbb{E}_0^\alpha(\mathbb{D}_h) \cap \mathbb{E}_1^\alpha(\mathbb{H}_h)$ and for each $\alpha \geq 0$. We have

$$
F = (0,\, f,\, 0)^T, \quad f \in E_0^\alpha(\mathbb{R}), \quad a_1(U_h) = b_1(U_h) = 0,
$$

and thus

$$
q_1 x_h = \gamma\tau^2\sigma(U_h), \quad \xi_h = \gamma\tau^2\rho(U_h), \tag{24}
$$

$$
Q_h F \stackrel{def}{=} F_h = \frac{1}{N_1}(0, -\gamma\tau^2 \sin q_1, -\sin q_1 v)^T f. \tag{25}
$$

Furthermore, X_h is given as

$$
X_h(t,\, v) = \phi(t+v) - \frac{1}{N_1}\int_0^t f(s) \sin q_1(t+v-s)ds, \tag{26}
$$

and we have to satisfy

$$
x_h(t) = X_h(t,0) = \frac{\gamma\tau^2}{q_1}\int_0^1 \sin q_1(1-s)[X_h(t,s) + X_h(t,-s)]ds,
$$

hence X_h may now be written as follows

$$
\begin{aligned}
X_h(t,v) &= x_h(t+v) + \frac{1}{N_1}\int_t^{t+v} f(s)\sin q_1(t+v-s)ds \qquad (27)\\
&= x_h(t+v) + \frac{1}{N_1}\int_0^v f(t+v-s)\sin(q_1 s)ds, \qquad (28)
\end{aligned}
$$

which leads to

$$
\frac{\partial}{\partial v} X_h(t,v) = \dot{x}_h(t+v) + \frac{q_1}{N_1}\int_0^v f(t+v-s)\cos(q_1 s)ds.
$$

Hence, there exists a constant c independent of $\alpha \in (-\alpha_0, \alpha_0)$ such that

$$
||X_h||_{E_0^\alpha(C^1[-1,+1])} \leq ||x_h||_{E_1^\alpha} + c||f||_{E_0^\alpha}. \tag{29}
$$

Now we take the Fourier transform of (23). For being able to do it, we first assume $\alpha < 0$ (i.e. the function f and the unknown U_h decay exponentially at infinity). We then obtain an expression for $\widehat{U}_h$ analytic with respect to $k \in B_\alpha = \{k \in \mathbb{C}; |\operatorname{Im} k| < \alpha\}$, taking values in $\mathbb{D}_h$, and which is solution of

$$(ik\mathbb{I} - L_{\gamma,\tau})\widehat{U}_h(k) = Q_h\widehat{F}(k). \tag{30}$$

For $\alpha > 0$, we use the distributions in S'_α (see Appendix 1), and for $\alpha = 0$ the tempered distributions in S'. Henceforth, set $S' = S'_0$. In such spaces, we cannot use the formula we established in Section 3 for the resolvent, since we have no right to divide by $N(ik;\gamma,\tau)$ (see Proposition 4 of Appendix 1), contrary to the case when $\alpha < 0$, where Fourier transforms are analytic.

For any α, using properties shown in Appendix 1 for $\alpha > 0$, the Fourier transform of $X_h(\cdot, v)$, given by (28), yields

$$\begin{aligned}
\widehat{\xi}_h(k) &= ik\widehat{x}_h(k), \\
\widehat{X}_h(k,v) &= e^{ikv}\widehat{x}_h(k) + \frac{\widehat{f}(k)}{N_1}\int_0^v e^{ik(v-s)}\sin(q_1 s)ds, \\
N(ik;\gamma,\tau)\widehat{x}_h(k) &= -\frac{\gamma\tau^2}{N_1}\widehat{f}(k)[-\sin q_1 + 2q_1(q_1^2 - k^2)^{-1}(\cos k - \cos q_1)],
\end{aligned}$$

and, after noticing that $N(ik;\gamma,\tau) = 2\gamma\tau^2(\cos k - \cos q_1) + k^2 - q_1^2$, and $N_1 = q_1 - \gamma\tau^2 \sin q_1$, this leads to

$$N(ik;\gamma,\tau)[\widehat{x}_h(k) + \widehat{H}(k;\gamma,\tau)\widehat{f}(k)] = 0, \tag{31}$$

where $\widehat{H}$ is defined by the identity

$$\frac{1}{N(ik;\gamma,\tau)} = \frac{q_1}{N_1(k^2 - q_1^2)} + \widehat{H}(k;\gamma,\tau).$$

Now, via Proposition 4 of Appendix 1, equation (31) leads to

$$\widehat{x}_h(k) + \widehat{H}(k;\gamma,\tau)\widehat{f}(k) = \begin{cases} a_+\delta_{q_1} + a_-\delta_{-q_1} \text{ in } S'_\alpha,\ \alpha \geq 0 \\ 0 \text{ for } \alpha < 0 \end{cases}$$

with $a_\pm$ to be determined.

We notice that $k \mapsto \widehat{H}$ is analytic in the strip B_{p_0}, tending to 0 as $1/k^2$ at infinity. So we have the following

Lemma 3 *The function $k \mapsto \widehat{H}(k;\gamma,\tau)$ is the Fourier transform of a function $t \mapsto H(t;\gamma,\tau) \in H^1_{-\delta}$, for any $\delta < p_0$, where $H^1_{-\delta}$ is the space of g such that $t \mapsto g(t)e^{\delta|t|} \in H^1(\mathbb{R})$. Moreover, for $f \in E_0^\alpha$, and $\alpha \in (-\alpha_0, \alpha_0)$, $\alpha_0 < \delta$, then $H(\cdot;\gamma,\tau) * f \in E_1^\alpha$ and there is a constant C independent of α such that*

$$||H(\cdot;\gamma,\tau) * f||_{E_1^\alpha} \leq C||f||_{E_0^\alpha}.$$

Proof: Let us suppress the parameter (γ, τ) for the moment in H and $\widehat{H}$. For $|\operatorname{Im} k| \leq \delta < p_0$, we have

$$(1 + |k|^2)|\widehat{H}(k)| \leq const,$$

hence, $k \mapsto (1 + |k|^2)^{1/2}\widehat{H}(k) \in L^2(\mathbb{R})$ holds. Now, for $0 \leq \delta < p_0$ we have

$$\begin{aligned} e^{\delta t}H(t) &= \frac{1}{2\pi}e^{\delta t}\int_{\mathbb{R}} e^{it(i\delta+s)}\widehat{H}(i\delta + s)ds \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{its}\widehat{H}(i\delta + s)ds, \end{aligned}$$

which implies (by Plancherel)

$$||e^{\delta(\cdot)}H(\cdot)||_{L^2} = \frac{1}{\sqrt{2\pi}}||\widehat{H}(i\delta + \cdot)||_{L^2}.$$

Moreover,

$$\frac{d}{dt}[e^{\delta t}H(t)] = \frac{1}{2\pi}\int_{\mathbb{R}} ise^{its}\widehat{H}(i\delta + s)ds,$$

hence (by Plancherel)

$$||\frac{d}{dt}[e^{\delta(\cdot)}H(\cdot)]||_{L^2} = \frac{1}{\sqrt{2\pi}}||i(\cdot)e^{it(\cdot)}\widehat{H}(i\delta + \cdot)||_{L^2}$$

i.e., doing the same estimate with $-\delta$, we get $t \mapsto e^{\delta|t|}H(t) \in H^1(\mathbb{R})$.

Now consider for $-\delta < \alpha < \delta$

$$\begin{aligned} ||\dot{H} * f||_{E_0^\alpha} &= \sup_{t\in\mathbb{R}} e^{-\alpha|t|}|\int_{\mathbb{R}} \dot{H}(t-\tau)f(\tau)d\tau| \\ &\leq ||f||_{E_0^\alpha}\sup_{t\in\mathbb{R}}\int_{\mathbb{R}} e^{-\alpha|t|+\alpha|\tau|-\delta|t-\tau|}|e^{\delta|t-\tau|}\dot{H}(t-\tau)|d\tau \\ &\leq ||f||_{E_0^\alpha}||e^{\delta|\cdot|}\dot{H}(\cdot)||_{L^2}\left(\sup_{t\in\mathbb{R}}\int_{\mathbb{R}} e^{2[\alpha(|\tau|-|t|)-\delta|t-\tau|]}d\tau\right)^{1/2} \\ &\leq \frac{c}{\sqrt{\delta - |\alpha|}}||f||_{E_0^\alpha}. \end{aligned}$$

This estimate completes Proposition 5 of Appendix 1, and the lemma is proved.

Now, let us define $\widetilde{U}_h = (\widetilde{x}_h, \widetilde{\xi}_h, \widetilde{X}_h)$ with

$$\begin{aligned} \widetilde{x}_h(t) &= -[H(\cdot;\gamma,\tau) * f](t), \\ \widetilde{\xi}_h(t) &= \frac{d}{dt}\widetilde{x}_h(t), \\ \widetilde{X}_h(t,v) &= \widetilde{x}_h(t+v) + \frac{1}{N_1}\int_0^v f(t+v-s)\sin(q_1 s)ds. \end{aligned}$$

Due to (29), it is clear that $\widetilde{U}_h \in \mathbb{E}_0^\alpha(\mathbb{D})$ for $\alpha \in (-\alpha_0, \alpha_0)$, with an estimate

$$||\widetilde{U}_h||_{\mathbb{E}_0^\alpha(\mathbb{D})\cap\mathbb{E}_1^\alpha(\mathbb{H})} \leq C(\alpha)||f||_{\mathbb{E}_0^\alpha} \tag{32}$$

and C is bounded on $(-\alpha_0, \alpha_0)$. Moreover $\widetilde{U}_h$ is a solution of (23). For showing this we first notice that the first and third equation of (23) are easily verified by construction. Now, the Fourier transform of the second equation is just identity (31) for $\widehat{\widetilde{x}}_h$. It results that the second equation of (23) is satisfied.

For $\alpha < 0$, we have by construction $P_1\widehat{\widetilde{U}}_h = 0$, hence $P_1\widetilde{U}_h = 0$. Let us show that for $\alpha \geq 0$, $P_1\widetilde{U}_h = 0$ also holds, since this implies formally exactly the same computations (see lemma 2 for the definition of P_1). Indeed, it is sufficient to show that $a_1(\widetilde{U}_h) = 0$, because this implies $b_1(\widetilde{U}_h) = 0$ by differentiating $\sigma(\widetilde{U}_h)$ with respect to t and integrating by parts. Taking the Fourier transform of $a_1(\widetilde{U}_h)$ (analytic in a strip for $\alpha < 0$, in S' for $\alpha = 0$, in S'_α for $\alpha > 0$), we obtain, due to the properties shown in Proposition 2 of Appendix 1,

$$\begin{aligned}&\mathcal{F}\left(\int_0^1 \sin q_1(1-v)\left[\int_0^v [f(t+u)+f(t-u)]\sin q_1(v-u)du\right]dv\right)(k)\\ &= \widehat{f}(k)\int_0^1 \sin q_1(1-v)\left[\int_0^v 2\cos ku \sin q_1(v-u)du\right]dv,\end{aligned}$$

which is the basic identity for showing that $\mathcal{F}[a_1(\widetilde{U}_h)] \in S'_\alpha$ is proportional to

$$\begin{aligned}&\widehat{\widetilde{x}}_h(k)\left[1-\frac{2\gamma\tau^2}{q_1}\int_0^1 \sin q_1(1-v)\cos kv dv\right] -\\ &- \widehat{f}(k)2\gamma\tau^2 N_1^{-1}\int_0^1 \sin q_1(1-v)\frac{\cos kv-\cos q_1}{q_1^2-k^2}dv\end{aligned}$$

with $\widehat{\widetilde{x}}_h(k) = -\widehat{H}(k;\gamma,\tau)\widehat{f}(k)$. It results that $\widehat{f}(k)$ is a factor of a quantity, now independent of the choice of space for f, i.e. independent of α. Since we know that $a_1(\widetilde{U}_h) = 0$ for $\alpha < 0$, the independence with respect to α shows that $a_1(\widetilde{U}_h) = 0$ for $\alpha \geq 0$ also, and thus $P_1\widetilde{U}_h = 0$ holds for all $\alpha \in (-\alpha_0,\, \alpha_0)$.

For $\alpha \geq 0$, the full solution U_h of (31) is obtained by adding to $\widetilde{x}_h$ a linear combination of the form $b_+\exp(iq_1t) + b_-\exp(-iq_1t)$, with $b_\pm = a_\pm/2\pi$, (see Proposition 3 of Appendix 1). But, since $\widetilde{U}_h \in E_0^\alpha(\mathbb{D}_h)$, we conclude $P_1U_h = 0$ if and only if $b_\pm = 0$. Thus, we have finally

Lemma 4 *Assume $f \in E_0^\alpha$, for $\alpha \in (-\alpha_0, \alpha_0), \alpha_0 < \delta < p_0$, then the system (23) has a unique solution $U_h \in E_0^\alpha(\mathbb{D}_h)\cap E_1^\alpha(\mathbb{H}_h)$, and the linear map $E_0^\alpha(\mathbb{R}) \ni f \mapsto U_h \in E_0^\alpha(\mathbb{D}) \cap E_1^\alpha(\mathbb{H})$ is bounded uniformly in $\alpha \in (-\alpha_0, \alpha_0)$.*

Thus, we have verified the assumptions of Theorem 3 in ([20], p. 133) with the special nonlinearity of $M_\tau(U) = \tau^2(0, g(U), 0)^T$. Therefore, there exists a neighborhood Ω of 0 in $\mathbb{D}$ and, for each $(\gamma,\tau) \in \Delta_0$, a mapping $h \in C_b^k(\mathbb{D}_c;\, \mathbb{D}_h)$, where $\mathbb{D}_c = P_1\mathbb{D}$, $\mathbb{D}_h = Q_h\mathbb{D}$, with $h(0;\gamma,\tau) = 0$, $Dh(0;\gamma,\tau) = 0$, such that

(i) if $\widetilde{U}_c : \mathbb{R} \to \mathbb{D}_c$ is any solution of

$$\partial_t U_c = L_{\gamma,\tau} U_c + P_1 M_\tau [U_c + h(U_c;\gamma,\tau)] \tag{33}$$

with $\widetilde{U}_c(t) \in \Omega_c$ for all $t \in \mathbb{R}$, then $\widetilde{U} = \widetilde{U}_c + h(\widetilde{U}_c;\gamma,\tau)$ solves (6).

(ii) if $\widetilde{U} : \mathbb{R} \to \mathbb{D}$ solves (6), and $\widetilde{U}(t) \in \Omega$ for all $t \in \mathbb{R}$, then

$$\widetilde{U}_h(t) = h(\widetilde{U}_c(t);\gamma,\tau), \quad t \in \mathbb{R},$$

holds, and $\widetilde{U}_c(t)$ solves (33).

Theorem 5 *For any $(\gamma,\tau) \in \Delta_0$, and for U near the origin in $\mathbb{D}$, the system (6) reduces to a two dimensional reversible smooth vector field. Moreover, the set of solutions near 0 of (6) constitutes a one parameter family of periodic orbits, bifurcating from 0.*

Corollary 6 *For any $(\gamma,\tau) \in \Delta_0$, the set of solutions near 0 of equation (2) is a one parameter family of periodic solutions, bifurcating from 0 (the parameter is the amplitude of the oscillations).*

It then results that, for $(\gamma,\tau) \in \Delta_0$, the only small amplitude travelling waves of the original problem (1), belong to a family of time-periodic waves bifurcating from 0.

Proof of the theorem. Once we reduced our problem into the two-dimensional reversible smooth vector field for U_c (33), with a linear part having the simple pair of eigenvalues $\pm iq_1$, the result is known as the Devaney-Lyapunov theorem (see [4]). In fact, it is just a consequence of the implicit function theorem.

5 Reduction near the first critical curve

In this section we define $\Gamma_0' \subset \Gamma_0 =$ boundary of Δ_0 as the set of parameters (γ_0,τ_0) such that $\sum_0 L_{\gamma_0,\tau_0} = \{\pm iq_1, \pm iq_0\}$, where q_0 and q_1 are positive and $\pm iq_1$ are simple, $\pm iq_0$ are double eigenvalues of L_{γ_0,τ_0}. Γ_0' is obviously dense in Γ_0. Thus, we are faced with the simplest possible bifurcation of our problem.

Let us proceed as in the previous section. We have

$$N(iq_0;\gamma_0,\tau_0) = \partial_\lambda N(iq_0;\gamma_0,\tau_0) = N(iq_1;\gamma_0,\tau_0) = 0.$$

The eigenprojection P_1 on the two-dimensional subspace, spanned by the eigenvectors belonging to $\pm iq_1$, was already given in the previous section. We compute the eigenprojection P_0 on the four-dimensional subspace, spanned by the eigenvectors and generalized eigenvectors belonging to $\pm iq_0$. This projection is again given by the sum of the two coefficients of $(\lambda \pm iq_0)^{-1}$ in the Laurent expansion (see [10]) of the resolvent operator $(\lambda\mathbb{I} - L_{\gamma_0,\tau_0})^{-1}$ near the double poles $\pm iq_0$. We obtain the following

Lemma 7 *Assume $(\gamma_0,\tau_0)\in\Gamma_0'$, and $\sum_0 L_{\gamma_0,\tau_0}=\{\pm iq_0,\pm iq_1\}$, where iq_0 is the double eigenvalue, then the spectral projection P_c on the six-dimensional subspace belonging to $\sum_0 L_{\gamma_0,\tau_0}$, is given as $P_c=P_0+P_1$ where P_0 and P_1 are projections of rank 4 resp. 2, commuting with L_{γ_0,τ_0}, such that $P_0P_1=P_1P_0=0$. They are explicitly defined, for $U=(x,\xi,X)^T\in\mathbb{H}$, as follows:*

$$
\begin{aligned}
(P_1U)_0 &= N_1^{-1}a_1(U), \quad (P_1U)_1 = q_1N_1^{-1}b_1(U),\\
(P_1U)_2(v) &= N_1^{-1}[a_1(U)\cos q_1v + b_1(U)\sin q_1 v],
\end{aligned}
$$

$$
\begin{aligned}
(P_0U)_0 &= -\frac{2q_0}{3N_0^2}a_0(U)-\frac{2}{N_0}c_0(U),\\
(P_0U)_1 &= -\left(\frac{2q_0^2}{3N_0^2}+\frac{2}{N_0}\right)b_0(U)-\frac{2q_0\gamma_0\tau_0^2}{N_0}\widehat{\rho}_0(U),
\end{aligned}
$$

$$
\begin{aligned}
(P_0U)_2(v) = (P_0U)_0\cos q_0v &- \left(\frac{2q_0}{3N_0^2}b_0(U)+\frac{2\gamma_0\tau_0^2}{N_0}\widehat{\rho}_0(U)\right)\sin q_0v+\\
&-\frac{2b_0(U)}{N_0}v\cos q_0v+\frac{2a_0(U)}{N_0}v\sin q_0 v,
\end{aligned}
$$

where

$$
\begin{aligned}
&q_j^2=\tau_0^2(1+2\gamma_0-2\gamma_0\cos q_j), \quad j=0,1, \quad q_0=\gamma_0\tau_0^2\sin q_0,\\
&N_1=q_1-\gamma_0\tau_0^2\sin q_1\neq 0, \quad N_0=\gamma_0\tau_0^2\cos q_0-1\neq 0,
\end{aligned}
$$

$$
\begin{aligned}
a_j(U) &= q_jx-\gamma_0\tau_0^2\sigma_j(U), \quad j=0,1,\\
b_j(U) &= \xi-\gamma_0\tau_0^2\rho_j(U),\\
c_0(U) &= x-\gamma_0\tau_0^2\widehat{\sigma}_0(U),\\
\sigma_j(U) &= \int_0^1 \sin q_j(1-s)[X(s)+X(-s)]ds, \quad j=0,1,\\
\rho_j(U) &= \int_0^1 \cos q_j(1-s)[X(s)-X(-s)]ds, \quad j=0,1,\\
\widehat{\sigma}_0(U) &= \int_0^1 (1-s)\cos q_0(1-s)[X(s)+X(-s)]ds,\\
\widehat{\rho}_0(U) &= \int_0^1 (1-s)\sin q_0(1-s)[X(s)-X(-s)]ds.
\end{aligned}
$$

The reader can check easily that $P_0P_1=P_1P_0=0$ follows from the 4 identities

$$\begin{aligned}
q_0 - 2\gamma_0\tau_0^2 \int_0^1 \sin q_0(1-s)\cos(q_1 s)ds = 0,\\
1 - 2\gamma_0\tau_0^2 \int_0^1 (1-s)\cos q_0(1-s)\cos(q_1 s)ds = 0,\\
q_1 - 2\gamma_0\tau_0^2 \int_0^1 \cos q_0(1-s)\sin(q_1 s)ds = 0,\\
\int_0^1 (1-s)\sin q_0(1-s)\sin(q_1 s)ds = 0.
\end{aligned}$$

A necessary and sufficient condition for U to be in the hyperbolic invariant subspace $\mathbb{H}_h$ is that the following 6 conditions are realised

$$a_j(U) = b_j(U) = c_0(U) = \widehat{\rho}_0(U) = 0, \quad j = 0, 1.$$

To prepare application of [20], we have to solve the affine linear system, associated with (6) for the hyperbolic part. Set $Q_h = \mathbb{I} - P_0 - P_1$, $U_h = Q_h U$, then we have to solve

$$\partial_t U_h = L_{\gamma_0,\tau_0} U_h + Q_h F \tag{34}$$

for $U_h \in \mathbb{E}_0^\alpha(\mathbb{D}_h) \cap \mathbb{E}_1^\alpha(\mathbb{H}_h)$ and for each $\alpha \in (-\alpha_0, \alpha_0)$. We have

$$\begin{aligned}
F &= (0, f, 0)^T, \quad f \in E_0^\alpha(\mathbb{R}), \text{ and } F_h = Q_h F,\\
(F_h)_0 &= 0,\\
(F_h)_1 &= \{-q_1 N_1^{-1} + (3N_0^2)^{-1}(3\gamma_0^2\tau_0^4 - 3 - q_0^2)\} f,\\
(F_h)_2(v) &= \left\{-N_1^{-1}\sin q_1 v + 2q_0(3N_0^2)^{-1}\sin q_0 v + 2N_0^{-1} v\cos q_0 v\right\} f.
\end{aligned}$$

The component X_h is now given by

$$\begin{aligned}
X_h(t,v) &= \phi(t+v) + \widetilde{X}_h(t,v),\\
\widetilde{X}_h(t,v) &= \int_0^t f(s)\left(\frac{2q_0}{3N_0^2}\sin q_0(t+v-s) + \frac{2(t+v-s)}{N_0}\cos q_0(t+v-s)\right)ds +\\
&\quad - N_1^{-1}\int_0^t \sin q_1(t+v-s) f(s) ds,
\end{aligned}$$

and

$$x_h(t) = \phi(t) + \int_0^t f(s)\left(\frac{2q_0}{3N_0^2}\sin q_0(t-s) + \frac{2(t-s)}{N_0}\cos q_0(t-s)\right)ds+ \tag{35}$$
$$- N_1^{-1}\int_0^t \sin q_1(t-s)f(s)ds,$$

$$X_h(t,v) = x_h(t+v) + N_1^{-1}\int_t^{t+v} \sin q_1(t+v-s)f(s)ds+ \tag{36}$$
$$- \int_t^{t+v} f(s)\left(\frac{2q_0}{3N_0^2}\sin q_0(t+v-s) + \frac{2(t+v-s)}{N_0}\cos q_0(t+v-s)\right)ds$$
$$= x_h(t+v) + N_1^{-1}\int_0^v \sin(q_1 s)f(t+v-s)ds+$$
$$- \int_0^v f(t+v-s)\left(\frac{2q_0}{3N_0^2}\sin q_0 s + \frac{2s}{N_0}\cos q_0 s\right)ds,$$

which leads to

$$\frac{\partial}{\partial v}X_h(t,v) = \dot{x}_h(t+v) + q_1 N_1^{-1}\int_0^v \cos(q_1 s)f(t+v-s)ds+$$
$$- \int_0^v f(t+v-s)\left(\left(\frac{2q_0^2}{3N_0^2} + \frac{2}{N_0}\right)\cos q_0 s - \frac{2s}{N_0}\sin q_0 s\right)ds.$$

Hence, there exists a constant c independent of $\alpha \in (-\alpha_0, \alpha_0)$ such that

$$||X_h||_{E_0^\alpha(C^1[-1,+1])} \leq ||x_h||_{E_1^\alpha} + c||f||_{E_0^\alpha} \tag{37}$$

holds again. Now we take the Fourier transform of (34). For being able to do it, we proceed as in the previous section. For $\alpha < 0$, we obtain an expression for $\widehat{U}_h$ analytic with respect to $k \in B_\alpha = \{k \in \mathbb{C}; |\operatorname{Im} k| < \alpha\}$, taking values in $\mathbb{D}_h$ and which is solution of

$$(ik\mathbb{I} - L_{\gamma,\tau})\widehat{U}_h(k) = Q_h\widehat{F}(k). \tag{38}$$

For $\alpha \geq 0$ we need to use the distributions in S'_α (see Appendix 1).

For any α, we have

$$\widehat{\xi}_h(k) = ik\widehat{x}_h(k),$$
$$\widehat{X}_h(k,v) = e^{ikv}\widehat{x}_h(k) + \frac{\widehat{f}(k)}{N_1}\int_0^v e^{ik(v-s)}\sin(q_1 s)ds+$$
$$- \widehat{f}(k)\int_0^v e^{ik(v-s)}\left(\frac{2q_0}{3N_0^2}\sin(q_0 s) + \frac{2s}{N_0}\cos q_0 s\right)ds,$$
$$N(ik;\gamma_0,\tau_0)\widehat{x}_h(k) = -\widehat{f}(k)\left(\frac{-q_1}{N_1} + \frac{3\gamma_0^2\tau_0^4 - 3 - q_0^2}{3N_0^2} + \right.$$
$$\left. + \gamma_0\tau_0^2\int_0^1 2\cos k(1-s)\left[\frac{1}{N_1}\sin q_1 s - \frac{2q_0}{3N_0^2}\sin q_0 s - \frac{2s}{N_0}\cos q_0 s\right]ds\right).$$

After using the definitions of N_0, N_1 and the fact that q_1 (resp q_0) is a simple (resp. double) root of $N(ik;\gamma_0,\tau_0)=0$, this leads, after elementary computations, to

$$N(ik;\gamma_0,\tau_0)[\widehat{x}_h(k)+\widehat{H}(k;\gamma_0,\tau_0)\widehat{f}(k)]=0, \tag{39}$$

where $\widehat{H}$ is defined by the identity

$$\frac{1}{N(ik;\gamma_0,\tau_0)}=\frac{q_1}{N_1(k^2-q_1^2)}-\frac{2(k^2+q_0^2)}{N_0(k^2-q_0^2)^2}-\frac{2q_0^2}{3N_0^2(k^2-q_0^2)}+\widehat{H}(k;\gamma_0,\tau_0). \tag{40}$$

The function $\mathbb{C}\ni k\longmapsto N(ik;\gamma_0,\tau_0)$ is entire, and $\pm q_1$ (resp. $\pm q_0$) are the unique simple (resp. double) roots of $N(ik;\gamma_0,\tau_0)=0$ in a strip B_{p_0} where $p_0>\alpha_0$ was defined in Lemma 1 (i). N behaves as k^2 at infinity in B_{p_0}. Notice that $\mathbb{C}\ni k\mapsto\widehat{H}(k;\gamma_0,\tau_0)$ is analytic in the strip B_{p_0} and tends to 0 as $1/k^2$ for $|k|\to\infty$. It results by the lemma shown at previous section, that $\widehat{H}$ is the Fourier transform of a function $\mathbb{R}\ni t\mapsto H(t;\gamma_0,\tau_0)\in H^1_{-\delta}$, for any $\delta<p_0$.

It results from Proposition 4, 3 and 5 of Appendix 1, that the solution of (39) reads

$$x_h(t)+[H(\cdot;\gamma_0,\tau_0)*f](t)=\begin{cases} a_1^+e^{iq_1t}+a_1^-e^{-iq_1t}+(a_0^++itb_0^+)e^{iq_0t}+ \\ \qquad +(a_0^--itb_0^-)e^{-iq_0t},\ \text{for } \alpha\geq 0, \\ =0,\ \text{for } \alpha<0, \end{cases}$$

$a_0^\pm, a_1^\pm, b_0^\pm$ being arbitrary constants. Now, define $\widetilde{U}$ as in the previous section, based on the new $\widetilde{x}_h=-H(\cdot;\gamma_0,\tau_0)*f$, and formula (36) for $\widetilde{X}_h$. The same argument as in the previous section, using Proposition 2 of Appendix 1, shows that $P_1(\widetilde{U}_h)=P_0(\widetilde{U}_h)=0$ independently of α. Then, for $\alpha\in(-\alpha_0,\alpha_0)$, $\alpha_0<\delta<p_0$, we have $a_0^\pm=a_1^\pm=b_0^\pm=0$.

Lemma 8 *Assume $f\in E_0^\alpha$, for $\alpha\in(-\alpha_0,\alpha_0)$, $\alpha_0<\delta<p_0$, then the affine system (34) has a unique solution $U_h\in E_0^\alpha(\mathbb{D}_h)\cap E_1^\alpha(\mathbb{H}_h)$, and the linear map $E_0^\alpha\ni f\longmapsto U_h\in E_0^\alpha(\mathbb{D})\cap E_1^\alpha(\mathbb{H})$ is bounded uniformly in $\alpha\in(-\alpha_0,\alpha_0)$.*

So, as in the previous section, we have verified the assumptions of Theorem 3 in ([20] p.133), and we are now able to use a reduction on a 6-dimensional center manifold.

6 Normal form near $\widetilde{\Gamma}_0$

As we have observed, Γ_0' is dense in Γ_0. Exceptional points on Γ_0 are the cusp points, where there is one pair of triple eigenvalues, and the angular points, where there are two pairs of double eigenvalues and one pair of simple eigenvalues. In what follows, we exclude points of the parameter plane which are close to points of Γ_0 where the ratio q_1/q_0 takes the values 1 (cusps), 1/2, 2, 1/3, 3 corresponding to strong resonances. We also exclude neighborhoods of the angular points of Γ_0. As a consequence we consider only those points $(\gamma,\tau)\in\Gamma_0$ near points where $(q_1/q_0)(\gamma,\tau)$ is close to a rational number r/s such that $r+s\geq 5$. This set of points is denoted by $\widetilde{\Gamma}_0\subset\Gamma_0$.

We stay in the parameter plane near the part $\widetilde{\Gamma}_0$ of Γ_0 where neighborhoods of strong resonances are avoided. For the computation of the normal form we need to define, for every point near $\widetilde{\Gamma}_0$, the nearest weak resonance. For any rational number r/s let us define the subset of $\widetilde{\Gamma}_0$

$$I_{r/s}=\{(\gamma,\tau)\in\widetilde{\Gamma}_0;|\frac{q_1(\gamma,\tau)}{q_0(\gamma,\tau)}|<\varepsilon_{r+s},\ \text{and}\ \frac{q_1(\gamma,\tau)}{q_0(\gamma,\tau)}=\frac{r'}{s'}\ \text{implies}\ r'+s'\geq r+s\}.$$

It is clear that $\widetilde{\Gamma}_0$ is the union of $I_{r/s}$ for $r/s\in\mathbb{Q}_+^*\backslash\{1,2,3,1/2,1/3\}$. We then compute the normal form for $q_1/q_0=r/s$ and we shall play on (γ,τ) to cover the full neighborhood of $\widetilde{\Gamma}_0$. The linear operator on the 6-dimensional central subspace has the form

$$L^{(0)}=\begin{pmatrix} iq_0 & 1 & 0 & 0 & 0 & 0\\ 0 & iq_0 & 0 & 0 & 0 & 0\\ 0 & 0 & iq_1 & 0 & 0 & 0\\ 0 & 0 & 0 & -iq_0 & 1 & 0\\ 0 & 0 & 0 & 0 & -iq_0 & 0\\ 0 & 0 & 0 & 0 & 0 & -iq_1 \end{pmatrix}$$

in the basis $\zeta_0,\widetilde{\zeta}_0,\zeta_1,\overline{\zeta}_0,\overline{\widetilde{\zeta}}_0,\overline{\zeta}_1$ defined by

$$\begin{aligned}\zeta_0&=(1,iq_0,e^{iq_0v})^T,\\ \widetilde{\zeta}_0&=(0,1,ve^{iq_0v})^T,\\ \zeta_1&=(1,iq_1,e^{iq_1v})^T,\end{aligned}$$

and which satisfies

$$\begin{aligned}L_{\gamma_0,\tau_0}\zeta_0&=iq_0\zeta_0, & S\zeta_0&=\overline{\zeta}_0,\\ L_{\gamma_0,\tau_0}\widetilde{\zeta}_0&=iq_0\widetilde{\zeta}_0+\zeta_0, & S\widetilde{\zeta}_0&=-\overline{\widetilde{\zeta}}_0,\\ L_{\gamma_0,\tau_0}\zeta_1&=iq_1\zeta_1, & S\zeta_1&=\overline{\zeta}_1.\end{aligned}$$

It is easy to check that the projection P_0+P_1 may now be defined as follows

$$(P_0+P_1)U=A\zeta_0+B\widetilde{\zeta}_0+C\zeta_1+c.c.=U_0,$$

with

$$\begin{aligned}A&=\frac{iq_0}{3N_0^2}[b_0(U)+ia_0(U)]+\frac{i}{N_0}[\gamma_0\tau_0^2\widehat{\rho}_0(U)+ic_0(U)],\\ B&=-N_0^{-1}[b_0(U)+ia_0(U)],\\ C&=1/2N_1^{-1}[a_1(U)-ib_1(U)].\end{aligned}$$

The structure of the *reversible normal form* corresponding to the linear operator $L^{(0)}$ is computed in Appendix 2. It is shown in particular that the reduced 6-dimensional system, with its normal form written at order $r+s-2$, takes the following form

$$\frac{dA}{dt} = iq_0 A + B + iAP(u_1,u_2,u_4) + O(|A|+|B|+|C|)^{r+s-1}, \tag{41}$$

$$\frac{dB}{dt} = iq_0 B + iBP(u_1,u_2,u_4) + AQ(u_1,u_2,u_4) + O(|A|+|B|+|C|)^{r+s-1}, \tag{42}$$

$$\frac{dC}{dt} = iq_1 C + iCR(u_1,u_2,u_4) + O(|A|+|B|+|C|)^{r+s-1}, \tag{43}$$

where $u_1 = A\overline{A}, u_2 = i/2(A\overline{B} - \overline{A}B), u_4 = C\overline{C}$, and P, Q, R are polynomials with smoothly parameter dependent *real* coefficients for (γ,τ) in the neighborhood of any $(\gamma_0,\tau_0) \in I_{r/s}$. The $0th$ order coefficients in P, Q, R correspond to the critical linear part of system (6). We notice that the normal form, truncated at order $r+s-2$, contains all solutions of the classical 1:1 resonant normal form (just consider solutions with $C=0$).

Let us specify the main coefficients of system (41,42,43). We have at first orders

$$\begin{aligned} P(u_1,u_2,u_4) &= a_1(\gamma,\tau) + a_2 u_1 + a_3 u_2 + a_4 u_4, \\ Q(u_1,u_2,u_4) &= b_1(\gamma,\tau) + b_2 u_1 + b_3 u_2 + b_4 u_4, \\ R(u_1,u_2,u_4) &= c_1(\gamma,\tau) + c_2 u_1 + c_3 u_2 + c_4 u_4, \end{aligned}$$

Coefficients a_1, b_1, c_1 cancel for $(\gamma,\tau) = (\gamma_0,\tau_0)$, and may be easily computed by using the property that

$$\begin{aligned} &iq_0 \pm \sqrt{b_1(\gamma,\tau)} + ia_1(\gamma,\tau), \\ &iq_1 + ic_1(\gamma,\tau) \end{aligned}$$

and their complex conjugate, are the six eigenvalues of the operator $L_{\gamma,\tau}$ for (γ,τ) close to (γ_0,τ_0). Notice that, $b_1(\gamma,\tau) = 0$ on Γ_0 and we have $b_1(\gamma,\tau) > 0$ on the side Δ_0 of the curve Γ_0.

Now, as for the 1:1 resonance case, the most important coefficient is b_2, which we compute below.

Let us denote the basic differential equation (6) as follows

$$\frac{dU}{dt} = L_{\gamma_0,\tau_0}U + (\gamma-\gamma_0)L^{(1,0)}U + (\tau-\tau_0)L^{(0,1)} + M_{2,0}(U,U) + M_{3,0}(U,U,U) + ...$$

with

$$\begin{aligned} L^{(1,0)}U &= \tau_0^2(0, -2x + X^1 + X^{-1}, 0)^T, \\ L^{(0,1)}U &= 2\tau_0(0, -(1+2\gamma_0)x + \gamma_0(X^1 + X^{-1}), 0)^T, \\ M_{2,0}(U,U) &= \tau_0^2(0, ax^2, 0)^T, \\ M_{3,0}(U,U,U) &= \tau_0^2(0, bx^3, 0)^T. \end{aligned}$$

The Taylor expansion of the 6-dimensional center manifold reads

$$
\begin{aligned}
U = {} & A\zeta_0 + B\widetilde{\zeta}_0 + C\zeta_1 + \overline{A\zeta}_0 + \overline{B}\overline{\widetilde{\zeta}}_0 + \overline{C\zeta}_1 + \\
& + \sum (\gamma-\gamma_0)^m (\tau-\tau_0)^n A^{r_0} B^{\widetilde{r}_0} C^{r_1} \overline{A}^{s_0} \overline{B}^{\widetilde{s}_0} \overline{C}^{s_1} \Phi^{(m,n)}_{r_0\widetilde{r}_0 r_1 s_0 \widetilde{s}_0 s_1}
\end{aligned} \tag{44}
$$

where the sum does not contain terms with $m=n=0,\ r_0+\widetilde{r}_0+r_1+s_0+\widetilde{s}_0+s_1=1$, and we have in a classical way

$$
\begin{aligned}
(2iq_0\mathbb{I}-L_{\gamma_0})\Phi^{(0,0)}_{200000} &= M_{2,0}(\zeta_0,\zeta_0), \\
-L_{\gamma_0}\Phi^{(0,0)}_{100100} &= 2M_{2,0}(\zeta_0,\overline{\zeta}_0), \\
ia_2\zeta_0 + b_2\widetilde{\zeta}_0 + (iq_0\mathbb{I}-L_{\gamma_0})\Phi^{(0,0)}_{200100} &= 2M_{2,0}(\overline{\zeta}_0,\Phi^{(0,0)}_{200000}) + 2M_{2,0}(\zeta_0,\Phi^{(0,0)}_{100100}) + \\
&\quad + 3M_{3,0}(\zeta_0,\zeta_0,\overline{\zeta}_0).
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\Phi^{(0,0)}_{200000} &= K_1(1,2iq_0,e^{2iq_0v})^T, \\
\Phi^{(0,0)}_{100100} &= 2a(1,0,1)^T, \\
\Phi^{(0,0)}_{200100} &= ia_2\widetilde{\zeta}_0 + \phi\zeta_0 + \frac{b_2}{2}(0,0,v^2e^{iq_0v})^T,
\end{aligned}
$$

with a_2 and ϕ still unknown, and

$$
\begin{aligned}
K_1 &= a[1-4q_0^2\tau_0^{-2}(1-\gamma_0^{-1}\tau_0^{-2})]^{-1}, \\
-N_0b_2 &= \tau_0^2\{2a^2[1-4q_0^2\tau_0^{-2}(1-\gamma_0^{-1}\tau_0^{-2})]^{-1}+4a^2+3b\}.
\end{aligned} \tag{45}
$$

Notice that $\gamma_0\tau_0^2>1$ due to (16) , and that N_0 may take any sign since it changes its sign at the cusp points of DEC, hence there are situations in the parameter plane such that the coefficient b_2 is negative. For the truncated normal form at cubic order, we have solutions with $C=0$, corresponding to a flat extra oscillatory part, and reducing to the solutions of the classical 1:1 reversible resonance vector field. We know that for $b_2<0$ there is a one parameter (a "circle") family of orbits homoclinic to 0 (see for instance [9])

$$
\begin{aligned}
A &= r_0(t)e^{i(q_0t+\psi(t)+\theta)}, \quad B = r_1(t)e^{i(q_0t+\psi(t)+\theta)}, \quad C=0 \\
r_0(t) &= \sqrt{\frac{2b_1(\gamma,\tau)}{-b_2}}\left(\cosh[t\sqrt{b_1(\gamma,\tau)}]\right)^{-1}, \\
r_1(t) &= \frac{dr_0(t)}{dt}, \\
\psi(t) &= a_1(\gamma,\tau)t + 2\frac{a_2}{b_2}\sqrt{b_1(\gamma,\tau)}\tanh(t\sqrt{b_1(\gamma,\tau)}),
\end{aligned}
$$

where $\theta\in\mathbb{R}$. Two of them are reversible: $\theta=0$ or π. For the full vector field (41,42,43), we are now able to use in particular the results of E.Lombardi [12]: under the non resonance assumptions which are realized here, there exists a family of pairs of reversible solutions of (2) homoclinic to periodic solutions of exponentially small amplitude. This means that this type of solutions, which are mainly given by the above mentioned reversible orbits homoclinic to 0, now contain an oscillating part in C which cannot be annihilated, whose size is $O(e^{-c/b_1^{1/2}})$ hence exponentially small in the bifurcation parameter $b_1(\gamma,\tau)$. So it remains a "phonon" at infinity, the central "localized" part of the solution being of order $\sqrt{b_1(\gamma,\tau)}$. More precisely, the principal parts of these "localized" travelling waves are obtained up to order $O[b_1(\gamma,\tau)]$ (resp. $O[\sqrt{b_1(\gamma,\tau)}]$) for $r+s \geq 6$ (resp. $r+s=5$), in replacing in the center manifold expansion (44) of U, amplitudes A, B, C by the above explicit expressions (see [5]).

Theorem 9 *For (γ,τ) in a neighborhood of the curve Γ_0, except near exceptional points (cusps, angular points and strong resonances), and for U near the origin in $\mathbb{D}$, the system (6) reduces to a 6-dimensional reversible vector field, with a fixed point at the origin and a linear part possessing a pair of double eigenvalues $\pm iq_0$ and a pair of simple eigenvalues $\pm iq_1$. The bifurcation parameter is $b_1 = dist[(\gamma,\tau),\Gamma_0], (counted > 0$ in $\Delta_0)$. All generic bifurcating (periodic, quasiperiodic, homoclinic,...) small bounded solutions of this 6-dim reversible vector field correspond to "small" travelling waves, solutions of (2).In particular, for (γ,τ) in the open set where $b_2 < 0$ [see(45)], there are travelling waves which are localized in space, with exponentially small oscillating tails, called* "nanopterons" *(following J.P.Boyd's denomination [2]).*

Appendix 1. Construction of a suitable distribution space

Given $\alpha > 0$ and $B_\alpha := \{z \in \mathbb{C}/|\operatorname{Im} z| < \alpha\}$, define the space $S_\alpha$as follows

$$S_\alpha = \{f : B_\alpha \to \mathbb{C}/f \text{ holomorphic in } B_\alpha, q_{m,p}(f) < \infty, (m,p) \in \mathbb{N}^2\},$$

where $q_{m,p}(f) = \sup_{z \in B_\alpha} |z^m f^{(p)}(z)| e^{\alpha|\operatorname{Re} z|}$, and where $\mathbb{N}$ is the set of integers starting at 0.

The pair $(S_\alpha, q_{m,p})$ defines a Fréchet space. Notice that $(\cosh \alpha z)^{-1} \in S_\alpha$ if $\alpha^2 < \pi/2$, so S_α is nontrivial and we have $S_\alpha \subset S$ (space of rapidly decaying functions).

Proposition 1: The Fourier transform $\mathcal{F}$ defines a bijection on S_α, being continuous in both directions.

Proof: For any $\phi \in S_\alpha$, we first show that $\mathcal{F}\phi =: \widehat{\phi}$ belongs to S_α. Let us notice that for any $p, m \geq 0$, and $k \in B_\alpha$ we have

$$i^{p+m} k^m \widehat{\phi}^{(p)}(k) = \int_{\mathbb{R}} x^p e^{-ikx} \phi^{(m)}(x) dx.$$

Now, take $k = k_r + ik_i$ and choose $k_r > 0$ and $z = x + i(\varepsilon - \alpha)$, $0 < \varepsilon < \alpha$, - for $k_r < 0$, take $z = x + i(\alpha - \varepsilon)$ and argue analogously - then, one obtains

$$i^{p+m} k^m \widehat{\phi}^{(p)}(k) e^{\alpha k_r} = e^{\varepsilon k_r} e^{i(\varepsilon-\alpha)k_i} \int_{z \in \mathbb{R}+i(\varepsilon-\alpha)} e^{-ik_r x} z^p \phi^{(m)}(z) e^{x k_i} dz,$$

whence it follows

$$|k^m \widehat{\phi}^{(p)}(k)| e^{\alpha k_r} \leq \pi e^{\varepsilon k_r} C_{p,m}(\phi),$$

where $C_{p,m}(\phi) = \sup\limits_{z \in B_\alpha} (1 + |\operatorname{Re} z|^2)|z^p \phi^{(m)}(z)| e^{\alpha |\operatorname{Re} z|} < \infty$ is independent of ε. The limit $\varepsilon \to 0_+$ then yields

$$q_{m,p}(\widehat{\phi}) \leq \pi [q_{p,m}(\phi) + q_{p+2,m}(\phi)].$$

Therefore, to each $\phi \in S_\alpha$, there exists a unique $\widehat{\phi} = \mathcal{F}\phi \in S_\alpha$, and the map $\phi \mapsto \widehat{\phi}$ is continuous. The surjectivity of this map follows by applying the inverse Fourier transform; and the above estimate gives the continuity in both directions.

Now, define the dual space S'_α of linear continuous forms on S_α and provide it with the weak topology, i.e. pointwise convergence. Then $\mathcal{F}'$ -which we denote by $\mathcal{F}$ again - is again a bijection on S'_α, and it is continuous in both directions. Moreover, we have $S' \subset S'_\alpha$, where S' is the set of tempered distributions.

Proposition 2: Given $\alpha > 0, f \in E_0^\alpha(\mathbb{R})$ *and* $r \in C^0[0,1]$; *then*

(i) $f \in S'_\alpha$ *via* $< f, \phi > := \int_{\mathbb{R}} f(t)\phi(t)dt$, *for any* $\phi \in S_\alpha$, *and the embedding* $E_0^\alpha \hookrightarrow S'_\alpha$ *is continuous.*

(ii) $[\mathcal{F}f(\cdot + v)](k) = e^{ikv}(\mathcal{F}f)(k), v \in \mathbb{R}$.

(iii) $h(t) := \int_0^1 r(s) f(t+s) ds \in S'_\alpha$ *and*

$$(\mathcal{F}h)(k) = (\mathcal{F}f)(k) \int_0^1 r(s) e^{iks} ds.$$

Proof:

Ad(i). For every $\phi \in S_\alpha$ the following inequality is valid

$$| < f, \phi > | = | \int_{\mathbb{R}} f(t)\phi(t)dt| \leq \pi [q_{0,0}(\phi) + q_{2,0}(\phi)].||f||_{E_0^\alpha}.$$

Ad(ii). This identity is obtained, similar to the case of tempered distributions

$$< \mathcal{F}f(\cdot + v), \phi > := < f((\cdot + v), \widehat{\phi} > = \int_{\mathbb{R}} f(t+v)\widehat{\phi}(t)dt$$
$$= \int_{\mathbb{R}} f(t)\widehat{\phi}(t-v)dt = \int_{\mathbb{R}} f(t) \left[\int_{\mathbb{R}} \phi(s) e^{-i(t-v)s} ds \right] dt =$$
$$= \int_{\mathbb{R}} f(t) \mathcal{F}[e^{iv(\cdot)}\phi](t) dt = < \mathcal{F}f, e^{iv(\cdot)}\phi > = < e^{iv(\cdot)}\mathcal{F}f, \phi > .$$

Add(iii) The inclusion $E_0^\alpha \subset S'_\alpha$ is obvious. Now, let

$$h_n(t) = \sum_{j=1}^{n} r(s_j) f(t+s_j)\Delta s_j \in S'_\alpha$$

be any Riemann sum for $\int_0^1 r(s)f(t+s)ds$. Then we have

$$| < h_n - h, \phi > | \leq [q_{0,0}(\phi) + q_{2,0}(\phi)] \int_{\mathbb{R}} \frac{e^{-\alpha|t|}}{1+t^2} |h(t) - h_n(t)| dt.$$

The integrand tends pointwise to 0 as $n \to \infty$ and is dominated by an integrable function, thus

$$\lim_{n\to\infty} < h_n, \phi > = < h, \phi >, \phi \in S_\alpha$$

holds. Similarly, we conclude

$$< \mathcal{F}h_n, \phi > = < h_n, \mathcal{F}\phi > \underset{n\to\infty}{\to} < \mathcal{F}h, \phi >$$

and the left side converges to the expression on the right side of the assertion (iii) as $n \to \infty$. The proposition is proved.

Proposition 3: For any $f \in S'_\alpha$

$$[\mathcal{F}(Df)](k) = ik(\mathcal{F}f)(k)$$

holds. Moreover $\mathcal{F}(e^{iqt}) = 2\pi\delta_q$, and $\mathcal{F}(ite^{iqt}) = -2\pi\delta'_q$.

Proof: Same proof as in S'.

Proposition 4: Let K be an analytic and polynomially bounded function in the strip B_δ where $\delta > \alpha$. Assume that K has a finite number of roots z_j with multiplicity $m_j, j = 1, 2, ...N$, in the strip B_α. Then the kernel in S'_α of the linear operator $f \to Kf$ is formed by all linear combinations of the form $\sum_{j=1}^{N}\sum_{k=1}^{m_j} a_{jk}\delta_{z_j}^{(k)}$ with arbitrary $a_{jk} \in \mathbb{C}$ (where δ_q is the Dirac distribution in q which is trivially in S'_α, and $\delta_q^{(m)}$ is the m^{th} derivative of δ_q).

Proof: Assume first that all roots are simple. For $f \in$ kernel defined above, and for any ϕ in S_α, we have $0 = < Kf, \phi > = < f, K\phi >$ since $K\phi \in S_\alpha$. This means that $< f, \psi > = 0$ for all ψ in S_α which cancel at simple roots $z_j, j = 1, 2, ...N$. Now, any $\phi \in S_\alpha$ may be decomposed as the sum of $N+1$ functions in S_α

$$\phi(z) = \sum_{p=1}^{N} \frac{\phi(z_p)\prod_{j\neq p}(z - z_j)}{\cosh\alpha(z - z_p)\prod_{j\neq p}(z_p - z_j)} + \psi(z)$$

where ψ has simple roots in $z_j, j = 1, 2, ...N$, and $< f, \phi > = \sum_{p=1}^{N} a_p\phi(z_p)$. This proves the Proposition 4 for simple roots.

Assume now that $z_1, ...z_r$ are double roots, and $z_{r+1}, ...z_N$ simple roots of $K = 0$ in the strip B_α. We conclude from the result above, that

$$\left[\prod_{p=1}^{r}(z - z_p)\right] f = \sum_{j=1}^{N} a_j \delta_{z_j}.$$

Hence, for any ϕ in S_α, we have $< f, \left[\prod_{p=1}^{r}(z - z_p)\right] \phi >= \sum_{j=1}^{N} a_j \phi(z_j)$. This means that for any ψ in S_α having simple roots in $z_p, p = 1, ...r$, we have

$$< f, \psi >= \sum_{j=1}^{r} c_j \psi'(z_j) + \sum_{j=r+1}^{N} b_j \psi(z_j),$$

where $b_j = a_j \left[\prod_{p=1}^{r}(z_j - z_p)\right]^{-1}, c_j = a_j \left[\prod_{p=1, p\neq j}^{r}(z_j - z_p)\right]^{-1}$. Let us take any $\phi \in S_\alpha$, we have the decomposition $\phi(z) = \sum_{p=1}^{r} \phi(z_p)\chi_p(z) + \psi(z)$, with $\chi_p(z_q) = 0$ if $q \neq p$, and $= 1$ if $q = p$ where $p \in \{1, ...r\}$. Now, $z_1, ...z_r$ are simple roots of ψ, hence we have

$$< f, \phi >= \sum_{p=1}^{N} \alpha_p \phi(z_p) + \sum_{p=1}^{r} c_p \phi'(z_p),$$

with

$$\alpha_p =< f, \chi_p > - \sum_{j=r+1}^{N} b_j \chi_p(z_j) - \sum_{j=1}^{r} c_j \chi_p'(z_j), \quad p = 1, ...r,$$
$$\alpha_p = b_p, \qquad p = r + 1, ...N.$$

Therefore, Proposition 4 is proved for roots at most double. For roots of arbitrary order, the proof is left to the reader.

Proposition 5. Let $H \in E_0^{-\delta}$ and $g \in E_0^{\alpha}$, with $\delta > \alpha \geq 0$, then we have
i) $H * g \in E_0^{\alpha}$, with $||H * g||_{0,\alpha} \leq 2(\delta - \alpha)^{-1}||H||_{0,-\delta}||g||_{0,\alpha}$
ii) $\mathcal{F}(H * g) = \widehat{H}.\widehat{g}$ where $\widehat{H} = \mathcal{F}H$ is the Fourier transform in the usual sense of functions, and $\widehat{g}$ and $\mathcal{F}(H * g)$ are Fourier transforms in S'_α for $\alpha > 0$, in S' for $\alpha = 0$.

Proof: i) comes from the inequality

$$\int_{\mathbb{R}} e^{\alpha(|s|-|t|)-\delta|t-s|} ds \leq 2(\delta - \alpha)^{-1}.$$

Now, for $\alpha > 0$, $\mathcal{F}(H * g) \in S'_\alpha$ and satisfies $\forall \varphi \in S_\alpha$,

$$
\begin{aligned}
&< \mathcal{F}(H * g), \varphi >=< H * g, \mathcal{F}\varphi > \\
&= \int\int\int H(t-s)g(s)e^{-ikt}\varphi(k)dkdtds \\
&= \int g(s)\int e^{-iks}\varphi(k)\widehat{H}(k)dkds =< g, \mathcal{F}(\varphi.\widehat{H}) > \\
&=< \widehat{g}, \varphi.\widehat{H} >=< \widehat{H}.\widehat{g}, \varphi > .
\end{aligned}
$$

We noticed, in this calculation, that $\varphi.\widehat{H} \in S_\alpha$ because $\widehat{H}$ is analytic in the strip $B_\delta \supset B_\alpha$, and bounded in B_α. As a corollary, this shows that $\mathcal{F}^{-1}(\widehat{H}.\widehat{g}) = H * g$ in E_0^α.

For $\alpha = 0$, $H * g \in E_0^0 = C_b^0(\mathbb{R}), \mathcal{F}(H * g) \in S'$, and all equalities above hold for $\phi \in S$.

Appendix 2. Reversible normal form associated with $L^{(0)}$

As indicated for instance in ([7], p.18 and 23-24) we need to solve

$$
\begin{aligned}
D_{U_0}N(U_0).L^{(0)*}U_0 &= L^{(0)*}N(U_0), \\
SN &= -N \circ S,
\end{aligned}
$$

where $N = (N_0, \widetilde{N}_0, N_1, \overline{N}_0, \overline{\widetilde{N}}_0, \overline{N}_1)^T$, and

$$
S(A, B, C, \overline{A}, \overline{B}, \overline{C})^T = (\overline{A}, -\overline{B}, \overline{C}, A, -B, C)^T.
$$

Moreover N has polynomial components of an arbitrarily fixed degree in variables $(A, B, C, \overline{A}, \overline{B}, \overline{C})$. Let us define the linear differential operator

$$
\begin{aligned}
\mathcal{D}^* f = -iq_0 A\frac{\partial f}{\partial A} + (-iq_0 B + A)\frac{\partial f}{\partial B} - iq_1 C\frac{\partial f}{\partial C} + \\
+ iq_0\overline{A}\frac{\partial f}{\partial \overline{A}} + (iq_0\overline{B} + \overline{A})\frac{\partial f}{\partial \overline{B}} + iq_1\overline{C}\frac{\partial f}{\partial \overline{C}},
\end{aligned}
$$

then we must verify

$$
\begin{aligned}
\mathcal{D}^* N_0 &= -iq_0 N_0, \\
\mathcal{D}^* \widetilde{N}_0 &= -iq_0 \widetilde{N}_0 + N_0, \\
\mathcal{D}^* N_1 &= -iq_1 N_1.
\end{aligned}
$$

Independent first integrals of $\mathcal{D}^* f = 0$, are

$$
u_1 = A\overline{A}, u_2 = i/2(A\overline{B} - \overline{A}B), u_3 = iq_0 B/A + \ln A, u_4 = C\overline{C}, u_5 = A^r\overline{C}^s,
$$

where we assumed that $\frac{q_1}{q_0} = \frac{r}{s}$. We observe that

$$\overline{A} = \frac{u_1}{A}, \quad B = \frac{A}{iq_0}(u_3 - \ln A), \quad \overline{B} = \frac{2u_2}{iA} + \frac{u_1(u_3 - \ln A)}{iq_0 A},$$
$$C = u_4 u_5^{-1/s} A^{r/s}, \quad \overline{C} = u_5^{1/s} A^{-r/s},$$

hence, a polynomial in variables $(A, B, C, \overline{A}, \overline{B}, \overline{C})$ can be expressed as a function of variables $(A, u_1, u_2, u_3, u_4, u_5)$, polynomial in (u_1, u_2, u_3, u_4), with coefficient functions of (A, u_5), the dependence in u_5 being with polynomials of $(u_5)^{\pm 1/s}$. Now, considering polynomial solutions of $\mathcal{D}^* f = 0$, it results easily, with the variables $(A, u_1, u_2, u_3, u_4, u_5)$, that f is independent of A, i.e. $f(A, B, C, \overline{A}, \overline{B}, \overline{C}) = \phi(u_1, u_2, u_3, u_4, u_5)$, where

$$\phi(u_1, u_2, u_3, u_4, u_5) = \sum \phi_{r_1 r_2 r_3 r_4 r_5} u_1^{r_1} u_2^{r_2} u_3^{r_3} u_4^{r_4} u_5^{r_5/s} \quad \text{(finite sum)}$$

with integers $r_j \geq 0, j = 1, 2, 3, 4$, and $r_5 \geq 0$ or < 0. We can first assert that ϕ is independent of u_3. This is due to the occurence of $\ln A$ at some power in $u_3^{r_3}$, and a study at infinity shows that ϕ cannot behave polynomially in A if u_3 occurs in ϕ. Now, an examination of the exponents of $\overline{C}$ and A (making $\overline{B} = 0$) in ϕ leads to the conditions

$$r_4 + r_5 \geq 0,$$
$$sr_1 + rr_5 \text{ is a positive multiple of } s.$$

Hence, $r_5 = k_5 s$, with $rk_5 \geq -r_1, sk_5 \geq -r_4$. It results that in case $k_5 > 0$, one has a monomial $u_1^{r_1} u_2^{r_2} u_4^{r_4} u_5^{k_5}$ while in case $k_5 < 0$, one has a monomial $u_1^{r_1'} u_2^{r_2} u_4^{r_4'} \overline{u}_5^{-k_5}$, with $r_1' = r_1 + rk_5, r_4' = r_4 + sk_5$. Finally the polynomial solutions of $\mathcal{D}^* f = 0$ can be written as

$$f = P_0(u_1, u_2, u_4) + u_5 P_1(u_1, u_2, u_4, u_5) + \overline{u}_5 P_2(u_1, u_2, u_4, \overline{u}_5),$$

where P_j are polynomials in their arguments. Notice that, if one has in addition $f \circ S = \pm \overline{f}$, then polynomials P_j have real or pure imaginary coefficients.

Let us now solve $\mathcal{D}^* N_0 = -iq_0 N_0, N_0 \circ S = -\overline{N}_0$.

We observe that $\mathcal{D}^*(\overline{A} N_0) = 0$, hence

$$\overline{A} N_0 = \phi_0(u_1, u_2, u_4) + u_5 \phi_1(u_1, u_2, u_4, u_5) + \overline{u}_5 \phi_2(u_1, u_2, u_4, \overline{u}_5),$$

and u_1 should be a factor of the polynomials ϕ_0 and ϕ_1. Finally one obtains, after using the reversibility condition,

$$N_0 = iA[P_0(u_1, u_2, u_4) + u_5 P_1(u_1, u_2, u_4, u_5) + \overline{u}_5 P_2(u_1, u_2, u_4, \overline{u}_5)]$$
$$+ i\overline{A}^{r-1} C^s P_3(u_2, u_4, \overline{u}_5),$$

where P_0, P_1, P_2, P_3 have *real coefficients*. Let us consider the equation $\mathcal{D}^* \widetilde{N}_0 = -iq_0 \widetilde{N}_0 + N_0$, and observe that $\mathcal{D}^*(\overline{A} \widetilde{N}_0) = \overline{A} N_0, \mathcal{D}^*(\overline{A} B) = u_1, \mathcal{D}^*(\overline{A}^{r-1} \overline{B} C^s) = \overline{u}_5$, hence (using reversibility again)

$$
\begin{aligned}
\overline{A}\widetilde{N}_0 = {} & i\overline{A}B[P_0(u_1,u_2,u_4)+u_5P_1(u_1,u_2,u_4,u_5)+\overline{u}_5P_2(u_1,u_2,u_4,\overline{u}_5)]+\\
& +A\overline{A}[Q_0(u_1,u_2,u_4)+u_5Q_1(u_1,u_2,u_4,u_5)+\overline{u}_5Q_2(u_1,u_2,u_4,\overline{u}_5)]+\\
& +i\overline{A}^{r-1}\overline{B}C^sP_3(u_2,u_4,\overline{u}_5)+\overline{A}^rC^sQ_3(u_2,u_4,\overline{u}_5).
\end{aligned}
$$

If $r=1$, making $\overline{A}=0$, leads to

$$
0=\overline{B}C^sP_3(u_2,u_4,0),
$$

and $\overline{u}_5$ is factor of P_3 if $r=1$. Finally, we also have $\mathcal{D}^*(\overline{C}N_1)=0$, then the normal form reads

$$
\begin{aligned}
N_0 = {} & iA[P_0(u_1,u_2,u_4)+u_5P_1(u_1,u_2,u_4,u_5)+\overline{u}_5P_2(u_1,u_2,u_4,\overline{u}_5)]+\\
& +i\overline{A}^{r-1}C^sP_3(u_2,u_4,\overline{u}_5),\\
\widetilde{N}_0 = {} & iB[P_0(u_1,u_2,u_4)+u_5P_1(u_1,u_2,u_4,u_5)+\overline{u}_5P_2(u_1,u_2,u_4,\overline{u}_5)]+\\
& +A[Q_0(u_1,u_2,u_4)+u_5Q_1(u_1,u_2,u_4,u_5)+\overline{u}_5Q_2(u_1,u_2,u_4,\overline{u}_5]+\\
& +i\overline{A}^{r-2}\overline{B}C^sP_3(u_2,u_4,\overline{u}_5)+\overline{A}^{r-1}C^sQ_3(u_2,u_4,\overline{u}_5),\\
N_1 = {} & iC[R_0(u_1,u_2,u_4)+u_5R_1(u_1,u_2,u_4,u_5)+\overline{u}_5R_2(u_1,u_2,u_4,\overline{u}_5)]+\\
& +i\overline{C}^{s-1}A^rR_3(u_1,u_2,u_5),
\end{aligned}
$$

where all polynomials have real coefficients and where $\overline{u}_5$ is in factor in P_3 when $r=1$.

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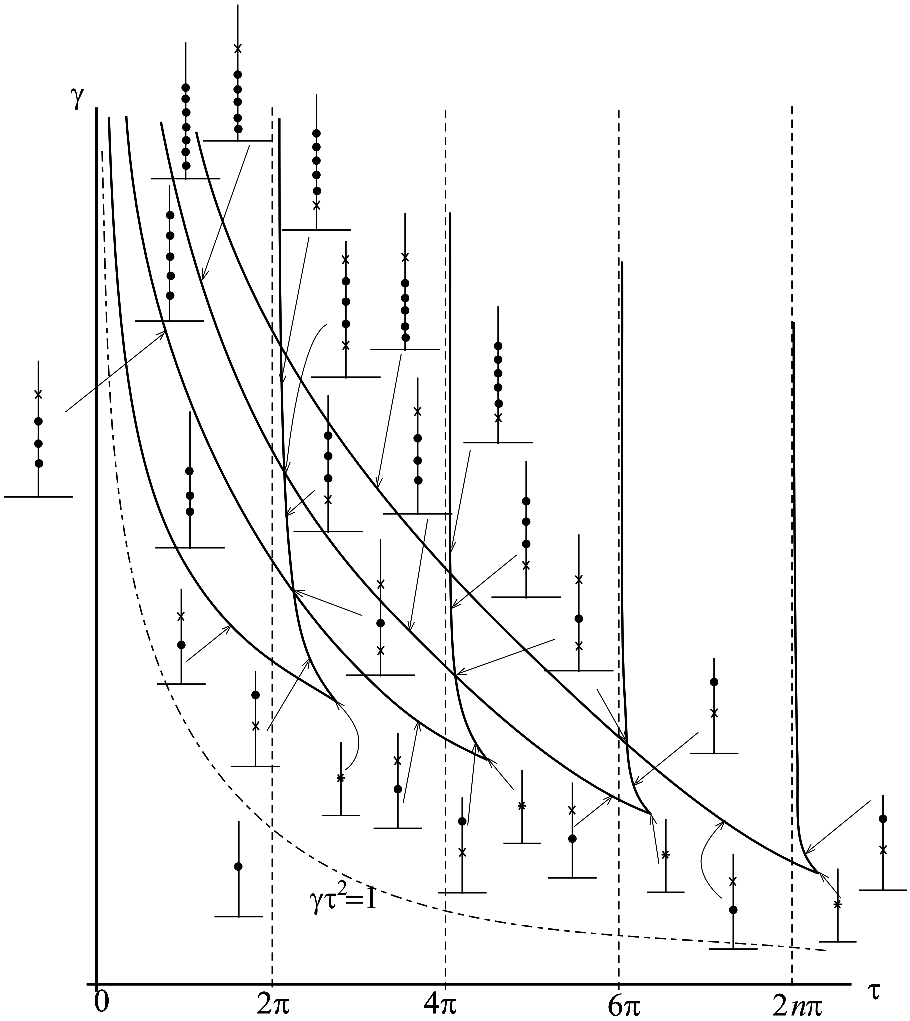

Figure 1: Pure imaginary eigenvalues of $L_{\gamma,\tau}$ (upper half). Dots are simple eigenvalues. A simple cross and a double cross respectively means double or triple eigenvalue.