Heteroclinic for a 6-dimensional reversible system occurring in orthogonal domain walls in convection

Gérard Iooss
Laboratoire J.A.Dieudonné, I.U.F., Université Côte d'Azur, CNRS,
Parc Valrose, 06108 Nice cedex 2, France
iooss.gerard@orange.fr

Abstract

A six-dimensional reversible normal form system occurs in Bénard-Rayleigh convection between parallel planes, when we look for domain walls intersecting orthogonally (see Buffoni et al [1]). On the truncated system, we prove analytically the existence, local uniqueness, and analyticity in parameters, of a heteroclinic connection between two equilibria, each corresponding to a system of convective rolls. We prove that the 3-dimensional unstable manifold of one equilibrium, intersects transversally the 3-dimensional stable manifold of the other equilibrium, both manifolds lying on a 5-dimensional invariant manifold. We also study the linearized operator along the heteroclinic, allowing to prove (in [8]) the persistence under perturbation, of the heteroclinic obtained in [1].

Key words: Reversible dynamical systems, Invariant manifolds, Bifurcations, Heteroclinic connection, Domain walls in convection

1 Introduction and Results

In this work we study the following 6th order reversible system

$$\begin{aligned} A^{(4)} &= A(1-A^2-gB^2) \\ B'' &= \varepsilon^2 B(-1+gA^2+B^2), \end{aligned} \tag{1}$$

where A and B are real functions of $x \in \mathbb{R}$. This system occurs in the search for domain walls intersecting orthogonally, in a fluid dynamic problem such as the Bénard-Rayleigh convection between parallel horizontal plates (see subsection 1.1 and all details in [1]). The heteroclinic we are looking for, corresponds to the connection between rolls on one side and rolls oriented orthogonally on the other side. The system (1) has been also introduced by Manneville and Pomeau in [13], obtained after formal physical considerations using symmetries.

We would like to find analytically a heteroclinic connection ($g > 1$, ε small) such that

$$\begin{aligned} A_*(x), B_*(x) &> 0, \\ (A_*(x), B_*(x)) &\to \begin{cases} (1,0) \text{ as } x \to -\infty \\ (0,1) \text{ as } x \to +\infty \end{cases}. \end{aligned}$$

By a variational argument Boris Buffoni et al [1] prove the existence of such an heteroclinic orbit, for any $g > 1$, and ε small enough. This type of elegant proof does not unfortunately allow to prove the persistence of such heteroclinic curve under reversible perturbations of the vector field. This is our motivation for producing analytic arguments, proving such an existence, uniqueness and smoothness in parameters (ε, g) of this orbit, however for limited values $1 < g \leq 2$, fortunately including physical interesting ones. Then we study the linearized operator along the heteroclinic curve, allowing to attack the problem of existence of orthogonal domain walls in convection (see [8] and Remark 22).

1.1 Origin of system (1)

The Bénard-Rayleigh convection problem is a classical problem in fluid mechanics. It concerns the flow of a three-dimensional viscous fluid layer situated between two horizontal parallel plates and heated from below. Upon increasing the difference of temperature between the two plates, the simple conduction state looses stability at a critical value of the temperature difference corresponding to a critical value $\mathcal{R}_c$ of the Rayleigh number. Beyond the instability threshold, a convective regime develops in which patterns are formed, such as convective rolls, hexagons, or squares. Observed patterns are often accompanied by defects.

We start with the Navier-Stokes-Boussinesq (N-S-B) *steady* system of PDE's, applying spatial dynamics with x as "time" (as introduced by K.Kirchgässner in [12], adapted for N-S equations in [9], and more generally in [6]) and considering solutions $2\pi/k$ periodic in y (coordinate parallel to the wall). We show in [1] that near criticality a 12-dimensional center manifold reduction to a reversible system applies for $(\mathcal{R}, k)$ close to $(\mathcal{R}_c, k_c)$, $\mathcal{R}$ being the Rayleigh number, and k_c the critical wave number. This high dimension of the center manifold may be explained as follows. Due to the equivariance of the system under horizontal shifts, the eigenvectors of the linearized problem are of the form $\exp i(\pm k_1 x \pm k_2 y)$, the factor being only function of $k^2 = k_1^2 + k_2^2$ (invariance under rotations). It results that, for eigenvectors independent of x corresponding to a 0 eigenvalue in the spatial dynamics formulation, the eigenvalue is double in general (make $\pm k_1 \to 0$). Now, at criticality, $k = k_c$ corresponds to two different values of k_2 merging towards k_c, which double the dimension, making a quadruple 0 eigenvalue with complex and complex-conjugate eigenvectors. Hence we already have a dimension 8 invariant subspace for the 0 eigenvalue, with two 4×4 Jordan blocks. This corresponds to convective rolls of amplitude A and $\overline{A}$ at $x = -\infty$. Now for eigenvectors independent of y corresponding to eigenvalues $\pm ik$ in the spatial dynamics formulation it is shown in [5] that they are simple, and give double eigenvalues $\pm ik_c$ for $k = k_c$ with amplitudes B and $\overline{B}$ respectively. Hence this adds 4 dimensions to the central space, so finally obtaining a 12-dimensional central space. Now we restrict the study to solutions invariant under reflection $y \to -y$ (the change y into $-y$ changing A in $\overline{A}$ and not changing B), which constitutes an *invariant subspace for the full system.* This restricts the study to *real* amplitudes A and the full system reduces to a 8-dimensional sub-center manifold, such that $A \in \mathbb{R}$ and $B \in \mathbb{C}$ are the amplitudes of the rolls respectively at $x = -\infty$, and $x = +\infty$. Moreover, for the full system, we keep

i) the reversibility symmetry:

$$(x, A, B) \to (-x, A, \overline{B}),$$

ii) the equivariance under shifts by half of a period in y direction, leading to the symmetry:

$$(A, B) \to (-A, B).$$

Now, in [1] we use a normal form reduction up to cubic order, and rewrite the system as one 4th order real differential equation for A, and a second order complex differential equation for B. In addition to the above symmetries, the normal form commutes with the symmetry:

$$(A, B) \to (A, Be^{i\phi}), \text{ for any } \phi \in \mathbb{R}.$$

Handling the full N-S-B equations, in [1] the authors show that the study leads to a small perturbation of the reduced system of amplitude equations (1). More precisely, after a suitable scaling (see [1]), and denoting by $(\varepsilon^2 A_0, \varepsilon^2 B_0)$ rescaled amplitudes $(A, Be^{-ik_c x})$, and after a rescaling of the coordinate x, we obtain the system

$$\begin{aligned} A_0^{(4)} &= k_- A_0'' + A_0\left(1 - \frac{k_-^2}{4} - A_0^2 - g|B_0|^2\right) + \widehat{f}, \\ B_0'' &= \varepsilon^2 B_0(-1 + gA_0^2 + |B_0|^2) + \widehat{g}, \end{aligned} \tag{2}$$

where ε^4 is a rescaling proportional to $\mathcal{R} - \mathcal{R}_c$, the coefficient $g > 1$ is function of the Prandtl number and is the same as introduced and computed in ([5]), k_- comes from the freedom left to the wave number of the rolls at $-\infty$, defined as

$$k = k_c(1 + \varepsilon^2 k_-),$$

and $\widehat{f}$ and $\widehat{g}$ are perturbation terms, smooth functions of their arguments, coming

i) from the rest of the cubic normal form, at least of order ε^2 for $\widehat{f}$, and at least of order ε^3 for $\widehat{g}$;

ii) from higher order terms not in normal form, and not autonomous (because of the introduction of $Be^{-ik_c x}$ rescaled as $\varepsilon^2 B_0$ in (2)), and of order ε^4 for $\widehat{f}$, and of order ε^6 for $\widehat{g}$. Without k_-, $\widehat{f}$, and $\widehat{g}$, this is the system (1), with $B_0 \in \mathbb{C}$

replacing B, and B^2 replaced by $|B_0|^2$. The truncation leading to (1) allows to take B real, since the phase of B_0 does not play any role in the dynamics for (1). The two different wave numbers of the rolls, close to the critical value k_c are left free for the full problem, however they do not appear in the present proof of the heteroclinic, even though they are important for the final proof of existence of the orthogonal domain walls (see Remark 22 in section 7). It should be noticed that the system (2), without $\widehat{f}$ and $\widehat{g}$, was obtained a long time ago by Pomeau-Manneville in [13], however they did not deal with the full N-S-B system, and only considered cases with identical wave numbers at infinities, while it is shown in [8] that some cubic terms, inexistent in [13], as $\varepsilon^2(A_0^2A_0'' - A_0A_0'^2)$ in $\widehat{f}$ and $i\varepsilon^3B_0A_0A_0'$ in $\widehat{g}$ are crucial for the determination of the solutions of the full problem, with different wave numbers at infinities (see Remark 22).

1.2 Sketch of the method and results

Let us now consider the system (1). The equilibrium $(A, B) = (0, 1)$ of the system (1) gives an approximation of convection rolls parallel to the wall (periodic in the x direction, with fixed phase) bifurcating for Rayleigh numbers $\mathcal{R} > \mathcal{R}_c$ close to $\mathcal{R}_c$, whereas the equilibrium $(A, B) = (1, 0)$ of the system (1) gives the same convection rolls (periodic in the y direction) rotated by an angle $\pi/2$ with the phase fixed by the imposed reflection symmetry. A heteroclinic orbit connecting these two equilibria provides then an approximation of orthogonal domain walls (see Figure 1).

We set $\delta = (g-1)^{1/2}$. The idea here might be to use the arc of equilibria $A^2 + B^2 = 1$, which exists for $\delta = 0$, connecting end points $M_- = (1, 0)$ and $M_+ = (0, 1)$, and to prove that for suitable values of δ (> 0 but close to 0), the 3-dimensional unstable manifold of M_- intersects transversally the 3-dimensional stable manifold of M_+, both staying on a 5 dimensional invariant manifold $\mathcal{W}_\delta$. However, for $\delta = 0$ the situation in M_+ is very degenerated, with a quadruple 0 eigenvalue for the linearized operator, while it is only a double eigenvalue in M_-. Then for δ close to 0, a 5-dimensional center-stable invariant manifold starting from M_+ needs to intersect a four-dimensional center-unstable manifold starting from M_-. Unfortunately, we are not able to prove this. Moreover, for δ close to 0, we cannot prove that the 3-dimensional unstable manifold of M_- exists from $B = 0$ until B reaches a value close enough to 1. In fact the physically interesting values of δ are not close to 0. So that we prefer to play with ε.

The strategy here consists to keep in mind that, after changing the coordinate x in $\overline{x} = \varepsilon x$, we obtain the new system

$$\begin{aligned} \varepsilon^4 \frac{d^4A}{d\overline{x}^4} &= A(1 - A^2 - gB^2) \\ \frac{d^2B}{d\overline{x}^2} &= B(B^2 - 1), \end{aligned} \tag{3}$$

and the limit $\varepsilon \to 0$ of the system (1) is singular, and gives indeed a non smooth heteroclinic solution such that

(i) for x running from $-\infty$ to 0, then (A, B) varies from $(1,0)$ to $(0, \frac{1}{\sqrt{g}})$ on the ellipse $A^2 + gB^2 = 1$, while

(ii) for x running from 0 to $+\infty$, then (A, B) varies from $(0, \frac{1}{\sqrt{g}})$ to $(0,1)$, satisfying, in the original coordinate x, the differential equation (see the first integral (5)).

$$B' = \frac{\varepsilon}{\sqrt{2}}(1 - B^2).$$

We might think to use Fenichel's theorems [3] on the system (1) for ε close to 0, at least for the first part of the proof, for $x \in (-\infty.0]$. However the set of equilibria (here $A^2 + gB^2 = 1$) is not normally hyperbolic at the end point $A = 0$, and this is still not close enough to $B = 1$.

The major difficulty in the proof of Theorem 1 is to prove the existence of the 3-dim unstable manifold of M_- until A reaches a neighborhood of 0, and to prove the existence of the 3-dim stable manifold of M_+ until B reaches a neighborhood of $1/\sqrt{g} = 1/\sqrt{1+\delta^2}$. The usual proofs of existence of such invariant manifolds give only local results, so we need to use here the additional property that we have a first integral of the system, expressing that both invariant 3-dim manifolds lie on a 5-dimensional invariant manifold. We are then able to extend sufficiently the domain of existence of these manifolds, as graphs with respect to B. Indeed we prove the following

Theorem 1 *Let us choose $0 < \delta_0 < 1/3$, then for $\delta_0 \leq \delta \leq 1$, η_0 such that $\varepsilon^{1/3} = [(1+\delta^2)\eta_0^2 - 1]^{1/2}$, and for ε small enough, the 3-dim unstable manifold of M_- intersects transversally the 3-dim stable manifold of M_+, except maybe for a finite number of values of δ. The connecting curve which is obtained is unique (see Remark 3). Moreover its dependency in parameters (ε, δ) is analytic. In addition we have $B(x)$ and $B'(x) > 0$ on $(-\infty, +\infty)$, the principal part of $B(x)$ being given*

i) for $x \in (-\infty, 0]$, by

$$\begin{aligned} B_0(x) &= \frac{1}{(1+\frac{\delta^2}{2})^{1/2}\cosh(x_0 - \varepsilon\delta x)}, \\ \cosh x_0 &= \frac{1}{B_{00}(1+\frac{\delta^2}{2})^{1/2}}, \\ B_{00} &= B_0(0) = (1 - \eta_0^2\delta^2)^{1/2}, \end{aligned}$$

ii) for $x \in [0, +\infty)$, by

$$B_0(x) = \frac{\tanh(\varepsilon x/\sqrt{2}) + B_{00}}{1 + B_{00}\tanh(\varepsilon x/\sqrt{2})}.$$

For $x \to -\infty$ we have $(A-1, A', A'', A''', B, B') \to 0$ at least as $e^{\varepsilon\delta x}$, while for $x \to +\infty$, $(A, A', A'', A''') \to 0$ at least as $e^{-\sqrt{\frac{\delta}{2}}x}$, and $(B-1, B') \to 0$ at least as $e^{-\sqrt{2}\varepsilon x}$.

In section 4 we prove at Lemma 9 the existence of the unstable manifold (graph with respect to B) of $M_- = (1,0)$ until a neighborhood of $(A,B) = (0, 1/\sqrt{1+\delta^2})$ with no restriction on the choice of δ, except $\delta \geq \delta_0 > 0$.

In section 5 we prove at Lemma 14 the existence of the stable manifold (graph with respect to B) of $M_+ = (0,1)$ until (backward direction) a neighborhood of $(0, 1/\sqrt{1+\delta^2})$. Here there is a restriction $\delta \leq 1$, for being able to reach the end point.

In section 6 we prove the transverse intersection of the two manifolds, except maybe for a finite set of values of δ. This ends the proof of Theorem 1. We claim that our proof uses only elementary analysis, as implicit function theorem in various function spaces (see [2]), theory of differential equations as developped in [4], and a classical property of analytic functions.

In section 7 we give in Lemma 21 the properties of the linearized operator along the heteroclinic, which are necessary to prove a persistence result under a reversible perturbation for the heteroclinic in the 8-dimensional space (with $B \in \mathbb{C}$). This allows to prove the existence of orthogonal domain walls in convection [8].

Remark 2 *It should be noticed that we show at Lemma 18 that, in the middle of the heteroclinic,* $A(0) = \mathcal{O}(\sqrt{\varepsilon})$ *and* $A(x)$ *oscillates, staying of this size very close to* 0 *for* $x \in (0, +\infty)$*, while* $B(0)$ *is very close to* $1/\sqrt{g}$ *and* $B(x)$ *grows monotonically until* 1.

Remark 3 *Using symmetries of the system:* $A \mapsto \pm A$, $B \mapsto \pm B$ *and reversibility symmetry:* $(A(x), B(x)) \mapsto (A(-x), B(-x))$*, we find 8 heteroclinics. Two are connecting* M_- *to* M_+ *with opposite dynamics, two others connect* $-M_-$ *to* M_+*, two connect* M_- *to* $-M_+$*, and two connect* $-M_-$ *to* $-M_+$*. The one which interests us is the only one connecting* M_- *to* M_+ *with the dynamics running from* M_- *to* M_+*.*

Remark 4 *It should be noticed that the study made in [13] on the heteroclinic solution for the system (1) uses asymptotic analysis, suggesting the existence of the heteroclinic, later proved mathematically in [1]. We think that our result on the size of* $A(0)$ *is optimal (see Remark 2), since it results from exact computations at section 6.*

Remark 5 *Values of* δ *such that* $0.476 \leq \delta$ *include values obtained for* δ *in the Bénard-Rayleigh convection problem where* $g = 1 + \delta^2$ *is function of the Prandtl number* $\mathcal{P}$ *(as computed in [5]). With rigid-rigid, rigid-free, or free-free boundaries the minimum values of* g *are respectively* ($g_{\min} = 1.227$, 1.332, 1.423) *corresponding to* $\delta_{\min} = 0.476$, 0.576, 0.650*. The restriction in Theorem 1 corresponds to* $1 < g \leq 2$*. The eligible values for the Prandtl number are then respectively* $\mathcal{P} > 0.5308, > 0.6222, > 0.8078$*.*

Acknowledgement The author warmly thanks Mariana Haragus for her help in section 7, and her constant encouragements.

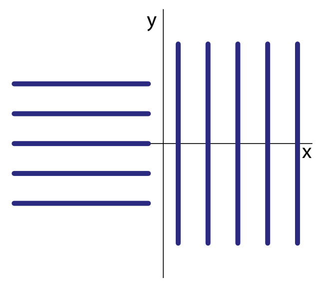

Figure 1: Orthogonal domain wall

2 Global invariant manifold $\mathcal{W}_\delta$

The coordinates in $\mathbb{R}^6$ are defined as

$$(A_0, A_1, A_2, A_3, B_0, B_1) = (A, A', A'', A''', B, B').$$

The first observation is that we have the first integral

$$\varepsilon^2(A'^2)'' - 3\varepsilon^2 A''^2 - B'^2 + \frac{\varepsilon^2}{2}(A^2 + B^2 - 1)^2 + \varepsilon^2\delta^2 A^2 B^2, \tag{4}$$

as noticed in [13], where an Euler-Lagrange equation is used (as used later in [1]). Then, our heteroclinic should satisfy

$$2\varepsilon^2 A_1 A_3 - \varepsilon^2 A_2^2 - B_1^2 + \frac{\varepsilon^2}{2}(A_0^2 + B_0^2 - 1)^2 + \varepsilon^2\delta^2 A_0^2 B_0^2 = 0. \tag{5}$$

Since our purpose is to find B_0 growing from 0 to 1, we extract the positive square root (needs to be justified later):

$$B_1 = \{2\varepsilon^2 A_1 A_3 - \varepsilon^2 A_2^2 + \frac{\varepsilon^2}{2}(A_0^2 + B_0^2 - 1)^2 + \varepsilon^2\delta^2 A_0^2 B_0^2\}^{1/2},$$

which defines a *5-dimensional invariant maniford* $\mathcal{W}_\delta$ valid for any $\delta > 0$, which should contain the heteroclinic curve that we are looking for.

For $\delta > 0$, we only find the singular points

$$\begin{aligned} (A_0, B_0) &= (\pm 1, 0),\ A_1 = A_2 = A_3 = B_1 = 0 \\ (A_0, B_0) &= (0, \pm 1),\ A_1 = A_2 = A_3 = B_1 = 0. \end{aligned} \tag{6}$$

For $\delta = 0$, all singular points belong to a circle of singular points:

$$A_0^2 + B_0^2 = 1,\ A_1 = A_2 = A_3 = B_1 = 0. \tag{7}$$

Remark 6 *We do not emphasize here on the hamiltonian structure of system (1) since this does not help our understanding. On the contrary, the reversibility property is inherited from the original physical problem and is still valid for the perturbed system (2). Moreover, if we consider perturbation terms as $\varepsilon^2(A_0^2 A_0'' - A_0 A_0'^2)$ in $\widehat{f}$ and $i\varepsilon^3 B_0 A_0 A_0'$ in $\widehat{g}$, we cannot find a new first integral as used in (5), while the system is still reversible.*

3 Linear study of the dynamics

3.1 Case $\delta > 0$ ($g > 1$)

3.1.1 Neighborhood of $M_- = (1, 0)$

The eigenvalues of the linearized operator at M_- are such that $\lambda^4 = -2$ or $\lambda^2 = \varepsilon^2 \delta^2$, hence

$$\pm 2^{-1/4}(1 \pm i),$$
$$\pm \varepsilon \delta.$$

This gives a 3-dimensional unstable manifold, and a 3-dimensional stable manifold.

3.1.2 Neighborhood of $M_+ = (0, 1)$

The eigenvalues of the linearized operator at M_+ are such that $\lambda^4 = -\delta^2$ or $\lambda^2 = 2\varepsilon^2$, hence defining $\delta' = \sqrt{\delta}$, the eigenvalues are

$$\pm 2^{-1/2}(1 \pm i)\delta',$$
$$\pm \varepsilon \sqrt{2}.$$

This gives again a 3-dimensional unstable manifold and a 3-dimensional stable manifold.

All this implies that the 3-dimensional unstable manifold starting at M_- and the 3-dimensional stable manifold starting at M_+ which are both included into the $5-$dimensional manifold $\mathcal{W}_\delta$ give a good hope for these two manifolds to intersect along a heteroclinic curve...provided that they still exist as graphs with repect to B, "far" from the end points M_+ and M_-. The idea is to show that this occurs when δ is not too small and at most 1.

The limit points $M_- = (1, 0)$ and $M_+ = (0, 1)$ have a degenerate situation for $\delta = 0$, because of the multiple 0 eigenvalue for the linearized operator. For $\delta = 0$, it is possible to build a family of 2-dim unstable invariant manifolds and a family of 2-dim stable manifolds along the arc of equilibria $A^2 + B^2 = 1$. For $\delta > 0$ and small, the perturbation gives two new 3-dim invariant manifolds, however their transversality is weaker and weaker as $B \to 1$. A "serious" study is then needed, for the object of our work. However the physical interest is for values of $\delta > 0$ not too small, which cancels the physical interest of such a difficult question.

4 Unstable manifold of M_-

4.1 Change of coordinates

Let us fix $0 < \delta_0 \leq 1/3$, and $\delta_1 > 1$, we assume, from now on

$$0 \leq B_0 \leq \sqrt{1 - \eta_0^2 \delta^2}, \quad \eta_0 > \frac{1}{\sqrt{1+\delta^2}} = \frac{1}{\sqrt{g}},$$

and define

$$\alpha \stackrel{def}{=} (\eta_0^2(1+\delta^2)-1)^{1/2}, \text{ assuming } \frac{\varepsilon^2}{\alpha^2} \leq \delta_0 \leq \delta \leq \delta_1, \tag{8}$$

and new coordinates

$$Z = (\widetilde{A_*} + \widetilde{A_0}, A_1, A_2, A_3, B_0, B_1)^t \tag{9}$$

where $A_0 = \widetilde{A_*}$ cancels $A_3' = A^{(4)}$ with

$$\widetilde{A_*}^2 \stackrel{def}{=} 1-(1+\delta^2)B_0^2, \ \widetilde{A_*} \geq \delta\alpha.$$

In the following α is a "small parameter", the relative size of which, with respect to ε is precized later.

Remark 7 *$\widetilde{A_*}$ is just the first part of the "singular" heteroclinic found for the system, singular for $\varepsilon = 0$ (3). The occurence of $\widetilde{A_*}$ is also linked with a formal computation of an expansion of the heteroclinic in powers of ε, which gives $\widetilde{A_*}$ as the principal part of A_0, valid for $B_0 < (1+\delta^2)^{-1/2} = 1/\sqrt{g}$. The hope is to build the unstable manifold until this limit value.*

Remark 8 *We choose the conditions on δ, $\delta_0 \leq \delta \leq \delta_1$ in the purpose to include known computed values of the coefficient $g = 1+\delta^2$, in the convection problems, with different boundary conditions (see Remark 5 and [5]).*

We prove below the main result of this section:

Lemma 9 *For ε small enough, $0 < \delta_0 < 1/3$, and δ_1 arbitrary,*

$$\begin{aligned} \delta &\in [\delta_0, \delta_1], \ \alpha^2 = \eta_0^2(1+\delta^2) - 1, \\ \varepsilon^2 &\leq \delta_0\alpha^2, \ \varepsilon = \alpha^3, \end{aligned}$$

the 3-dimensional unstable manifold of M_- exists for

$$0 \leq B_0(x) \leq (1-\eta_0^2\delta^2)^{1/2}, \ x \in (-\infty, 0].$$

It sits in $\mathcal{W}_\delta$, is analytic in (ε, δ), and for any $\delta^* < \delta$,

$$\begin{aligned} A_0 &= \widetilde{A_*} + B_0\mathcal{O}(\alpha^{1/2}\delta^{1/2}e^{\varepsilon\delta^* x}) \\ A_1 &= B_0\mathcal{O}(\alpha\delta e^{\varepsilon\delta^* x}) \\ A_2 &= B_0\mathcal{O}(\alpha\delta e^{\varepsilon\delta^* x}) \\ A_3 &= B_0\mathcal{O}(\alpha\delta e^{\varepsilon\delta^* x}), \end{aligned}$$

where

$$0 \leq 1 - \widetilde{A_*} \leq cB_0^2, \ \widetilde{A_*}|_{B_0=0} = 1, \ \widetilde{A_*} \geq \delta\alpha.$$

Remark 10 *We observe that A_0 reaches a value close to 0 since $\widetilde{A_*}$ reaches $\delta\alpha$ which is close to 0, while B_0 reaches $(1-\eta_0^2\delta^2)^{1/2}$ which is close to $1/(1+\delta^2)^{1/2} = 1/\sqrt{g}$, not close to 1.*

The system (1) becomes

$$
\begin{aligned}
\widetilde{A_0}' &= A_1 + \frac{(1+\delta^2)B_0}{\widetilde{A_*}} B_1 \\
A_1' &= A_2 \\
A_2' &= A_3 \\
A_3' &= -2\widetilde{A_*}^2 \widetilde{A_0} - 3\widetilde{A_*}\widetilde{A_0}^2 - \widetilde{A_0}^3 \\
B_0' &= B_1 \\
B_1' &= \varepsilon^2\delta^2 B_*(\widetilde{A_*}^2 - B_0^2) + 2\varepsilon^2(1+\delta^2)\widetilde{A_*}B_0\widetilde{A_0} + \varepsilon^2(1+\delta^2)B_0\widetilde{A_0}^2.
\end{aligned}
\tag{10}
$$

We expect that $\widetilde{A_0}, A_1, A_2, A_3, B_1$ stay small enough for $x \in (-\infty, 0]$, so we introduce the linear operator

$$
\boldsymbol{L}_\delta = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \frac{(1+\delta^2)B_0}{\widetilde{A_*}} \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-2\widetilde{A_*}^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
2\varepsilon^2(1+\delta^2)\widetilde{A_*}B_0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{11}
$$

The idea is to find new coordinates such that we are able to give nice estimates of the monodromy operator. Don't forget that the coefficients of $\boldsymbol{L}_\delta$ are functions of B_0.

The operator $\boldsymbol{L}_\delta$ has a double eigenvalue 0, and is such that the non zero eigenvalues satisfy

$$\lambda^4 - 2\varepsilon^2 B_0^2(1+\delta^2)^2\lambda^2 + 2\widetilde{A_*}^2 = 0. \tag{12}$$

The discriminant of (12) is

$$\Delta' = \varepsilon^4 B_0^4(1+\delta^2)^4 - 2\widetilde{A_*}^2.$$

Our assumption $B_0 \le \sqrt{1-\eta_0^2\delta^2}$ and $\frac{\varepsilon^2}{\alpha^2} \le \delta \le \delta_1$, in addition with the constraint

$$\frac{1}{\alpha} \ge (1+\delta^2)^2. \tag{13}$$

implies

$$-\Delta' \ge \widetilde{A_*}^2.$$

Then we have two pairs of complex eigenvalues

$$\lambda_\pm^2 = \varepsilon^2 B_0^2(1+\delta^2)^2 \pm i\sqrt{-\Delta'}.$$

We want to find new coordinates able to manage a new linear operator in the form of two independent blocs

$$\begin{pmatrix} \pm\lambda_r & \lambda_i \\ -\lambda_i & \pm\lambda_r \end{pmatrix} \tag{14}$$

for which the eigenvalues are

$$\pm\lambda_r \pm i\lambda_i,$$

where

$$\begin{aligned} 2\lambda_r^2 &= \sqrt{2}\widetilde{A_*} + \varepsilon^2 B_0^2(1+\delta^2)^2 \\ 2\lambda_i^2 &= \sqrt{2}\widetilde{A_*} - \varepsilon^2 B_0^2(1+\delta^2)^2 \\ \lambda_r^2 - \lambda_i^2 &= \varepsilon^2 B_0^2(1+\delta^2)^2 \\ \lambda_r^2 + \lambda_i^2 &= \sqrt{2}\widetilde{A_*} \\ 4\lambda_r^2\lambda_i^2 &= -\Delta'. \end{aligned} \tag{15}$$

The form of the linear operator as (14) is such that we are able to have good estimates for the monodromy operator associated with the linear operator, the coefficients of which are functions of $B_0 \in [0, \sqrt{1-\eta_0^2\delta^2}]$ (see Appendix A.1).

4.2 Estimates for the eigenvalues

First, notice that (15) and

$$\alpha \leq (1+\delta^2)^{-2}$$

imply

$$\lambda_r\lambda_i \geq \frac{\widetilde{A_*}}{2},$$

$$2^{1/4}\widetilde{A_*}^{1/2} \geq \lambda_r \geq \frac{\widetilde{A_*}^{1/2}}{2^{1/4}} \geq \frac{\alpha^{1/2}}{2^{1/4}}\sqrt{\delta}, \tag{16}$$

$$\frac{1}{2^{3/4}}\widetilde{A_*}^{1/2} \leq \lambda_i \leq \frac{\widetilde{A_*}^{1/2}}{2^{1/4}}, \tag{17}$$

while $\widetilde{A_*}$ varies from 1 to $\alpha\delta$.

4.3 New coordinates

The eigenvector and generalized eigenvector for the eigenvalue 0 are :

$$Z_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \widetilde{A_*} \\ 0 \end{pmatrix}, \; Z_1 = \begin{pmatrix} 0 \\ -(1+\delta^2)B_0 \\ 0 \\ 0 \\ 0 \\ \widetilde{A_*} \end{pmatrix}.$$

Now we denote by

$$V_r^+ \pm i\lambda_i V_i^+,\ V_r^- \pm i\lambda_i V_i^-$$

the eigenvectors belonging respectively to the eigenvalues

$$\lambda_r \pm i\lambda_i, \quad -\lambda_r \pm i\lambda_i$$

then we define

$$V_r^+ = \begin{pmatrix} -\frac{\lambda_r(\lambda_r^2-3\lambda_i^2)}{2\widetilde{A_*}^2} \\ 1 \\ \lambda_r \\ \lambda_r^2-\lambda_i^2 \\ -\frac{\lambda_r(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}} \\ -\frac{(\lambda_r^2-\lambda_i^2)^2}{(1+\delta^2)B_0\widetilde{A_*}} \end{pmatrix}, \quad V_i^+ = \begin{pmatrix} -\frac{3\lambda_r^2-\lambda_i^2}{2\widetilde{A_*}^2} \\ 0 \\ 1 \\ 2\lambda_r \\ -\frac{(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}} \\ -\frac{2\lambda_r(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}} \end{pmatrix},$$

and we define new coordinates in $\mathbb{R}^6$: $(x_1, x_2, y_1, y_2, B_0, z_1)$ defined by

$$\begin{pmatrix} \widetilde{A_0} \\ A_1 \\ A_2 \\ A_3 \\ 0 \\ B_1 \end{pmatrix} = B_0(x_1 V_r^+ + x_2\lambda_i V_i^+ + y_1 V_r^- + y_2 \lambda_i V_i^- + z_0 Z_0 + z_1 Z_1).$$

We observe that after eliminating z_0, we still have 6 coordinates, including B_0 as one of the new coordinates.

Remark 11 *We notice that we put B_0 in front of the new coordinates, as this results from the analysis, and shorten the computations.*

The coordinate change is non linear in B_0 and given explicitely by:

$$\begin{aligned} \widetilde{A_0} &= -B_0\frac{\lambda_r(\lambda_r^2-3\lambda_i^2)}{2\widetilde{A_*}^2}(x_1-y_1) - B_0\frac{\lambda_i(3\lambda_r^2-\lambda_i^2)}{2\widetilde{A_*}^2}(x_2+y_2) \\ A_1 &= B_0(x_1+y_1) - (1+\delta^2)B_0^2 z_1 \\ A_2 &= \lambda_r B_0(x_1-y_1) + \lambda_i B_0(x_2+y_2) \\ A_3 &= (\lambda_r^2-\lambda_i^2)B_0(x_1+y_1) + 2\lambda_r\lambda_i B_0(x_2-y_2) \\ 0 &= -\frac{(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}}A_2 + \widetilde{A_*}B_0 z_0 \\ B_1 &= -\frac{(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}}A_3 + \widetilde{A_*}B_0 z_1, \end{aligned} \tag{18}$$

which needs to be inverted. We obtain

$$
\begin{aligned}
B_0 x_1 &= \frac{(\lambda_r^2+\lambda_i^2)}{4\lambda_r}\widetilde{A_0} + \frac{3\lambda_r^2-\lambda_i^2}{4\lambda_r(\lambda_r^2+\lambda_i^2)}A_2 \\
&\quad + \frac{A_1}{2} + \frac{(1+\delta^2)B_*}{2\widetilde{A_*}}B_1 + \frac{(\lambda_r^2-\lambda_i^2)}{2\widetilde{A_*}^2}A_3,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\lambda_i B_0 x_2 &= -\frac{(\lambda_r^2+\lambda_i^2)}{4}\widetilde{A_0} - \frac{\lambda_r^2-3\lambda_i^2}{4(\lambda_r^2+\lambda_i^2)}A_2 \\
&\quad -\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r}\left(A_1 + \frac{(1+\delta^2)B_0}{\widetilde{A_*}}B_1\right) + \frac{1}{4\lambda_r}\left(1 - \frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right)A_3,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
B_0 y_1 &= -\frac{(\lambda_r^2+\lambda_i^2)}{4\lambda_r}\widetilde{A_0} - \frac{3\lambda_r^2-\lambda_i^2}{4\lambda_r(\lambda_r^2+\lambda_i^2)}A_2 \\
&\quad + \frac{A_1}{2} + \frac{(1+\delta^2)B_0}{2\widetilde{A_*}}B_1 + \frac{(\lambda_r^2-\lambda_i^2)}{2\widetilde{A_*}^2}A_3,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\lambda_i B_0 y_2 &= -\frac{(\lambda_r^2+\lambda_i^2)}{4}\widetilde{A_0} - \frac{\lambda_r^2-3\lambda_i^2}{4(\lambda_r^2+\lambda_i^2)}A_2 \\
&\quad +\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r}\left(A_1 + \frac{(1+\delta^2)B_0}{\widetilde{A_*}}B_1\right) - \frac{1}{4\lambda_r}\left(1 - \frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right)A_3,
\end{aligned}
\tag{22}
$$

$$
B_0 z_1 = \frac{(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}^2}A_3 + \frac{1}{\widetilde{A_*}}B_1 = \varepsilon^2 B_0(1+\delta^2)\frac{A_3}{\widetilde{A_*}^2} + \frac{1}{\widetilde{A_*}}B_1.
$$

Let us now define

$$
\begin{aligned}
X &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ Y = X = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \\
|X| &= |x_1| + |x_2|,\ |Y| = |y_1| + |y_2| \text{ (norms in } \mathbb{R}^2).
\end{aligned}
$$

Then, for ε small enough, we obtain the following useful estimates

$$
\begin{aligned}
\frac{\widetilde{A_*}^{1/2}}{2^{3/4}} &\le \lambda_r, \lambda_i < 2^{1/4}\widetilde{A_*}^{1/2},\quad \widetilde{A_*} \ge \delta\alpha \ge \frac{\varepsilon^2}{\alpha}, \\
|\widetilde{A_0}| &\le 3\frac{B_0}{\widetilde{A_*}^{1/2}}(|X|+|Y|), \\
|A_1| &\le B_0(|X|+|Y|) + 2B_0^2|z_1|, \\
|A_2| &\le 2B_0\widetilde{A_*}^{1/2}(|X|+|Y|), \\
|A_3| &\le 2B_0\widetilde{A_*}(|X|+|Y|), \\
|B_1| &\le 3\varepsilon^2 B_0^2(|X|+|Y|) + \widetilde{A_*}B_0|z_1|.
\end{aligned}
\tag{23}
$$

4.4 System with new coordinates

The system (10) writen in the new coordinates is computed in Appendix A.2. It takes the following form (quadratic and higher order terms are not explicited)

$$
\begin{aligned}
& x_1' = f_1 + \lambda_r x_1 + \lambda_i x_2 \\
& + B_1 \left[a_1 \widetilde{A_0} + c_1 A_2 + d_1 A_3 + e_1 \frac{B_1}{B_0} - \frac{1}{B_0} x_1 \right] \\
& - \varepsilon^2 \frac{(1+\delta^2)(2-\delta^2) B_0}{2\widetilde{A_*}} \widetilde{A_0}^2 - \varepsilon^2 \frac{(1+\delta^2) B_0}{2\widetilde{A_*}^2} \widetilde{A_0}^3,
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
x_2' &= f_2 - \lambda_i x_1 + \lambda_r x_2 + B_1 \left[-a_2 \widetilde{A_0} + b_2 A_1 + c_2 A_2 + d_2 A_3 + e_2 B_1 - \frac{1}{B_0} x_2 \right] \\
& - \frac{1}{4\lambda_r \lambda_i \widetilde{A_*} B_0} \left(3\widetilde{A_*}^2 - 2\varepsilon^4 B_0^4 (1+\delta^2)^4 \right) \widetilde{A_0}^2 - \frac{1}{4\lambda_r \lambda_i B_0} \left(1 - \frac{(\lambda_r^2 - \lambda_i^2)^2}{\widetilde{A_*}^2} \right) \widetilde{A_0}^3.
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
y_1' &= f_1 - \lambda_r y_1 + \lambda_i y_2 + \\
& + B_1 \left[-a_1 \widetilde{A_0} - c_1 A_2 + d_1 A_3 + e_1 \frac{B_1}{B_0} - \frac{1}{B_0} y_1 \right] \\
& - \varepsilon^2 \frac{(1+\delta^2)(2-\delta^2) B_0}{2\widetilde{A_*}} \widetilde{A_0}^2 - \varepsilon^2 \frac{(1+\delta^2) B_0}{2\widetilde{A_*}^2} \widetilde{A_0}^3,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
y_2' &= -f_2 - \lambda_i y_1 - \lambda_r y_2 + B_1 \left[-a_2 \widetilde{A_0} - b_2 A_1 + c_2 A_2 - d_2 A_3 + e_2 B_1 - \frac{1}{B_0} y_2 \right] \\
& + \frac{1}{4\lambda_r \lambda_i \widetilde{A_*} B_0} \left(3\widetilde{A_*}^2 - 2\varepsilon^4 B_0^4 (1+\delta^2)^4 \right) \widetilde{A_0}^2 + \frac{1}{4\lambda_r \lambda_i B_0} \left(1 - \frac{(\lambda_r^2 - \lambda_i^2)^2}{\widetilde{A_*}^2} \right) \widetilde{A_0}^3,
\end{aligned}
\tag{27}
$$

with

$$
f_1 = \frac{\varepsilon^2 \delta^2 B_0 (1+\delta^2)(\widetilde{A_*}^2 - B_0^2)}{2\widetilde{A_*}},
$$

$$
f_2 = -\frac{\varepsilon^2 \delta^2 B_0 (1+\delta^2)(\lambda_r^2 - \lambda_i^2)(\widetilde{A_*}^2 - B_0^2)}{4\lambda_r \lambda_i \widetilde{A_*}},
$$

coefficients a_j, b_j, c_j, d_j, e_j are defined and estimated in Appendix A.2 in (90,91), (92,93,94), (95,96), (97,98). Here $\widetilde{A_0}, A_1, A_2, A_3, B_1$ should be replaced by their (linear) expressions (18) in coordinates $(x_1, x_2, y_1, y_2, z_1)$ with coefficients functions of B_0. The system above should be completed by the differential equations for z_1' and $B_0'(= B_1)$. In fact we will replace the equation for z_1' by the direct resolution of the first integral (5) with respect to z_1 (see below) using

$$
B_1 = -\varepsilon^2 (1+\delta^2) B_0 \frac{A_3}{\widetilde{A_*}} + \widetilde{A_*} B_0 z_1. \tag{28}
$$

4.5 Resolution of (5) with respect of $z_1(X, Y, B_0)$

For extending the validity (as a graph with respect to B_0) for the existence of the unstable manifold of M_- we need to replace the differential equation for z_1' by the expression of z_1 given by the first integral (5). This leads to (see (28))

$$\begin{aligned} B_1^2 &= \{\widetilde{A_*} B_0 z_1 - \varepsilon^2 \frac{B_0(1+\delta^2)}{\widetilde{A_*}} A_3\}^2 = 2\varepsilon^2 A_1 A_3 - \varepsilon^2 A_2^2 + \\ & \frac{\varepsilon^2}{2}(-\delta^2 B_0^2 + 2\widetilde{A_*}\widetilde{A_0} + \widetilde{A_0}^2)^2 + \varepsilon^2\delta^2(\widetilde{A_*} + \widetilde{A_0})^2 B_0^2, \end{aligned}$$

hence

$$\begin{aligned} \widetilde{A_*}^2 z_1^2 &= \varepsilon^2\delta^2 \widetilde{A_*}^2 (1 + \frac{\delta^2 B_0^2}{2\widetilde{A_*}^2}) + \frac{2\varepsilon^2}{B_0} A_3(x_1 + y_1) - \frac{\varepsilon^4(1+\delta^2)^2}{\widetilde{A_*}^2} A_3^2 - \frac{\varepsilon^2}{B_0^2} A_2^2 + \\ & + \frac{2\varepsilon^2 \widetilde{A_*}^2}{B_0^2} \widetilde{A_0}^2 + \frac{2\varepsilon^2 \widetilde{A_*}}{B_0^2} \widetilde{A_0}^3 + \frac{\varepsilon^2}{2B_0^2} \widetilde{A_0}^4, \end{aligned} \tag{29}$$

where we may observe on the r.h.s., that

$$\frac{\delta^2}{2\widetilde{A_*}^2} < \frac{1}{2\alpha^2},$$

hence

$$\varepsilon^2\delta^2 \le \varepsilon^2\delta^2(1 + \frac{\delta^2 B_0^2}{2\widetilde{A_*}^2}) \le \varepsilon^2\delta^2(1 + \frac{1}{2\alpha^2}),$$

which is independent of (X, Y). Moreover there is no linear part in (X, Y). For further estimates, we make a new scaling

$$(X, Y) = \alpha\delta(\overline{X}, \overline{Y}), \;\; z_1 = \varepsilon\delta\overline{z_1}. \tag{30}$$

We notice that (23) implies (c is a generic constant, independent of (ε, α))

$$|\frac{2\varepsilon^2}{B_0} A_3(x_1 + y_1)| \le c\varepsilon^2\alpha^2\delta^2 \widetilde{A_*}(|\overline{X}| + |\overline{Y}|)^2$$

$$|\frac{\varepsilon^4(1+\delta^2)^2}{\widetilde{A_*}^2} A_3^2| \le c\varepsilon^4\alpha^2\delta^2(|\overline{X}| + |\overline{Y}|)^2$$

$$\frac{\varepsilon^2}{B_0^2} A_2^2 \le c\varepsilon^2\alpha^2\delta^2 \widetilde{A_*}(|\overline{X}| + |\overline{Y}|)^2)$$

$$\frac{2\varepsilon^2 \widetilde{A_*}^2}{B_0^2} \widetilde{A_0}^2 \le c\varepsilon^2\alpha^2\delta^2 \widetilde{A_*}(|\overline{X}| + |\overline{Y}|)^2$$

$$|\frac{2\varepsilon^2 \widetilde{A_*}}{B_0^2} \widetilde{A_0}^3| \le \frac{c\varepsilon^2\alpha^3\delta^3}{\widetilde{A_*}^{1/2}}(|\overline{X}| + |\overline{Y}|)^3$$

$$\frac{\varepsilon^2}{2B_0^2}\widetilde{A_0}^4 \leq \frac{c\varepsilon^2\alpha^4\delta^4}{\widetilde{A_*}^2}(|\overline{X}|+|\overline{Y}|)^4,$$

so that the factors in the estimates are such that

$$\frac{c\varepsilon^2\alpha^2\delta^2\widetilde{A_*}}{\varepsilon^2\delta^2\widetilde{A_*}^2} \leq c\frac{\alpha^2}{\widetilde{A_*}},\ \frac{c\varepsilon^2\alpha^4\delta^4}{\varepsilon^2\delta^2\widetilde{A_*}^4} \leq c\frac{\alpha^2}{\widetilde{A_*}^2},\ \frac{c\varepsilon^2\alpha^3\delta^3}{\varepsilon^2\delta^2\widetilde{A_*}^{5/2}} \leq c\frac{\alpha^{5/2}\delta^{3/4}}{\widetilde{A_*}^2},$$

c being independent of ε, α and $\delta \in [\delta_0, \delta_1]$. Now defining $\overline{z_{10}}$ such that

$$1 \leq \overline{z_{10}}(B_0) \stackrel{def}{=} (1+\frac{\delta^2 B_0^2}{2\widetilde{A_*}^2})^{1/2} \leq \frac{1}{\alpha},\ \text{for } \alpha \leq 1/\sqrt{2}, \tag{31}$$

It results that

$$\overline{z_1}^2 = \overline{z_{10}}^2 + \mathcal{O}\left(\frac{\alpha^2}{\widetilde{A_*}}(|\overline{X}|+|\overline{Y}|)^2 + \frac{\alpha^{5/2}}{\widetilde{A_*}^2}(|\overline{X}|+|\overline{Y}|)^3 + \frac{\alpha^2}{\widetilde{A_*}^2}(|\overline{X}|+|\overline{Y}|)^4\right)$$

and using

$$B_0^2 = \frac{1-\widetilde{A_*}^2}{1+\delta^2}$$

we also have

$$\frac{1}{\overline{z_{10}}^2} = \frac{2\widetilde{A_*}^2}{2\widetilde{A_*}^2+\delta^2 B_0^2} \leq \frac{2(1+\delta^2)\widetilde{A_*}^2}{\delta^2} \leq \frac{c\widetilde{A_*}^2}{\delta^2},$$

so that (taking the positive square root as for (5))

$$\begin{aligned}\overline{z_1} &= \overline{z_{10}}(B_0)\left\{1+\frac{\widetilde{A_*}^2}{\delta^2}\mathcal{O}\left(\frac{\alpha^2}{\widetilde{A_*}}(|\overline{X}|+|\overline{Y}|)^2 + \frac{\alpha^{5/2}}{\widetilde{A_*}^2}(|\overline{X}|+|\overline{Y}|)^3 + \frac{\alpha^2}{\widetilde{A_*}^2}(|\overline{X}|+|\overline{Y}|)^4\right)\right\}^{1/2} \\ &= \overline{z_{10}}(B_0)\left\{1+\mathcal{O}[\alpha^2(|\overline{X}|+|\overline{Y}|)^2]\right\}^{1/2},\ \text{for } |\overline{X}|+|\overline{Y}| \leq \rho,\ \rho \text{ fixed},\end{aligned}$$

and taking the square root, we obtain

$$\overline{z_1} = \overline{z_{10}}(B_0) + \mathcal{Z}(\overline{X},\overline{Y},B_0) \tag{32}$$

with, using (31),

$$\mathcal{Z}(\overline{X},\overline{Y},B_0) = \mathcal{O}(\alpha(|\overline{X}|+|\overline{Y}|)^2,$$

$\mathcal{Z}(\overline{X},\overline{Y},B_0)$ being defined in the ball

$$|\overline{X}|+|\overline{Y}| \leq \rho,$$

provided that ε is small enough and where ρ is of order 1, not necessarily small with respect to α. Moreover $\mathcal{Z}$ is analytic in its arguments and is at least quadratic in $(\overline{X},\overline{Y})$.

Since z_1 contains $\overline{z_{10}}$ which is independent of $(\overline{X},\overline{Y})$, the new system for $(\overline{X},\overline{Y})$ has new "constant terms" and "linear terms", appearing as perturbations of the former ones.

4.6 System where z_1 is eliminated

Now we stay on the 5-dimensional invariant manifold (5) and we need to express the new differential system in terms of $(\overline{X}, \overline{Y}, B_0)$. The new system is computed in Appendix A.3. We obtain (notice that B_0 is in factor of the "constant" terms)

$$\begin{aligned} \overline{X}' &= \mathbf{L}_0\overline{X} + B_0\mathcal{F}_0 + \mathcal{L}_{01}(\overline{X}, \overline{Y}) + \mathcal{B}_{01}(\overline{X}, \overline{Y}), \\ \overline{Y}' &= \mathbf{L}_1\overline{Y} + B_0\mathcal{F}_1 + \mathcal{L}_{11}(\overline{X}, \overline{Y}) + \mathcal{B}_{11}(\overline{X}, \overline{Y}), \end{aligned} \tag{33}$$

which should be completed by an equation for B_0' (see (28) in terms of $(\overline{X}, \overline{Y}, B_0)$), and where

$$\mathbf{L}_0 = \begin{pmatrix} \lambda_r & \lambda_i \\ -\lambda_i & \lambda_r \end{pmatrix}, \quad \mathbf{L}_1 = \begin{pmatrix} -\lambda_r & \lambda_i \\ -\lambda_i & -\lambda_r \end{pmatrix},$$

with the following estimates, for terms independent of $(\overline{X}, \overline{Y})$

$$|\mathcal{F}_0| + |\mathcal{F}_1| \leq \frac{c\varepsilon^2}{\alpha^4}, \tag{34}$$

for terms which are linear in $(\overline{X}, \overline{Y})$

$$|\mathcal{L}_{01}(\overline{X}, \overline{Y})| + |\mathcal{L}_{11}(\overline{X}, \overline{Y})| \leq c\frac{\varepsilon}{\alpha^2}(|\overline{X}| + |\overline{Y}|), \tag{35}$$

and for terms at least quadratic in $(\overline{X}, \overline{Y})$, choosing α small enough and for

$$|\overline{X}| + |\overline{Y}| \leq \rho,$$

we obtain

$$|\mathcal{B}_{01}(\overline{X}, \overline{Y})| + |\mathcal{B}_{11}(\overline{X}, \overline{Y})| \leq c(\alpha + \frac{\varepsilon^2}{\alpha^2})(|\overline{X}| + |\overline{Y}|)^2. \tag{36}$$

We are now ready to formulate the search for the unstable manifold of M_-.

4.7 Integral formulation for solutions bounded as $x \to -\infty$

Let us introduce the monodromy operators associated with the linear operators $\mathbf{L}_0, \mathbf{L}_1$ which have non constant coefficients (functions of B_0 (see [4]):

$$\begin{aligned} \frac{\partial}{\partial x}S_0(x,s) &= \mathbf{L}_0 S_0(x,s),\ S_0(x,s_1)S_0(s_1,s_2) = S_0(x,s_2),\ S_0(x,x) = \mathbb{I}, \\ \frac{\partial}{\partial x}S_1(x,s) &= \mathbf{L}_1 S_1(x,s),\ S_1(x,s_1)S_1(s_1,s_2) = S_1(x,s_2),\ S_1(x,x) = \mathbb{I}. \end{aligned}$$

The coefficients of operators $\mathbf{L}_0, \mathbf{L}_1$ are functions of B_0, so we need the Lemma 26 in Appendix A.1, with the following estimates, valid for $0 \leq B_0 \leq \sqrt{1 - \eta_0^2\delta^2}$, $\alpha \leq (1 + \delta^2)^{-2}$:

$$||\boldsymbol{S}_0(x,s)|| \leq e^{\sigma(x-s)}, \ -\infty < x < s \leq 0, \tag{37}$$

$$||\boldsymbol{S}_1(x,s)|| \leq e^{-\sigma(x-s)}, \ -\infty < s < x \leq 0, \tag{38}$$

with

$$\sigma = \frac{\alpha^{1/2}\delta^{1/2}}{2^{1/4}}.$$

We are looking for solutions of (33) which stay bounded for $x \to -\infty$. Then, thanks to estimates (37) (38), the system (33) may be formulated as

$$\begin{aligned} \overline{X}(x) &= \mathbf{S}_0(x,0)\overline{X}_0 + \int_0^x \mathbf{S}_0(x,s)G_0(s)ds \\ \overline{Y}(x) &= \int_{-\infty}^x \mathbf{S}_1(x,s)G_1(s)ds \end{aligned} \tag{39}$$

$$\begin{aligned} G_0(s) &\stackrel{def}{=} B_0\mathcal{F}_0 + \mathcal{L}_{01}(\overline{X},\overline{Y}) + \mathcal{B}_{01}(\overline{X},\overline{Y}), \\ G_1(s) &\stackrel{def}{=} B_0\mathcal{F}_1 + \mathcal{L}_{11}(\overline{X},\overline{Y}) + \mathcal{B}_{11}(\overline{X},\overline{Y}) \end{aligned}$$

where $\overline{X},\overline{Y}$ and B_0 are bounded and continuous functions of s, B_0 tending towards 0 as $s \to -\infty$.

4.8 Strategy

The idea is

i) solve (39) with respect to $(\overline{X},\overline{Y})$ in function of $(\overline{X}_0, B_0)$;

ii) solve the integro-differential equation for B_0, with $B_0|_{x=0} = B_0(0)$, $B_0(-\infty) = 0$.

Then the 3-dimensional unstable manifold of M_- is given (see [4]) by $\overline{Y}|_{x=0}, z_1|_{x=0}$ in terms of $\overline{X}_0, B_0(0)$. The result will be valid for an interval $[0, \sqrt{1-\eta_0^2\delta^2}]$ for B_0 and it appears that A_0 is then very close to 0 at the end point. The hope is that this should allow to obtain an intersection with the 3-dim stable manifold of M_+ which computation should be valid for B_0 in the interval $[\sqrt{1-\eta_0^2\delta^2}, 1]$.

4.9 Resolution for $(\overline{X},\overline{Y})$

Let us define, for $\kappa > 0$ the function space

$$C^0_\kappa = \{\overline{X} \in C^0(-\infty,0]; \overline{X}(x)e^{-\kappa x} \text{ is bounded}\}$$

equiped with the norm

$$||\overline{X}||_\kappa = \sup_{(-\infty,0)} |\overline{X}(x)e^{-\kappa x}|.$$

We observe that, provided that $\kappa < \sigma$

$$|\int_{-\infty}^x \boldsymbol{S}_1(x,s)e^{\kappa s}ds| \leq \frac{e^{\kappa x}}{\kappa+\sigma}$$

$$|\boldsymbol{S}_0(x,0)e^{-\kappa x}| \leq e^{(\sigma-\kappa)x}, \quad x \leq 0,$$
$$|\int_0^x \mathbf{S}_0(x,s)e^{\kappa s}ds| \leq \frac{e^{\kappa x}}{\sigma-\kappa}, \quad x \leq 0.$$

Let us choose
$$\kappa \leq \frac{\sigma}{2},$$
then
$$|\int_{-\infty}^x \boldsymbol{S}_1(x,s)e^{\kappa s}ds| \leq \frac{e^{\kappa x}}{\sigma} = 2^{1/4}\frac{e^{\kappa x}}{\alpha^{1/2}\delta^{1/2}},$$
$$|\int_0^x \mathbf{S}_0(x,s)e^{\kappa s}ds| \leq 2^{5/4}\frac{e^{\kappa x}}{\alpha^{1/2}\delta^{1/2}}, \quad x \leq 0.$$

Let us assume that
$$||B_0||_\kappa \leq m \tag{40}$$
holds with m independent of ε, which needs to be proved at next subsection. Hence, the analytic implicit function theorem (see [2]) applies for $(\overline{X},\overline{Y})$ in a neighborhood of 0 in the function space C^0_κ, provided that we can choose $\kappa \leq \frac{\sigma}{2}$ and $||\overline{X}||_\kappa + ||\overline{Y}||_\kappa \leq \rho$. Using the above estimates for coefficients, we obtain for $x \in (-\infty, 0]$

$$|\overline{X}(x)e^{-\kappa x}| \leq |\overline{X}_0| + \frac{2^{5/4}}{\alpha^{1/2}\delta^{1/2}}||B_0\mathcal{F}_0 + \mathcal{L}_{01}(\overline{X},\overline{Y}) + \mathcal{B}_{01}(\overline{X},\overline{Y})||_\kappa,$$

hence

$$||\overline{X}||_\kappa \leq |\overline{X}_0| + \frac{2^{5/4}}{\alpha^{1/2}\delta^{1/2}}||B_0\mathcal{F}_0 + \mathcal{L}_{01}(\overline{X},\overline{Y}) + \mathcal{B}_{01}(\overline{X},\overline{Y})||_\kappa, \tag{41}$$

and in the same way

$$||\overline{Y}||_\kappa \leq \frac{2^{1/4}}{\alpha^{1/2}\delta^{1/2}}||B_0\mathcal{F}_1 + \mathcal{L}_{11}(\overline{X},\overline{Y}) + \mathcal{B}_{11}(\overline{X},\overline{Y})||_\kappa. \tag{42}$$

Remark 12 *The choice of κ is governed by the behavior of $B_0(x)$ as $x \to -\infty$, which is studied at next subsection.*

For ε small enough, estimates on $\mathcal{F}_1$, $\mathcal{B}_{11}$, (42) and $||\overline{X}||_\kappa + ||\overline{Y}||_\kappa \leq \rho$, we obtain, for $S \overset{def}{=} ||\overline{X}||_\kappa + ||\overline{Y}||_\kappa$

$$S \leq |\overline{X}_0| + c[\frac{\varepsilon^2 m}{\alpha^{9/2}} + \frac{S\varepsilon}{\alpha^{5/2}} + (\alpha^{1/2} + \frac{\varepsilon^2}{\alpha^{5/2}})S\rho]$$

so that for $\delta_0 \leq \delta \leq \delta_1$ and choosing

$$\varepsilon = \alpha^3, \tag{43}$$

and ε small enough and ρ (nor necessarily small), such that

$$(1+\rho+\rho\varepsilon)\varepsilon^{1/6} \leq (1+2\rho)\varepsilon^{1/6}$$

we obtain

$$S \leq (1 + c'\varepsilon^{1/6})|\overline{X}_0| + c\varepsilon^{1/2}m,$$

which leads finally to X and Y in C^0_κ, depending analytically on $(\overline{X}_0, B_0) \in \mathbb{R}^2 \times C^0_\kappa$, and such that

$$||\overline{Y}||_\kappa \leq c(m\varepsilon^{1/2} + \varepsilon^{1/6}|\overline{X}_0|), \tag{44}$$

$$||\overline{X}||_\kappa \leq (1 + c\varepsilon^{1/6})|\overline{X}_0| + c\varepsilon^{1/2}m, \tag{45}$$

where c is a number independent of $\varepsilon = \alpha^3$ small enough, and we assume m (see (40)) of order 1, $S \leq \rho$, where ρ is fixed arbitrarily, of order 1.

4.10 Resolution for B_0

We intend to solve the part of our system for B_0 with $B_0(0) = B_0|_{x=0}$.

We notice from (28), (32) and (23) that

$$\begin{aligned}
B_1 &= \varepsilon\delta\widetilde{A_*}B_0\left\{\overline{z_{10}}(B_0) + \mathcal{Z}(\overline{X}, \overline{Y}, B_0)\right\} - \varepsilon^2\alpha\delta(1+\delta^2)\frac{B_0}{\widetilde{A_*}}\overline{A_3} \\
\overline{A_3} &= B_0[\varepsilon^2 B_0^2(1+\delta^2)^2(\overline{x_1} + \overline{y_1}) + 2\lambda_r\lambda_i(\overline{x_2} - \overline{y_2})], \\
\varepsilon\alpha(1+\delta^2)\frac{\overline{A_3}}{\widetilde{A_*}^2} &\leq \frac{4\varepsilon\alpha^2\delta}{\widetilde{A_*}}(|\overline{X}| + |\overline{Y}|) \leq 4\varepsilon\alpha(|\overline{X}| + |\overline{Y}|),
\end{aligned}$$

so that it is clear that (see above estimates for $\mathcal{Z}$)

$$\overline{B_1} > 0 \text{ for } B_0 \in (0, \sqrt{1 - \eta_0^2\delta^2}), |\overline{X}| + |\overline{Y}| \leq \rho, \tag{46}$$

which justifies to take the positive square root for B_1 in (5). This is coherent with the study of the linearized system near M_- : Indeed the principal part of the differential equation for B_0 is

$$B_0' = \varepsilon\delta B_0\widetilde{A_*}\overline{z_{10}}(B_0)$$

which may be integrated as

$$\begin{aligned}
B_0^2 &= \frac{1}{(1+\frac{\delta^2}{2})\cosh^2(x_0 - \varepsilon\delta x)}, \qquad (47) \\
\cosh x_0 &= \frac{1}{B_0(0)(1+\frac{\delta^2}{2})^{1/2}},
\end{aligned}$$

which satisfies $B_0 = 0$ for $x = -\infty$, and $B_0 = B_0(0)$ for $x = 0$. More precisely the differential equation for B_0 is now (after replacing $(\overline{X}, \overline{Y})$ by the expression found at previous subsection)

$$B_0' = \varepsilon\delta\widetilde{A_*}B_0\overline{z_{10}}(B_0)[1 + f(B_0)] \tag{48}$$

where $f(B_0)$ is a non local analytic function of B_0 in C^0_κ, such that

$$||f(B_0)||_\kappa \leq c\alpha^2\rho.$$

Remark 13 *We may notice that we might replace* $c\alpha^2\rho$ *in the estimate above, by*

$$c\alpha^2\rho e^{\kappa x} \to 0 \text{ as } x \to -\infty,$$

since X *and* $Y \in C^0_\kappa$.

We are looking for the solution such that $B_0 = 0$ for $x = -\infty$, and $B_0(0) \leq \sqrt{1-\eta_0^2\delta^2}$ for $x = 0$. We can rewrite (48) as

$$\frac{2B_0B_0'}{B_0^2\widetilde{A_*}\overline{z_{10}}(B_0)} = 2\varepsilon\delta[1+f(B_0)]. \tag{49}$$

We now introduce the variable v :

$$v = \frac{1-\sqrt{1-(1+\frac{\delta^2}{2})B_0^2}}{1+\sqrt{1-(1+\frac{\delta^2}{2})B_0^2}},\ B_0^2 = \frac{1}{1+\frac{\delta^2}{2}}\frac{4v}{(1+v)^2},$$

so that

$$(\ln v)' = 2\varepsilon\delta[1+f(B_0)].$$

We observe that for x running from $-\infty$ to 0,

$$w = \ln v \text{ is increasing from } -\infty \text{ to } w_0 = \ln v_0 < 0.$$

Now let us define h continuous in its argument and such that

$$\begin{aligned} h(w) &= f(B_0), \\ B_0 &= \frac{1}{\left(1+\frac{\delta^2}{2}\right)^{1/2}}\frac{2e^{w/2}}{(1+e^w)}, \end{aligned}$$

and let us find an a priori estimate for the solution $B_0(x)$, for $x \in (-\infty, 0]$. We obtain by simple integration

$$\int_0^x \frac{w'(s)}{1+h(w)(s)}ds = 2\varepsilon\delta x.$$

For α small enough we have

$$1-c\alpha^2\rho \leq \frac{1}{1+h(w)} \leq 1+c\alpha^2\rho,$$

hence (since $w < w_0$, and $x < 0$)

$$(w_0-w)(1-c\alpha^2\rho) \leq -2\varepsilon\delta x \leq (w_0-w)(1+c\alpha^2\rho)$$

so that

$$\exp\left(\frac{-2\varepsilon\delta x}{1+c\varepsilon\rho}\right) \leq e^{w_0-w} \leq \exp\left(\frac{-2\varepsilon\delta x}{1-c\varepsilon\rho}\right)$$

and

$$v_0 \exp(\frac{2\varepsilon\delta}{1-c\alpha^2\rho}x) \leq v(x) \leq v_0 \exp(\frac{2\varepsilon\delta}{1+c\alpha^2\rho}x).$$

It finally results that we obtain an a priori estimate for

$$B_0(x) = \mathcal{B}_0(\overline{X}_0, B_0(0))(x) \in C^0_\kappa, \tag{50}$$

$$\begin{aligned}
\mathcal{B}_0(\overline{X}_0, B_0(0))(x) &= \frac{1}{\left(1+\frac{\delta^2}{2}\right)^{1/2}} \frac{2\sqrt{v(x)}}{(1+v(x))},\ x \in (-\infty, 0), \\
\frac{2\sqrt{v_0}\exp(\frac{\varepsilon\delta}{1-c\alpha^2\rho}x)}{1+v_0\exp(\frac{2\varepsilon\delta}{1-c\alpha^2\rho}x)} &\leq \left(1+\frac{\delta^2}{2}\right)^{1/2} \mathcal{B}_0 \leq \frac{2\sqrt{v_0}\exp(\frac{\varepsilon\delta}{1+c\alpha^2\rho}x)}{1+v_0\exp(\frac{2\varepsilon\delta}{1+c\alpha^2\rho}x)}, \qquad (51) \\
v_0 &= \frac{1-\sqrt{1-(1+\frac{\delta^2}{2})B_0^2(0)}}{1+\sqrt{1-(1+\frac{\delta^2}{2})B_0^2(0)}} < 1.
\end{aligned}$$

It remains to notice that we can choose

$$\kappa = \frac{\varepsilon\delta}{1+c\alpha^2\rho}$$

in the proof for $(\overline{X}, \overline{Y})$, which needs to satisfy

$$\kappa \leq \frac{\sigma}{2} = \frac{\alpha^{1/2}\sqrt{\delta}}{2^{5/4}}. \tag{52}$$

We have already chosen $\varepsilon = \alpha^3$ hence

$$\kappa \leq \varepsilon\delta = \delta\alpha^3 \leq \frac{\alpha^{1/2}\sqrt{\delta}}{2^{5/4}}$$

for α small enough, and (52) is satisfied. The a priori estimate for B_0 allows to prove that there is a unique solution of the integro-differential equation (49) which satisfies the estimate (51) (see [4]). Since B_0 is in factor in $\widetilde{A_0}, A_1, A_2, A_3, B_1$ the behavior for $x \to -\infty$ of the coordinates of the unstable manifold, is governed by the behavior of B_0. The estimates indicated in Lemma 9 results from (23), (30), (31), (45), (44), (52) with $\kappa = \varepsilon\delta_*$. This ends the proof of Lemma 9.

Let us define the hyperplane H_0

$$B_0 = (1-\eta_0^2\delta^2)^{1/2}.$$

4.11 Intersection of the unstable manifold with H_0

We need to give precisely the intersection of the unstable manifold with the hyperplane $B_0 = \sqrt{1-\eta_0^2\delta^2}$. This gives a two-dimensional manifold lying in the 4-dimensional manifold $\mathcal{W}_g \cap H_0$. Taking into account of

$$\begin{aligned}
\widetilde{A_*} &= \delta\alpha \\
\lambda_r, \lambda_i &\sim \frac{\delta^{1/2}\alpha^{1/2}}{2^{1/4}},\ \varepsilon = \alpha^3, \\
\overline{z_{10}} &\sim \frac{B_{00}}{\alpha\sqrt{2}},\ B_{00} = \sqrt{1-\eta_0^2\delta^2}, \\
|\overline{Y}(0)| &= \mathcal{O}(\alpha^{1/2}|\overline{X_0}| + B_{00}\alpha^{3/2}),
\end{aligned}$$

we obtain a two-dimensional intersection which is tangent to a plane (parameters $\overline{x_1}, \overline{x_2}$) with principal part given by

$$\begin{aligned}
A_0 &= \delta\alpha + \frac{\alpha^{1/2}\delta^{1/2}}{2^{3/4}} B_{00}(\overline{x_1} - \overline{x_2}) + \mathcal{O}(\alpha|\overline{X}_0| + \alpha^2 B_{00}) \\
A_1 &= \alpha\delta B_{00}\overline{x_1} - \frac{\alpha^2\delta}{\sqrt{2}} B_{00} + \mathcal{O}(\alpha^{3/2}|\overline{X}_0| + \alpha^{5/2}B_{00}) \\
A_2 &= \frac{\delta^{3/2}}{2^{1/4}} B_{00}\alpha^{3/2}(\overline{x_1} + \overline{x_2}) + \mathcal{O}(\alpha^2|\overline{X}_0| + \alpha^3 B_{00}) \\
A_3 &= \sqrt{2}\delta^2 B_{00}\alpha^2\overline{x_2} + \mathcal{O}(\alpha^{5/2}|\overline{X}_0| + \alpha^{7/2}B_{00}) \\
B_{00} &\sim (1+\delta^2)^{-1/2},
\end{aligned} \tag{53}$$

with

$$|\overline{x_1}| + |\overline{x_2}| \leq \rho, \quad \delta_0 \leq \delta \leq \delta_1, \quad \varepsilon = \alpha^3,\ \alpha^2 = \eta_0^2(1+\delta^2) - 1 > 0,$$

and where we do not write B_1 since we know that this manifold lies in the 5 dimensional manifold $\mathcal{W}_g$.

5 Stable manifold of M_+

We show the following

Lemma 14 *For ε small enough, , $\delta_0 \leq \delta \leq 1$, the 3-dimensional stable manifold of M_+ is included in the 5-dimensional manifold $\mathcal{W}_g$, exists for A_0, A_1, A_2, A_3 in a ball of small radius η (independent of ε), is analytic in parameters (ε, δ), and reaches $B_0(0) = B_{00} \stackrel{def}{=} \sqrt{1-\eta_0^2\delta^2}$, with $\eta_0^2(1+\delta^2) = 1 + \varepsilon^{2/3}$. Moreover as $x \to +\infty$, $(A_0, A_1, A_2, A_3) \to 0$ as $exp(-\sqrt{\frac{\delta}{2}}x)$, $(B_0 - 1, B_1) \to 0$ as* $\exp(-\sqrt{2}\varepsilon x)$,

$$\begin{aligned}
v(x) \stackrel{def}{=} \frac{B_0(x) - 1}{\delta^{1/2}} &\simeq -\frac{(1 - B_0(0))(1 - \tanh(\varepsilon x/\sqrt{2})}{1 + B_0(0)\tanh(\varepsilon x/\sqrt{2})}, \\
|v(0)| &\leq 0.293.
\end{aligned} \tag{54}$$

The idea is first to adapt new coordinates (with a fixed basis in this section), such that we are able to use monodromy operators with easy estimates in the formulation of the search for the 3-dimensional stable manifold of M_+.

Let us define

$$\delta' = \delta^{1/2},$$

and choose a new basis

$$V_r^- = \begin{pmatrix} 1 \\ -\frac{\delta'}{\sqrt{2}} \\ 0 \\ \frac{\delta'^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \; V_i^- = \begin{pmatrix} 0 \\ -\frac{\delta'}{\sqrt{2}} \\ \delta'^2 \\ -\frac{\delta'^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix},$$

$$V_r^+ = \begin{pmatrix} 1 \\ \frac{\delta'}{\sqrt{2}} \\ 0 \\ -\frac{\delta'^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}, \; V_i^+ = \begin{pmatrix} 0 \\ \frac{\delta'}{\sqrt{2}} \\ \delta'^2 \\ \frac{\delta'^3}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix},$$

$$W_1^- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -\varepsilon\sqrt{2} \end{pmatrix}, \; W_1^+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ \varepsilon\sqrt{2} \end{pmatrix},$$

for defining new coordinates $(x_1, x_2, y_1, y_2, z_0, z_1)$ such that

$$Z = (0,0,0,0,1,0)^t + \delta' x_1 V_r^- + \delta' x_2 V_i^- + \delta' y_1 V_r^+ + \delta' y_2 V_i^+ + \delta' z_0 W_1^- + \delta' z_1 W_1^+$$

hence

$$\begin{aligned}
A_0 &= \delta'(x_1 + y_1) \\
A_1 &= -\frac{\delta'^2}{\sqrt{2}}(x_1 - y_1 + x_2 - y_2) \\
A_2 &= \delta'^3(x_2 + y_2) \\
A_3 &= \frac{\delta'^4}{\sqrt{2}}(x_1 - y_1 - x_2 + y_2) \\
B_0 &= 1 + \delta'(z_0 + z_1) \\
B_1 &= -\varepsilon\sqrt{2}\delta'(z_0 - z_1).
\end{aligned} \tag{55}$$

A simple inversion leads to

$$
\begin{aligned}
x_1 &= \frac{A_0}{2\delta'} - \frac{A_1}{2\sqrt{2}\delta'^2} + \frac{A_3}{2\sqrt{2}\delta'^4} \\
x_2 &= -\frac{A_1}{2\sqrt{2}\delta'^2} + \frac{A_2}{2\delta'^3} - \frac{A_3}{2\sqrt{2}\delta'^4} \\
y_1 &= \frac{A_0}{2\delta'} + \frac{A_1}{2\sqrt{2}\delta'^2} - \frac{A_3}{2\sqrt{2}\delta'^4} \\
y_2 &= \frac{A_1}{2\sqrt{2}\delta'^2} + \frac{A_2}{2\delta'^3} + \frac{A_3}{2\sqrt{2}\delta'^4} \\
z_0 &= \frac{B_0 - 1}{2\delta'} - \frac{B_1}{2\varepsilon\delta'\sqrt{2}} \\
z_1 &= \frac{B_0 - 1}{2\delta'} + \frac{B_1}{2\varepsilon\delta'\sqrt{2}}.
\end{aligned}
$$

Let us define

$$
\begin{aligned}
u &= x_1 + y_1 = \frac{1}{\delta'} A_0 \qquad (56) \\
v &= z_0 + z_1 = \frac{1}{\delta'}(B_0 - 1),
\end{aligned}
$$

then system (1) reads as

$$
\begin{aligned}
A_0' &= A_1, \\
A_1' &= A_2, \\
A_2' &= A_3, \\
A_3' &= -A_0\left(\delta^2 + 2\delta^2 v + \delta u^2 + (1+\delta^2)v^2\right), \\
v' &= \frac{1}{\delta'} B_1, \\
B_1' &= \varepsilon^2(1+\delta' v)\left(2\delta' v + \delta v^2 + (1+\delta^2)\delta u^2\right).
\end{aligned}
$$

With variables (55) this leads to the new 6-dimensional system

$$
\begin{aligned}
x_1' &= -\frac{\delta'}{\sqrt{2}}(x_1 + x_2) - \frac{\delta' u g(u,v)}{2\sqrt{2}}, \\
x_2' &= \frac{\delta'}{\sqrt{2}}(x_1 - x_2) + \frac{\delta' u g(u,v)}{2\sqrt{2}}, \\
\\
y_1' &= \frac{\delta'}{\sqrt{2}}(y_1 + y_2) + \frac{\delta' u g(u,v)}{2\sqrt{2}}, \\
y_2' &= -\frac{\delta'}{\sqrt{2}}(y_1 - y_2) - \frac{\delta' u g(u,v)}{2\sqrt{2}},
\end{aligned}
$$

$$
\begin{aligned}
z_0' &= -\varepsilon\sqrt{2}z_0 - \frac{\varepsilon\delta'}{2\sqrt{2}}f(u,v), \\
z_1' &= \varepsilon\sqrt{2}z_1 + \frac{\varepsilon\delta'}{2\sqrt{2}}f(u,v),
\end{aligned}
$$

$$
\begin{aligned}
g(u,v) &= u^2 + 2\delta v + (1+\delta^2)v^2 \\
f(u,v) &= 3v^2 + \delta' v^3 + (1+\delta^2)(1+\delta' v)u^2,
\end{aligned}
$$

where the linear part is as expected.

For finding the stable manifold of M_+ we put the system in an integral form, looking for solutions tending to 0 as $x \to +\infty$. Defining

$$
G = \begin{pmatrix} g \\ -g \end{pmatrix}, \quad \mathbf{L} = \frac{\delta'}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},
$$

we obtain the system, where $x \geq 0$,

$$
\begin{aligned}
X(x) &= e^{-\mathbf{L}x}X_0 - \frac{\delta'}{2\sqrt{2}}\int_0^x e^{-\mathbf{L}(x-s)}u(s)G(u,v)(s)ds, \\
Y(x) &= -\frac{\delta'}{2\sqrt{2}}\int_x^{+\infty} e^{\mathbf{L}(x-s)}u(s)G(u,v)(s)ds, \qquad (57)
\end{aligned}
$$

$$
\begin{aligned}
z_0(x) &= e^{-\varepsilon\sqrt{2}x}z_{00} - \frac{\varepsilon\delta'}{2\sqrt{2}}\int_0^x e^{-\varepsilon\sqrt{2}(x-s)}f(u,v)(s)ds, \\
z_1(x) &= -\frac{\varepsilon\delta'}{2\sqrt{2}}\int_x^{+\infty} e^{\varepsilon\sqrt{2}(x-s)}f(u,v)(s)ds. \qquad (58)
\end{aligned}
$$

We notice that

$$
e^{\mathbf{L}x} = e^{\frac{\delta' x}{\sqrt{2}}} \begin{pmatrix} \cos\frac{\delta' x}{\sqrt{2}} & \sin\frac{\delta' x}{\sqrt{2}} \\ -\sin\frac{\delta' x}{\sqrt{2}} & \cos\frac{\delta' x}{\sqrt{2}} \end{pmatrix}, \qquad (59)
$$

$$
||e^{-\mathbf{L}x}|| \leq e^{-\frac{\delta' x}{\sqrt{2}}}, \; x \geq 0.
$$

The 3-dimensional stable manifold is obtained in expressing $(Y(0), z_1(0))$ as function of (X_0, z_{00}).

Let us define for this section

$$
C_\kappa^0 = \{X \in C^0[0,+\infty); X(x)e^{\kappa x} \text{ is bounded}\}
$$

equiped with the norm

$$
||X||_\kappa = \sup_{(0,+\infty)} |X(x)e^{\kappa x}|.
$$

Using (59), the system (57,58) gives remarquably two scalar equations with unknown functions (u,v). We obtain:

$$u(x) = e^{-\frac{\delta' x}{\sqrt{2}}}u_0(x) - \frac{\delta'}{2}\int_0^\infty e^{-\frac{\delta'|x-s|}{\sqrt{2}}}\cos[\frac{\delta'|x-s|}{\sqrt{2}} - \frac{\pi}{4}]u(s)g(u,v)(s)ds, \quad (60)$$

$$v(x) = e^{-\varepsilon\sqrt{2}x}z_{00} - \frac{\varepsilon\delta'}{2\sqrt{2}}\int_0^\infty e^{-\varepsilon\sqrt{2}|x-s|}f(u,v)(s)ds \quad (61)$$

with

$$u_0(x) = x_{10}\cos\frac{\delta' x}{\sqrt{2}} - x_{20}\sin\frac{\delta' x}{\sqrt{2}}.$$

We may observe that we have the *explicit solution of* (61) for $u \equiv 0$. Indeed

$$v(x) = e^{-\varepsilon\sqrt{2}x}z_{00} - \frac{\varepsilon\delta'}{2\sqrt{2}}\int_0^\infty e^{-\varepsilon\sqrt{2}|x-s|}f(0,v)(s)ds$$

corresponds to look for v such that

$$\begin{aligned} v' &= (z_0' + z_1') = -\varepsilon\sqrt{2}(z_0 - z_1) \\ z_0' - z_1' &= -\varepsilon\sqrt{2}v - \frac{\varepsilon\delta'}{\sqrt{2}}(3v^2 + \delta' v^3), \end{aligned}$$

hence

$$v'' = 2\varepsilon^2 v + \varepsilon^2\delta'(3v^2 + \delta' v^3),\ v, v' \underset{x\to+\infty}{\to} 0,$$

which gives

$$v'^2 = \frac{\varepsilon^2}{2}v^2(2+\delta' v)^2.$$

It results that

$$v' = -\frac{\varepsilon}{\sqrt{2}}v(2+\delta' v) \quad (62)$$

since v grows for $x > 0$ and $v < 0$ and $|\delta' v| = 1 - B_0 < 1$ implies

$$2 + \delta' v > 1. \quad (63)$$

Finally we obtain

$$v(x) = \frac{v_0 e^{-\varepsilon\sqrt{2}x}}{1 + v_0\frac{\delta'}{2}(1 - e^{-\varepsilon\sqrt{2}x})}.$$

5.1 Using the first integral (5)

Assuming some estimates needing to be checked at the end, the strategy here is to first solve with respect to v in using the first integral (5) and an implicit function argument. Hence v becomes function of (X, Y, v_0). Then, we solve the scalar equation (60) with respect to u using again the implicit function theorem (section 5.2). This step imposes a restriction on the choice of δ now in $[\delta_0, 1]$. Finally in section 5.3, using (57) we obtain X, Y then function of X_0, z_{00}.

Instead of the differential equations for z_0 and z_1 (or v) we use the first integral (5), for extending the domain of validity for the stable manifold of M_+ as a graph with respect to B_0, and in using (56),

$$\begin{aligned} B_1^2 &= \frac{\varepsilon^2}{2}[(B_0^2-1)^2 + 2\delta(1+\delta^2)u^2(2\delta' v + \delta v^2) \\ &\quad + \delta^2 u^4 + 8\delta^3(x_1y_1 - x_2y_2)]. \end{aligned}$$

Taking the square root gives the traces of the stable and the unstable manifolds on $\mathcal{W}_\delta$. The stable manifold needs satisfy $B_1 = B_0' > 0$, since $B_0 < 1$ for $x = 0$, and $B_0 = 1$ for $x = \infty$. Assuming that the sign of B_1 does not change in the interval, we obtain

$$B_1 = \frac{\varepsilon}{\sqrt{2}}(1-B_0^2)\left(1 + \frac{2\delta'(1+\delta^2)u^2}{(2v+\delta' v^2)} + \frac{\delta u^4 + 8\delta^2(x_1y_1 - x_2y_2)}{(2v+\delta' v^2)^2}\right)^{1/2} \tag{64}$$

Remark 15 *We notice that this implies that $v < 0$, $v' > 0$, and $|v(x)|_{\max} = |v_0| = |z_{00} + z_1(0)|$ is then $\mathcal{O}(\alpha^2)$ close to $h(\delta)$, where*

$$h(\delta) = \frac{1}{\sqrt{\delta}}\left(1 - \frac{1}{\sqrt{1+\delta^2}}\right) \sim \frac{1}{\sqrt{\delta}}(1 - B_{00}).$$

We observe that

$$B_1 = B_0' = \frac{\varepsilon}{\sqrt{2}}[1 - B_0^2]$$

may be easily integrated on $(0, +\infty)$ with $B_0(\infty) = 1$), and moreover leads to

$$z_0 - z_1 = \frac{1}{2}v(2 + \delta' v) \tag{65}$$

i.e.

$$z_1 = -\frac{\delta'}{4}(z_0 + z_1)^2 < 0 \tag{66}$$

which is the solution of (58) for $u = 0$. We notice that (65) is exactly (62). It results that (64) may be written as

$$v' = -\varepsilon v\sqrt{2}(1 + \frac{\delta'}{2}v)\left(1 + \frac{2\delta'(1+\delta^2)u^2}{(2v+\delta' v^2)} + \frac{\delta u^4 + 8\delta^2(x_1y_1 - x_2y_2)}{(2v+\delta' v^2)^2}\right)^{1/2}.$$

Let us assume that

$$|X(x)|, |Y(x)|, |u(x)| \le \gamma |v(x)|, \;\; x \in [0, +\infty) \tag{67}$$

with

$$0 < \gamma < \frac{1}{6\delta}.$$

Using (63), this implies that for $|X|, |Y|$ satisfying (67) we have

$$\frac{2\delta'(1+\delta^2)u^2}{(2v+\delta' v^2)} \le 2\gamma\delta'(1+\delta^2)|u|,$$

$$\frac{\delta u^4 + 8\delta^2(x_1y_1 - x_2y_2)}{(2v + \delta' v^2)^2} \le \delta\gamma^2 u^2 + \frac{4}{9}.$$

It results that for $\gamma|u|$ such that

$$\gamma|u| \le \frac{1}{3\delta'}[12(1+\delta^2) + \sqrt{2}]^{-1} \tag{68}$$

then

$$\frac{2\delta'(1+\delta^2)u^2}{(2v+\delta' v^2)} + \frac{\delta u^4 + 8\delta^2(x_1y_1 - x_2y_2)}{(2v + \delta' v^2)^2} < \frac{1}{2},$$

and the square root is analytic in v with

$$v' = -\varepsilon\sqrt{2}v(1 + \frac{\delta'}{2}v)[1 + \mathcal{Z}(X, Y, v)], \quad |\mathcal{Z}| \le 1/4. \tag{69}$$

Then we can integrate the integro-differential equation, as in section 4.10. We introduce the new variable w as

$$\begin{aligned} w' &= \frac{v'}{v(1 + (\delta'/2)v)}, \\ w &= \ln\left(\frac{-v}{1 + (\delta'/2)v}\right), \\ v &= -\frac{e^w}{1 + \frac{\delta'}{2}e^w}; \end{aligned}$$

w decreases from w_0 to $-\infty$ for $x \in (0, \infty)$, while v grows from $v_0 < 0$ to 0.

$$\begin{aligned} \mathcal{Z}(X, Y, v) &= h(X, Y, w), \\ |h| &\le 1/4. \end{aligned}$$

We then obtain, by simple integration

$$\varepsilon\sqrt{2}x(1 - 1/4) \le w_0 - w(x) \le \varepsilon\sqrt{2}x(1 + 1/4).$$

Remark 16 *The constant* $1/4$ *above may later be replaced by*

$$ce^{-\delta' x/\sqrt{2}}$$

since we show later that $|X|$ *and* $|Y|$ *lie in* $C^0_{\delta'/\sqrt{2}}$.

We deduce the estimate

$$v_0 \frac{1 - \tanh(\frac{\varepsilon x(1-1/4)}{\sqrt{2}})}{1 + B_0(0)\tanh(\frac{\varepsilon x(1-1/4)}{\sqrt{2}})} \le v(x) \le v_0 \frac{1 - \tanh(\frac{\varepsilon x(1+1/4)}{\sqrt{2}})}{1 + B_0(0)\tanh(\frac{\varepsilon x(1+1/4)}{\sqrt{2}})} \tag{70}$$

where

$$v_0 = \frac{B_0(0) - 1}{\delta'} < 0.$$

The a priori estimate for v obtained in (70) allows to prove (see [4]) the existence and uniqueness of a solution for (69), provided that (67) is satisfied on the whole interval $x \in [0, \infty)$.

5.2 Estimate for u

Let us first show that for $\delta \in (0,5)$, $x \in [0,+\infty)$ and for $\gamma|u| \leq c_1(\delta)$ where c_1 is defined in (74), then

$$\begin{aligned} g(u,v) &= u^2 + v[2\delta + (1+\delta^2)v] < 0 \\ |g(u,v)| &\leq 2\delta|v|. \end{aligned} \tag{71}$$

Using (67), we observe that (71) is valid as soon as

$$\gamma|u| + (1+\delta^2)|v_0| \leq 2\delta. \tag{72}$$

We wish to reach $v_0 = \delta'^{-1}(B_{00} - 1)$ where B_{00} is the value of B_0 we have reached with the unstable manifold of M_-. So, next computations should be valid for v_0 such that $|v_0| \leq |\delta'^{-1}(1 - B_{00})|$.

Using

$$\eta_0^2 \sim \frac{1}{1+\delta^2},$$

since α is as close to 0 as we wish, we obtain

$$1 - B_{00} = 1 - (1 - \eta_0^2\delta^2)^{1/2} = 1 - \frac{1}{\sqrt{1+\delta^2}} + \mathcal{O}(\alpha^2).$$

Then conditions (72) and (68) lead to

$$\gamma|u| \leq c_1(\delta), \tag{73}$$

with

$$c_1(\delta) = \min\{2\delta - \frac{1}{\sqrt{\delta}}[(1+\delta^2) - \sqrt{1+\delta^2}], \frac{1}{3\sqrt{\delta}}[12(1+\delta^2) + \sqrt{2}]^{-1}\}. \tag{74}$$

We observe that

$$c_1(\delta) > 0 \text{ for } \delta \in (0,5),\ c_1(\delta) \sim 2\delta \text{ for } \delta \text{ close to } 0.$$

Now the estimates (70) of v for $x \in [0,+\infty)$ lead to

$$|v_0| \frac{e^{\frac{-5\varepsilon\sqrt{2}x}{4}}}{1 - \frac{|v_0\delta'|}{2}(1 - e^{\frac{-5\varepsilon\sqrt{2}x}{4}})} \leq |v(x| \leq |v_0| \frac{e^{\frac{-3\varepsilon\sqrt{2}x}{4}}}{1 - \frac{|v_0\delta'|}{2}(1 - e^{\frac{-3\varepsilon\sqrt{2}x}{4}})},$$

so that

$$|v_0| e^{\frac{-5\varepsilon\sqrt{2}x}{4}} \leq |v(x)| \leq \frac{|v_0|}{1 - \frac{|v_0\delta'|}{2}} e^{\frac{-3\varepsilon\sqrt{2}x}{4}} \tag{75}$$

and to reach B_{00} we need to satisfy

$$h(\delta) e^{\frac{-5\varepsilon\sqrt{2}x}{4}} \leq |v(x)| \leq c_0(\delta) e^{\frac{-3\varepsilon\sqrt{2}x}{4}} \tag{76}$$

with

$$c_0(\delta) = \frac{2\delta^{3/2}}{(\sqrt{1+\delta^2}+1)^2}.$$

Let us consider (60), then (71) and (75) lead to

$$|u(x)e^{\frac{5\varepsilon\sqrt{2}x}{4}}| \leq |X_0|e^{(\frac{-\delta'}{\sqrt{2}}+\frac{5\varepsilon\sqrt{2}}{4})x}+\frac{\delta\delta'|v_0|}{1-\frac{|v_0\delta'|}{2}}\int_0^\infty e^{\frac{-\delta'|x-s|}{\sqrt{2}}+\frac{5\varepsilon\sqrt{2}(x-s)}{4}}|u(s)e^{\frac{5\varepsilon\sqrt{2}s}{4}}|e^{\frac{-3\varepsilon\sqrt{2}s}{4}}ds.$$

We have

$$\int_0^\infty e^{\frac{-\delta'|x-s|}{\sqrt{2}}+\frac{5\varepsilon\sqrt{2}(x-s)}{4}}e^{\frac{-3\varepsilon\sqrt{2}s}{4}}ds \leq \frac{2\sqrt{2}e^{\frac{-3\varepsilon\sqrt{2}x}{4}}}{\delta'(1-16\varepsilon^2/\delta')},$$

so that

$$||u||_{\frac{5\varepsilon\sqrt{2}}{4}} \leq |X_0| + \frac{2\sqrt{2}\delta|v_0|}{(1-\frac{|v_0\delta'|}{2})(1-16\varepsilon^2/\delta')}||u||_{\frac{5\varepsilon\sqrt{2}}{4}}.$$

We notice that, for $|v_0| \leq \frac{1}{\delta'}(1-B_{00})$

$$\frac{2\sqrt{2}\delta|v_0|}{(1-\frac{|v_0\delta'|}{2})(1-16\varepsilon^2/\delta')} \leq c_2(\delta)$$

with

$$c_2(\delta) = \frac{2\sqrt{2}\delta c_0(\delta)}{(1-16\varepsilon^2/\delta')},$$

and we observe that

$$2\sqrt{2}\delta c_0(\delta) = \frac{4\sqrt{2}\delta^{5/2}}{(\sqrt{1+\delta^2}+1)^2} < k < 1 \text{ for } \delta \in [\delta_0, 1],$$

so that, since δ_0 is arbitrarily fixed > 0, and ε small enough, for any v satisfying (75) with $|v_0| \leq \frac{1}{\delta'}(1-B_{00})$, by a fixed point argument, we obtain a unique $u \in C^0_{\frac{5\varepsilon\sqrt{2}}{4}}$ such that

$$||u||_{\frac{5\varepsilon\sqrt{2}}{4}} \leq \frac{1}{1-k}|X_0|.$$

Remark 17 *Notice that we neglected terms of order α^2 in the estimates leading to the calculus of k which is stricly <1. However, because of the flexibility of choice for δ, we may check that for ε (i.e. α) small enough the choice $\delta \in [\delta_0, 1]$ is valid.*

5.3 End of the proof of Lemma 14

We may now estimate X and Y given by (57), and we need to check that (67) is satisfied. Now, from (57) we obtain

$$\begin{aligned}
||X||_{\frac{5\varepsilon\sqrt{2}}{4}} &\leq |X_0| + \frac{\delta\delta'|v_0|}{(\delta'-\varepsilon)(1-\frac{|v_0\delta'|}{2})}||u||_{\frac{5\varepsilon\sqrt{2}}{4}} \\
&\leq |X_0|\left(1+\frac{\delta\delta'|v_0|}{(1-k)(\delta'-\varepsilon)(1-\frac{|v_0\delta'|}{2})}\right) \\
&\leq |X_0|\left(1+\frac{\delta' k}{2\sqrt{2}(1-k)(\delta'-\varepsilon)}\right),
\end{aligned}$$

$$\begin{aligned}
||Y||_{\frac{5\varepsilon\sqrt{2}}{4}} &\leq \frac{\delta\delta'|v_0|}{(1-k)(\delta+4\varepsilon)(1-\frac{|v_0\delta'|}{2})}|X_0| \\
&\leq \frac{\delta' k}{2\sqrt{2}(1-k)(\delta'+4\varepsilon)}|X_0|.
\end{aligned}$$

We observe that

$$1+\frac{\delta' k}{2\sqrt{2}(1-k)(\delta'-\varepsilon)} < \frac{1}{1-k},$$

hence, using (75) for a lower bound for $v(x)$ we see that for X_0 such that

$$|X_0| \leq (1-k)\gamma|v_0|,$$

then conditions (67) are realized, and (73) is satisfied as soon as

$$|X_0| \leq \frac{(1-k)}{\gamma}c_1(\delta).$$

This ends the proof of the existence, uniqueness and analyticity in parameters of the solution of (57), (60) and (61). Then (58) allows to find z_0, z_1 with

$$\begin{aligned}
||z_0||_{\varepsilon\sqrt{2}} &\leq |z_{00}| + \frac{\varepsilon\delta'[3+(1+\delta^2)\gamma^2]}{2\sqrt{2}}\int_0^x v^2(s)e^{\varepsilon\sqrt{2}s}ds \\
&\leq |z_{00}| + \frac{\delta'[3+(1+\delta^2)\gamma^2]}{2}\left(\frac{v_0}{1-\frac{|v_0|\delta'}{2}}\right)^2,
\end{aligned}$$

$$||z_1||_{\varepsilon\sqrt{2}} \leq \frac{\delta'[3+(1+\delta^2)\gamma^2]}{10}\left(\frac{v_0}{1-\frac{|v_0|\delta'}{2}}\right)^2.$$

We notice that

$$|z_1(0)| \leq \frac{\delta'[3+(1+\delta^2)\gamma^2]}{10}\frac{c_0(\delta)}{1-\frac{|v_0|\delta'}{2}}|v_0|,$$

and for $\delta \in [\delta_0, 1]$

$$\frac{\delta'[3+(1+\delta^2)\gamma^2]}{10} \frac{c_0(\delta)}{1-\frac{|v_0|\delta'}{2}} \leq 0.0402[3+2\gamma^2]$$

which is $< 1/4$ for a good choice of γ (still free choice). Hence

$$|z_1(0)| \leq \frac{1}{4}(|z_{00}| + |z_1(0)|)$$

leads to

$$\begin{aligned} |z_1(0)| &\leq \frac{1}{3}|z_{00}|, \\ |v_0| &= |z_{00} + z_1(0)| \leq \frac{4}{3}|z_{00}|. \end{aligned}$$

This ends the proof of existence, uniqueness and analyticity in parameters of the stable manifold of M_+. The exponential estimates declared in Lemma 14 follow from the linear study of section 3 as $x \to +\infty$. The asymptotic expression of $v(x)$ follows from (70) after replacing $1/4$ by the better estimate $Ce^{-\sqrt{\frac{\delta}{2}}x}$. The bound for v_0 comes from $h(\delta)$ with $\delta = 1$. This ends the proof of Lemma 14.

5.4 Intersection of the stable manifold with H_0

We need to compute the intersection of the 3-dimensional stable manifold of M_+ with the hyperplane H_0 defined by

$$B_0 = \sqrt{1 - \eta_0^2 \delta^2}. \tag{77}$$

We then obtain a 2-dimensional sub-manifold living in the 4-dimensional manifold $\mathcal{W}_\delta \cap H_0$. We have by construction

$$\begin{aligned} A_0 &= \delta^{1/2}(x_{10} + y_{10}), \\ A_1 &= -\frac{\delta}{\sqrt{2}}(x_{10} + x_{20} - y_{10} - y_{20}) \\ A_2 &= \delta^{3/2}(x_{20} + y_{20}) \\ A_3 &= \frac{\delta^2}{\sqrt{2}}(x_{10} - x_{20} - y_{10} + y_{20}), \end{aligned} \tag{78}$$

where y_{10} and y_{20} are expressed in function of $X_0 = (x_{10}, x_{20})$, with the restriction

$$|x_{10}| + |x_{20}| \leq \eta.$$

Below, we need to express the tangent plane to the intersection of the stable manifold with the hyperplane H_0. This is given by (see (57))

$$Y_0 = -\frac{\delta\delta'}{\sqrt{2}} \int_0^\infty e^{-\mathbf{L}s} \begin{pmatrix} 1 \\ -1 \end{pmatrix} u(s)v(s)ds,$$

with

$$\begin{aligned}
u(x) &= e^{-\frac{\delta' x}{\sqrt{2}}}u_0(x) - \delta\delta' \int_0^\infty e^{-\frac{\delta'|x-s|}{\sqrt{2}}} \cos[\frac{\delta'|x-s|}{\sqrt{2}} - \frac{\pi}{4}]u(s)v(s)ds, \\
u_0(x) &= x_{10}\cos\frac{\delta' x}{\sqrt{2}} - x_{20}\sin\frac{\delta' x}{\sqrt{2}}, \\
v(x) &= \frac{v_0 e^{-\varepsilon\sqrt{2}x}}{1+v_0\frac{\delta'}{2}(1-e^{-\varepsilon\sqrt{2}x})},\ v_0 = \frac{1}{\delta'}(B_{00}-1),
\end{aligned}$$

so that

$$u(x) = l(v_0,\delta,x)X_0 \in C^0_{\frac{5\varepsilon\sqrt{2}}{4}}$$

is linear in X_0, hence

$$Y_0 = \mathcal{L}_1(v_0,\delta)X_0, \tag{79}$$

with a 2x2 matrix $\mathcal{L}_1$ depending analytically of $\delta \in [\delta_0, 1]$.

6 Intersection of the two manifolds

In this section we prove the following

Lemma 18 *For ε small enough, and for $\delta_0 \le \delta \le 1$, except maybe for a finite number of values, the unstable manifold of M_- intersects transversally the stable manifold of M_+ along the heteroclinic solution. Moreover for $x = 0$ we have the estimates*

$$\begin{aligned}
A_0(0) &= \mathcal{O}(B_{00}\varepsilon^{1/2}) \\
A_1(0) &= \mathcal{O}(B_{00}\varepsilon^{1/2}) \\
A_2(0) &= \mathcal{O}(B_{00}\varepsilon^{2/3}) \\
A_3(0) &= \mathcal{O}(B_{00}\varepsilon^{5/6}).
\end{aligned} \tag{80}$$

We need to study the intersection of the plane (53) tangent to the unstable manifold of M_-, with the plane tangent to the stable manifold of M_+ given by (78), satisfying (79).

We then find a linear system with 4 unknowns $(\overline{x_1}^{(u)}, \overline{x_2}^{(u)}, x_{10}^{(s)}, x_{20}^{(s)})$, with the restrictions

$$|x_{10}^{(s)}| + |x_{20}^{(s)}| \le \eta,\ |\overline{x_1}^{(u)}| + |\overline{x_2}^{(u)}| \le \rho.$$

We then have

$$\begin{aligned}
(x_{10}^{(s)} + y_{10}^{(s)}) &= \delta'\alpha + \frac{\alpha^{1/2}}{2^{3/4}}B_{00}(\overline{x_1}^{(u)} - \overline{x_2}^{(u)}) \\
-(x_{10}^{(s)} + x_{20}^{(s)} - y_{10}^{(s)} - y_{20}^{(s)}) &= \sqrt{2}\alpha B_{00}\overline{x_1}^{(u)} - \alpha^2 B_{00} \\
(x_{20}^{(s)} + y_{20}^{(s)}) &= \frac{\alpha^{3/2}}{2^{1/4}}B_{00}(\overline{x_1}^{(u)} + \overline{x_2}^{(u)}) \\
(x_{10}^{(s)} - x_{20}^{(s)} - y_{10}^{(s)} + y_{20}^{(s)}) &= 2\alpha^2 B_{00}\overline{x_2}^{(u)},
\end{aligned} \tag{81}$$

where we need to express $(y_{10}^{(s)}, y_{20}^{(s)})$ as a linear function of $(x_{10}^{(s)}, x_{20}^{(s)})$ (see (79). Let us define

$$X_0^{(s)} = \begin{pmatrix} x_{10}^{(s)} \\ x_{20}^{(s)} \end{pmatrix},\ Y_0^{(s)} = \begin{pmatrix} y_{10}^{(s)} \\ y_{20}^{(s)} \end{pmatrix},\ \overline{X}^{(u)} = \begin{pmatrix} \overline{x_1}^{(u)} \\ \overline{x_2}^{(u)} \end{pmatrix},$$

then we have

$$X_0^{(s)} = \begin{pmatrix} \frac{\delta'}{2}\alpha + \frac{\alpha^2 B_{00}}{4} \\ \frac{\alpha^2 B_{00}}{4} \end{pmatrix} + \boldsymbol{M}_1 \overline{X}^{(u)},$$

$$Y_0^{(s)} = \begin{pmatrix} \frac{\delta'}{2}\alpha - \frac{\alpha^2 B_{00}}{4} \\ -\frac{\alpha^2 B_{00}}{4} \end{pmatrix} + \boldsymbol{M}_2 \overline{X}^{(u)},$$

with

$$\begin{aligned} \boldsymbol{M}_1 &= \frac{\alpha^{1/2} B_{00} 2^{1/4}}{4} \begin{pmatrix} 1 - 2^{1/4}\alpha^{1/2} & -1 + 2^{3/4}\alpha^{3/2} \\ -2^{1/4}\alpha^{1/2} + \sqrt{2}\alpha & \sqrt{2}\alpha - 2^{3/4}\alpha^{3/2} \end{pmatrix}, \\ \boldsymbol{M}_2 &= \frac{\alpha^{1/2} B_{00} 2^{1/4}}{4} \begin{pmatrix} 1 + 2^{1/4}\alpha^{1/2} & -1 - 2^{3/4}\alpha^{3/2} \\ 2^{1/4}\alpha^{1/2} + \sqrt{2}\alpha & \sqrt{2}\alpha + 2^{3/4}\alpha^{3/2} \end{pmatrix}. \end{aligned}$$

The matrix $\boldsymbol{M}_2$ is invertible with

$$M_2^{-1} = \frac{4}{\alpha^{1/2} B_{00} 2^{1/4} \det(M_2')} \begin{pmatrix} \sqrt{2}\alpha + 2^{3/4}\alpha^{3/2} & 1 + 2^{3/4}\alpha^{3/2} \\ -2^{1/4}\alpha^{1/2} - \sqrt{2}\alpha & 1 + 2^{1/4}\alpha^{1/2} \end{pmatrix}$$

$$\begin{aligned} \det(M_2') &= [\sqrt{2}\alpha(1 + 2^{1/4}\alpha^{1/2})^2 + (1 - 2^{3/4}\alpha^{3/2})(2^{1/4}\alpha^{1/2} + \sqrt{2}\alpha)] \\ &= 2^{1/4}\alpha^{1/2} + 2\sqrt{2}\alpha + \mathcal{O}(\alpha^{3/2}). \end{aligned}$$

It results that

$$\boldsymbol{M}_1 \boldsymbol{M}_2^{-1} \sim \begin{pmatrix} 1 + \mathcal{O}(\alpha^{1/2}) & -2 + \mathcal{O}(\alpha^{1/2}) \\ -2^{3/2}\alpha + \mathcal{O}(\alpha^{3/2}) & -1 + \mathcal{O}(\alpha^{1/2}) \end{pmatrix}$$

$$X_0^{(s)} = \begin{pmatrix} 1 + \mathcal{O}(\alpha^{1/2}) & -2 + \mathcal{O}(\alpha^{1/2}) \\ -2^{3/2}\alpha + \mathcal{O}(\alpha^{3/2}) & -1 + \mathcal{O}(\alpha^{1/2}) \end{pmatrix} Y_0^{(s)} + \begin{pmatrix} \mathcal{O}(\alpha^{3/2}) \\ \sqrt{2}\delta'\alpha^2 + \mathcal{O}(\alpha^{5/2}) \end{pmatrix}. \tag{82}$$

Equation (82) represents a 2-dim affine plane expressing the 2 compatibility conditions of the system (81) while solving with respect to $(\overline{x_1}^{(u)}, \overline{x_2}^{(u)})$, and gives a condition on coordinates of the stable manifold. This affine plane needs to intersect the tangent plane to the stable manifold given by (79) with $Y_0^{(s)}$ expressed as a linear function of $X_0^{(s)}$.

We deduce that (79) combined with

$$X_0^{(s)} = \mathcal{L}_2(v_0, \delta) Y_0^{(s)} + \mathcal{O}(\alpha^{3/2})$$

leads to

$$Y_0^{(s)} = \mathcal{L}_1(v_0, \delta)\mathcal{L}_2(v_0, \delta) Y_0^{(s)} + \mathcal{O}(\alpha^{3/2}).$$

We notice that

$$\begin{aligned} v_0 &= \frac{1}{\sqrt{\delta}}(B_{00}-1), \\ B_{00} &= \sqrt{1-\eta_0^2\delta^2},\ \eta_0^2(1+\delta^2)=1+\alpha^2, \end{aligned}$$

so that the 2x2 matrix $\mathcal{L}_1(v_0,\delta)\mathcal{L}_2(v_0,\delta)$ is a function of α (which is as small as we wish), and depends analytically of $\delta\in[\delta_0,1]$. For δ small enough, the norm of $\mathcal{L}_1(v_0,\delta)$ is small, which does not allow an eigenvalue 1 for $\mathcal{L}_1(v_0,\delta)\mathcal{L}_2(v_0,\delta)$. Due to the analyticity in δ, the characteristic polynomial of this 2x2 matrix has analytic coefficients, and it is then clear that 1 might be an eigenvalue of this matrix operator only for isolated values of δ. It then results for α small enough, that for any $\delta\in[\delta_0,1]$ except maybe for a finite set of values, we obtain a unique solution

$$Y_0^{(s)}=\mathcal{O}(\alpha^{3/2}),$$

leading to

$$X_0^{(s)}=\mathcal{O}(\alpha^{3/2})$$

which is coherent with the condition $|x_{10}^{(s)}|+|x_{20}^{(s)}|\leq\eta$.

Moreover adding the 3 first equation of (81) gives

$$\begin{aligned} 0 &= \delta'\alpha-\alpha^2B_{00}+\frac{\alpha^{1/2}}{2^{3/4}}B_{00}(\overline{x_1}^{(u)}-\overline{x_2}^{(u)}) \\ &\quad +\sqrt{2}\alpha B_{00}\overline{x_1}^{(u)}+\frac{\alpha^{3/2}}{2^{1/4}}B_{00}(\overline{x_1}^{(u)}+\overline{x_2}^{(u)}) \end{aligned}$$

hence

$$\overline{x_2}^{(u)}=\overline{x_1}^{(u)}-\frac{2^{7/4}\alpha^{1/2}+2^{3/2}\alpha}{1-\alpha\sqrt{2}}\overline{x_1}^{(u)}+\frac{2^{3/4}(\alpha^{1/2}\delta'-\alpha^{3/2}B_{00})}{B_{00}(1-\alpha\sqrt{2})},$$

so that

$$\delta'\alpha+\frac{\alpha^{1/2}}{2^{3/4}}B_{00}(\overline{x_1}^{(u)}-\overline{x_2}^{(u)})\simeq-2\alpha B_{00}\overline{x_1}^{(u)}=\mathcal{O}(\alpha^{3/2}).$$

It results that

$$(\overline{x_1}^{(u)},\overline{x_2}^{(u)})=\mathcal{O}(\alpha^{1/2}),$$

which then satisfies the condition $|\overline{x_1}^{(u)}|+|\overline{x_2}^{(u)}|\leq\rho$. Finally, from (53) and since $\alpha^{3/2}=\sqrt{\varepsilon}$, we obtain

$$\begin{aligned} A_0(0) &= \mathcal{O}(B_{00}\sqrt{\varepsilon}) \\ A_1(0) &= \mathcal{O}(B_{00}\sqrt{\varepsilon}) \\ A_2(0) &= \mathcal{O}(B_{00}\varepsilon^{2/3}) \\ A_3(0) &= \mathcal{O}(B_{00}\varepsilon^{5/6}) \end{aligned} \tag{83}$$

which are the estimates announced at Lemma 18. The uniqueness of the intersection of the tangent planes between the unstable manifold of M_- and the stable manifold of M_+ proves that it is transverse while they both sit on $\mathcal{W}_\delta$ and cross the hyperplane (77). Since it is the transverse intersection of two manifolds, depending analytically on parameters (ε, δ), the resulting curve depends analytically on these parameters.

We observe that, along this intersection, and by construction, $B_1(x) = B_0'(x) > 0$. Its principal part on $(-\infty, 0]$ is given by (47) with $B_0(0) = B_{00} = \sqrt{1 - \eta_0^2\delta^2}$, and on $[0, +\infty)$ by (54).

The Theorem 1 is then proved.

Moreover, for the heteroclinic solution, we can improve the a priori estimates given at Lemma 9. Taking into account the size of variables for $x = 0$, we have now

Corollary 19 *For $x \in (-\infty, 0]$ and choosing $\delta^* < \delta$, there exists $c > 0$ independent of ε small enough, such that the heteroclinic curve satisfies*

$$\begin{aligned} |\widetilde{A_0}(x)| &\leq c\varepsilon^{1/3} B_0(x) e^{\varepsilon\delta^* x} \\ |A_1(x)|, |A_2(x)|, |A_3(x)| &\leq c\varepsilon^{1/2} B_0(x) e^{\varepsilon\delta^* x}. \end{aligned}$$

We also give estimates for $x > 0$. Using (60) and playing on the flexibility of choice for δ, we can find $\chi < 1$ independent of ε, such that for $\delta \in [\delta_0, 1]$ we have

$$||u||_{\frac{\delta'}{\sqrt{2}}(1-\chi)} \leq |X_0| + \frac{2\sqrt{2}\delta c_0(\delta)}{\chi(2-\chi)} ||u||_{\frac{\delta'}{\sqrt{2}}(1-\chi)},$$

with

$$\frac{2\sqrt{2}\delta c_0(\delta)}{\chi(2-\chi)} < k' < 1.$$

Hence

$$||u||_{\frac{\delta'}{\sqrt{2}}(1-\chi)} \leq \frac{1}{1-k'}|X_0|$$

and as above we see that there exists $C > 0$ such that

$$||X||_{\frac{\delta'}{\sqrt{2}}(1-\chi)} + ||Y||_{\frac{\delta'}{\sqrt{2}}(1-\chi)} \leq C|X_0|,$$

so that, using (55) and (83) we have the following

Corollary 20 *For $x \in [0, +\infty)$ and choosing $\delta^* < \delta$, there exists $c > 0$ independent of ε small enough, such that the heteroclinic curve satisfies*

$$|A_0^{(m)}(x)| \leq c\varepsilon^{1/2} e^{-\sqrt{\frac{\delta^*}{2}}x}, \ m = 0, 1, 2, 3.$$

7 Study of the linearized operator

Let us redefine the heteroclinic connection we found at Theorem 1 as

$$(A_*(x), B_*(x)) \subset \mathbb{R}^2$$

with

$$1 < 1 + \delta_0^2 \leq g = 1 + \delta^2 \leq 2,$$

and where we know that, for ε small enough

$$\begin{array}{rcl} B_*(x) & > & 0,\ B_*'(x) > 0 \\ (A_*(x), B_*(x)) & \to & \left\{ \begin{array}{l} (1,0) \text{ as } x \to -\infty \\ (0,1) \text{ as } x \to +\infty \end{array} \right. , \end{array}$$

at least as $e^{\varepsilon\delta x}$ for $x \to -\infty$, and at least as $e^{-\sqrt{2}\varepsilon x}$ for $x \to +\infty$.

The system (1) is now considered with B_0 complex valued, so in (1) B^2 is replaced by $|B|^2$.

For being able to prove any persistence result under reversible perturbations of system (1) in $\mathbb{R}^4 \times \mathbb{C}^2$ we need to study the linearized operator at the above heteroclinic solution. We follow the lines of [5].

The linearized operator is given by

$$\begin{array}{rcl} A^{(4)} & = & (1 - 3A_*^2 - gB_*^2)A - gA_*B_*(B + \overline{B}), \\ B'' & = & \varepsilon^2(-1 + gA_*^2 + 2B_*^2)B + 2\varepsilon^2 gA_*B_*A + \varepsilon^2 B_*^2\overline{B}. \end{array}$$

Taking real and imaginary parts for B :

$$B = C + iD,$$

we then obtain the linearized system

$$\begin{array}{rcl} -A^{(4)} + (1 - 3A_*^2 - gB_*^2)A - 2gA_*B_*C & = & 0, \\ \frac{1}{\varepsilon^2}C'' + (1 - gA_*^2 - 3B_*^2)C - 2gA_*B_*A & = & 0, \\ \frac{1}{\varepsilon^2}D'' + (1 - gA_*^2 - B_*^2)D & = & 0. \end{array}$$

Notice that the equation for D decouples, so that we can split the linear operator in an operator $\mathcal{M}_g$ acting on (A, C) and an operator $\mathcal{L}_g$ acting on D :

$$\mathcal{M}_g \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} -A^{(4)} + (1 - 3A_*^2 - gB_*^2)A - 2gA_*B_*C \\ \frac{1}{\varepsilon^2}C'' + (1 - gA_*^2 - 3B_*^2)C - 2gA_*B_*A \end{pmatrix},$$

$$\mathcal{L}_g D = \frac{1}{\varepsilon^2}D'' + (1 - gA_*^2 - B_*^2)D.$$

Let us define the Hilbert spaces

$$L_\eta^2 = \{u; u(x)e^{\eta|x|} \in L^2(\mathbb{R})\},$$

$$\begin{aligned}\mathcal{D}_0 &= \{(A,C)\in H^4_\eta\times H^2_\eta; A\in H^4_\eta, C\in\mathcal{D}_1\}\\ \mathcal{D}_1 &= \{C\in H^2_\eta; \varepsilon^{-2}||C''||_{L^2_\eta}+\varepsilon^{-1}||C'||_{L^2_\eta}+||C||_{L^2_\eta}\overset{def}{=}||C||_{\mathcal{D}_1}<\infty\}\end{aligned}$$

equiped with natural scalar products. Below, we prove the following

Lemma 21 *Except maybe for a set of isolated values of g, the kernel of $\mathcal{M}_g$ in L^2_η is one dimensional, span by (A'_*, B'_*), and its range has codimension 1, L^2-orthogonal to (A'_*, B'_*). $\mathcal{M}_g$ has a pseudo-inverse acting from L^2_η to $\mathcal{D}_0$ for any $\eta>0$ small enough, with bound independent of ε.*

The operator $\mathcal{L}_g$ has a trivial kernel, and its range which has codimension 1, is L^2- orthogonal to B_* ($B_*\notin L^2$). $\mathcal{L}_g$ has a pseudo-inverse acting from L^2_η to $\mathcal{D}_1$ for $\eta>0$ small enough, with bound independent of ε.

Remark 22 *The above Lemma is useful for proving the persistence under reversible perturbations, as indicated in (2), of our heteroclinic. This is done in [8] and appears to be more difficult than the symmetric case solved in [5]. Indeed, it is needed to introduce two different wave numbers for the two convective rolls at $\pm\infty$. In [8] it is shown that the component on the kernel of $\mathcal{M}_g$ corresponds to a phase shift of rolls parallel to the wall, while the codimension 2 of the range implies that each wave number is function not only of the amplitude of rolls but also of the above shift. This then leads to a one parameter family of domain walls, for any fixed small amplitude ε^2.*

7.1 Asymptotic operators

Let us define the operators obtained when $x=\pm\infty$:

$$\mathcal{M}^-_\infty\begin{pmatrix}A\\C\end{pmatrix}=\begin{pmatrix}-A^{(4)}-2A\\ \varepsilon^{-2}C''-(g-1)C\end{pmatrix},$$

$$\mathcal{M}^+_\infty\begin{pmatrix}A\\C\end{pmatrix}=\begin{pmatrix}-A^{(4)}-(g-1)A\\ \varepsilon^{-2}C''-2C\end{pmatrix},$$

$$\begin{aligned}\mathcal{L}^-_\infty D &= \varepsilon^{-2}D''-(g-1)D,\\ \mathcal{L}^+_\infty D &= \varepsilon^{-2}D''.\end{aligned}$$

Notice that all these operators are negative. Furthermore, their spectra in $L^2(\mathbb{R})$ are such that

$$\begin{aligned}\sigma(\mathcal{M}^-_\infty) &= (-\infty,-c_-],\ c_-=\max\{2,(g-1)\}>0,\\ \sigma(\mathcal{M}^+_\infty) &= (-\infty,-c_+],\ c_+=c_-,\\ \sigma(\mathcal{L}^-_\infty) &= (-\infty,-(g-1)],\\ \sigma(\mathcal{L}^+_\infty) &= (-\infty,0].\end{aligned}$$

Operators $\mathcal{M}_g$ and $\mathcal{L}_g$ are respectively relatively compact perturbations of the corresponding asymptotic operators $\mathcal{M}_\infty$ and $\mathcal{L}_\infty$ defined as

$$\mathcal{M}_\infty = \begin{cases} \mathcal{M}_\infty^-, & x<0 \\ \mathcal{M}_\infty^+, & x>0 \end{cases}, \quad \mathcal{L}_\infty = \begin{cases} \mathcal{L}_\infty^-, & x<0 \\ \mathcal{L}_\infty^+, & x>0 \end{cases},$$

Their essential spectrum, i.e. the set of $\lambda \in \mathbb{C}$ for which $\lambda - \mathcal{M}_g$ (resp. $\lambda - \mathcal{L}_g$) is not Fredholm with index 0, is equal to the essential spectrum of $\mathcal{M}_\infty$ (resp. $\mathcal{L}_\infty$) (see [11]). The latter spectra are found from the spectra of $\mathcal{M}_\infty^\pm$ and $\mathcal{L}_\infty^\pm$:

$$\begin{aligned} \sigma_{ess}(\mathcal{M}_\infty) &= (-\infty, -c_+], \\ \sigma_{ess}(\mathcal{L}_\infty) &= (-\infty, 0]. \end{aligned}$$

In particular, this implies that 0 does not belong to the essential spectrum of $\mathcal{M}_g$, so that the operator $\mathcal{M}_g$ is Fredholm with index 0. Moreover operators $\mathcal{M}_\infty$ and $\mathcal{L}_\infty$ are self adjoint negative operators in L^2, and $\mathcal{M}_\infty$ has a bounded inverse [11].

$$||\mathcal{M}_\infty^{-1}||_{L^2} \leq \frac{1}{c_+}.$$

This last property remains valid in exponentially weighted spaces, with weights $e^{\eta|x|}$, and η sufficiently small, since this acts as a small perturbation of the differential operator (see [10] section 3.1).

We show at section 7.3.1 that the kernel of $\mathcal{M}_g$ is one-dimensional (except for a finite set of values of g), spanned by $(A'_*, B'_*) \stackrel{def}{=} U_*$ with a range orthogonal to U_* in L^2. Let us define the projections Q_0 on $U_*^\perp$ and P_0 on U_* , which are orthogonal projections in L^2, then we need to solve in L^2_η

$$\mathcal{M}_g u = f$$

in decomposing

$$u = zU_* + v, \quad v = Q_0 u,$$

$$(\mathcal{M}_\infty + \mathcal{A}_g)v = Q_0 f$$

and we need to satisfy the compatibility condition

$$\langle f, U_* \rangle = 0,$$

while z is arbitrary and we obtain for v :

$$(\mathbb{I} + \mathcal{M}_\infty^{-1}\mathcal{A}_g)v = \mathcal{M}_\infty^{-1} Q_0 f,$$

where the operator $\mathcal{M}_\infty^{-1}\mathcal{A}_g$ is now a compact operator for which -1 is not an eigenvalue, since $v \in U_*^\perp$. It results that there is a number c independent of ε such that

$$||v||_{L^2_\eta} \leq c||f||_{L^2_\eta}.$$

From the form of operator $\mathcal{M}_g$ and using interpolation properties, we obtain for $v = (A, C)$

$$||(A, C)||_{\mathcal{D}_0} \leq c||f||_{L^2_\eta}$$

with a certain c independent of ε.

7.2 Properties of $\mathcal{L}_g$

Notice that $\mathcal{L}_g$ is self adjoint in $L^2(\mathbb{R})$ and that

$$\mathcal{L}_g B_* = 0, \quad \text{but } B_* \notin L^2(\mathbb{R}).$$

This property allows to solve explicitely the equation $\mathcal{L}_g u = f \in L^2_\eta$ with respect to $u \in L^2_\eta$ (using variation of constants method), and shows that it has a unique solution, provided that

$$\int_{\mathbb{R}} f B_* dx = 0.$$

We obtain

$$\begin{aligned} u(x) &= \int_x^\infty \frac{\varepsilon^2 B_*(x)}{B_*^2(s)} F(s) ds \\ \text{with } F(s) &= \int_s^\infty f(\tau) B_*(\tau) d\tau \text{ for } s \geq 0 \\ &= -\int_{-\infty}^s f(\tau) B_*(\tau) d\tau \text{ for } s \leq 0. \end{aligned}$$

By Fubini's theorem we can write for $x \geq 0$

$$u(x) = \varepsilon^2 B_*(x) \int_x^\infty f(\tau) B_*(\tau) \left(\int_x^\tau \frac{ds}{B_*^2(s)} \right) d\tau$$

and, for $x \leq 0$

$$\begin{aligned} u(x) &= -\varepsilon^2 B_*(x) \int_{-\infty}^x f(\tau) B_*(\tau) \left(\int_x^0 \frac{ds}{B_*^2(s)} \right) d\tau \\ &\quad -\varepsilon^2 B_*(x) \int_x^0 f(\tau) B_*(\tau) \left(\int_\tau^0 \frac{ds}{B_*^2(s)} \right) d\tau. \end{aligned}$$

The asymptotic properties of $B_*(x)$ at $\pm\infty$ imply, for $x \geq 0$

$$|u(x)| e^{\eta x} \leq C\varepsilon^2 \int_x^\infty |f(\tau) e^{\eta\tau}| (\tau - x) e^{-\eta(\tau - x)} d\tau,$$

and for $x \leq 0$

$$\begin{aligned} |u(x)| e^{-\eta x} &\leq \frac{C\varepsilon^2}{2\varepsilon\delta} \int_{-\infty}^x |f(\tau) e^{-\eta\tau}| e^{-(\eta+\varepsilon\delta)(x-\tau)} d\tau \\ &\quad + \frac{C\varepsilon^2}{2\varepsilon\delta} \int_x^0 |f(\tau) e^{-\eta\tau}| e^{(\eta-\varepsilon\delta)(\tau - x)} d\tau. \end{aligned}$$

The bound

$$||u||_{L^2_\eta} \leq c_2 ||f||_{L^2_\eta}$$

follows from classical convolution results between functions in L^2 and functions in L^1, since

$$\begin{aligned}\int_{-\infty}^{0} e^{(\eta-\varepsilon\delta)\tau} d\tau &= \frac{1}{\eta-\varepsilon\delta}, \\ \int_{0}^{\infty} \tau e^{-\eta\tau} d\tau &= \frac{1}{\eta^2}.\end{aligned}$$

Then, we choose $\eta = \frac{1}{2}\varepsilon\delta$, so that the pseudo-inverse of $\mathcal{L}_g$ has a bounded inverse in L^2_η :

$$||\widetilde{\mathcal{L}_g}^{-1}|| \leq c_2,$$

where c_2 is independent of ε. Using the form of $\mathcal{L}_g$ we obtain easily

$$||u||_{\mathcal{D}_1} \leq c_3 ||f||_{L^2_\eta}$$

with c_3 independent of ε.

Remark 23 *The choice made for η is such that*

$$\eta < \varepsilon\delta, \quad \eta < \varepsilon\sqrt{2},$$

for values of δ for which Theorem 1 is valid. This means that as $x \to -\infty$ $(A_* - 1,\, B_*)$, and, as $x \to +\infty$ $(A_*, B_* - 1)$ tend exponentially to 0 faster than $e^{-\eta|x|}$.

In fact, $\mathcal{L}_g$ has the same properties as the operator $\mathcal{M}_i$ in the proof of Lemma 7.3 in [5], see also [7]: $\mathcal{L}_g$ is Fredholm with index -1, when acting in L^2_η, for η small enough. $\mathcal{L}_g$ has a trivial kernel, and its range is orthogonal to B_*, with the scalar product of $L^2(\mathbb{R})$.

7.3 Properties of $\mathcal{M}_g$

We saw that $\mathcal{M}_g$ is Fredholm with index 0. Furthermore the derivative of the heteroclinic solution belongs to its kernel:

$$\begin{aligned}\mathcal{M}_g \begin{pmatrix} A'_* \\ B'_* \end{pmatrix} &= \begin{pmatrix} -A_*^{(5)} + A'_* - (A_*^3)' - gB_*^2 A'_* - gA_*(B_*^2)' \\ \varepsilon^{-2} B'''_* + [B'_* - gA_*^2 B'_* - (B_*^3)' - gB_*(A_*^2)'] \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{84}$$

We show below (see section 7.3.1) that the kernel of $\mathcal{M}_g$, is one dimensional, then this implies that the range of $\mathcal{M}_g$ needs satisfy the orthogonality with only one element. In fact, because of selfadjointness in L^2, the range of $\mathcal{M}_g$ is orthogonal in $L^2(\mathbb{R})$ to

$$(A'_*, B'_*) \in L^2_\eta.$$

7.3.1 Dimension of $\ker \mathcal{M}_g$

Any element $\zeta(x)$ in the kernel lies, by definition, in L^2_η, hence $\zeta(x)$ tends towards 0 exponentially at $\pm\infty$. Near $x = \pm\infty$ the vector $\zeta(x) \sim \zeta_\pm(x)$ should verify

$$\mathcal{M}^\pm_\infty \zeta_\pm(x) = 0$$

where there are only 2 possible good dimensions (on each side). This gives a bound $= 2$ to the dimension of the kernel of $\mathcal{M}_g$. Let us show that *dimension 2 of* $\ker \mathcal{M}_g$ *implies non uniqueness of the heteroclinic*, which contradicts Theorem 1, hence the only possibility is that the dimension is one.

Let us choose arbitrarily g_0 and assume that the kernel of $\mathcal{M}_{g_0}$ consists in

$$\zeta_0(x), \zeta_1(x)$$

where $\zeta_0 = (A'_*, B'_*)|_{g_0}$ and let us decompose a solution of (1) in the neighborhood of g_0 as

$$U = \mathbf{T}_a(U_*^{(g_0)} + a_1\zeta_1 + Y), \tag{85}$$

where $\mathbf{T}_a$ represents the shift $x \mapsto x + a$, where a, $a_1 \in \mathbb{R}$, and Y belongs to a subspace transverse to $\ker \mathcal{M}_{g_0}$. Let us denote by $\mathbf{Q}_0$ and $\mathbf{P}_0 = \mathbb{I} - \mathbf{Q}_0$, projections, respectively on the range of $\mathcal{M}_{g_0}$, and on a complementary subspace ($\mathbf{Q}_0$ may be built in using the eigenvectors ζ_0^*, ζ_1^* of the adjoint operator $\mathcal{M}^*_{g_0}$). Let us denote by

$$\mathcal{F}(U, g) = 0$$

the system (1) where we look for an heteroclinic U for $g \neq g_0$. Then, we have

$$\begin{aligned} \mathcal{F}(U_*^{(g_0)}, g_0) &= 0, \\ D_U\mathcal{F}(U_*^{(g_0)}, g_0) &= \mathcal{M}_{g_0}, \end{aligned}$$

and since

$$\mathcal{M}_{g_0}\zeta_j = 0, \ j = 0, 1,$$

using the equivariance under operator $\mathbf{T}_a$, we obtain (denoting $\mathcal{F}_0 = \mathcal{F}(U_*^{(g_0)}, g_0)$ and $[..]^{(2)}$ the argument of a quadratic operator)

$$\begin{aligned} 0 &= \mathcal{M}_{g_0}Y + (g - g_0)\partial_g\mathcal{F}_0 + \frac{1}{2}D^2_{UU}\mathcal{F}_0[a_1\zeta_1 + Y]^{(2)} + \\ &\quad + \mathcal{O}(|g - g_0|[|g - g_0| + |a_1| + ||Y||] + ||Y||^3). \end{aligned}$$

The projection Q_0 of this equation allows to use the implicit function theorem to solve with respect to Y and then obtain a unique solution

$$Y = \mathcal{Y}(a_1, g),$$

with

$$\begin{aligned} \mathcal{Y} &= -(g - g_0)\widetilde{\mathcal{M}_{g_0}}^{-1}\mathbf{Q}_0\partial_g\mathcal{F}_0 - \frac{1}{2}\widetilde{\mathcal{M}_{g_0}}^{-1}\mathbf{Q}_0 D^2_{UU}\mathcal{F}_0[a_1\zeta_1]^{(2)} + \\ &\quad + \mathcal{O}(|g - g_0|(|g - g_0| + |a_1|) + |a_1|^3)). \end{aligned}$$

Then projecting on the complementary space, (only one equation since we work in the subspace orthogonal to ζ_0^*), we may observe (see the proof below) that $\mathbf{P}_0\partial_{g_0}\mathcal{F}_0 = 0$ and then obtain the "bifurcation" equation as

$$q(a_1, g - g_0) = \mathcal{O}((|g - g_0| + |a_1|)^3),$$

where the function q is quadratic in its arguments and

$$q|_{g=g_0}\zeta_1 = \frac{1}{2}\mathbf{P}_0 D^2_{UU}\mathcal{F}_0[a_1\zeta_1]^{(2)}.$$

This equation is just at main order a second degree equation in a_1 depending on $g - g_0$. Provided that the discriminant is not 0, the generic number of solutions is 2 or 0. If the discriminant is 0 for $g = g_0$, we just go a little farther in g, and obtain a non zero discriminant, since the discriminant cannot stay $= 0$. Indeed the heteroclinic is analytic in g and if the discriminant were identically 0, this would mean that we have a double root for any g, contradicting the transversality for all g, except a finite number, of the intersection of the two manifolds (unstable one of M_-, stable one of M_+). Hence, this is true except for a set of isolated values of g. We can then use the implicit function theorem for finding corresponding solutions for the system with higher order terms. In fact we already know a solution, corresponding to $U_*^{(g)} = U_*^{(g_0)} + (g - g_0)\partial_g U_*^{(g_0)} +$ $h.o.t.$ which corresponds to specific values for a_1 and Y, of order $\mathcal{O}(g - g_0)$. It then results that there is at least another solution of order $\mathcal{O}(g - g_0)$, so that there exists another heteroclinic, in the neighborhood of the known one (then in contradiction with Theorem 1).

Remark 24 *The above proof with only 1 dimension in the Kernel, provides* $Y = -(g - g_0)\widetilde{\mathcal{M}_{g_0}}^{-1}\partial_g\mathcal{F}_0 + \mathcal{O}((g - g_0)^2)$, *which gives a unique heteroclinic. Since we found only one heteroclinic, this shows that the kernel is of dimension 1.*

7.3.2 Proof of $\mathbf{P}_0\partial_g\mathcal{F}_0 = 0$

Lemma 25 *Any* (u, v) *in the kernel of* $\mathcal{M}_g$ *satisfies*

$$\int_{\mathbb{R}} A_*B_*(B_*u + A_*v)dx = 0,$$

and $\partial_g\mathcal{F}_0(U_*, g) = (A_*B_*^2, A_*^2B_*)$ *belongs to the range of* $\mathcal{M}_g$, *hence* $\mathbf{P}_0\partial_g\mathcal{F}_0 = 0$.

Proof.

Differentiating with respect to g the system (1) verified by the heteroclinic, we obtain

$$\mathcal{M}_g \begin{pmatrix} \partial_g A_* \\ \partial_g B_* \end{pmatrix} = \begin{pmatrix} A_*B_*^2 \\ A_*^2B_* \end{pmatrix} = \partial_g\mathcal{F}_0(U_*, g),$$

hence $(A_*B_*^2, A_*^2B_*)$ belongs to the range of $\mathcal{M}_g$. When $(u,v) \in \ker \mathcal{M}_g$, then $(u,v) \in \ker \mathcal{M}_g^*$ where $\mathcal{M}_g = \mathcal{M}_g^*$, when the adjoint is computed with the scalar product of L^2, hence

$$\int_{\mathbb{R}} A_*B_*(B_*u + A_*v)dx = 0. \tag{86}$$

Hence, the eigenvectors ζ_0^*, ζ_1^* of the adjoint $\mathcal{M}_g^*$ (the orthogonal of this 2-dimensional eigenspace is the range of $\mathcal{M}_g$), are orthogonal to $\partial_g \mathcal{F}_0 = (A_*B_*^2, A_*^2B_*)|_{g_0}$ in L^2.

A Appendix

A.1 Monodromy operator

Let us prove the estimate for the monodromy operators. We prove the following

Lemma 26 *For $\eta_0\delta \le A_* \le 1$, and $\alpha^{-1} \ge (1+\delta^2)^2$ and the following estimates hold*

$$\begin{aligned} ||\boldsymbol{S}_0(x,s)|| &\le e^{\sigma(x-s)}, \; -\infty < x < s \\ ||\boldsymbol{S}_1(x,s)|| &\le e^{-\sigma(x-s)}, \; -\infty < s < x \end{aligned}$$

with

$$\sigma = \frac{\alpha^{1/2}\delta^{1/2}}{2^{1/4}}.$$

We start with the system

$$\begin{aligned} x_1' &= \lambda_r x_1 + \lambda_i x_2 \\ x_2' &= -\lambda_i x_1 + \lambda_r x_2 \end{aligned}$$

where λ_r and λ_i are functions of x. When $\eta_0\delta \le A_* \le 1$, $\alpha^{-1} \ge (1+\delta^2)^2$, we have, for ε small enough (see (16))

$$\lambda_r \ge \frac{\alpha^{1/2}\delta^{1/2}}{2^{1/4}} = \sigma.$$

Now we have

$$(x_1^2 + x_2^2)' = 2\lambda_r(x_1^2 + x_2^2)$$

hence

$$(x_1^2 + x_2^2)(x) = e^{\int_s^x 2\lambda_r(\tau)d\tau}(x_1^2 + x_2^2)(s),$$

which, for $x < s$, leads to

$$\sqrt{(x_1^2 + x_2^2)(x)} \le e^{\sigma(x-s)}\sqrt{(x_1^2 + x_2^2)(s)}.$$

The proof is then done for the operator $\boldsymbol{S}_0$. The estimate for $\boldsymbol{S}_1$ is obtained in the same way.

Remark 27 *We have*

$$\boldsymbol{S}_0(x,s) = e^{\int_s^x \lambda_r(\tau)d\tau} \begin{pmatrix} \cos(\int_s^x \lambda_i(\tau)d\tau) & \sin(\int_s^x \lambda_i(\tau)d\tau) \\ -\sin(\int_s^x \lambda_i(\tau)d\tau) & \cos(\int_s^x \lambda_i(\tau)d\tau) \end{pmatrix}.$$

A.2 Computation of the system with new coordinates

Let us look for the system (10) writen in the new coordinates, first in forgetting quadratic and higher orders terms

$$\begin{aligned}
B_0 x_1' &= \frac{(\lambda_r^2+\lambda_i^2)}{4\lambda_r}\left(A_1+\frac{(1+\delta^2)B_0B_1}{\widetilde{A_*}}\right)+\frac{3\lambda_r^2-\lambda_i^2}{4\lambda_r(\lambda_r^2+\lambda_i^2)}A_3 \\
&\quad +\frac{A_2}{2}+\frac{(1+\delta^2)}{2\widetilde{A_*}}B_0^2\varepsilon^2\left(\delta^2(\widetilde{A_*}^2-B_0^2)+2(1+\delta^2)\widetilde{A_*}\widetilde{A_0}\right)-(\lambda_r^2-\lambda_i^2)\widetilde{A_0} \\
&= B_0f_1+\frac{(\lambda_r^2+\lambda_i^2)}{4\lambda_r}B_0(x_1+y_1)+\frac{A_2}{2}+\frac{1}{4\lambda_r}A_3,
\end{aligned}$$

$$\begin{aligned}
\lambda_i B_0 x_2' &= -\frac{(\lambda_r^2+\lambda_i^2)}{4}\left(A_1+\frac{(1+\delta^2)B_0B_1}{\widetilde{A_*}}\right)-\frac{\lambda_r^2-3\lambda_i^2}{4(\lambda_r^2-\alpha)}A_3 \\
&\quad -\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r}\left(A_2+\frac{(1+\delta^2)B_0^2\varepsilon^2}{\widetilde{A_*}}\delta^2(\widetilde{A_*}^2-B_0^2)\right) \\
&\quad -\frac{1}{4\lambda_r}(\lambda_r^2+\lambda_i^2)^2\widetilde{A_0} \\
&= \lambda_i B_0 f_2-\frac{(\lambda_r^2+\lambda_i^2)}{4}B_0(x_1+y_1)-\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r}A_2 \\
&\quad +\frac{1}{4}A_3-\frac{1}{4\lambda_r}(\lambda_r^2+\lambda_i^2)^2\widetilde{A_0},
\end{aligned}$$

with

$$f_1=\frac{\varepsilon^2\delta^2B_0(1+\delta^2)(\widetilde{A_*}^2-B_0^2)}{2\widetilde{A_*}},$$

$$f_2=-\frac{\varepsilon^2\delta^2B_0(1+\delta^2)(\lambda_r^2-\lambda_i^2)(\widetilde{A_*}^2-B_*^2)}{4\lambda_r\lambda_i\widetilde{A_*}},$$

hence

$$\begin{aligned}
x_1' &= f_1+\lambda_r x_1+\lambda_i x_2, \\
x_2' &= f_2-\lambda_i x_1+\lambda_r x_2,
\end{aligned} \tag{87}$$

and in the same way

$$\begin{aligned}
y_1' &= f_1-\lambda_r y_1+\lambda_i y_2, \\
y_2' &= -f_2-\lambda_i y_1-\lambda_r y_2, \\
z_1' &= \frac{2\varepsilon^2\delta^2(\widetilde{A_*}^2-B_0^2)}{\widetilde{A_*}}=\frac{2f_1}{(1+\delta^2)B_0}, \\
B_*' &= -\frac{(\lambda_r^2-\lambda_i^2)}{(1+\delta^2)B_0\widetilde{A_*}}A_3+\widetilde{A_*}B_0z_1.
\end{aligned} \tag{88}$$

We notice that the following estimates hold

$$
\begin{aligned}
|f_1| &\leq \frac{B_0\varepsilon^2\delta^2}{\widetilde{A_*}} \leq \frac{B_0\varepsilon^2\delta}{\alpha}, \\
|f_2| &\leq \frac{B_0\varepsilon^4\delta^2}{\widetilde{A_*}^2} \leq B_0\varepsilon^2\delta.
\end{aligned}
\tag{89}
$$

A.2.1 Full system in new coordinates

We intend to derive the full system (1) with coordinates $(x_1, x_2, y_1, y_2, B_0, z_1)$. Differentiating (19) and (20) we see that we respectively need to add to the previous expressions (87) for x_1' and x_2'

$$
\frac{1}{B_0}\left\{\left(\frac{\widetilde{A_*}}{2\sqrt{2}\lambda_r}\right)'\widetilde{A_0} + \left(\frac{(3\lambda_r^2-\lambda_i^2)}{4\sqrt{2}\lambda_r\widetilde{A_*}}\right)' A_2 + \varepsilon^2\left(\frac{(1+\delta^2)^2B_0^2}{2\widetilde{A_*}^2}\right)' A_3 + \left(\frac{(1+\delta^2)B_0}{2\widetilde{A_*}}\right)' B_1\right\}
$$
$$
-\varepsilon^2\frac{(1+\delta^2)^2B_0}{2\widetilde{A_*}^2}[3\widetilde{A_*}\widetilde{A_0}^2 + \widetilde{A_0}^3] + \frac{B_0\varepsilon^2(1+\delta^2)^2\widetilde{A_0}^2}{2\widetilde{A_*}} - \frac{B_1}{B_0}x_1.
$$

and

$$
\frac{1}{B_0}\left\{-\left(\frac{\widetilde{A_*}}{2\sqrt{2}\lambda_i}\right)'\widetilde{A_0} - \left(\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r\lambda_i}\right)' A_1 - \left(\frac{(\lambda_r^2-3\lambda_i^2)}{4\sqrt{2}\lambda_i\widetilde{A_*}}\right)' A_2 + \left(\frac{\varepsilon^2(1+\delta^2)^3B_0^3}{4\lambda_r\lambda_i\widetilde{A_*}}\right)' B_1\right\}
$$
$$
+\frac{1}{B_0}\left(\frac{1}{4\lambda_r\lambda_i}\left[1-\frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right]\right)' A_3 - \frac{1}{4\lambda_r\lambda_i B_0}\left(1-\frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right)[3\widetilde{A_*}\widetilde{A_0}^2 + \widetilde{A_0}^3]
$$
$$
-\frac{\varepsilon^4B_0^3(1+\delta^2)^4}{4\lambda_r\lambda_i\widetilde{A_*}}\widetilde{A_0}^2 - \frac{B_1}{B_0}x_2.
$$

We then arrive to the system (24,25,26,27).

We observe that (using (13))

$$
\widetilde{A_*}' = -\frac{(1+\delta^2)B_0}{\widetilde{A_*}}B_1
$$

$$
(\lambda_r^2)' = -\frac{(1+\delta^2)B_0B_1}{\sqrt{2}\widetilde{A_*}}(1-\varepsilon^2\sqrt{2}(1+\delta^2)\widetilde{A_*})
$$

$$
(\lambda_i^2)' = -\frac{(1+\delta^2)B_0B_1}{\sqrt{2}\widetilde{A_*}}(1+\varepsilon^2\sqrt{2}(1+\delta^2)\widetilde{A_*})
$$

$$
\left(\frac{\widetilde{A_*}}{2\sqrt{2}\lambda_r}\right)' = a_1B_0B_1, \quad |a_1| \leq \frac{c}{\widetilde{A_*}^{3/2}}, \tag{90}
$$

$$
\left(\frac{\widetilde{A_*}}{2\sqrt{2}\lambda_i}\right)' = a_2B_0B_1, \quad |a_2| \leq \frac{c}{\widetilde{A_*}^{3/2}}, \tag{91}
$$

$$\left(-\frac{(\lambda_r^2-\lambda_i^2)}{4\lambda_r\lambda_i}\right)' = b_2 B_0 B_1, \quad |b_2| \le \frac{c\varepsilon^2}{\widetilde{A_*}^3}, \tag{92}$$

$$\left(\frac{(3\lambda_r^2-\lambda_i^2)}{4\sqrt{2}\lambda_r\widetilde{A_*}}\right)' = c_1 B_0 B_1, \quad |c_1| \le \frac{c}{\widetilde{A_*}^{5/2}}, \tag{93}$$

$$\left(-\frac{(\lambda_r^2-3\lambda_i^2)}{4\sqrt{2}\lambda_i\widetilde{A_*}}\right)' = c_2 B_0 B_1, \quad |c_2| \le \frac{c}{\widetilde{A_*}^{5/2}}, \tag{94}$$

$$\varepsilon^2\left(\frac{(1+\delta^2)^2 B_0^2}{2\widetilde{A_*}^2}\right)' = d_1 B_0 B_1, \quad |d_1| \le \frac{c}{\widetilde{A_*}^3}, \tag{95}$$

$$\left(\frac{1}{4\lambda_r\lambda_i}\left[1-\frac{(\lambda_r^2-\lambda_i^2)^2}{\widetilde{A_*}^2}\right]\right)' = d_2 B_0 B_1, \quad |d_2| \le \frac{c}{\widetilde{A_*}^3}, \tag{96}$$

$$\left(\frac{(1+\delta^2)B_0}{2\widetilde{A_*}}\right)' = e_1 B_1, \quad |e_1| \le \frac{c}{\widetilde{A_*}^3} \tag{97}$$

$$\left(\frac{\varepsilon^2(1+\delta^2)^3 B_0^2}{4\lambda_r\lambda_i\widetilde{A_*}}\right)' = e_2 B_0 B_1, \quad |e_2| \le \frac{c}{\widetilde{A_*}^3}, \tag{98}$$

with c independent of ε and $\delta \in [\delta_0, \delta_1]$.

A.3 Elimination of z_1

A.3.1 System after scaling

After the scaling (30) our system (24,25,26,27) takes the form

$$\begin{aligned}
\overline{X}' &= \mathbf{L}_0\overline{X} + B_0\overline{F_0} + \mathbf{B}_{01}(\overline{X},\overline{Y}) + \overline{z_1}\boldsymbol{M}_{01}(\overline{X},\overline{Y}) \\
&\quad + \overline{z_1}^2 B_0\boldsymbol{n}_0 + \boldsymbol{C}_{01}(\overline{X},\overline{Y}), \\
\overline{Y}' &= \mathbf{L}_1\overline{Y} + B_0\overline{F_1} + \mathbf{B}_{11}(\overline{X},\overline{Y}) + \overline{z_1}\boldsymbol{M}_{11}(\overline{X},\overline{Y}) \\
&\quad + \overline{z_1}^2 B_0\boldsymbol{n}_1 + \boldsymbol{C}_{11}(\overline{X},\overline{Y}),
\end{aligned}$$

where $\overline{F_0}, \overline{F_1}, \boldsymbol{n}_0, \boldsymbol{n}_1$ are two-dimensional vectors $\mathbf{M}_{01}, \mathbf{M}_{11}$ are linear operators in $(\overline{X},\overline{Y})$, $\mathbf{B}_{01}, \mathbf{B}_{11}$ are quadratic and $\mathbf{C}_{01}, \mathbf{C}_{11}$ are cubic in $(\overline{X},\overline{Y})$, all functions of B_0. More precisely we have

$$\overline{F_0} = \begin{pmatrix} \frac{f_1}{\alpha\delta B_0} \\ \frac{f_2}{\alpha\delta B_0} \end{pmatrix}, \ \overline{F_1} = \begin{pmatrix} \frac{f_1}{\alpha\delta B_0} \\ -\frac{f_2}{\alpha\delta B_0} \end{pmatrix}, \ |\overline{F_j}| \le c\frac{\varepsilon^2}{\alpha^2},$$

$$\boldsymbol{n}_0 = \frac{\varepsilon^2\delta}{\alpha}\begin{pmatrix} e_1\widetilde{A_*}^2 \\ e_2\widetilde{A_*}^2 B_0 - b_2(1+\delta^2)\widetilde{A_*}B_0^2 \end{pmatrix},$$

$$\boldsymbol{M}_{01}(\overline{X},\overline{Y}) = \varepsilon\delta\begin{pmatrix} m_{01}(\overline{X},\overline{Y}) \\ m_{02}(\overline{X},\overline{Y}) \end{pmatrix},$$

$$m_{01}(\overline{X},\overline{Y}) = \widetilde{A_*}B_0\left(a_1\overline{\widetilde{A_0}} + c_1\overline{A_2} + (d_1 - 2e_1(1+\delta^2)\varepsilon^2\frac{B_0}{\widetilde{A_*}})\overline{A_3} - \frac{\overline{x_1}}{B_0}\right),$$

$$m_{02}(\overline{X},\overline{Y}) = \widetilde{A_*}B_0\left(-a_2\overline{\widetilde{A_0}} + c_2\overline{A_2} + (d_2 - 2e_2(1+\delta^2)\varepsilon^2\frac{B_0^2}{A_*})\overline{A_3} - \frac{\overline{x_2}}{B_0}\right) + \widetilde{A_*}B_0^2 b_2(\overline{x_1}+\overline{y_1}) + (1+\delta^2)^2\varepsilon^2\frac{B_0^3}{\widetilde{A_*}}b_2\overline{A_3},$$

$$\mathbf{B}_{01}(\overline{X},\overline{Y}) = \alpha\delta\begin{pmatrix} b_{01}(\overline{X},\overline{Y}) \\ b_{02}(\overline{X},\overline{Y}) \end{pmatrix},$$

$$b_{01}(\overline{X},\overline{Y}) = -\varepsilon^2\frac{(1+\delta^2)(2-\delta^2)B_0}{2\widetilde{A_*}}\overline{\widetilde{A_0}}^2 + e_1\frac{\varepsilon^4(1+\delta^2)^2B_0}{\widetilde{A_*}^2}\overline{A_3}^2 - \varepsilon^2\frac{(1+\delta^2)B_0}{\widetilde{A_*}}\overline{A_3}[a_1\overline{\widetilde{A_0}} + c_1\overline{A_2} + d_1\overline{A_3} - \frac{\overline{x_1}}{B_0}],$$

$$b_{02}(\overline{X},\overline{Y}) = -\frac{1}{4\lambda_r\lambda_i\widetilde{A_*}B_0}\left(3\widetilde{A_*}^2 - 2\varepsilon^4B_0^4(1+\delta^2)^4\right)\overline{\widetilde{A_0}}^2 + e_2\frac{\varepsilon^4(1+\delta^2)B_0^2}{\widetilde{A_*}^2}\overline{A_3}^2 - \varepsilon^2\frac{(1+\delta^2)B_0}{\widetilde{A_*}}\overline{A_3}[-a_2\overline{\widetilde{A_0}} + b_2B_0(\overline{x_1}+\overline{y_1}) + c_2\overline{A_2} + d_2\overline{A_3} - \frac{\overline{x_2}}{B_0}],$$

$$\boldsymbol{C}_{01}(\overline{X},\overline{Y}) = \alpha^2\delta^2\overline{\widetilde{A_0}}^3\begin{pmatrix} -\varepsilon^2\frac{(1+\delta^2)B_0}{2\widetilde{A_*}^2} \\ -\frac{1}{4\lambda_r\lambda_iB_0}\left(1 - \frac{\varepsilon^4B_0^4(1+\delta^2)^4}{\widetilde{A_*}^2}\right) \end{pmatrix}.$$

$\boldsymbol{n}_1$, $\boldsymbol{M}_{11}$, $\boldsymbol{B}_{11}$, $\boldsymbol{C}_{11}$ are deduced respectively from $\boldsymbol{n}_0$, $\boldsymbol{M}_{01}, \boldsymbol{B}_{01}, \boldsymbol{C}_{01}$ in changing $(a_1, c_1, b_2, d_2, e_2)$ into their opposite.

A.3.2 System after elimination of z_1

Let us replace $\overline{z_1}$ by $\overline{z_{10}} + \mathcal{Z}(\overline{X},\overline{Y},B_0)$ in the differential system for $(\overline{X},\overline{Y})$. The new system becomes (notice that B_0 is in factor of the "constant" terms)

$$\overline{X}' = \mathbf{L}_0\overline{X} + B_0\mathcal{F}_0 + \mathcal{L}_{01}(\overline{X},\overline{Y}) + \mathcal{B}_{01}(\overline{X},\overline{Y}),$$
$$\overline{Y}' = \mathbf{L}_1\overline{Y} + B_0\mathcal{F}_1 + \mathcal{L}_{11}(\overline{X},\overline{Y}) + \mathcal{B}_{11}(\overline{X},\overline{Y}),$$

which is (33) with

$$\mathcal{F}_0 = \overline{F_0} + \overline{z_{10}}^2\boldsymbol{n}_0,$$
$$\mathcal{L}_{01}(\overline{X},\overline{Y}) = \overline{z_{10}}\boldsymbol{M}_{01}(\overline{X},\overline{Y}),$$

$$\mathcal{B}_{01}(\overline{X},\overline{Y}) = \mathbf{B}_{01}(\overline{X},\overline{Y}) + \mathcal{Z}(\overline{X},\overline{Y})\boldsymbol{M}_{01}(\overline{X},\overline{Y}) + \boldsymbol{C}_{01}(\overline{X},\overline{Y}) + 2\overline{z_{10}}\mathcal{Z}(\overline{X},\overline{Y})B_0\boldsymbol{n}_0 + \mathcal{Z}(\overline{X},\overline{Y})^2B_0\boldsymbol{n}_0.$$

In using estimates (23), (90) to (98), it is straightforward to check that

$$|\mathcal{F}_0| + |\mathcal{F}_1| \leq \frac{c\varepsilon^2}{\alpha^4},$$

$$|\boldsymbol{M}_{01}(\overline{X},\overline{Y})| \leq c\frac{\varepsilon\delta}{\widetilde{A_*}}(|\overline{X}| + |\overline{Y}|),$$

hence

$$|\mathcal{L}_{01}(\overline{X},\overline{Y})| + |\mathcal{L}_{11}(\overline{X},\overline{Y})| \leq c\frac{\varepsilon}{\alpha^2}(|\overline{X}| + |\overline{Y}|).$$

For higher order terms we have

$$\begin{aligned}
|\mathbf{B}_{01}(\overline{X},\overline{Y})| &\leq c\alpha(|\overline{X}| + |\overline{Y}|)^2, \\
|2\overline{z_{10}}\mathcal{Z}(\overline{X},\overline{Y})\boldsymbol{n}_0| &\leq c\frac{\varepsilon^2}{\alpha^2}(|\overline{X}| + |\overline{Y}|)^2, \\
|\mathcal{Z}(\overline{X},\overline{Y})\boldsymbol{M}_{01}(\overline{X},\overline{Y})| &\leq c\varepsilon(|\overline{X}| + |\overline{Y}|)^3, \\
|\mathcal{Z}(\overline{X},\overline{Y})^2\boldsymbol{n}_0| &\leq c\varepsilon^2(|\overline{X}| + |\overline{Y}|)^4, \\
|\boldsymbol{C}_{01}(\overline{X},\overline{Y})| &\leq c\alpha(|\overline{X}| + |\overline{Y}|)^3,
\end{aligned}$$

hence, choosing α small enough and for

$$|\overline{X}| + |\overline{Y}| \leq \rho, \tag{99}$$

we obtain

$$|\mathcal{B}_{01}(\overline{X},\overline{Y})| + |\mathcal{B}_{11}(\overline{X},\overline{Y})| \leq c(\alpha + \frac{\varepsilon^2}{\alpha^2})(|\overline{X}| + |\overline{Y}|)^2.$$

References

[1] B.Buffoni, M.Haragus, G.Iooss. Heteroclinic orbits for a system of amplitude equations for orthogonal domain walls. J.Diff.Equ,2023. https://doi.org/10.1016/j.jde.2023.01.026.

[2] J.Dieudonné. Eléments d'Analyse. vol1. Gauthier-Villars, Paris 1969.

[3] N.Fenichel. Geometric singular perturbation theory for ordinary differential equations. J. Diff. Equ. 31 (1979), 1, 53-98.

[4] J.Hale. Ordinary differential equations. Wiley, New York, 1969.

[5] M.Haragus, G.Iooss. Bifurcation of symmetric domain walls for the Bénard-Rayleigh convection problem. Arch. Rat. Mech. Anal. 239(2), 733-781, 2020.

[6] M.Haragus, G.Iooss. Local Bifurcations, Center Manifolds, and Normal forms in Infinite Dimensional Dynamical Systems. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, 2011.

[7] M.Haragus, A.Scheel. Grain boundaries in the Swift-Hohenberg equation. Europ. J. Appl. Math. 23 (2012), 737-759.

[8] G.Iooss. Existence of orthogonal domain walls in Bénard-Rayleigh convection. Submitted to JMFM 2024

[9] G.Iooss, A.Mielke, Y.Demay. Theory of steady Ginzburg-Landau equation in hydrodynamic stability problems. Eur. J. Mech. B/fluids 8, 1989, 229-268.

[10] T.Kapitula, K.Promislow. Spectral and Dynamical Stability of Nonlinear Waves. Springer series, Appl. Math. Sci. 185. 2013.

[11] T.Kato. Perturbation theory for linear operators. Classics in Maths. Springer-Verlag, Berlin, 1995 (1st ed. in 1966).

[12] K.Kirchgässner. Wave-solutions of reversible systems and applications. J. Diff. Equ. 45, 1982, 113-127.

[13] P.Manneville, Y.Pomeau. A grain boundary in cellular structures near the onset of convection. Phil. Mag. A, 1983, 48, 4, 607-621.