Gravity travelling waves for two superposed fluid layers, one being of infinite depth: a new type of bifurcation

By Gérard Iooss[1,2], Eric Lombardi[2], Shu Ming Sun[3]

[1] *IUF*, [2] *INLN, UMR CNRS-UNSA 6618, 1361 route des Lucioles, 06560 Valbonne, France*
[3] *Math. Dept., Virginia Tech, Blacksburg VA 24061, USA*

In this paper, we study the travelling gravity waves in a system of two layers of perfect fluids, the bottom one being infinitely deep, the upper one having a finite thickness h. We assume that the flow is potential, and the dimensionless parameters are the ratio between densities $\rho = \rho_2/\rho_1$ and $\lambda = gh/c^2$. We study special values of the parameters such that $\lambda(1-\rho)$ is near 1^-, where a bifurcation of a new type occurs. We formulate the problem as a spatial reversible dynamical system, where $U = 0$ corresponds to a uniform state (velocity c in a moving reference frame), and we consider the linearized operator around 0. We show that its spectrum contains the entire real axis (essential spectrum), with in addition a double eigenvalue in 0, a pair of simple imaginary eigenvalues $\pm i\lambda$ at a distance $O(1)$ from 0, and for $\lambda(1-\rho)$ above 1, another pair of simple imaginary eigenvalues tending towards 0 as $\lambda(1-\rho) \to 1^+$. When $\lambda(1-\rho) \leq 1$ this pair disappears into the essential spectrum. The rest of the spectrum lies at a distance at least $O(1)$ from the imaginary axis. We show in this paper that for $\lambda(1-\rho)$ close to 1^-, there is a family of periodic solutions like in the Lyapunov-Devaney theorem (despite the resonance due to the point 0 in the spectrum). Moreover, showing that the full system can be seen as a perturbation of the Benjamin-Ono equation, coupled with a nonlinear oscillation, we also prove the existence of a family of homoclinic connections to these periodic orbits, provided that these ones are not too small.

Keywords: nonlinear water waves, travelling waves, bifurcation theory, infinite dimensional reversible dynamical systems, normal forms with essential spectrum, homoclinic orbits, solitary waves with polynomial decay

This paper is dedicated to Klaus Kirchgässner on the occasion of his 70th birthday.

Contents

1. Position of the problem

Let us consider two layers of perfect fluids (densities ρ_1 (bottom layer), ρ_2 (upper layer)), assuming that there is no surface tension, neither at the free surface nor at the interface, and assuming that the flow is potential. The thickness at rest of the upper layer is h while the bottom one has infinite thickness (see figure 1). We are interested in travelling waves of horizontal velocity c. The dimensionless parameters are $\rho = \rho_2/\rho_1 < 1$, and $\lambda = \frac{gh}{c^2}$ (inverse of (Froude number)^2).

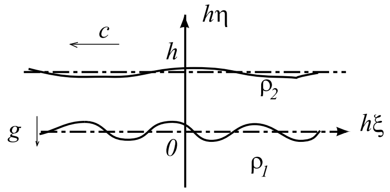

Figure 1. Two layers, the bottom one being of infinite depth

The existence of a family of periodic travelling waves, for generic values of these parameters is known (Iooss 1999). Below, we study special values of the parameters such that $\lambda(1-\rho)$ is near 1, where a singularity of a new type occurs. Indeed, we formulate the problem as a spatial *reversible dynamical system*

$$\frac{dU}{dx} = F(\rho, \lambda; U), \quad U(x) \in \mathbb{D}, \tag{1.1}$$

where $\mathbb{D}$ is an appropriate infinite dimensional Banach space including the boundary conditions and suitable decay in the η coordinate (see section 2), and where $U = 0$ corresponds to a uniform state (velocity c in a moving reference frame). The galilean invariance of the physical problem induces a mirror symmetry of the system in the moving frame. This symmetry leads to the reversibility of system (1.1), i.e. to the existence of a linear symmetry S which anticommutes with the vector field $F(\rho, \lambda; \cdot)$.

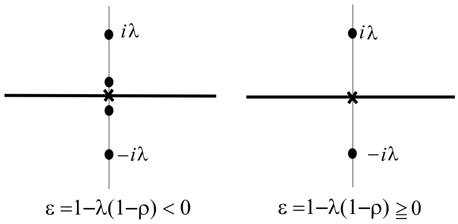

Figure 2. Spectrum of L_ε

Considering the linearized operator around 0

$$L_\varepsilon = D_U F(\rho, \lambda; 0)$$

with $\varepsilon = 1 - \lambda(1-\rho)$, we show that its spectrum contains the entire real line (essential spectrum), with in addition a double eigenvalue in 0, a pair of simple imaginary eigenvalues $\pm i\lambda$ (where λ is defined above) at a distance $O(1)$ from 0 when ε is near 0, and for ε below 0, another pair of simple imaginary eigenvalues tending towards 0 as $\varepsilon \to 0^-$. When $\varepsilon \geq 0$, this pair completely disappears into the essential spectrum! (see figure 2). The rest of the spectrum consists of a discrete set of eigenvalues situated at a distance at least $O(1)$ from the imaginary axis.

For one or several layers of finite depth, the study of travelling waves may as well be formulated as an infinite dimensional reversible dynamical system (Kirchgässner 1988; Dias & Iooss 2001). In these cases, the existence of travelling waves can be obtained via a center manifold reduction (see for example (Mielke 1988)) which leads to a finite dimensional reversible O.D.E. studied near a resonant fixed point, i.e. a fixed point at which all the eigenvalues of the differential lie on the imaginary axis, for a critical value of the set of parameters. For instance, for one layer of finite depth in presence of gravity and surface tension, the existence of true solitary waves have been obtained

i) for a Froude number close to 1, and a Bond number larger than 1/3 (Amick & Kirchgässner 1989). In this case the reduced O.D.E. is two-dimensional and admits a 0^2 resonant fixed point (see figure 3).

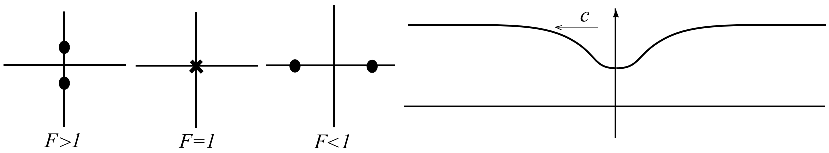

Figure 3. (left) 0^2 resonance for a Bond number b >1/3, and a Froude number F close to 1, and (right) shape of the solitary waves for F < 1.

Remark: In all the diagrams of the paper, concerning the spectrum of a linear operator, a point means a simple eigenvalue, and a cross means a double eigenvalue.

ii) True solitary waves have also been obtained for a Bond number b less than 1/3 and a Froude number F close to a critical value F $= C(b)$ (see for instance (Iooss & Kirchgässner 1990; Iooss & Pérouème 1993)), near which the reduced O.D.E. is 4-dimensional and admits a $(i\omega)^2$ resonance (also called 1:1 resonance) (see figure 4).

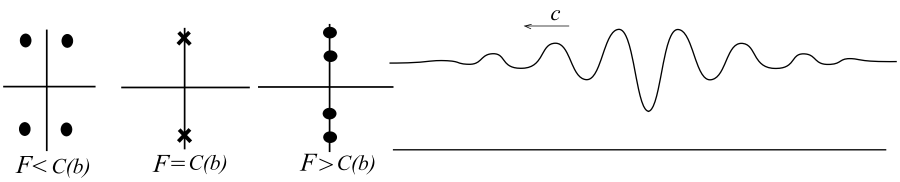

Figure 4. (left) $(i\omega)^2$ resonance for F near $C(b)$, and (right) shape of one of the two types of solitary waves for $b < 1/3$, F $< C(b)$.

iii) For a Froude number close to 1, and a Bond number less than 1/3, the reduced O.D.E. is 4-dimensional and admits a $0^2 i\omega$ resonant fixed point. In this case, for F > 1 and $b < 1/3$ periodic travelling waves and generalized solitary waves asymptotic at infinity to each of these periodic waves, have been obtained provided that the amplitude of the ripples is larger than an exponentially small quantity (as function of $F-1$) ((Sun & Shen 1993; Lombardi 1997)), (see figure 5). The non existence of true solitary waves has also been proved by Sun (1999) for a Froude number F close to 1^+, and a Bond number b near $1/3^-$.

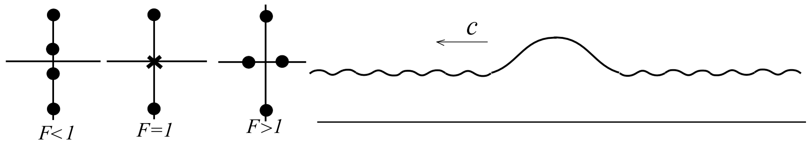

Figure 5. (left) $0^2 i\omega$ resonance, and (right) shape of the generalized solitary waves for $b<1/3, F>1$.

In these three cases the solitary (resp. the generalized solitary waves) are obtained as homoclinic connection to 0 (resp. to a periodic orbit) for the dynamical system. In all cases, the homoclinic connections have an exponential decay rate at infinity, given by the spectral gap of the linearized operator near the imaginary axis.

For the cases with an infinitely deep layer, the situation is more intricate, and in particular no center manifold reduction can be performed because the linearized operators have no spectral gap near the imaginary axis: the entire real line lies in the essential spectrum of the linearized operator.

A first example is the problem of the existence of solitary waves for one layer of infinite depth, in presence of gravity and surface tension. In this case, the problem may be formulated as a spatial reversible dynamical system in infinite dimensions, such that the linearized operator at the origin (which corresponds to the rest state), has an essential spectrum composed with the entire real line, with in addition 4 eigenvalues in the spectrum making a $(i\omega)^2$ resonance for a critical value of the parameter μ ($\sim b/F^2$ for a very large depth h) (see figure 6).

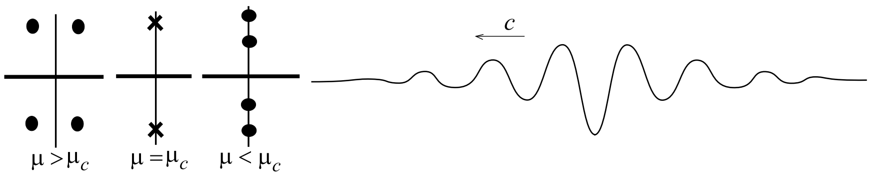

Figure 6. (left) $\mathbb{R}(i\omega)^2$ resonance, and (right) shape of one type of solitary waves for $\mu > \mu_c$.

For $\mu > \mu_c$, the existence of true solitary waves has been obtained by Iooss & Kirrmann (1996). These homoclinics have their principal part at finite distance, given by the 4-dimensional critical part of the vector field corresponding to the $(i\omega)^2$ resonance, while at infinity they have a polynomial decay induced by the essential spectrum. This is a major difference with the finite depth case, for which the decay is exponential (given by the spectral gap).

A second example is a system of two superposed layers, the bottom one being infinitely deep, and the upper one being bounded by a rigid horizontal top, with no interfacial tension (see figure 7).

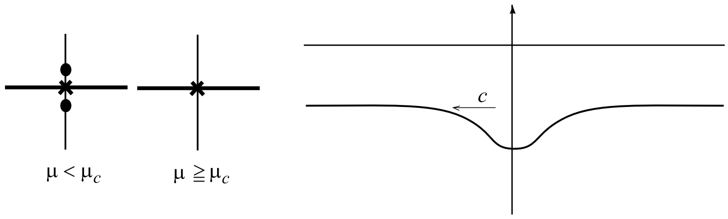

Figure 7. (left) $\mathbb{R}00^2$ resonance, and (right) shape of the internal solitary wave in the two-layer system for $\mu > \mu_c$ (bottom layer infinitely deep).

This problem was first studied on a model equation derived from the Euler equations with a long-wave approximation, by Benjamin (1967), Davis & Acrivos (1967), and Ono (1975). The now called Benjamin-Ono equation is non local and reads

$$\mathcal{H}(u') + u - u^2 = 0, \tag{1.2}$$

where $\mathcal{H}$ is the Hilbert transform, and u is a scalar function. This equation admits an homoclinic connection to 0, given explicitly by

$$u_h(\tau) = \frac{2}{1+\tau^2}. \tag{1.3}$$

All the other solutions of equation (1.2) have been described by Amick & Toland (1991). For the full Euler equations, the existence of the solitary waves with polynomial decay at infinity, has been obtained in this case independently by Amick (1994) and Sun (1997). More precisely, they both proved that, for $\mu > \mu_c$ and close to μ_c (we can just play on the velocity c of the wave), the form of the interface for the solitary wave satisfies

$$Z(x) = \mu u_h(\mu x) + \mu^2 u_1(\mu x)$$

where

$$\sup_{\tau\in\mathbb{R}}(1+|\tau|)\left|\frac{d^j u_1}{d\tau^j}(\tau)\right| \leq K_j, \quad j = 0, 1, 2, ..$$

Therefore, the solitary wave solution (1.3) of the Benjamin-Ono equation (1.2) gives the first order approximation of a solitary wave solution of the full Euler equations. Neither the approach of Amick, nor the one of Sun was based on a dynamical system approach. However, we observe that the problem may be formulated as a reversible dynamical system, for which the differential at the origin (which corresponds to the rest state) admits the entire real line as essential spectrum, a zero eigenvalue, and a pair of simple imaginary eigenvalues for $\mu < \mu_c$ tending towards 0 as $\mu \to \mu_c^-$. When $\mu \geq \mu_c$ this pair completely disappears in the essential spectrum (see figure 7).

A third example of problem involving an infinitely deep layer, is the one we consider in this paper, which was described at the beginning of the introduction: two layers, the bottom one infinitely deep, no surface tension, no interfacial tension. As already mentioned, this problem takes the form (1.1)

$$\frac{dU}{dx} = F(\rho, \lambda; U), \quad U(x) \in \mathbb{D},$$

and the spectrum of the linearized vector field

$$L_\varepsilon = D_U F(\rho, \lambda; 0) \qquad \text{with } \varepsilon = 1 - \lambda(1-\rho)$$

has the behavior described at figure 2 ($\mathbb{R}00^2(i\lambda)$ resonance here).

From now on, we consider ρ as fixed and we use ϵ as our bifurcation parameter (instead of λ). So in all what follows, $\lambda := \lambda_\varepsilon$ is seen as a function of ε.

Moreover we denote by ξ_0 and ξ_1 the two eigenvectors belonging to the 0 eigenvalue

$$L_\varepsilon \xi_0 = 0, \qquad L_\varepsilon \xi_1 = 0,$$

which come from the existence of a two parameters family of trivial solutions corresponding to a flow where each layer moves freely horizontally with different velocities. We also denote by ζ_ε and $\overline{\zeta}_\varepsilon$ the two eigenvectors belonging to the simple eigenvalues $\pm\lambda_\varepsilon$, i.e.

$$L_\varepsilon \zeta_\varepsilon = i\lambda_\varepsilon \zeta_\varepsilon, \qquad L_\varepsilon \overline{\zeta}_\varepsilon = -i\lambda_\varepsilon \overline{\zeta}_\varepsilon.$$

We observe on figure 2 that the behavior of the spectrum of L_ε is the same as the one of the previous example, with in addition an extra pair of simple eigenvalues lying on the imaginary axis (not close to 0). These additional eigenvalues $\pm i\lambda$ lead to a competition between the oscillatory dynamics they induce, and the Benjamin-Ono type of dynamics induced by the essential spectrum with the 0 eigenvalue.

In this paper, we first show that this extra pair of eigenvalues $\pm i\lambda$ induces the existence of a family of periodic solutions (of arbitrary small size), like in the Lyapunov-Devaney theorem (despite the resonance due to the point 0 in the spectrum) : for each $\varepsilon > 0$, the linearized problem possesses a four parameter family of periodic solutions

$$u_0\xi_0 + v_0\xi_1 + A_0 e^{is}\zeta_\varepsilon + \overline{A}_0 e^{-is}\overline{\zeta}_\varepsilon, \qquad (u_0, v_0) \in \mathbb{R}^2, \ A_0 \in \mathbb{C}$$

which are circles of radii $|A_0|$ centered at $u_0\xi_0 + v_0\xi_1$. For all these circles, the spatial frequency of the corresponding periodic solution is λ_ε. We prove in this paper that periodic solution of the *nonlinear* problem are obtained by an analytic perturbation of the graph $\widehat{p}^{(1)}$ and of the frequency $\gamma^{(1)}$ of the periodic solution of the *linearized* problem. This can be summed up in the following

Theorem A. *For any $M > 0$, there exists $\varepsilon_0 > 0$ such that for any $(u_0, v_0, A_0, \varepsilon) \in \mathbb{R}^2 \times \mathbb{C} \times \mathbb{R}$ satisfying*

$$|u_0| + |v_0| + |A_0| \leq M, \quad 0 < \varepsilon < \varepsilon_0,$$

equ. (1.1) admits a family of periodic solutions $U = p_{A_0,u_0,v_0,\varepsilon}$ *bifurcating from 0, with*

$$p_{A_0,u_0,v_0,\varepsilon}(x) = \widehat{p}_{A_0,u_0,v_0,\varepsilon}(s)$$

and

$$\begin{aligned}\widehat{p}_{A_0,u_0,v_0,\varepsilon}(s) &= \varepsilon\left(u_0\xi_0 + v_0\xi_1 + A_0 e^{is}\zeta_\varepsilon + \overline{A}_0 e^{-is}\overline{\zeta}_\varepsilon\right) + \widehat{p}^{(1)}_{A_0,u_0,v_0,\varepsilon}(s)\\ s &= \left(\lambda + \gamma^{(1)}\right)x,\end{aligned}$$

where $\xi_0, \xi_1, \zeta_\varepsilon, \overline{\zeta}_\varepsilon$ *are the eigenvectors defined above and where the two perturbation terms* $\widehat{p}^{(1)}_{A_0,u_0,v_0,\varepsilon}$ *and* $\gamma^{(1)}$ *possess the following converging power series in* $\varepsilon, u_0, v_0, A, \overline{A}$

$$\begin{aligned}\widehat{p}^{(1)}_{A_0,u_0,v_0,\varepsilon} &= \sum_{\substack{p+q\geq 1\\ 2\leq n+m+p+q\leq r+1}} \varepsilon^{r+1} u_0^n v_0^m A_0^p \overline{A}_0^q e^{i(p-q)s} Y_{nmpqr}\\ \gamma^{(1)} &= \sum_{1\leq n+m+2p\leq r} \gamma_{nmpr} u_0^n v_0^m |A_0|^{2p} \varepsilon^r \in \mathbb{R},\end{aligned}$$

where the coefficients Y_{nmpqr} *lie in* $\mathbb{D}$ *and* γ_{nmpr} *lie in* $\mathbb{R}$.

A precise definition of the space $\mathbb{D}$ is given at page 12 in section 2 and a more precise version of this theorem is given at section 6 (see theorem 6.3).

Remark i) Observe that the perturbation of the graph $\widehat{p}^{(1)}$ is quadratic in the amplitudes (u_0, v_0, A_0).

Remark ii) Contrary to Lyapunov-Devaney Theorem, there is here a factor ε scaling the amplitudes. This is due to the resonance induced by the two small eigenvalues (for $\varepsilon < 0$) diving in the essential spectrum for $\varepsilon > 0$.

Remark iii) When A_0 is real the periodic solution $p_{A_0,u_0,v_0,\varepsilon}$ is reversible, i.e.

$$Sp_{A_0,u_0,v_0,\varepsilon}(x) = p_{A_0,u_0,v_0,\varepsilon}(-x) \qquad \text{for all } x \in \mathbb{R}.$$

On the other hand, A_0 complex corresponds to a phase shift in x.

In addition, one might expect the existence of a Benjamin-Ono like soliton, induced by the essential spectrum with eigenvalue 0, as for the problem with two layers, one being deep, and with a rigid top. However, the coexistence of this Benjamin-Ono type of dynamics with an oscillatory mode induced by the pair of simple imaginary eigenvalues causes the appearance of oscillations at infinity for the solutions. Such a coexistence of an oscillatory dynamics and a hyperbolic dynamics also occurs in the $0^2 i\omega$ resonance (see figure 5) for which it is proved in (Lombardi 2000) that there are generically no homoclinic connections to 0, whereas there are always homoclinic connections to periodic orbits, until they are exponentially small. We expect a result of the same type here, i.e. non existence of true solitary waves and existence of generalized solitary waves with exponentially small ripples at infinity. In this paper we prove a weaker result, i.e. the existence of reversible homoclinic connections to the periodic solutions found at theorem A, provided that the size of the limiting periodic orbit is not too small (at least of order $\varepsilon^{5/2}$). The proof of the existence of homoclinic connections to exponentially small periodic orbits is done in a forthcoming paper (Lombardi & Iooss 2001). In the theorem below, we consider A_0 real positive, which corresponds to a specific choice of the origin of x on the periodic solution. Moreover, for simplicity of the analysis we restrict our attention to the homoclinic connections to periodic solutions of family with $u_0 = v_0 = 0$. The same theorem is expected to be true for u_0 and v_0 near 0.

Theorem B. *For any* $0 < \alpha \leq 1/2$, *there exist* δ, δ_0, $\varepsilon_0 > 0$, *such that for* $0 < \varepsilon < \varepsilon_0$, *and* $\delta_0 \varepsilon^{2-\alpha} < A_0 < \delta$, *equ. (1.1) has two reversible homoclinic connections* $U^{(j)}_{A_0,\varepsilon}$ $(j = 1, 2)$ *to each periodic solution* $p_{A_0,0,0,\varepsilon}$ *found at theorem A, which satisfy*

$$U^{(j)}_{A_0,\varepsilon}(x) = p_{A_0,0,0,\varepsilon}\Big(x + \phi_j \rho \arctan(\varepsilon x/\rho)\Big) - \frac{2\varepsilon}{3} u_h(\varepsilon x/\rho)\xi_0 + \mathcal{O}\left(\frac{\varepsilon^{2-\alpha} + A_0\varepsilon}{1 + \varepsilon|x|}\right).$$

where u_h *is the Benjamin-Ono homoclinic connection given by (1.3).*

Remark i) The two distinct phase shift ϕ_j depend on (ε, A_0), and $(\phi_1 - \phi_2)\rho\frac{\pi}{2}$ tends towards half of the period of the limiting periodic orbits as its radius goes to 0. The proof of theorem B is the object of section 10.

Remark ii) Observe once more that, since there is no spectral gap (the entire real line is the essential spectrum), the decay rate at infinity is polynomial, and not exponential as it is the case for the finite dimensional reversible bifurcations (resonances 0^2, $(i\omega)^2$, $0^2 i\omega$) obtained for finitely deep layers.

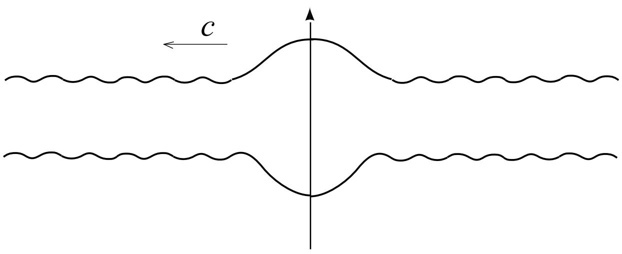

Figure 8. shape of generalized solitary waves in the two layer system

Remark iii) At leading order, the shape of the free surface $Z(x)$ and interface $Z_I(x)$ are given by (see figure 8)

$$Z(x) = 1 + \frac{2}{3}\varepsilon^2(1-\rho)u_h(\varepsilon x/\rho),$$
$$Z_I(x) = -\frac{2}{3}\varepsilon^2 \rho u_h(\varepsilon x/\rho).$$

The heart of the proof of theorem B is the following proposition, which ensures that, up to an appropriate change of variables, the full Euler equations (1.1) are equivalent to a Benjamin-Ono equation, coupled with a nonlinear oscillator equation, with higher order terms.

Proposition C. *Provided that suitable decay conditions in* x *on the solution* U *are satisfied, there exists an appropriate non local change of variables and a scaling*

$$U = \Upsilon_\varepsilon(A, u, Y), \qquad \underline{x} = \varepsilon x,$$

with $(A, u, Y)(\underline{x}) \in \mathbb{C}\times\mathbb{R}\times\mathbb{D}$ *such that, close to the origin, equ. (1.1) is equivalent to the reversible system*

$$
\begin{aligned}
&\frac{dA}{d\underline{x}} = i\frac{A}{\varepsilon}\left[\lambda_\varepsilon + \gamma(u, Y, |A|^2, \varepsilon)\right] + R_A(A, \overline{A}, u, Y),\\
&\rho\mathcal{H}\left(\frac{du}{d\underline{x}}\right) + u + \frac{3}{2}u^2 = \mathcal{B}_\varepsilon(A, \overline{A}, u, Y),\\
&Y = \mathcal{T}(A, u, Y),
\end{aligned}
$$

where the reversibility means the commutation of the system with the symmetry $\widehat{S} : (A(x), u(x), Y(x)) \mapsto (\overline{A}(-x), u(-x), SY(-x))$; *where* $\gamma \in \mathbb{R}$, *and where the local rest* R_A *and the nonlocal rest* $\mathcal{B}_\varepsilon$ *are small in suitable norms, and* $\mathcal{T}$ *is a known smooth nonlocal, nonlinear operator, such that* $Id - \mathcal{T}$ *is invertible with respect to* Y.

A more precise version of this proposition is given in section 8 (lemma 8.5).

Remark i) Roughly speaking, A corresponds to the amplitude of the oscillatory mode, u corresponds to the amplitude along the 0-eigenvector ξ_0, and Y corresponds to the rest of the spectrum.

Remark ii) The required decay conditions are such that u and Y tends towards 0 in $1/\underline{x}^2$ while A tends to A_0 in $1/\underline{x}$ as $|\underline{x}| \to \infty$.

Remark iii) The norms we use are Hölder norms in $\underline{x}$, with the above decay rates.

Remark iv) This proposition may be seen as a "rigorous derivation of the Benjamin-Ono equation" in this context. It also clearly shows the competition between the oscillatory dynamics and the Benjamin-Ono type of dynamics. We should finally notice that the present problem is numerically studied by Părău & Dias (2001), with lot of information on the shapes of the free surface and interface.

In what follows, after showing at section 2 how this problem may be formulated as a reversible dynamical system in a suitable space, and making a precise study of the resolvent of the linearized operator on the imaginary axis (sections 3, 5), we prove the existence of a three parameter family of periodic solutions (theorem 6.3, section 6), and we find an *infinite dimensional normal form*, where the whole family of periodic solutions appears trivially, and where a special treatment is needed of a priori reasoning terms coming from the point 0 in the essential spectrum of the linearized operator (section 7 and Appendix Normal Form). The homoclinic of Benjamin-Ono type also appears on this normal form, as an approximate solution (section 9), and we are able to prove, close to the Benjamin-Ono (false) homoclinic solution, the existence of a pair of reversible (i.e. symmetric) solutions homoclinic to every periodic solution, provided their size is not too small (theorem 10.1). These homoclinics differ mainly by a phase shift at infinity, and take physically the form indicated at figure 8.

2. Formulation as a dynamical system

The domain of the flow can be transformed into two superposed horizontal strips in using the (conformal) transformation defined below. The complex potential in layer j is denoted by $w_j(\xi + i\eta)$ and the complex velocity (in dimensionless form)

$w'_j(\xi + i\eta) = u_j - iv_j$. The Euler equations are expressed here by the fact that w_j is analytic in $\zeta = \xi + i\eta$. The kinematic conditions at the free surface and interface between the two fluids are

$$u_2\widetilde{Z}'(\xi) - v_2 = 0 \text{ at } \eta = 1 + \widetilde{Z}(\xi) \text{ (free surface)},$$

$$u_2\widetilde{Z}'_I(\xi) - v_2 = u_1\widetilde{Z}'_I(\xi) - v_1 = 0 \text{ at } \eta = \widetilde{Z}_I(\xi) \text{ (interface)}.$$

The Bernoulli first integrals at the free surface, and at the interface, express the continuity of the pressure:

$$\frac{1}{2}(u_2^2 + v_2^2) + \lambda\widetilde{Z} = \widetilde{c}_1 \text{ at } \eta = 1 + \widetilde{Z}(\xi) \text{ (free surface)},$$

$$\frac{1}{2}(u_1^2 + v_1^2) - \frac{\rho}{2}(u_2^2 + v_2^2) + \lambda(1-\rho)\widetilde{Z}_I = \widetilde{c}_2 \text{ at } \eta = \widetilde{Z}_I(\xi) \text{ (interface)},$$

where the parameters are $\rho = \rho_2/\rho_1 < 1$, and $\lambda = \frac{gh}{c^2}$, and $\widetilde{c}_1$ and $\widetilde{c}_2$ are arbitrary constants. For formulating our problem as a dynamical system, we first transform the unknown domain into a strip. There are different ways for such a change of coordinates. We choose the one used by Levi-Civita (1925). Its advantage is that it leads to a weakly nonlinear problem. The new unknown are $\alpha_j + i\beta_j$, $j = 1, 2$, which are analytic functions of $w_j = x_j + iy$, where x_j is the velocity potential in the layer j, and y is the stream function, and where

$$w'_j(\xi + i\eta) = e^{\beta_j - i\alpha_j},$$

the free surface is given by $y = 1$, and the interface by $y = 0$. The region of the flow is $-\infty < y < 0$ for fluid 1, and $0 < y < 1$ for fluid 2. One difficulty is that the x coordinate is not the same in each strip! In fact we have

$$\frac{dx_2}{dx_1} = e^{\beta_{20} - \beta_{10}}$$

where $\beta_{20} - \beta_{10}$ is the value of $\beta_2 - \beta_1$ taken at the interface $y = 0$.

We have to choose as the basic x coordinate the one given by the bottom layer (x_1) which then introduces a factor in the Cauchy-Riemann equations of the upper layer. In such a formulation, the unknown is defined by

$$[U(x)]\,(y) = (\beta_{10}(x), \beta_{21}(x), \alpha_1(x,y), \beta_1(x,y), \alpha_2(x,y), \beta_2(x,y))^t$$

and the system has the form

$$\frac{dU}{dx} = F(\rho, \lambda; U) \tag{2.1}$$

with

$$F(\rho,\lambda;U) = \begin{cases} -\lambda(1-\rho)e^{-3\beta_{10}}\sin\alpha_{20} - \rho e^{3(\beta_{20}-\beta_{10})}\frac{\partial\alpha_2}{\partial y}|_{y=0} \\ -\lambda e^{-3\beta_{21}+\beta_{20}-\beta_{10}}\sin\alpha_{21} \\ \left.\begin{array}{l} \frac{\partial\beta_1}{\partial y} \\ -\frac{\partial\alpha_1}{\partial y} \end{array}\right\} \quad y \in (-\infty, 0) \\ \left.\begin{array}{l} \frac{\partial\beta_2}{\partial y}e^{\beta_{20}-\beta_{10}} \\ -\frac{\partial\alpha_2}{\partial y}e^{\beta_{20}-\beta_{10}} \end{array}\right\} \quad y \in (0,1) \end{cases} \tag{2.2}$$

where we denote by α_{20}, β_{10} and β_{20} the traces of (resp.) $\alpha_2, \beta_1, \beta_2$ at $y=0$, and α_{21}, β_{21} the traces of α_2 and β_2 at $y=1$. Here we choose the basic space

$$\mathbb{H} = \mathbb{R}^2 \times C^0_1(\mathbb{R}^-) \times C^0_{\mathrm{lim},1}(\mathbb{R}^-) \times \{C^0(0,1)\}^2$$

and the domain of the operator F is:

$$\begin{aligned}\mathbb{D} = \ & \mathbb{R}^2 \times C^1_1(\mathbb{R}^-) \times C^1_{\mathrm{lim},1}(\mathbb{R}^-) \times \{C^1(0,1)\}^2 \\ & \cap \{\alpha_{10} = \alpha_{20}, \beta_{10} = \beta_1|_{y=0}, \beta_{21} = \beta_2|_{y=1}\},\end{aligned} \tag{2.3}$$

where we define the Banach spaces

$$\begin{aligned}C^0_\nu(\mathbb{R}^-) &= \{f \in C^0(\mathbb{R}^-); |f(y)|(1+|y|)^\nu < \infty\}, \quad \nu > 0,\\ C^1_\nu(\mathbb{R}^-) &= \{f \in C^0_\nu(\mathbb{R}^-), f' \in C^0_\nu(\mathbb{R}^-)\},\\ C^0_{\mathrm{lim},\nu}(\mathbb{R}^-) &= \{f \in C^0(\mathbb{R}^-); \exists l \in \mathbb{R},\ |f(y)-l|(1+|y|)^\nu < \infty\},\\ C^1_{\mathrm{lim},\nu}(\mathbb{R}^-) &= \{f \in C^0_{\mathrm{lim},\nu}(\mathbb{R}^-); f' \in C^0_\nu(\mathbb{R}^-)\},\end{aligned}$$

and we take for $(a,b,f_1,g_1,f_2,g_2)^t = V \in \mathbb{H}$, the norm

$$||V||_{\mathbb{H}} = |a| + |b| + ||f_1||_{1,\infty} + ||g_1||^{\mathrm{lim}}_{1,\infty} + ||f_2||_\infty + ||g_2||_\infty,$$

with

$$\begin{aligned}||f||_{\nu,\infty} &\stackrel{def}{=} \sup_{y\in\mathbb{R}^-} \left(|f(y)|(1+|y|)^\nu\right), \quad ||f||_\infty \stackrel{def}{=} \sup_y |f(y)|,\\ ||g||^{\mathrm{lim}}_{\nu,\infty} &\stackrel{def}{=} \sup_{y\in\mathbb{R}^-} |g(y)| + \sup_{y\in\mathbb{R}^-} \left(|g(y)-l|(1+|y|)^\nu\right).\end{aligned}$$

The definition of the norm in $\mathbb{D}$ is similar, in adding the norms of f'_j and g'_j.

The reversibility symmetry reads:

$$SU = (\beta_{10}, \beta_{21}, -\alpha_1, \beta_1, -\alpha_2, \beta_2)^t. \tag{2.4}$$

We notice that the system (2.1-2.2) has the two-parameter set of "trivial" solutions

$$\beta_1 = \beta_{10}, \quad \beta_2 = \beta_{21}, \quad \alpha_1 = \alpha_2 = 0,$$

which correspond to the sliding of one layer over the other, with different velocities. The system (2.1-2.2) should be completed by the following two Bernoulli first integrals (interface and free surface), when the integrals are convergent:

$$\begin{aligned}\int_{-\infty}^0 (e^{-\beta_1}\cos\alpha_1 - e^{-\lim\beta_1})dy + \int_0^1 (e^{-\beta_2}\cos\alpha_2 - 1)dy + \frac{1}{2\lambda}(e^{2\beta_{21}} - 1) = c_1,\\ \lambda(1-\rho)\int_{-\infty}^0 (e^{-\beta_1}\cos\alpha_1 - e^{-\lim\beta_1})dy + \frac{1}{2}(e^{2\beta_{10}} - 1) - \frac{\rho}{2}(e^{2\beta_{20}} - 1) = c_2,\end{aligned}$$

which give the two first components of (2.1-2.2) after differentiation. However, since we do not impose a priori that α_1 and $(\beta_1 - \lim\beta_1)$ tend towards 0 fast enough at infinity, we cannot consider both these first integrals, but only a suitable combination of them

$$c = \frac{1}{2}(e^{2\beta_{10}} - 1) - \frac{1-\rho}{2}(e^{2\beta_{21}} - 1) - \frac{\rho}{2}(e^{2\beta_{20}} - 1) + \\ - \lambda(1-\rho)\int_0^1 (e^{-\beta_2}\cos\alpha_2 - 1)dy. \tag{2.5}$$

Notice that the interface and free surface, expressed in the new coordinates satisfy the following expressions:

$$Z_I(x) = \int_{-\infty}^0 (e^{-\beta_1}\cos\alpha_1 - e^{-\lim\beta_1})dy,$$

$$1 + Z(x) = 1 + \int_{-\infty}^0 (e^{-\beta_1}\cos\alpha_1 - e^{-\lim\beta_1})dy + \int_0^1 (e^{-\beta_2}\cos\alpha_2 - 1)dy,$$

provided the integrals are convergent, and (preferably)

$$\frac{dZ_I}{dx} = e^{-\beta_{10}}\sin\alpha_{10},$$
$$\frac{dZ}{dx} = e^{\beta_{20}-\beta_{10}-\beta_{21}}\sin\alpha_{21}.$$

In principle we might choose to treat this problem on a codimension-2 manifold, instead of expressing the two first components of (2.2) above. It appears that it is easier to work as we do at present, just keeping in mind that there are two arbitrary constants which may be fixed. We notice that the system is *still linear in the unbounded strip* $y < 0$ (Cauchy-Riemann equations). It is no longer linear in the bounded strip $y \in (0,1)$, but the dependency in y is still occurring in similar "linear terms", the multiplicator being only function of x.

3. The linearized Problem

Let us fix ρ and define ε by

$$\lambda(1-\rho) = 1 - \varepsilon \tag{3.1}$$

and rewrite $F(\rho, \lambda; U) = L_\varepsilon U + N(\varepsilon; U)$, where all linear terms are in $L_\varepsilon U$. The linearized system then reads

$$\frac{dU}{dx} = L_\varepsilon U \tag{3.2}$$

in $\mathbb{H}$. The following lemma describes the spectral properties of L_ε :

Lemma 3.1. *(a) The spectrum of* L_ε *acting in* $\mathbb{H}$ *is* symmetric with respect to both axis *of the complex plane. It is composed*

(i) with the entire real line, which constitutes the essential spectrum*, every real* $\sigma \neq 0$ *being such that* $(\sigma\mathbb{I} - L_\varepsilon)$ *is injective, but has a non closed range and 0 is a double eigenvalue;*

(ii) with isolated eigenvalues $\sigma = ik$ *of finite multiplicities, given by the roots of the dispersion relation* $\Delta[(sgn\,\mathrm{Re}\,k)k, \varepsilon] = 0$ *where, for* $\mathrm{Re}\,k > 0$

$$\Delta(k,\varepsilon) \equiv [\lambda(\varepsilon) - k]\Delta_1(k,\varepsilon),$$
$$\Delta_1(k,\varepsilon) = [\rho k - (1-\varepsilon)]\sinh k + k\cosh k$$

holds.

(b) For any ε, 0 is a double eigenvalue, associated with the two eigenvectors

$$\xi_0 = (0,1,0,0,0,1)^t,$$
$$\xi_1 = (1,0,0,1,0,0)^t,$$

satisfying

$$S\xi_0 = \xi_0, \quad S\xi_1 = \xi_1.$$

(c) For any ε, there is a pair of simple eigenvalues $ik = \pm i\lambda$, associated with the eigenvectors ζ_ε and $\overline{\zeta}_\varepsilon$ such that

$$\zeta_\varepsilon = (1, e^{\lambda}, -ie^{\lambda y}, e^{\lambda y}, -ie^{\lambda y}, e^{\lambda y})^t, \quad S\zeta_\varepsilon = \overline{\zeta}_\varepsilon. \tag{3.3}$$

(d) For $\varepsilon \geq 0$ the only eigenvalues with 0 real part are $ik = \pm i\lambda$, and 0, whereas for $\varepsilon < 0$, in addition to the above imaginary eigenvalues, there is another pair of simple eigenvalues, tending towards 0 as $\varepsilon \to 0^-$ (see figure 2).

(e) For k real and $|k|$ large enough, we have the following estimate (uniform estimate for ε near 0)

$$||(ik\mathbb{I} - L_\varepsilon)^{-1}||_{\mathcal{L}(\mathbb{H})} \leq C/|k|. \tag{3.4}$$

Proof. The spectrum of L_ε is *symmetric with respect to both axis* of the complex plane, because of reversibility. Let us look for eigenvalues denoted by ik where k is complex. The linearized problem for system (2.1-2.2) is given by the linear operator L_ε acting in $\mathbb{H}$, with domain $\mathbb{D}$, and defined by

$$L_\varepsilon U = \begin{pmatrix} -(1-\varepsilon)\alpha_{20} - \rho\frac{\partial\alpha_2}{\partial y}|_{y=0} \\ -\frac{1-\varepsilon}{1-\rho}\alpha_{21} \\ \frac{\partial\beta_1}{\partial y} \\ -\frac{\partial\alpha_1}{\partial y} \\ \frac{\partial\beta_2}{\partial y} \\ -\frac{\partial\alpha_2}{\partial y} \end{pmatrix}. \tag{3.5}$$

Looking at the eigenvalues ik such that $\operatorname{Re} k > 0$ leads to eigenvectors of the form

$$\zeta = (1, \frac{\lambda}{k}e^{k}, -ie^{ky}, e^{ky}, -i\frac{\lambda}{k}e^{ky}, \frac{\lambda}{k}e^{ky})^t +$$
$$\frac{k-\lambda}{2\rho k}\left[(1+\rho)(0, e^{k}, 0, 0, -ie^{ky}, e^{ky})^t + (1-\rho)(0, e^{-k}, 0, 0, ie^{-ky}, e^{-ky})^t\right],$$

and the dispersion relation

$$\Delta[(sgn \operatorname{Re} k)k, \varepsilon] = 0$$

has the form (for $\operatorname{Re} k > 0$):

$$\Delta(k,\varepsilon) \equiv [\lambda(\varepsilon) - k]\Delta_1(k,\varepsilon), \tag{3.6}$$
$$\Delta_1(k,\varepsilon) = [\rho k - (1-\varepsilon)]\sinh k + k\cosh k, \tag{3.7}$$

An interesting property is that there is an explicit pair of simple eigenvalues $\pm i\lambda$ associated with eigenvectors ζ_ε and $\overline{\zeta}_\varepsilon$ defined by (3.3).

The study of (3.6) shows that there is another pair of simple eigenvalues on the imaginary axis $\pm ik_1$ *if and only if* $\varepsilon < 0$, moreover, we have $k_1(\varepsilon) = -\varepsilon/\rho - \varepsilon^2/3\rho^3 + O(\varepsilon^3)$. This pair of eigenvalues tends towards 0 as $\varepsilon \to 0^-$, and *disappears for* $\varepsilon > 0$ (see figure 2).

Notice that 0 is always an eigenvalue, associated with the eigenvectors ξ_0 and ξ_1 given in the lemma. These eigenvectors correspond to the *existence of the two-parameter family of solutions* $\beta_{21}\xi_0 + \beta_{10}\xi_1$ of the nonlinear system (2.1,2.2).

Contrary to the paper (Iooss 1999), where we assumed essentially that ε was not close to 0, the object of the present paper is to study what are the solutions of (2.1) for ε close to 0 and positive.

The study made in (Iooss 1999) on the resolvent operator $(ik\mathbb{I} - L_\varepsilon)^{-1}$ is made with another choice for space $\mathbb{H}$. We show below at section 5 (on a rescaled formulation) that the estimate (3.4), for $k \in \mathbb{R}$ and $|k|$ large enough, stays valid with our new choice of spaces.

As it is shown in (Iooss 1999), in addition to the above eigenvalues in the spectrum of L_ε, the spectrum contains the entire real axis which constitutes the *"essential spectrum"*. With our choice of basic space $\mathbb{H}$, any real $\sigma \neq 0$ is not an eigenvalue, and it is such that the range of $(\sigma\mathbb{I} - L_\varepsilon)$ is not closed. The double eigenvalue $\sigma = 0$ is embedded into the essential spectrum. Moreover it is easy to check that, even for $\varepsilon = 0$, there is no generalized eigenvector despite of the fact that when $\varepsilon = 0^-$ two simple eigenvalues dive into the real line through 0! (perturbation theory is no longer valid in this situation, with an essential spectrum containing 0). We may also observe, that the operator L_ε has a non closed range, whose closure has codimension two (see lemma 5.5 at section 5.(e)).

The divergence of the resolvent operator when $k \to 0$ is worse for $\varepsilon = 0$, than for $\varepsilon \neq 0$, as can be seen on formula (40) of (Iooss 1999), and in subsequent expressions for the operators occurring in this formula. For understanding better this singularity, we need now to make an adapted computation, tracking the dependency in ε, for ε near 0. In what follows, we only consider the case $\varepsilon > 0$. The case $\varepsilon < 0$ would give two new small frequencies, and the conjecture is that the solutions of the problem would be analogous to the ones of the corresponding four dimensional reversible vector field with two pairs of simple eigenvalues on the imaginary axis, leading to tori of periodic solutions.

4. Rescaling for $\varepsilon \gtrsim 0$

(*a*) *Dynamical system formulation*

Let us consider the case when $1 - \lambda(1-\rho) = \varepsilon > 0$ is close to 0. The study of the resolvent operator for k near 0, leads to rescaling of x and to a specific scaling for y depending on whether $-\infty < y < 0$, or $0 < y < 1$. Moreover, the form of the underlying homoclinic in our problem (analogous to the Benjamin-Ono homoclinic) also leads to a rescaling of U. So, we rescale our system as follows:

$$\varepsilon x = \underline{x}, \varepsilon y = \underline{y}, \text{ for } y \in (-\infty, 0),$$
$$U = \varepsilon \underline{U}.$$

Then system (2.1)

$$\frac{dU}{dx} = L_\varepsilon U + N(\varepsilon; U)$$

now reads

$$\frac{d\underline{U}}{d\underline{x}} = \mathcal{L}_\varepsilon \underline{U} + \mathcal{N}(\varepsilon; \underline{U}), \tag{4.1}$$

where

$$\mathcal{L}_\varepsilon \underline{U} = \begin{pmatrix} \varepsilon^{-1}\left\{-(1-\varepsilon)\alpha_{20} - \rho\frac{\partial\alpha_2}{\partial y}|_{y=0}\right\} \\ -\frac{\varepsilon^{-1}(1-\varepsilon)}{1-\rho}\alpha_{21} \\ \frac{\partial\beta_1}{\partial y} \\ -\frac{\partial\alpha_1}{\partial y} \\ \varepsilon^{-1}\frac{\partial\beta_2}{\partial y} \\ -\varepsilon^{-1}\frac{\partial\alpha_2}{\partial y} \end{pmatrix}$$

and we have in $\mathbb{H}$

$$\mathcal{N}(\varepsilon; \underline{U}) = \varepsilon^{-2} N(\varepsilon; \varepsilon\underline{U}) = N_\varepsilon^{(2)}(\underline{U}, \underline{U}) + O(\varepsilon||\underline{U}||_{\mathbb{D}}^3),$$

where we denote again $\underline{U} = (\beta_{10}, \beta_{21}, \alpha_1, \beta_1, \alpha_2, \beta_2)^t$ and

$$N_\varepsilon^{(2)}(\underline{U}, \underline{U}) = N_0^{(2)}(\underline{U}, \underline{U}) + \varepsilon N_1^{(2)}(\underline{U}, \underline{U}) \tag{4.2}$$

$$\begin{aligned} N_0^{(2)}(\underline{U}, \underline{U}) &= (a^{(2)}, b^{(2)}, 0, 0, f_2^{(2)}, g_2^{(2)})^t, \\ N_1^{(2)}(\underline{U}, \underline{U}) &= (-3\alpha_{20}\beta_{10}, -b^{(2)}, 0, 0, 0, 0)^t, \\ a^{(2)} &= 3\alpha_{20}\beta_{10} + 3\rho g_{20}^{(2)}, \\ b^{(2)} &= (\rho-1)^{-1}\alpha_{21}(-3\beta_{21} + \beta_{20} - \beta_{10}), \\ f_2^{(2)} &= \frac{\partial\beta_2}{\partial y}(\beta_{20} - \beta_{10}), \\ g_2^{(2)} &= -\frac{\partial\alpha_2}{\partial y}(\beta_{20} - \beta_{10}). \end{aligned}$$

The domain of the operator $\mathcal{L}_\varepsilon$ acting in $\mathbb{H}$, is still $\mathbb{D}$. However, we observe that the nonlinear operator $\mathcal{N}(\varepsilon; \cdot)$ is analytic from $\widetilde{\mathbb{D}}$ into $\mathbb{H}$, where $\widetilde{\mathbb{D}}$ is larger than $\mathbb{D}$. Such a space $\widetilde{\mathbb{D}}$ is used in Appendix Normal Form, where α_1 and β_1 are allowed to grow as $\underline{y} \to -\infty$, while their derivatives lie in C_1^0. Moreover, it is used in all section 8 (see the choice we make for the space $B_{\mathbb{D},w}^{\alpha}$), that the nonlinear terms do not contain derivatives of α_1 and β_1, but just their traces at $\underline{y} = 0$. This explains why we don't need to estimate $\underline{y}-$derivatives of α_1 and β_1 in Appendices Resolvent ∞, and Resolvent 0.

We have a pair of simple eigenvalues $\pm i\lambda/\varepsilon$ for $\mathcal{L}_\varepsilon$, with eigenvectors $\underline{\zeta}_\varepsilon$ and $\overline{\underline{\zeta}}_\varepsilon$, and

$$\underline{\zeta}_\varepsilon = (1, e^\lambda, -ie^{\lambda\underline{y}/\varepsilon}, e^{\lambda\underline{y}/\varepsilon}, -ie^{\lambda y}, e^{\lambda y})^t, \tag{4.3}$$

and ξ_0 and ξ_1 are still the eigenvectors belonging to the eigenvalue 0 of $\mathcal{L}_\varepsilon$.

For later use, let us define the symmetry $\widehat{S}$ by

$$\left(\widehat{S}\underline{U}\right)(x) = S\underline{U}(-x).$$

(b) *Nonlocal formulation*

We shall exploit at sections 6 and 9 the form of the third and fourth components of system (2.1,2.2), which are just the Cauchy-Riemann equations for α_1, β_1 in the half plane $y < 0$. The other components of the right hand side are expressed completely in terms of $(\beta_{10}, \alpha_2, \beta_2)$. Provided that α_1 and β_1 tends towards 0 as $y \to -\infty$, we write the relationship between β_{10} and $\alpha_{20} = \alpha_{10}$, with the help of Hilbert transform (hence *non local in x*). This simplifies a lot in some sense the analysis, but we pay this by *loosing (for a moment) the "dynamical system" formulation.* Indeed, consider the system

$$\frac{\partial \alpha_1}{\partial x} = \frac{\partial \beta_1}{\partial y},$$
$$\frac{\partial \beta_1}{\partial x} = -\frac{\partial \alpha_1}{\partial y},$$

in the half plane $y < 0$, with a decaying (to 0) condition as $y \to -\infty$. In the case of a suitable decaying condition as $x \to \pm\infty$, we can write for example

$$\beta_1(x, y) = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{(x-\xi)}{y^2 + (x-\xi)^2}\alpha_{10}(\xi)d\xi,$$

which leads to $\beta_{10} + \mathcal{H}(\alpha_{10}) = 0$, if we define the Hilbert transform of a function g by

$$(\mathcal{H}g)(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{g(\xi)}{x-\xi}d\xi,$$

which is OK for instance for $g \in L^2(\mathbb{R})$. Now, we notice that this formula is still valid for functions which are periodic in x, i.e for $g \in L^2_\sharp(\mathbb{R})$ (a.e. periodic, locally square integrable), with the convention that $\mathcal{H}a = 0$ for any constant a.

In the following, we restrict our attention to solutions of (2.1,2.2) which are either *periodic*, or *asymptotic to a $x-$periodic* solution, or *tending to* 0 *at infinity.* So, we choose function spaces such that the Hilbert transform $\mathcal{H}$ of α_{10} exists. When they exist, the Fourier transforms satisfy

$$\widehat{(\mathcal{H}g)}(k) = -i(sgnk)\widehat{g}(k).$$

Notice that if we define $\widehat{s}x = -x$ the reflection in $\mathbb{R}$, and for any $\lambda \in \mathbb{R}$, then

$$(\mathcal{H}g \circ \widehat{s})(x) = -(\mathcal{H}g) \circ \widehat{s}(x),$$
$$(\mathcal{H}e^{i\lambda(\cdot)})(x) = -i(sgn\lambda)e^{i\lambda x}, \quad \text{for } \lambda \neq 0$$

hold.

Let us denote by $\mathcal{P}$ the projection: $(a, b, f_1, g_1, f_2, g_2)^t \mapsto (a, b, f_2, g_2)^t$, then we define

$$W = \mathcal{P}\underline{U},$$

Then system (2.1,2.2) now reads

$$\frac{dW}{d\underline{x}} = \mathcal{L}_\varepsilon^{(\mathcal{P})}W + \mathcal{N}^{(\mathcal{P})}(\varepsilon; W), \tag{4.4}$$

$$\mathcal{D}(W) = 0, \tag{4.5}$$

provided that $\mathcal{D}(W)$ is well defined, where

$$\mathcal{L}_\varepsilon^{(\mathcal{P})} = \varepsilon^{-1}\mathcal{A}_0 + \mathcal{A}_1,$$

$$\mathcal{D}(W) \stackrel{def}{=} \beta_{10} - [\beta_{10}] + \mathcal{H}\alpha_{20},$$

where $[\beta_{10}]$ is the average of β_{10} and where

$$\mathcal{A}_0 W = \begin{pmatrix} -\alpha_{20} - \rho\frac{\partial\alpha_2}{\partial y}|_{y=0} \\ -\frac{1}{1-\rho}\alpha_{21} \\ \frac{\partial\beta_2}{\partial y} \\ -\frac{\partial\alpha_2}{\partial y} \end{pmatrix},$$

$$\mathcal{A}_1 W = \begin{pmatrix} \alpha_{20} \\ \frac{1}{1-\rho}\alpha_{21} \\ 0 \\ 0 \end{pmatrix},$$

and $\mathcal{N}^{(\mathcal{P})}(\varepsilon; W) = \mathcal{P}\mathcal{N}(\varepsilon; \underline{U})$.

Let us define the new spaces

$$\mathbb{H}^{\mathcal{P}} = \mathbb{R}^2 \times \{C^0(0,1)\}^2,$$
$$\mathbb{D}^{\mathcal{P}} = \mathbb{R}^2 \times \{C^1(0,1)\}^2 \cap \{\beta_{21} = \beta_2|_{y=1}\},$$

then, the domain of the operator $\mathcal{L}_\varepsilon^{(\mathcal{P})}$ acting in $\mathbb{H}^{\mathcal{P}}$, is still $\mathbb{D}^{\mathcal{P}}$.

We have now eigenvectors $\zeta_\varepsilon^{(\mathcal{P})}$, $\overline{\zeta}_\varepsilon^{(\mathcal{P})}$, $\xi_0^{(\mathcal{P})}$, $\xi_1^{(\mathcal{P})}$ with

$$\zeta_\varepsilon^{(\mathcal{P})} = (1, e^{\lambda}, -ie^{\lambda y}, e^{\lambda y})^t \in \mathbb{D}^{\mathcal{P}}, \tag{4.6}$$
$$\xi_0^{(\mathcal{P})} = (0,1,0,1)^t \in \mathbb{D}^{\mathcal{P}}, \tag{4.7}$$
$$\xi_1^{(\mathcal{P})} = (1,0,0,0)^t \in \mathbb{D}^{\mathcal{P}}, \tag{4.8}$$

and

$$\mathcal{L}_\varepsilon^{(\mathcal{P})}\zeta_\varepsilon^{(\mathcal{P})} = i\,(\lambda/\varepsilon)\,\zeta_\varepsilon^{(\mathcal{P})},$$
$$\mathcal{L}_\varepsilon^{(\mathcal{P})}\overline{\zeta}_\varepsilon^{(\mathcal{P})} = -i\,(\lambda/\varepsilon)\,\overline{\zeta}_\varepsilon^{(\mathcal{P})},$$
$$\mathcal{L}_\varepsilon^{(\mathcal{P})}\xi_0^{(\mathcal{P})} = 0, \quad \mathcal{L}_\varepsilon^{(\mathcal{P})}\xi_1^{(\mathcal{P})} = 0.$$

Finally, we observe that the Bernoulli first integral (2.5) gives a first integral for our remaining rescaled variables (independent of α_1 and β_1):

$$\begin{aligned} h(\varepsilon; W) \stackrel{def}{=} & \frac{\varepsilon^{-1}}{2}(e^{2\varepsilon\beta_{10}} - 1) - \frac{\rho\varepsilon^{-1}}{2}(e^{2\varepsilon\beta_{20}} - 1) - \frac{(1-\rho)\varepsilon^{-1}}{2}(e^{2\varepsilon\beta_{21}} - 1) + \\ & - (1-\varepsilon)\varepsilon^{-1}\int_0^1 (e^{-\varepsilon\beta_2}\cos\varepsilon\alpha_2 - 1)dy. \end{aligned} \tag{4.9}$$

Denoting

$$D_W h(\varepsilon; 0)W \stackrel{def}{=} \xi_\varepsilon^*(W),$$

we then have

$$h(\varepsilon; W) = \xi_\varepsilon^*[W(x)] + O(\varepsilon||W(x,\cdot)||^2_{\mathbb{H}^{\mathcal{P}}}) = c_3 \text{ (indep of } x) \tag{4.10}$$

where for any $V = (a, b, f_1, g_1, f_2, g_2)^t \in \mathbb{H}_\nu$, the linear form $\xi_\varepsilon^* \in \mathbb{H}^{\mathcal{P}*} \cap \mathbb{H}^*$ is defined by

$$\xi_\varepsilon^*(V) = a - \rho g_{20} - (1-\rho)b + (1-\varepsilon)\int_0^1 g_2(y)dy. \tag{4.11}$$

Moreover, we observe that the following identities hold

$$\begin{aligned}
\xi_\varepsilon^*(\xi_0) = -\varepsilon, \quad \xi_\varepsilon^*(\xi_1) = 1, \quad \xi_\varepsilon^*(\underline{\zeta}_\varepsilon) = \xi_\varepsilon^*(\overline{\underline{\zeta}_\varepsilon}) = 0,\\
\xi_\varepsilon^*(SV) = \xi_\varepsilon^*(V), \quad \text{for all } V \in \mathbb{H},\\
\xi_\varepsilon^*(\mathcal{L}_\varepsilon^{(\mathcal{P})}W) = \xi_\varepsilon^*(\mathcal{L}_\varepsilon \underline{U}) = 0, \text{ for all } W \in \mathbb{D}^{\mathcal{P}}, \underline{U} \in \mathbb{D}.
\end{aligned} \tag{4.12}$$

5. Resolvent operator of $\mathcal{L}_\varepsilon$

This section is devoted to the study of the resolvent operator $(ik - \mathcal{L}_\varepsilon)^{-1}$ for $\varepsilon > 0$ and small enough. In subsection 5.(a) we give explicit formulas and we use them in further subsections for obtaining estimates of the resolvent for $|k|$ large (subsection 5.(b)), near the poles $ik = \pm i\lambda/\varepsilon$ (subsection 5.(c)), and near 0 (subsection 5.(d)).

(*a*) *Explicit formulas for the resolvent*

Here we solve the resolvent equation

$$\begin{aligned}
(ik - \mathcal{L}_\varepsilon)\underline{U} = V,\\
(a, b, f_1, g_1, f_2, g_2)^t = V \in \mathbb{H} \text{ is given}
\end{aligned}$$

where we look for $\underline{U} \in \mathbb{D}$. We then need to solve the following system

$$\begin{aligned}
&ik\beta_{10} + \varepsilon^{-1}(1-\varepsilon)\alpha_{10} + \varepsilon^{-1}\rho\alpha_2'(0) = a\\
&ik\beta_{21} + \varepsilon^{-1}(1-\varepsilon)(1-\rho)^{-1}\alpha_{21} = b\\
&\left.\begin{aligned} ik\alpha_1 - \beta_1' = f_1 \\ ik\beta_1 + \alpha_1' = g_1 \end{aligned}\right\} \ \underline{y} \in (-\infty, 0)\\
&\left.\begin{aligned} ik\alpha_2 - \varepsilon^{-1}\beta_2' = f_2 \\ ik\beta_2 + \varepsilon^{-1}\alpha_2' = g_2 \end{aligned}\right\} \quad y \in (0,1).
\end{aligned}$$

Solving this linear ordinary differential equation for $k > 0$ and $\Delta(\varepsilon k, \varepsilon) \neq 0$ we get

$$\begin{aligned}
\alpha_1(\underline{y}) &= \underline{A}e^{k\underline{y}} + H_1[f_1, g_1](k, \underline{y}),\\
\beta_1(\underline{y}) &= i\underline{A}e^{k\underline{y}} + K_1[f_1, g_1](k, \underline{y}),\\
\alpha_2(y) &= \underline{A}\cosh \varepsilon ky + \underline{B}\sinh \varepsilon ky + \underline{H}_2[f_2, g_2](k, y),\\
\beta_2(y) &= i\underline{A}\sinh \varepsilon ky + i\underline{B}\cosh \varepsilon ky + i\underline{K}_2[f_2, g_2](k, y),
\end{aligned}$$

where (for $k \in \mathbb{R}\backslash\{0\}$)

$$\begin{aligned}
H_1[f,g](k,\underline{y}) &= \frac{i}{2}\int_{-\infty}^{0}[(sgnk)f(\tau)+ig(\tau)]e^{|k|(\tau+\underline{y})}d\tau+\\
&-\frac{i}{2}\int_{-\infty}^{0}[(sgnk)f(\tau)+ig(\tau)sgn(\underline{y}-\tau)]e^{-|k||\underline{y}-\tau|}d\tau,\\
K_1[f,g](k,\underline{y}) &= -\frac{1}{2}\int_{-\infty}^{0}[f(\tau)+i(sgnk)g(\tau)]e^{|k|(\tau+\underline{y})}d\tau+\\
&-\frac{1}{2}\int_{-\infty}^{0}[f(\tau)sgn(\underline{y}-\tau)+i(sgnk)g(\tau)]e^{-|k||\underline{y}-\tau|}d\tau,
\end{aligned}$$

$$\begin{aligned}
\underline{H}_2[f,g](k,y) &= \varepsilon\int_0^1[if(\tau)H_{21}(y,\tau)-g(\tau)H_{22}(y,\tau)]d\tau,\\
\underline{K}_2[f,g](k,y) &= \varepsilon\int_0^1[if(\tau)K_{21}(y,\tau)-g(\tau)K_{22}(y,\tau)]d\tau,
\end{aligned}$$

with (for $k>0$)

$$\begin{aligned}
&H_{21}(y,\tau)=H_{21}(\tau,y)=(\sinh\varepsilon k)^{-1}\sinh\varepsilon k(y-1)\sinh(\varepsilon k\tau),\ \text{ for } 0<\tau<y<1,\\
&H_{22}(y,\tau)=K_{21}(\tau,y)=(\sinh\varepsilon k)^{-1}\begin{cases}\sinh\varepsilon k(y-1)\cosh(\varepsilon k\tau), \text{for } 0<\tau<y<1,\\ \sinh(\varepsilon ky)\cosh\varepsilon k(\tau-1), \text{for } 0<y<\tau<1,\end{cases}\\
&K_{22}(y,\tau)=K_{22}(\tau,y)=(\sinh\varepsilon k)^{-1}\cosh\varepsilon k(y-1)\cosh(\varepsilon k\tau),\ \text{ for } 0<\tau<y<1.
\end{aligned}$$

$$\begin{aligned}
\Delta(\varepsilon k,\varepsilon)\underline{A} &= (\varepsilon k\cosh\varepsilon k-(1-\varepsilon)(1-\rho)^{-1}\sinh\varepsilon k)\underline{a}_1+\rho\varepsilon k\underline{b}_1,\\
\Delta(\varepsilon k,\varepsilon)\underline{B} &= ((1-\varepsilon)(1-\rho)^{-1}\cosh\varepsilon k-\varepsilon k\sinh\varepsilon k)\underline{a}_1+[\varepsilon k-(1-\varepsilon)]\underline{b}_1,
\end{aligned}$$

$$\begin{aligned}
\underline{a}_1 &= \varepsilon a+i\varepsilon k\int_{-\infty}^{0}[f_1(\tau)+ig_1(\tau)]e^{k\tau}d\tau-\varepsilon\rho g_{20}+\\
&-\rho\varepsilon^2k(\sinh\varepsilon k)^{-1}\int_0^1[if_2(\tau)\sinh\varepsilon k(\tau-1)-g_2(\tau)\cosh\varepsilon k(\tau-1)]d\tau,\\
\underline{b}_1 &= \varepsilon b+\varepsilon^2k(\sinh\varepsilon k)^{-1}\int_0^1[if_2(\tau)\sinh(\varepsilon k\tau)-g_2(\tau)\cosh(\varepsilon k\tau)]d\tau.
\end{aligned}$$

Lemma 5.1. *For $\Delta(\varepsilon|k|,\varepsilon)\neq 0$, α_1 and $\beta_1 \in C^1_1(\mathbb{R}^-)$, and α_2 and $\beta_2 \in C^1(0,1)$.*

Proof. For $\Delta(\varepsilon|k|,\varepsilon)\neq 0$ the above formulas insure that α_2 and β_2 are bounded in $C^1(0,1)$. We also check that α_1 and $\beta_1 \in C^1(\mathbb{R}^-)$, and more precisely, we show that indeed α_1 and $\beta_1 \in C^1_1(\mathbb{R}^-)$ (hence $\beta_1 \in C^1_{\text{lim},1}(\mathbb{R}^-)$) thanks to the estimate

$$||\int_{-\infty}^{0} f(\tau)e^{-k|y-\tau|}d\tau||_{1,\infty}\leq \frac{c}{k}||f||_{1,\infty},\quad \text{as } k\to\infty, \tag{5.1}$$

which results from the two elementary estimates

$$e^{ky}(1+|y|)\begin{cases}\leq 1 \quad \text{for } k\geq 1,\\ \leq c(1+k^{-1}) \quad \text{for } k<1,\end{cases},\quad y<0, \tag{5.2}$$

$$\int_0^x e^{-k(x-\tau)}\left(\frac{1+x}{1+\tau}\right)d\tau = \frac{1}{k}\left[1-e^{-kx}(1+x)\right]+O(1/k^2).$$

(b) Estimate of the resolvent for $|k|$ large

The following lemma gives estimates of the resolvent for k real and $|k|$ large enough:

Lemma 5.2. *For $\varepsilon|k|>M$ (M large enough) and k real, there exists $c>0$ such that we have the estimates*

$$\begin{aligned}||(ik\mathbb{I}-\mathcal{L}_\varepsilon)^{-1}||_{\mathcal{L}(\mathbb{H})} &\leq \frac{c}{|k|},\\ ||(ik\mathbb{I}-\mathcal{L}_\varepsilon)^{-1}||_{\mathcal{L}(\mathbb{H},\mathbb{D})} &\leq c.\end{aligned} \tag{5.3}$$

Moreover, if we only consider the components in $\mathbb{D}^{\mathcal{P}}$, then

$$||\mathcal{P}(ik\mathbb{I}-\mathcal{L}_\varepsilon)^{-1}||_{\mathcal{L}(\mathbb{H},\mathbb{D}^{\mathcal{P}})} \leq c\varepsilon.$$

Proof. For $|k|$ large, let us follow the proof made in (Iooss 1999). It is easy to show that, for $|k|$ large enough

$$\Delta(\varepsilon|k|,\varepsilon) = -\frac{\rho+1}{2}(\varepsilon k)^2 e^{\varepsilon|k|}+\lambda\varepsilon|k|e^{\varepsilon|k|}+O(\varepsilon^2k^2e^{-\varepsilon|k|}),$$

$$\begin{aligned}|\underline{a}_1|+|\underline{b}_1| &\leq c\varepsilon(|a|+|b|+||f_1||_\infty+||g_1||_\infty+||f_2||_\infty+||g_2||_\infty),\\ |\underline{A}|+|\underline{B}| &\leq \frac{c}{|k|}(|a|+|b|+||f_1||_\infty+||g_1||_\infty+||f_2||_\infty+||g_2||_\infty),\end{aligned}$$

holds. From the estimates (5.1,5.2), we deduce that for $\varepsilon|k|>M$, α_1 and β_1 satisfy

$$||\alpha_1||_{1,\infty}+||\beta_1||_{1,\infty}^{\lim} \leq \frac{c}{|k|}.$$

So, in using the same proof as in (Iooss 1999), we obtain the following identities (here $k>0$)

$$\begin{aligned}\alpha_2(y) &= \frac{a_1}{\Delta(\varepsilon k,\varepsilon)}[\varepsilon k\cosh\varepsilon k(1-y)-\lambda\sinh\varepsilon k(1-y)]+\\ &+\frac{b_1}{\Delta(\varepsilon k,\varepsilon)}\{[\varepsilon k-(1-\varepsilon)]\sinh\varepsilon ky+\rho\cosh\varepsilon ky\}+H_2[f_2,g_2](k,y),\\ \beta_2(y) &= \frac{ia_1}{\Delta(\varepsilon k,\varepsilon)}[\lambda\cosh\varepsilon k(1-y)-\varepsilon k\sinh\varepsilon k(1-y)]+\\ &+\frac{ib_1}{\Delta(\varepsilon k,\varepsilon)}\{[\varepsilon k-(1-\varepsilon)]\cosh\varepsilon ky+\rho\varepsilon k\sinh\varepsilon ky\}+iK_2[f_2,g_2](k,y),\end{aligned}$$

which finally allow to get (5.3).

The last part of the lemma comes from an examination of the consequence of differentiating with respect to y [this introduces a factor $\varepsilon|k|$ only in the components (α_2,β_2)].

(c) Study of the resolvent near the poles $ik = \pm i\lambda/\varepsilon$

Near $k = \lambda/\varepsilon$, we have (uniformly in ε)

$$\frac{1}{\Delta(\varepsilon k, \varepsilon)} = \frac{de^{\lambda}}{\lambda(\lambda - \varepsilon k)} + O(1),$$

with

$$d = (1 - \rho + \rho e^{2\lambda})^{-1}.$$

It then results immediately the following

Lemma 5.3. *For k in the neighborhood of $\pm\lambda/\varepsilon$ we have in $\mathcal{L}(\mathbb{H}, \mathbb{D})$*

$$(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1} = \frac{\zeta_\varepsilon}{i(k - \frac{\lambda}{\varepsilon})}\zeta_\varepsilon^* + O(1),$$
$$(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1} = \frac{\overline{\zeta}_\varepsilon}{i(k + \frac{\lambda}{\varepsilon})}\overline{\zeta}_\varepsilon^* + O(1),$$

where $O(1)$ is uniform in k and ε and, for any $V = (a, b, f_1, g_1, f_2, g_2)^t \in \mathbb{H}$

$$\zeta_\varepsilon^*(V) = d\left\{a - \rho g_{20} + \rho e^{\lambda} b + \rho\lambda \int_0^1 [if_2(y) - g_2(y)]e^{\lambda y}dy + \right.$$
$$\left. + \lambda \int_{-\infty}^0 [if_1(\varepsilon\tau) - g_1(\varepsilon\tau)]e^{\lambda\tau}d\tau\right\}. \tag{5.4}$$

The linear form ζ_ε^* satisfies for any $U \in \mathbb{D}$,

$$\zeta_\varepsilon^*(\overline{\zeta}_\varepsilon) = 0,\ \zeta_\varepsilon^*(\mathcal{L}_\varepsilon U) = \frac{i\lambda}{\varepsilon}\zeta_\varepsilon^*(U),\ \zeta_\varepsilon^*(\xi_0) = 0,\ \zeta_\varepsilon^*(\xi_1) = 0,$$
$$\zeta_\varepsilon^*(SV) = \overline{\zeta}_\varepsilon^*(V), \quad \text{for all } V \in \mathbb{H}.$$

Moreover, we have the following better estimates in $\mathcal{L}(\mathbb{H}, \mathbb{D}^{\mathcal{P}})$

$$\mathcal{P}(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1} = \frac{\zeta_\varepsilon^{(\mathcal{P})}}{i(k - \frac{\lambda}{\varepsilon})}\zeta_\varepsilon^* + O(\varepsilon),$$
$$\mathcal{P}(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1} = \frac{\overline{\zeta}_\varepsilon^{(\mathcal{P})}}{i(k + \frac{\lambda}{\varepsilon})}\overline{\zeta}_\varepsilon^* + O(\varepsilon),$$

where $O(\varepsilon)$ is uniform in k.

Defining the projection π_ε commuting with $\mathcal{L}_\varepsilon$

$$\pi_\varepsilon = \mathbb{I} - \zeta_\varepsilon\zeta_\varepsilon^* - \overline{\zeta}_\varepsilon\overline{\zeta}_\varepsilon^*,$$

and restricting the resolvent to the subspace $\ker\zeta_\varepsilon^* \cap \ker\overline{\zeta}_\varepsilon^* = range(\pi_\varepsilon)$, we have for any $\delta > 0$, the existence of c independent of k and ε such that, for $\varepsilon|k| > \delta$

$$||(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1}\pi_\varepsilon||_{\mathcal{L}(\mathbb{H})} \leq c/|k|,$$
$$||(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1}\pi_\varepsilon||_{\mathcal{L}(\mathbb{H},\mathbb{D})} \leq c,$$
$$||\mathcal{P}(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1}\pi_\varepsilon||_{\mathcal{L}(\mathbb{H},\mathbb{D}^{\mathcal{P}})} \leq c\varepsilon.$$

We notice that $\zeta_\varepsilon^* \in \mathbb{H}^*$ (i.e. is a linear bounded form on $\mathbb{H}$), C^0 function of ε [because f_1 and $g_1 \in C^0(\mathbb{R}^-)$], while ζ_ε^* is analytic in ε, if restricted to $\mathbb{H}^{\mathcal{P}}$ (recall that $\lambda = (1-\varepsilon)(1-\rho)^{-1}$). For obtaining the estimate of the remaining term of the resolvent, of order $O(1)$ in $\mathcal{L}(\mathbb{H},\mathbb{D})$, we use the fact that $e^{\frac{\lambda y}{\varepsilon}}$ has a norm in $C_1^1(\mathbb{R}^-)$ of order $1/\varepsilon$.

(d) Study of the resolvent near 0

Let us define $\widetilde{\rho}$ and $\widetilde{\Delta}(|k|,\varepsilon)$ by

$$\widetilde{\rho} = \frac{\rho-\varepsilon}{1-\varepsilon} = 1-\frac{1}{\lambda} = \rho - \frac{\varepsilon}{\lambda},$$
$$\Delta = \Delta(\varepsilon|k|,\varepsilon) = \varepsilon^2|k|\lambda\widetilde{\Delta}(|k|,\varepsilon),$$
$$\widetilde{\Delta}(|k|,\varepsilon) = 1+\widetilde{\rho}|k| + (\frac{1}{3}-\frac{\rho}{\lambda})\varepsilon k^2 + O[\varepsilon^2k^2(1+|k|)].$$

The following lemma describes the resolvent near 0:

Lemma 5.4. *For any $V=(a,b,f_1,g_1,f_2,g_2)^t \in \mathbb{H}$, and any $k \in \mathbb{R}\backslash\{0\}$, we have the following asymptotic expansion for $\varepsilon \to 0^+$, which is uniformly valid in $\mathbb{H}$ for $\varepsilon(1+|k|)<\delta$:*

$$\begin{aligned}(ik-\mathcal{L}_\varepsilon)^{-1}V = &\left\{-(i\varepsilon k\widetilde{\Delta}(|k|,\varepsilon))^{-1}[\xi_{\varepsilon,k}^*(V)] + [\Xi^*(V)]\,(k)\right\}\xi_0+ \\ &+\frac{1}{1+\widetilde{\rho}|k|}\xi_\varepsilon^*(V)\,(y\xi_2-\widetilde{\rho}\chi_k) + \frac{\varepsilon}{1+\widetilde{\rho}|k|}\left[\frac{b}{\lambda}-\int_0^1 g_2(\tau)d\tau\right]\chi_k+ \\ &+\varepsilon\widetilde{\Phi}(f_2,g_2) + [\Phi(f_1,g_1)]\,(k) + O\left(\frac{\varepsilon|k|}{1+|k|}||V||_{\mathbb{H}}\right),\end{aligned} \tag{5.5}$$

where a uniformly valid estimate in $\mathbb{D}$ is obtained in replacing the $O(\cdot)$ term, by $O\left(\frac{\varepsilon|k|^2}{1+|k|}||V||_{\mathbb{H}}\right)$ for components α_1 and β_1, and by $O\left(\frac{\varepsilon^2|k|^2}{1+|k|}||V||_{\mathbb{H}}\right)$ in α_2 and β_2, (no change for the two first components). In the above expression, we use the notations

$$[\xi_{\varepsilon,k}^*(V)] = \xi_\varepsilon^*(V) + ik\int_{-\infty}^0 [f_1(\tau)+i(sgnk)g_1(\tau)]e^{|k|\tau}d\tau, \tag{5.6}$$

$$[\Xi^*(V)]\,(k) = (1+\widetilde{\rho}|k|)^{-1}\left\{i(sgnk)\left[\frac{b}{\lambda}-\int_0^1 g_2(\tau)d\tau\right] + \int_0^1 [\tau(1-\varepsilon)-\rho]f_2(\tau)d\tau\right\},$$

$$\begin{aligned}\chi_k &= \left(i(sgnk),0,e^{|k|\underline{y}},i(sgnk)e^{|k|\underline{y}},1,0\right)^t \in \mathbb{D}, \\ y\xi_2 &= (0,0,0,0,y,0)^t \in \mathbb{D}, \\ \widetilde{\Phi}(f_2,g_2) &= \left(0,0,0,0,\int_0^y g_2(\tau)d\tau, \int_y^1 f_2(\tau)d\tau\right)^t \in \mathbb{D},\end{aligned}$$

$$\begin{aligned}[\Phi(f_1,g_1)]\,(k) = &\left(-\frac{1}{1+\widetilde{\rho}|k|}\int_{-\infty}^0 [f_1(\tau)+i(sgnk)g_1(\tau)]e^{|k|\tau}d\tau, 0, \widetilde{H}_1[f_1,g_1](k,\underline{y}),\right. \\ &\left.\widetilde{K}_1[f_1,g_1](k,\underline{y}), \frac{ik(y-\widetilde{\rho})}{1+\widetilde{\rho}|k|}\int_{-\infty}^0 [f_1(\tau)+i(sgnk)g_1(\tau)]e^{|k|\tau}d\tau, 0\right)^t,\end{aligned}$$

$$\widetilde{H}_1[f_1,g_1](k,\underline{y}) = H_1[f_1,g_1](k,\underline{y}) - \frac{i\widetilde{\rho}|k|}{1+\widetilde{\rho}|k|}\int_{-\infty}^{0}[(sgnk)f_1(\tau)+ig_1(\tau)]e^{|k|(\tau+\underline{y})}d\tau,$$
$$\widetilde{K}_1[f_1,g_1](k,\underline{y}) = K_1[f_1,g_1](k,\underline{y}) + \frac{\widetilde{\rho}|k|}{1+\widetilde{\rho}|k|}\int_{-\infty}^{0}[f_1(\tau)+i(sgnk)g_1(\tau)]e^{|k|(\tau+\underline{y})}d\tau.$$

Notice that in this expansion we do not expand with respect to k, to keep track of the exponentials [the choice of the scaling is crucial here, and notice that $||\chi_k||_{\mathbb{D}} = O(1+|k|)$].

Notice also that the principal part for $k \to 0$ is given by

$$(ik-\mathcal{L}_\varepsilon)^{-1}V \sim -(i\varepsilon k)^{-1}[\xi_\varepsilon^*(V)]\xi_0,$$

which corresponds to the pole at the double 0 eigenvalue, but only in the direction of ξ_0. Moreover *the projection* $-\varepsilon^{-1}\xi_0\xi_\varepsilon^*$ *on the eigenvector* ξ_0 *becomes singular as* $\varepsilon \to 0$, since its norm diverges. We also notice that if $\xi_\varepsilon^*(V) = 0$, there is still a jump in the resolvent as k crosses 0. Let us define the following linear form, only bounded if g_1 decays sufficiently fast as $\underline{y} \to -\infty$

$$\eta_\varepsilon^*(V) = \int_0^1 g_2(\tau)d\tau + \varepsilon^{-1}\int_{-\infty}^{0} g_1(\tau)d\tau - \frac{b}{\lambda}. \tag{5.7}$$

The jump in the resolvent then disappears if $\eta_\varepsilon^*(V) = 0$ (see the β_1 and β_2 components). Indeed, if we impose a decay towards 0 as $\underline{y} \to -\infty$, the two conditions $\xi_\varepsilon^*(V) = 0$ and $\eta_\varepsilon^*(V) = 0$ are necessary conditions for V being in the range of $\mathcal{L}_\varepsilon$.

In fact, in the following we do not use the projection $-\varepsilon^{-1}\xi_0\xi_\varepsilon^*$ because of its diverging norm as $\varepsilon \to 0$. Let us define the uniformly bounded linear forms p_0^* and p_1^* defined for any $V = (a,b,f_1,g_1,f_2,g_2)^t \in \mathbb{H}$, by

$$p_0^*(V) = g_{21} = g_2|_{y=1}, \tag{5.8}$$
$$p_1^*(V) = a. \tag{5.9}$$

We check that

$$p_0^*(\xi_0) = 1, \quad p_0^*(\xi_1) = 0, \quad p_0^*(\underline{\zeta}_\varepsilon) = p_0^*(\overline{\underline{\zeta}}_\varepsilon) = e^\lambda,$$
$$p_1^*(\xi_0) = 0, \quad p_1^*(\xi_1) = 1, \quad p_1^*(\underline{\zeta}_\varepsilon) = p_1^*(\overline{\underline{\zeta}}_\varepsilon) = 1.$$

Now, the linear operators $\xi_0 p_0^*$ and $\xi_1 p_1^*$ are projections on the subspaces, respectively spanned by ξ_0, and ξ_1, which are bounded uniformly in ε, as well in $\mathbb{H}$ as in $\mathbb{D}$.

Proof of lemma 5.4. For $\varepsilon \to 0^+$ we have (uniform estimates for $\varepsilon|k| < \delta$)

$$\widetilde{\Delta}(|k|,\varepsilon)^{-1} = (1+\widetilde{\rho}|k|)^{-1} + O\left\{\varepsilon|k|^2(1+|k|)^{-2}\right\}$$

and (below the formulas are for $k > 0$)

$$\Delta(\varepsilon k,\varepsilon)\underline{A} = \varepsilon^2 k[-\widetilde{\rho}\lambda a_{10} + \rho b_{10}] + O\left(\varepsilon^3 k^2 ||V||_{\mathbb{H}}\right),$$
$$\Delta(\varepsilon k,\varepsilon)\underline{B} = \varepsilon[\lambda a_{10} - (1-\varepsilon)b_{10}] +$$
$$+ \varepsilon^2 k[\lambda a_{11} + b_{10} - (1-\varepsilon)b_{11}] + O\left(\varepsilon^3 k^2 ||V||_{\mathbb{H}}\right),$$

$$a_{10} = a - \rho g_{20} + \rho \int_0^1 g_2(\tau)d\tau + ik\int_{-\infty}^0 [f_1(\tau) + ig_1(\tau)]e^{k\tau}d\tau,$$
$$a_{11} = -i\rho \int_0^1 (\tau - 1)f_2(\tau)d\tau,$$

$$b_{10} = b - \int_0^1 g_2(\tau)d\tau,$$
$$b_{11} = i\int_0^1 \tau f_2(\tau)d\tau,$$

$$H_2[f_2, g_2](k, y) = -\varepsilon y \int_0^1 g_2(\tau)d\tau + \varepsilon \int_0^y g_2(\tau)d\tau + O\left[\varepsilon^2 k(||f_2||_\infty + ||g_2||_\infty)\right],$$
$$iK_2[f_2, g_2](k, y) = -\frac{i}{k}\int_0^1 g_2(\tau)d\tau - \varepsilon \int_0^1 \tau f_2(\tau)d\tau + \varepsilon \int_y^1 f_2(\tau)d\tau + \\ + O\left[\varepsilon^2 k(||f_2||_\infty + ||g_2||_\infty)\right].$$

Then, for $k \in \mathbb{R}\backslash\{0\}$, and $\varepsilon(1 + |k|) < \delta$, we arrive to

$$\alpha_1(\underline{y}) = -\frac{\widetilde{\rho}}{1+\widetilde{\rho}|k|}[\xi_\varepsilon^*(V)]e^{|k|\underline{y}} - \frac{\varepsilon}{1+\widetilde{\rho}|k|}\left[\int_0^1 g_2(\tau)d\tau - \frac{b}{\lambda}\right]e^{|k|\underline{y}} + \\ + \widetilde{H}_1[f_1, g_1](k, \underline{y}) + O\left(\frac{\varepsilon|k|e^{|k|\underline{y}}}{1+|k|}||V||_{\mathbb{H}}\right),$$
$$\beta_1(\underline{y}) = -\frac{i(sgnk)\widetilde{\rho}}{1+\widetilde{\rho}|k|}[\xi_\varepsilon^*(V)]e^{|k|\underline{y}} - \frac{i(sgnk)\varepsilon}{1+\widetilde{\rho}|k|}\left[\int_0^1 g_2(\tau)d\tau - \frac{b}{\lambda}\right]e^{|k|\underline{y}} + \\ + \widetilde{K}_1[f_1, g_1](k, \underline{y}) + O\left(\frac{\varepsilon|k|e^{|k|\underline{y}}}{1+|k|}||V||_{\mathbb{H}}\right),$$

$$\alpha_2(y) = \frac{[y - \widetilde{\rho}]}{1+\widetilde{\rho}|k|}[\xi_{\varepsilon,k}^*(V)] + \varepsilon\int_0^y g_2(\tau)d\tau - \frac{\varepsilon}{1+\widetilde{\rho}|k|}\left[\int_0^1 g_2(\tau)d\tau - \frac{b}{\lambda}\right] + \\ + O\left\{\varepsilon|k|(1+|k|)^{-1}||V||_{\mathbb{H}}\right\},$$
$$\beta_2(y) = \frac{-1}{i\varepsilon k\widetilde{\Delta}}[\xi_{\varepsilon,k}^*(V)] - \frac{i(sgnk)}{1+\widetilde{\rho}|k|}\left[\int_0^1 g_2(\tau)d\tau - \frac{b}{\lambda}\right] + \\ + \left\{\frac{1}{1+\widetilde{\rho}|k|}\int_0^1[\tau(1-\varepsilon) - \rho]f_2(\tau)d\tau + \varepsilon\int_y^1 f_2(\tau)d\tau\right\} + O\left(\frac{\varepsilon|k|}{1+|k|}||V||_{\mathbb{H}}\right),$$

The linear form $\xi_{\varepsilon,k}^*$ is a bounded linear form in $\mathbb{H}$, uniformly bounded in k. If V has its components f_1 and g_1 sufficiently decaying as $\underline{y}\to -\infty$ then $\xi_{\varepsilon,k}^*(V) \sim \xi_\varepsilon^*(V)$ [see (4.11)] for $k = 0$. In fact, in a further section, we need to apply the resolvent to $V \in \mathbb{H}$ such that the components f_1 and g_1 decay exponentially fast (in $e^{\lambda\underline{y}/\varepsilon}$) as $\underline{y}\to -\infty$; in such cases we have a precise behavior of $\widetilde{H}_1[f_1, g_1]$, $\widetilde{K}_1[f_1, g_1]$, $[\xi_{\varepsilon,k}^*(V) - \xi_\varepsilon^*(V)]$ as $k \to 0$.

The estimates in the rests for α_1, β_1 and α_2, β_2 are uniform for $\varepsilon(1 + |k|) < \delta$, when $\varepsilon \to 0$. We then obtain the asymptotic expression for $(ik - \mathcal{L}_\varepsilon)^{-1}$, with estimates in $\mathbb{H}$ and in $\mathbb{D}$, and the lemma is proved.

(e) Study of the range of $\mathcal{L}_\varepsilon$

Let us give more details on the range of the operator $\mathcal{L}_\varepsilon$. We use extensively two projections $\underline{\pi}_\varepsilon$ and $\widetilde{\pi}_\varepsilon$ defined for any $V \in \mathbb{H}$ by

$$\begin{aligned}\underline{\pi}_\varepsilon V &= \pi_\varepsilon V - p_0^*(\pi_\varepsilon V)\xi_0,\\ \widetilde{\pi}_\varepsilon V &= \pi_\varepsilon V - p_0^*(\pi_\varepsilon V)\xi_0 - p_1^*(\pi_\varepsilon V)\xi_1,\end{aligned}$$

where we defined already π_ε at lemma 5.3. We also need in the following, the range of the reduced linear operator acting on the subspace $\widetilde{\pi}_\varepsilon\mathbb{H}$. Finally we need the range of $\mathcal{L}_\varepsilon$ on a subspace where the components f_1 and g_1 are rapidly decaying. For this result we need to introduce new spaces, which will be also useful at next section. So we define

$$C_\varepsilon^{k,\exp} = \{f : \mathbb{R}^- \to \mathbb{C}; e^{-\lambda y/2\varepsilon} f \in C^k(\mathbb{R}^-)\}$$

equipped with the norms

$$\begin{aligned}||f||_{0,\varepsilon}^{\exp} &= \sup_{y\in\mathbb{R}^-} |e^{-\lambda y/2\varepsilon} f(y)|, \text{ for } k = 0,\\ ||f||_{1,\varepsilon}^{\exp} &= \varepsilon||f'||_{0,\varepsilon}^{\exp} + ||f||_{0,\varepsilon}^{\exp}, \text{ for } k = 1,\end{aligned}$$

where we notice the factor ε in the second norm. We also notice that for ε small enough

$$||f||_{1,\infty} \leq ||f||_{0,\varepsilon}^{\exp}.$$

We also define the Banach spaces

$$\begin{aligned}\mathbb{K}_\varepsilon &= \{U \in \mathbb{H}; \alpha_1^*(U) \in C_\varepsilon^{0,\exp}, \beta_1^*(U) \in C_\varepsilon^{0,\exp}\},\\ \mathbb{E}_\varepsilon &= \{U \in \mathbb{D}; \alpha_1^*(U) \in C_\varepsilon^{1,\exp}, \beta_1^*(U) \in C_\varepsilon^{1,\exp}\},\\ \mathbb{F}_\varepsilon &= \{U \in \mathbb{H}; \alpha_1^*(U) \in C_\varepsilon^{1,\exp}, \beta_1^*(U) \in C_\varepsilon^{1,\exp}\},\end{aligned} \tag{5.10}$$

with the appropriate norms

$$\begin{aligned}||U||_{\mathbb{K}_\varepsilon} &= ||\mathcal{P}U||_{\mathbb{H}} + ||\alpha_1^*(U)||_{0,\varepsilon}^{\exp} + ||\beta_1^*(U)||_{0,\varepsilon}^{\exp},\\ ||U||_{\mathbb{E}_\varepsilon} &= ||\mathcal{P}U||_{\mathbb{D}} + ||\alpha_1^*(U)||_{1,\varepsilon}^{\exp} + ||\beta_1^*(U)||_{1,\varepsilon}^{\exp},\\ ||U||_{\mathbb{F}_\varepsilon} &= ||\mathcal{P}U||_{\mathbb{H}} + ||\alpha_1^*(U)||_{1,\varepsilon}^{\exp} + ||\beta_1^*(U)||_{1,\varepsilon}^{\exp},\end{aligned}$$

and where we denote for example by $\alpha_1^*(U)$ the α_1 component of U. We notice that ξ_0 and $\underline{\zeta}_\varepsilon \in \mathbb{E}_\varepsilon$ with a uniformly bounded norm, as $\varepsilon \to 0$, while $\xi_1 \notin \mathbb{E}_\varepsilon$. We also notice that we have the following continuous embeddings

$$\mathbb{E}_\varepsilon \hookrightarrow \mathbb{F}_\varepsilon \hookrightarrow \mathbb{K}_\varepsilon.$$

These spaces are useful in the sequel when we treat the rests originated from the nonlinear terms of the system. We observe that the 3rd and 4th components of vectors have an exponential decay in y, which is a strong restriction with respect to the decay or boundedness required in $\mathbb{H}$ and $\mathbb{D}$.

A consequence is that the projections π_ε and $\underline{\pi}_\varepsilon$ are bounded uniformly in ε in $\mathcal{L}(\mathbb{K}_\varepsilon)$, $\mathcal{L}(\mathbb{F}_\varepsilon)$, and in $\mathcal{L}(\mathbb{E}_\varepsilon)$, whereas this is not the case for the projection $\widetilde{\pi}_\varepsilon$.

Let us denote $\underline{U}$ and V as

$$\underline{U} = (\beta_{10}, \beta_{21}, \alpha_1, \beta_1, \alpha_2, \beta_2)^t \in \mathbb{D},$$
$$V = (a, b, f_1, g_1, f_2, g_2)^t \in \mathbb{H},$$

then we state

Lemma 5.5. *(i) The range of* $\mathcal{L}_\varepsilon$ *in* $\mathbb{H}$ *is the set of* V *such that* $\xi^*_\varepsilon(V) = 0$, *and*

$$\int_{\underline{y}}^0 f_1(\tau)d\tau \in C^0_{\lim,1}, \text{ and } g_1 \in C^0_1, \int_{\underline{y}}^0 g_1(\tau)d\tau \in C^0_{\lim,1},$$

$$\eta^*_\varepsilon(V) = 0,$$

where ξ^*_ε *and* η^*_ε *are defined in (4.11),(5.7).*

(ii) The range of $\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon$ *in* $\widetilde{\pi}_\varepsilon \mathbb{H}$ *is the set of* $Z \in \widetilde{\pi}_\varepsilon \mathbb{H}$, *such that*

$$\int_{\underline{y}}^0 f_1(\tau)d\tau \in C^0_{\lim,1}, \quad \int_{\underline{y}}^0 [g_1(\tau) - \lim g_1]d\tau \in C^0_{\lim,1},$$

and

$$\chi^*_\varepsilon(Z) = 0,$$

where the linear form χ^*_ε *is defined by*

$$\chi^*_\varepsilon(Z) \stackrel{def}{=} \int_{-\infty}^0 [g_1(\tau) - \lim g_1]d\tau + \varepsilon \left[\int_0^1 g_2(\tau)d\tau - b/\lambda\right] + $$
$$+ \widetilde{\rho}[\xi^*_\varepsilon(Z) - \lim g_1].$$

(iii) The range of $\underline{\pi}_\varepsilon \mathcal{L}_\varepsilon$ *in* $\underline{\pi}_\varepsilon \mathbb{K}_\varepsilon$ *is the set of* $Z \in \underline{\pi}_\varepsilon \mathbb{K}_\varepsilon$, *such that*

$$\chi^*_\varepsilon(Z) = 0,$$

where the linear form χ^*_ε *reduces to*

$$\chi^*_\varepsilon(Z) \stackrel{def}{=} \int_{-\infty}^0 g_1(\tau)d\tau + \varepsilon \left[\int_0^1 g_2(\tau)d\tau - b/\lambda\right] + \widetilde{\rho}\xi^*_\varepsilon(Z).$$

Proof of (i). This is straightforward, and it is the same as the proof in (Iooss 1999).

Proof of (ii). Looking at the range of $\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon$ means that we solve

$$\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon W = Z \in \widetilde{\pi}_\varepsilon \mathbb{H},$$

where

$$\zeta^*_\varepsilon(Z) = \overline{\zeta}^*_\varepsilon(Z) = p^*_0(Z) = p^*_1(Z) = 0,$$

and where we look for $W \in \widetilde{\pi}_\varepsilon \mathbb{D}$ such that

$$\zeta^*_\varepsilon(W) = \overline{\zeta}^*_\varepsilon(W) = p^*_0(W) = p^*_1(W) = 0.$$

Since $\underline{\zeta}_\varepsilon\zeta_\varepsilon^*$ is a projection commuting with $\mathcal{L}_\varepsilon$, this implies the existence of c_0 and c_1 such that

$$\mathcal{L}_\varepsilon W = Z + c_0\xi_0 + c_1\xi_1.$$

Hence we obtain

$$\begin{aligned}
\alpha_1(\underline{y}) &= -\int_0^{\underline{y}} g_1(\tau)d\tau + \varepsilon\left[\int_0^1 g_2(\tau)d\tau - b/\lambda\right] + \varepsilon\widetilde{\rho}c_0 - c_1\underline{y},\\
\beta_1(\underline{y}) &= \int_0^{\underline{y}} f_1(\tau)d\tau,\\
\alpha_2(y) &= \varepsilon\left[\int_y^1 g_2(\tau)d\tau - b/\lambda\right] + \varepsilon c_0(\widetilde{\rho} - y),\\
\beta_2(y) &= -\varepsilon\int_y^1 f_2(\tau)d\tau,
\end{aligned}$$

with

$$\varepsilon c_0 - c_1 = \xi_\varepsilon^*(Z).$$

It is then clear that the condition $\int_0^{\underline{y}}[g_1(\tau) + c_1]d\tau$ bounded, defines $c_1 = -\lim g_1$ (exists since $g_1 \in C^0_{\lim,1}$), hence c_0 is also uniquely defined, and the part (ii) of the lemma is straightforward.

Proof of (iii). We modify the proof made for (ii), in relaxing the conditions with p_1^* on Z and W. Instead we impose an exponential decay for α_1 and β_1. We look for W in $\underline{\pi}_\varepsilon\mathbb{E}_\varepsilon$ and

$$\underline{\pi}_\varepsilon\mathcal{L}_\varepsilon W = Z \in \underline{\pi}_\varepsilon\mathbb{K}_\varepsilon,$$

which leads to

$$\begin{aligned}
\alpha_1(\underline{y}) &= -\int_0^{\underline{y}} g_1(\tau)d\tau + \varepsilon\left[\int_0^1 g_2(\tau)d\tau - b/\lambda\right] + \widetilde{\rho}\xi_\varepsilon^*(Z),\\
\beta_1(\underline{y}) &= -\int_{-\infty}^{\underline{y}} f_1(\tau)d\tau,\\
\alpha_2(y) &= \varepsilon\left[\int_y^1 g_2(\tau)d\tau - b/\lambda\right] + \xi_\varepsilon^*(Z)(\widetilde{\rho} - y),\\
\beta_2(y) &= -\varepsilon\int_y^1 f_2(\tau)d\tau,
\end{aligned}$$

and the condition for having α_1 decaying to 0 (same exponential as f_1 and g_1) at infinity, is then $\chi_\varepsilon^*(Z) = 0$, where χ_ε^* is defined in the lemma, and we notice that

$$\chi_\varepsilon^* = \varepsilon\eta_\varepsilon^* + \widetilde{\rho}\xi_\varepsilon^*$$

[see (5.7)]. Hence, part (iii) of the lemma is proved. We observe in addition that the integrals $\int_{-\infty}^{\underline{y}} f_1(\tau)d\tau$ and $\int_{-\infty}^{\underline{y}} g_1(\tau)d\tau$ are bounded in $C_\varepsilon^{0,\exp}$, with $O(\varepsilon)$, hence

$$\xi_\varepsilon^*(Z) = -\widetilde{\rho}^{-1}\left[\int_{-\infty}^0 g_1(\tau)d\tau + \varepsilon\int_0^1 g_2(\tau)d\tau - \varepsilon b/\lambda\right]$$

verifies

$$|\xi_\varepsilon^*(Z)| \leq c\varepsilon||Z||_{\mathbb{K}_\varepsilon},$$

hence

$$||W||_{\mathbb{E}_\varepsilon} \leq c\varepsilon ||Z||_{\mathbb{K}_\varepsilon}.$$

6. Periodic solutions

In this section, we study periodic solutions of (4.1). First, we give at theorem 6.1, periodic solutions of (4.4,4.5) provided we define correctly the condition (4.5), i.e. without mentioning 3rd and 4th components of $\underline{U}$ [we denote these vectors by W or with an upper index $^{(\mathcal{P})}$]. The analyticity in ε disappears in incorporating the two missing components because of the non analyticity of $\underline{\zeta}_\varepsilon$ (see (4.3)) for ε near 0 contrary to $\underline{\zeta}_\varepsilon^{(\mathcal{P})}$ (see (4.6)). The result given at theorem 6.3 comes from the structure of the eigenvectors belonging to the eigenvalues $\pm i\lambda/\varepsilon$, and from the structure of the third and fourth equations in (4.1). Theorems 6.1 and 6.3 are analogous to the classical Lyapunov-Devaney theorem which ensures the existence of a one-parameter family of periodic solutions, bifurcating from a pair of simple imaginary eigenvalues for reversible vector fields in finite dimensions. In the present case, there are two extra difficulties: i) 0 lies in the spectrum of $\mathcal{L}_\varepsilon$ and is resonant with the pair of eigenvalues $\pm i\lambda/\varepsilon$, ii) 0 also lies in the essential spectrum of $\mathcal{L}_\varepsilon$, since the entire real line constitutes the essential spectrum.

In this section we use the spaces

$$\mathbb{H}^T_{\sharp,AS} = \{V \in H^1_{\sharp,T}(\mathbb{H}); \widehat{S}V = -V\},$$
$$\mathbb{D}^T_{\sharp,S} = \left\{U \in H^2_{\sharp,T}(\mathbb{H}) \cap H^1_{\sharp,T}(\mathbb{D}); \widehat{S}U = U\right\},$$

where, for any Banach space E, $H^m_{\sharp,T}(E) = \{u \in H^m_{loc}(E); u(s+T) - u(s) = 0$ in E for almost all $s\}$. The superscript T may be omitted if there is no ambiguity about the spatial period. We put in the above spaces an index S or an index AS when we restrict respectively to vector functions such that $\widehat{S}V = V$, or $\widehat{S}V = -V$. If $U \in \mathbb{D}_{\sharp,S}$, then $\mathcal{L}_\varepsilon U \in \mathbb{H}_{\sharp,AS}$, $\mathcal{N}(\varepsilon; U) \in \mathbb{H}_{\sharp,AS}$. Notice that if $W \in \mathbb{D}^{\mathcal{P}}_{\sharp,S}$, then $\mathcal{L}^{(\mathcal{P})}_\varepsilon W \in \mathbb{H}^{\mathcal{P}}_{\sharp,AS}$, $\mathcal{N}^{(\mathcal{P})}(\varepsilon; W) \in \mathbb{H}^{\mathcal{P}}_{\sharp,AS}$, and $\mathcal{D}(W) \in H^2_{\sharp,S}(\mathbb{R})$. Let us state the following

Theorem 6.1. *For any constant $M > 0$, there exists $\varepsilon_0 > 0$ such that for any $(u_0, v_0, A_0, \varepsilon) \in \mathbb{R}^2 \times \mathbb{C} \times \mathbb{R}$ satisfying*

$$|u_0| + |v_0| + |A_0| \leq M, \quad 0 < \varepsilon < \varepsilon_0,$$

there is a family of periodic solutions $p^{(\mathcal{P})}_{A_0,u_0,v_0,\varepsilon}$ of (4.4,4.5) in $\mathbb{D}^{\mathcal{P},\underline{T}}_\sharp$, bifurcating from 0, where $\underline{T}$ denotes the period, and

$$p^{(\mathcal{P})}_{A_0,u_0,v_0,\varepsilon}(\underline{x}) = \widehat{p}^{(\mathcal{P})}_{A_0,u_0,v_0,\varepsilon}(s)$$

possesses the following power series in $\varepsilon, u_0, v_0, A_0, \overline{A}_0$ converging in $\mathbb{D}^{\mathcal{P},2\pi}_\sharp$:

$$\widehat{p}^{(\mathcal{P})}_{A_0,u_0,v_0,\varepsilon}(s) = u_0\xi^{(\mathcal{P})}_0 + v_0\xi^{(\mathcal{P})}_1 + A_0 e^{is}\zeta^{(\mathcal{P})}_\varepsilon + \overline{A}_0 e^{-is}\overline{\zeta}^{(\mathcal{P})}_\varepsilon + \\ + \sum_{\substack{p+q\geq 1 \\ 2\leq n+m+p+q\leq r+1}} \varepsilon^r u_0^n v_0^m A_0^p \overline{A}_0^q e^{i(p-q)s} Y^{(\mathcal{P})}_{nmpqr}$$

where

$$s = \varepsilon^{-1}[\lambda+\gamma]\underline{x}, \quad \underline{T} = \frac{2\pi\varepsilon}{\lambda+\gamma},$$
$$\gamma = \sum_{1\leq n+m+2p\leq r} \gamma_{nmpr} u_0^n v_0^m |A_0|^{2p}\varepsilon^r \in \mathbb{R},$$
$$\zeta_\varepsilon^{(\mathcal{P})} = (1, e^\lambda, -ie^{\lambda y}, e^{\lambda y})^t,$$
$$\zeta_\varepsilon^*(Y_{nmpqr}^{(\mathcal{P})}) = 0 \text{ for } p = q+1, \quad p_0^*(Y_{nmppr}^{(\mathcal{P})}) = 0, \; p_1^*(Y_{nmppr}^{(\mathcal{P})}) = 0.$$

These solutions are reversible for A_0 real, *and we have* $SY_{nmpqr}^{(\mathcal{P})} = Y_{nmqpr}^{(\mathcal{P})} = \overline{Y}_{nmpqr}^{(\mathcal{P})}$.

Proof. Let us set $s = \varepsilon^{-1}(\lambda+\gamma)\underline{x}$, where $\lambda = (1-\varepsilon)/(1-\rho)$, where γ is close to 0, and $(\lambda+\gamma)\varepsilon^{-1}$ is the wave number in coordinate $\underline{x}$ of the periodic solution we are looking for. We then look for ($2\pi-$periodic) functions of s. Let us precise the norm we choose in the Banach space $H_\sharp^p(E)$, the space of ($2\pi-$periodic) functions such that their derivatives up to order p are in $L^2(\mathbb{R}/2\pi\mathbb{Z})$, taking values in the Banach space E:

$$||u||_{H_\sharp^p}^2 = \sum_{n\in\mathbb{Z}}(1+n^{2p})||u_n||_E^2$$

where

$$u_n = \frac{1}{2\pi}\int_0^{2\pi} u(s)e^{-nis}ds.$$

Let us define the linear operator $T_\varepsilon = \lambda\frac{d}{ds} - (\mathcal{A}_0 + \varepsilon\mathcal{A}_1)$ which maps $\mathbb{D}_\sharp^{\mathcal{P}}$ into $\mathbb{H}_\sharp^{\mathcal{P}}$. The basic tool is the following

Lemma 6.2. *For any given* V *in* $\mathbb{H}_\sharp^{\mathcal{P}}$, *such that*

$$\int_0^{2\pi} \zeta_\varepsilon^*[V(s)]e^{-is}ds = \int_0^{2\pi} \overline{\zeta}_\varepsilon^*[V(s)]e^{is}ds = \int_0^{2\pi} \xi_\varepsilon^*[V(s)]ds = 0,$$

there exists a unique W *in* $\mathbb{D}_\sharp^{\mathcal{P}}$ *such that*

$$T_\varepsilon W = V, \quad \mathcal{D}(W) = 0, \tag{6.1}$$

$$\int_0^{2\pi} \zeta_\varepsilon^*[W(s)]e^{-is}ds = \int_0^{2\pi} \overline{\zeta}_\varepsilon^*[W(s)]e^{is}ds = 0,$$
$$\int_0^{2\pi} p_0^*[W(s)]ds = \int_0^{2\pi} p_1^*[W(s)]ds = 0,$$

where ζ_ε^* *and* $\overline{\zeta}_\varepsilon^*$ *are given by (5.4). Moreover, the linear mapping* $V \mapsto W = \widetilde{T}_\varepsilon^{-1}V$ *is bounded:*

$$||\widetilde{T}_\varepsilon^{-1}V||_{\mathbb{D}_\sharp^{\mathcal{P}}} \leq c||V||_{\mathbb{H}_\sharp^{\mathcal{P}}}.$$

Important remark. In the above lemma the condition $\mathcal{D}(W) = 0$ only applies on Fourier coefficients of non-zero index. Indeed, for the average $[W]_0 = \frac{1}{2\pi}\int_0^{2\pi} W(s)ds$ we have no condition on β_{10}.

Proof of lemma 6.2. Expanding in Fourier series V and W, equation (6.1) gives for $n \neq 0, 1, -1$

$$W_n e^{ins} = [in\lambda\mathbb{I} - (\mathcal{A}_0 + \varepsilon\mathcal{A}_1)]^{-1} V_n e^{ins},$$
$$W_n \in \mathbb{D}^{\mathcal{P}} \cap \{\beta_{10} = isgn(n)\alpha_{20}\}.$$

Now using the resolvent estimate for $|k|$ large (see (5.3) at lemma 5.2 of section 5.(b)), this insures that, if we define

$$W' = \sum_{n\in\mathbb{Z}\setminus\{0,1,-1\}} W_n e^{nis} \in \mathbb{D}^{\mathcal{P}}_{\sharp}, \text{ then } ||W'||_{\mathbb{D}^{\mathcal{P}}_{\sharp}} \leq C_1 ||V||_{\mathbb{H}^{\mathcal{P}}_{\sharp}} \text{ holds.} \tag{6.2}$$

It then remains to study the equations $[in\lambda\mathbb{I} - (\mathcal{A}_0 + \varepsilon\mathcal{A}_1)]W_n = V_n$ for $n = 0, 1, -1$.

For $n = 0$, we have

$$-(\mathcal{A}_0 + \varepsilon\mathcal{A}_1)W_0 = V_0 \in \mathbb{H}^{\mathcal{P}}.$$

If we note $V_0 = (a, b, f_2, g_2)^t$, the following compatibility condition has to be satisfied (see lemma 5.5)

$$\xi^*_\varepsilon(V_0) = 0, \tag{6.3}$$

where ξ^*_ε is defined in (4.11). All solutions such that $W_0 \in \mathbb{D}^{\mathcal{P}}$ are given by

$$W_0 = \widetilde{W_0} + k_0\xi^{\mathcal{P}}_0 + k_1\xi^{\mathcal{P}}_1, \tag{6.4}$$
$$\widetilde{W_0} = \left(0, 0, -\int_y^1 g_2(\tau)d\tau + \frac{b}{\lambda}, \int_y^1 f_2(\tau)d\tau\right)^t,$$

where k_0 and k_1 are arbitrary. Then, provided that $\xi^*_\varepsilon(V_0) = 0$ holds, there is a unique solution $W_0 = \widetilde{W_0} = -\widetilde{(\mathcal{A}_0 + \varepsilon\mathcal{A}_1)}^{-1} V_0$ which satisfies

$$-(\mathcal{A}_0 + \varepsilon\mathcal{A}_1)\widetilde{W_0} = V_0,$$
$$p^*_0(\widetilde{W_0}) = 0,$$
$$p^*_1(\widetilde{W_0}) = 0,$$

and we have the estimate

$$||\widetilde{W_0}||_{\mathbb{D}^{\mathcal{P}}} \leq c||V_0||_{\mathbb{H}^{\mathcal{P}}}.$$

This means that we have a pseudo-inverse of $-\varepsilon\mathcal{L}_\varepsilon$ acting from $\mathbb{H}^{\mathcal{P}} \cap \ker \xi^*_\varepsilon$ towards $\mathbb{D}^{\mathcal{P}} \cap \ker p^*_0 \cap \ker p^*_1$ with a uniformly bounded norm with respect to ε.

For $n = 1$, we need to use the linear forms ζ^*_ε and $\overline{\zeta}^*_\varepsilon$: for any $V = (a, b, 0, 0, f_2, g_2)^t$ lying in $\mathbb{H}_\sharp$, we define

$$Z^*_\varepsilon(V) = \frac{1}{2\pi}\int_0^{2\pi} e^{-is}\zeta^*_\varepsilon[\mathcal{P}V(s)]ds.$$

It is easy to check that for any $W \in \mathbb{D}^{\mathcal{P}}_\sharp$

$$Z^*_\varepsilon[(\mathcal{A}_0 + \varepsilon\mathcal{A}_1)W] = i\lambda Z^*_\varepsilon(W),$$
$$\xi^*_0[(\mathcal{A}_0 + \varepsilon\mathcal{A}_1)W] \equiv 0 \text{ (as a function of } s),$$

and

$$Z_\varepsilon^*(e^{is}\zeta_\varepsilon^{(\mathcal{P})}) = 1\ , \quad Z_\varepsilon^*(\widehat{S}V) = \overline{Z}_\varepsilon^*(V).$$

The necessary and sufficient conditions for finding the components W_1 and W_{-1} of the solution W in (6.1) may be written as

$$\begin{aligned} Z_\varepsilon^*(V) &= \zeta_\varepsilon^*(V_1) = 0, \\ \overline{Z}_\varepsilon^*(V) &= \overline{\zeta}_\varepsilon^*(V_{-1}) = 0, \end{aligned}$$

and W_1 and W_{-1} are respectively defined up to arbitrary multiples of the eigenvectors $\zeta_\varepsilon^{(\mathcal{P})}$ and $\overline{\zeta}_\varepsilon^{(\mathcal{P})} \in \mathbb{D}^{\mathcal{P}}$.

We define now two projections P_ε and $\widetilde{P}_\varepsilon$, as well on $\mathbb{H}_\sharp^{\mathcal{P}}$ as on $\mathbb{D}_\sharp^{\mathcal{P}}$, as follows, for any V and $W \in \mathbb{H}_\sharp^{\mathcal{P}}$ (identified with V and $W \in \mathbb{H}_\sharp$ with 3rd and 4th components identically 0):

$$\begin{aligned} W &= u_0\xi_0^{(\mathcal{P})} + v_0\xi_1^{(\mathcal{P})} + Ae^{is}\zeta_\varepsilon^{(\mathcal{P})} + Be^{-is}\overline{\zeta}_\varepsilon^{(\mathcal{P})} + Y, \qquad (6.5)\\ A &= Z_\varepsilon^*(W), \quad B = \overline{Z}_\varepsilon^*(W), \\ u_0 &= p_0^*(W_0), \quad v_0 = p_1^*(W_0), \\ Z_\varepsilon^*(W) &\stackrel{def}{=} \frac{1}{2\pi}\int_0^{2\pi} \zeta_\varepsilon^*[W(s)]e^{-is}ds, \\ Z_\varepsilon^*(Y) &= \overline{Z}_\varepsilon^*(Y) = p_0^*(Y_0) = p_1^*(Y_0) = 0, \\ Y &\stackrel{def}{=} P_\varepsilon W, \end{aligned}$$

$$\begin{aligned} V &= C\chi_0 + A'e^{is}\zeta_\varepsilon^{(\mathcal{P})} + B'e^{-is}\overline{\zeta}_\varepsilon^{(\mathcal{P})} + \widetilde{V}, \qquad (6.6)\\ A' &= Z_\varepsilon^*(V), \quad B' = \overline{Z}_\varepsilon^*(V), \\ Z_\varepsilon^*(V) &\stackrel{def}{=} \frac{1}{2\pi}\int_0^{2\pi} \zeta_\varepsilon^*[V(s)]e^{-is}ds, \\ C &= \xi_\varepsilon^*(V_0) = \frac{1}{2\pi}\int_0^{2\pi} \xi_\varepsilon^*[V(s)]ds, \\ \chi_0 &= (1,0,0,0)^t, \quad \xi_\varepsilon^*(\chi_0) = 1, \\ Z_\varepsilon^*(\widetilde{V}) &= \overline{Z}_\varepsilon^*(\widetilde{V}) = \xi_\varepsilon^*(\widetilde{V}_0) = 0, \\ \widetilde{V} &\stackrel{def}{=} \widetilde{P}_\varepsilon V, \end{aligned}$$

where, for instance, Y_0 denotes the 0-Fourier component of Y. We check that

$$\mathcal{D}(e^{is}\zeta_\varepsilon^{\mathcal{P}}) = 0, \quad \mathcal{D}(e^{-is}\overline{\zeta}_\varepsilon^{\mathcal{P}}) = 0,$$

hence $\mathcal{D}(W) = 0$ implies $\mathcal{D}(P_\varepsilon W) = 0$. Finally, we just built above, a (uniformly bounded in ε) pseudo-inverse $\widetilde{T}_\varepsilon^{-1}$of T_ε, from $\widetilde{P}_\varepsilon\mathbb{H}_\sharp^{\mathcal{P}}$ onto $P_\varepsilon\mathbb{D}_\sharp^{\mathcal{P}} \cap \ker\mathcal{D}$, and the lemma is proved.

Proof of theorem 6.1. Let us now consider the system (4.4, 4.5) where we look for solutions in $\mathbb{D}_\sharp^{\mathcal{P}}$. We can rewrite this system under the form of the following equation in $\mathbb{H}_\sharp^{\mathcal{P}}$:

$$T_\varepsilon W = G(\varepsilon, \gamma, W), \quad \mathcal{D}(W) = 0, \qquad (6.7)$$

where

$$G(\varepsilon,\gamma,W) = -\gamma\frac{dW}{ds} + \varepsilon\mathcal{N}^{(\mathcal{P})}(\varepsilon;W). \tag{6.8}$$

We observe that, due to the form of the scaling, (6.7) may be written as

$$\begin{aligned} T_\varepsilon W' &= G'(\varepsilon,\gamma,W'), \quad \mathcal{B}(W') = 0, \qquad (6.9)\\ G'(\varepsilon,\gamma,W') &= -\gamma\frac{dW'}{ds} + N^{\mathcal{P}}(\varepsilon;W'), \end{aligned}$$

where $W' = \varepsilon W$, and $G'(\cdot,\cdot,\cdot)$ is analytic: $\mathbb{R}^2\times\mathbb{D}_\sharp^{\mathcal{P}} \to \mathbb{H}_\sharp^{\mathcal{P}}$ in the neighborhood of 0, and is such that

$$G'(\varepsilon,\gamma,0) = 0, \qquad D_{(\gamma,W')}G'(\varepsilon,0,0) = 0.$$

It is also easy to check that the following properties hold:

$$\widehat{S}T_\varepsilon = -T_\varepsilon\widehat{S},\ \ \widehat{S}G'(\varepsilon,\gamma,W') = -G'(\varepsilon,\gamma,\widehat{S}W'),\ \ \mathcal{D}(\widehat{S}W') = \mathcal{D}(W')\circ\widehat{s}$$

where $\widehat{s}\underline{x} = -\underline{x}$.

We want to apply the decompositions (6.5,6.6) to the system (6.9), with

$$\begin{aligned} W' &= u_0'\xi_0^{(\mathcal{P})} + v_0'\xi_1^{(\mathcal{P})} + A_0'e^{is}\zeta_\varepsilon^{(\mathcal{P})} + \overline{A}_0'e^{-is}\overline{\zeta}_\varepsilon^{(\mathcal{P})} + Y',\\ Y' &= P_\varepsilon W', \end{aligned}$$

since W' is real. This leads to the system

$$T_\varepsilon Y' = -\gamma\frac{dY'}{ds} + \widetilde{P}_\varepsilon[N^{(\mathcal{P})}(\varepsilon;u_0'\xi_0^{(\mathcal{P})} + v_0'\xi_1^{(\mathcal{P})} + A_0'e^{is}\zeta_\varepsilon^{(\mathcal{P})} + \overline{A}_0'e^{-is}\overline{\zeta}_\varepsilon^{(\mathcal{P})} + Y')], \tag{6.10}$$

$$0 = -i\gamma A_0' + Z_\varepsilon^*\left[N^{(\mathcal{P})}(\varepsilon;u_0'\xi_0^{(\mathcal{P})} + v_0'\xi_1^{(\mathcal{P})} + A_0'e^{is}\zeta_\varepsilon^{(\mathcal{P})} + \overline{A}_0'e^{-is}\overline{\zeta}_\varepsilon^{(\mathcal{P})} + Y')\right], \tag{6.11}$$

$$0 = \xi_\varepsilon^*\left[N^{(\mathcal{P})}(\varepsilon;u_0'\xi_0^{(\mathcal{P})} + v_0'\xi_1^{(\mathcal{P})} + A_0'e^{is}\zeta_\varepsilon^{(\mathcal{P})} + \overline{A}_0'e^{-is}\overline{\zeta}_\varepsilon^{(\mathcal{P})} + Y')\right]_0. \tag{6.12}$$

Remark. In the proof of the classical Lyapunov-Devaney theorem, there are two equations similar to (6.10,6.11), the other compatibility condition comes from the fact that in the present case, the operator $\mathcal{L}_\varepsilon$ has 0 for eigenvalue, which is resonant with $\pm i\lambda/\varepsilon$. We show below that, roughly speaking, (6.10) gives Y', (6.11) gives γ, and fortunately (6.12) is automatically satisfied because of all equivariances of our system.

Using the pseudo-inverse, we defined above, we have first from (6.10) in using the analytic implicit function theorem

$$Y' = \mathcal{Y}(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0')$$

where $\mathcal{Y}$ is analytic in its arguments and, because of the fact that the identity

$$N^{(\mathcal{P})}(\varepsilon;u_0\xi_0^{(\mathcal{P})} + v_0\xi_1^{(\mathcal{P})}) \equiv 0$$

holds, we have the estimate

$$\mathcal{Y}(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') = O[|A_0'|(|u_0'|+|v_0',|+|A_0'|)]$$

with the following principal part

$$\mathcal{Y}(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') \sim \widetilde{T}_\varepsilon^{-1}\widetilde{P}_\varepsilon N_0^{(2)}\left[(u_0'\xi_0+v_0'\xi_1+A_0'e^{is}\underline{\zeta}_\varepsilon+\overline{A}_0'e^{-is}\overline{\underline{\zeta}}_\varepsilon)^{(2)}\right],$$

where we notice that

$$\begin{aligned} N_0^{(2)}(\xi_0,\xi_0) &= 0,\\ N_0^{(2)}(\xi_1,\xi_1) &= 0,\\ N_0^{(2)}(\xi_0,\xi_1) &= 0,\\ N_0^{(2)}(\underline{\zeta}_\varepsilon,\overline{\underline{\zeta}}_\varepsilon) &= 0.\end{aligned}$$

Moreover, we have in addition the equivariance of (6.10,6.11,6.12) with respect to the symmetry $\widehat{S}$ and to the representation of the group $SO(2)$ given by $(\tau_\phi W')(s) = W'(s+\phi)$. We know that $S\xi_0 = \xi_0$, $S\underline{\zeta}_\varepsilon = \overline{\underline{\zeta}}_\varepsilon$ holds, then we have, due to the uniqueness of the solution

$$\begin{aligned} \tau_\phi\mathcal{Y}(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &= \mathcal{Y}(\varepsilon,\gamma,u_0',v_0',A_0'e^{i\phi},\overline{A}_0'e^{-i\phi}),\\ \widehat{S}\mathcal{Y}(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &= \mathcal{Y}(\varepsilon,\gamma,u_0',v_0',\overline{A}_0',A_0').\end{aligned}$$

Replacing Y' by $\mathcal{Y}(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0')$ in (6.11,6.12), these equations take the form

$$\begin{aligned} f_1(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &= 0 \text{ in } \mathbb{C},\\ f_0(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &= 0 \text{ in } \mathbb{R},\end{aligned}$$

satisfying, because of all equivariances (anticommutation of G' with $\widehat{S}$),

$$\begin{aligned} e^{i\phi}f_1(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &= f_1(\varepsilon,\gamma,u_0',v_0',A_0'e^{i\phi},\overline{A}_0'e^{-i\phi}),\\ -\overline{f}_1(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &= f_1(\varepsilon,\gamma,u_0',v_0',\overline{A}_0',A_0'),\\ f_0(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &= f_0(\varepsilon,\gamma,u_0',v_0',A_0'e^{i\phi},\overline{A}_0'e^{-i\phi}),\\ f_0(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &= -f_0(\varepsilon,\gamma,u_0',v_0',\overline{A}_0',A_0'),\end{aligned}$$

for any $\phi\in\mathbb{R}$. This leads to

$$\begin{aligned} f_1(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &= iA_0'g(\varepsilon,\gamma,u_0',v_0',|A_0'|^2),\\ f_0(\varepsilon,\gamma,u_0',v_0',A_0',\overline{A}_0') &\equiv 0,\end{aligned}$$

where $g(\varepsilon,\gamma,u_0',v_0',|A_0'|^2)$ is *real valued and analytic.*

Remark. We observe here that the compatibility condition (6.12) is automatically satisfied, thanks to the equivariances of the system.

More precisely, it can be checked that

$$g(\varepsilon,\gamma,u_0',v_0',|A_0'|^2)=-\gamma+O[(|u_0'|+|v_0'|+|A_0'|^2)].$$

and by implicit function theorem, we have:

$$\gamma=\Gamma(\varepsilon,u_0',v_0',|A_0'|^2).$$

We now perform the scaling

$$A_0'=\varepsilon A_0,\quad u_0'=\varepsilon u_0,\quad v_0',=\varepsilon v_0,\quad W'=\varepsilon W,$$

and the theorem follows directly, after noticing that monomials $\varepsilon^{r'}u_0'^n v_0'^m A_0'^p\overline{A}_0'^q$ become $\varepsilon^{r'+n+m+p+q}u_0^n v_0^m A_0^p\overline{A}_0^q$, which shows that the exponent of ε is $r=r'+n+m+p+q-1$, hence $n+m+p+q\leq r+1$. This ends the proof of theorem 6.1. We can now treat the two missing components of $\underline{U}$, which is stated in the following

Theorem 6.3. *For any constant $M>0$, there exists $\varepsilon_0>0$ such that for any $(u_0,v_0,A_0,\varepsilon)\in\mathbb{R}^2\times\mathbb{C}\times\mathbb{R}$ satisfying*

$$|u_0|+|v_0|+|A_0|\leq M,\quad 0<\varepsilon<\varepsilon_0,$$

there is a family of periodic solutions $\underline{U}=p_{A_0,u_0,v_0,\varepsilon}$ of (4.1) in $\mathbb{D}_\sharp^{\underline{T}}$, bifurcating from 0, where

$$p_{A_0,u_0,v_0,\varepsilon}(\underline{x})=\widehat{p}_{A_0,u_0,v_0,\varepsilon}(s)$$

possesses the following power series in $u_0,v_0,A,\overline{A},\varepsilon$ converging in $\mathbb{D}_\sharp^{2\pi}$:

$$\begin{aligned}\widehat{p}_{A_0,u_0,v_0,\varepsilon}(s)&=u_0\xi_0+v_0\xi_1+A_0e^{is}\underline{\zeta}_\varepsilon+\overline{A}_0e^{-is}\overline{\underline{\zeta}}_\varepsilon+\\&+\sum_{\substack{p+q\geq1\\2\leq n+m+p+q\leq r+1}}\varepsilon^r u_0^n v_0^m A_0^p\overline{A}_0^q e^{i(p-q)s}Y_{nmpqr}\end{aligned}$$

where

$$s=\varepsilon^{-1}[\lambda+\gamma]\,\underline{x},\quad \underline{T}=\frac{2\pi\varepsilon}{\lambda+\gamma},$$

$$\gamma(u_0,v_0,|A_0|^2,\varepsilon)=\sum_{1\leq n+m+2p\leq r}\gamma_{nmpr}u_0^n v_0^m|A_0|^{2p}\varepsilon^r\in\mathbb{R},$$

$$\underline{\zeta}_\varepsilon=(1,e^\lambda,-ie^{\lambda\underline{y}/\varepsilon},e^{\lambda\underline{y}/\varepsilon},-ie^{\lambda y},e^{\lambda y})^t,$$

$$\zeta_\varepsilon^*(Y_{nmpqr})=0 \text{ for } p=q+1,$$

$$p_0^*(Y_{nmppr})=p_1^*(Y_{nmppr})=0.$$

These solutions are reversible for A_0 real, *and we have $SY_{nmpqr}=Y_{nmqpr}=\overline{Y}_{nmpqr}$. Moreover, there exist $c>0$, and $K>0$ such that we have the estimates*

$$\varepsilon||Y_{nmpqr}||_{\mathbb{D}}+||\mathcal{P}Y_{nmpqr}||_{\mathbb{D}}+||Y_{nmpqr}||_{\mathbb{H}}\leq c\varepsilon_0^{-r}K^{n+m+p+q}.$$

We have in fact better estimates, due to the exponential decay of 3rd and 4th components as $\underline{y}\to-\infty$. This is stated as follows:

Corollary 6.4. *There exist $c'>0$ and $K'>0$ such that*

$$||Y_{nmpqr}||_{\mathbb{E}_\varepsilon} \leq c'\varepsilon_0^{-r} K'^{n+m+p+q}.$$

Remarks. With respect to theorem 6.1, we notice that we need to manage the relative loss of analyticity with respect to ε. This comes in particular, from the 3rd and 4th components of $\underline{\zeta}_\varepsilon$ where $e^{\lambda \underline{y}/\varepsilon}$ introduces this loss, even in C_1^0, when $\varepsilon \to 0$. Another remark is that the family of periodic solutions does depend on two constants, which is natural, because of the freedom we had from the beginning in our system. This is a consequence of the fact that we did not impose $\beta_1 \to 0$ as $\underline{y} \to -\infty$.

Proof of theorem 6.3. Because of the analyticity of the expansion at theorem 6.1, we already know that there exists $K>0$ such that

$$||\mathcal{P}Y_{nmpqr}||_{\mathbb{D}^{\mathcal{P}}} \leq c\varepsilon_0^{-r} K^{n+m+p+q}.$$

We just need now to compute the 3rd and 4th components of the solutions where the 4 other components are known, thanks to theorem 6.1. So, we consider the 3rd and 4th components of system (4.1), which are just the Cauchy-Riemann equations. It results immediately, that the mth Fourier coefficients are given for $l \neq 0$, by

$$\alpha_1^{(l)} = -i[sgn(l)]\beta_{10}^{(l)} e^{|l|(\lambda+\gamma)\underline{y}/\varepsilon},$$
$$\beta_1^{(l)} = \beta_{10}^{(l)} e^{|l|(\lambda+\gamma)\underline{y}/\varepsilon},$$

where $\beta_{10}^{(l)}$ is the first component of the $l-$th Fourier coefficient of the (already found) solution $\widehat{p}^{(\mathcal{P})}_{A,u_0,\varepsilon}(s)$. For $l=0$, we notice that, by construction $\beta_{10}^{(0)} = v_0$, hence we have $\beta_1^{(0)}(\underline{y}) = v_0$ and, $\alpha_1^{(0)}$ is given by

$$\alpha_1^{(0)}(\underline{y}) = \alpha_{20}^{(0)} = \sum_{\substack{p+q\geq 1 \\ 2\leq n+m+p+q\leq r+1}} \varepsilon^r u_0^n v_0^m |A_0|^{2p} \alpha_{20}^*(Y_{nmppr}).$$

For $l \neq 0$, we have for instance

$$\alpha_1^{(1)}(\underline{y}) = -iA\left[1 + \sum_{1\leq n+m+2q\leq r-2} \varepsilon^r u_0^n v_0^m |A|^{2q} \beta_{10}^*(Y_{n,m,q+1,q,r})\right] e^{(\lambda+\gamma)\underline{y}/\varepsilon}.$$

Now, we observe that $e^{|l|(\lambda+\gamma)\underline{y}/\varepsilon} = e^{|l|\lambda \underline{y}/\varepsilon}[\sum_{j\geq 0} \frac{1}{j!}\left[(|l|\gamma/\varepsilon)\underline{y}\right]^j$,

with $\gamma/\varepsilon = O(|A|^2 + |u_0| + |v_0|)$ and

$$\frac{1}{j!}||\left[(|l|\gamma/\varepsilon)\underline{y}\right]^j e^{|l|\lambda\underline{y}/\varepsilon}||_{1,\infty} \leq (\gamma/\lambda)^j\left[1 + \left(\frac{\varepsilon}{|l|\lambda}\right)\frac{(j+1)^{j+1}e^{-(j+1)}}{j!}\right],$$
$$\leq c(\gamma/\lambda)^j(1+\varepsilon\, j^{1/2}) \text{ for large } j.$$

Replacing γ by its expression, we deduce that $e^{|l|(\lambda+\gamma)\underline{y}/\varepsilon}$ may be expanded in powers of $(|A_0|^2, u_0, v_0)$ the series converging in $C^1(\mathbb{R}^-)$ and we have

$$||e^{|l|\lambda\underline{y}/\varepsilon}[\sum_{j\geq 0}\frac{1}{j!}\left[(|l|\gamma/\varepsilon)\underline{y}\right]^j||_{C^1} \leq c|l|/\varepsilon, \text{ for } \varepsilon \text{ small enough.}$$

Moreover, the factor of $e^{|l|\lambda \underline{y}/\varepsilon}$ is analytic in $(\varepsilon, |A_0|^2, u_0, v_0)$. This above estimate explains the factor $1/\varepsilon$ for the estimate of $||Y_{nmpqr}||_{\mathbb{D}}$. The estimates of the Corollary with the 3rd and 4th components in $C^{1,\exp}_{\varepsilon}$ follow easily, with K' larger than K, since we use a part of the exponential decay in the norm.

Now the orthogonality condition $\zeta^*_\varepsilon(Y^{(\mathcal{P})}_{npqr}) = 0$ for $p = q+1$, leads to $\zeta^*_\varepsilon(Y_{npqr}) = 0$ (since $i[sgn(p-q)]f_1 - g_1 = 0$). The result of theorem 6.3 then follows.

The Bernoulli first integral and periodic solutions

We already defined the Bernoulli first integral (4.9) of our nonlinear rescaled system. Any solution $\underline{U}(\underline{x})$ of system (4.1), satisfies $h[\varepsilon; \underline{U}(\underline{x})] = const.$ Moreover we already observed that

$$D_{\underline{U}} h(\varepsilon; 0)U = \xi^*_\varepsilon(U), \quad \text{for } U \in \mathbb{H}.$$

Let us consider the family of periodic solutions $p_{A_0,u_0,v_0,\varepsilon}(\underline{x})$ and compute the value of the Bernoulli first integral. We first observe that

$$h[\varepsilon; \underline{U}] - \xi^*_\varepsilon(\underline{U}) = \varepsilon \left[\beta_{10}^2 - \rho\beta_{20}^2 - (1-\rho)\beta_{21}^2 - \frac{1-\varepsilon}{2}\int_0^1 (\beta_2^2 - \alpha_2^2)dy \right] + O(\varepsilon^2 ||\mathcal{P}\underline{U}||^3_{\mathbb{H}^{\mathcal{P}}}),$$

and

$$\begin{aligned} \widehat{p}_{A_0,u_0,v_0,\varepsilon}(s) &= u_0\xi_0 + v_0\xi_1 + A_0 e^{is}\underline{\zeta}_\varepsilon + \overline{A}_0 e^{-is}\overline{\underline{\zeta}}_\varepsilon + \\ &\quad + \Phi_\varepsilon(A_0 e^{is}, \overline{A}_0 e^{-is}, u_0), \\ \Phi_\varepsilon(A_0 e^{is}, \overline{A}_0 e^{-is}, u_0) &= O\left[\varepsilon |A_0|(|A_0| + |u_0| + |v_0|)\right]. \end{aligned}$$

Since h is a constant, we have the following invariance property

$$h[\varepsilon; \widehat{p}_{A_0,u_0,v_0,\varepsilon}(s+\phi)] = h[\varepsilon; \widehat{p}_{A_0,u_0,v_0,\varepsilon}(s)],$$

hence, a straightforward consequence is that

$$h[\varepsilon; \widehat{p}_{A_0,u_0,v_0,\varepsilon}(s)] = \widetilde{h}(|A_0|^2, u_0, v_0, \varepsilon)$$

where $\widetilde{h}$ is analytic in its arguments. Now we use

$$\xi^*_\varepsilon(u_0\xi_0 + v_0\xi_1 + A_0 e^{is}\underline{\zeta}_\varepsilon + \overline{A}_0 e^{-is}\overline{\underline{\zeta}}_\varepsilon) = v_0 - \varepsilon u_0,$$

and we obtain, after a simple computation

Lemma 6.5. *The Bernoulli first integral $h(\varepsilon; \underline{U})$, evaluated on the family of periodic solutions found at theorem 6.3, satisfies*

$$\begin{aligned} h[\varepsilon; \widehat{p}_{A_0,u_0,v_0,\varepsilon}(s)] &= v_0 - \varepsilon[u_0 + h_\varepsilon(|A_0|^2, u_0, v_0)], \qquad (6.13)\\ h_\varepsilon(|A_0|^2, u_0, v_0) &= 2(1-\rho)(e^{2\lambda} - 1)|A_0|^2 + \frac{1}{2}(3-\varepsilon)u_0^2 - v_0^2 + \\ &\quad + O[\varepsilon(|u_0| + |v_0|)|A_0|^2 + \varepsilon^2|A_0|^4 + \varepsilon(|u_0| + |v_0|)^3], \end{aligned}$$

where h_ε is analytic in its arguments.

Remark. The coefficient $3/2$ in the expression of h_ε is the one which will occur in the Benjamin-Ono equation later on.

7. Normal form

In this section, we introduce new variables such that the system they satisfy has a nicer form than the original system. Indeed, on the new system the set of periodic solutions of theorem 6.3 appears trivially on a flat 4 dimensional manifold, and many terms of the system are transformed in such a way that for the study of solutions homoclinics to anyone of the periodic solutions above mentioned, the decay rate of the rests will be sufficiently good. As we shall see, this needs some (instructive) technical work.

In section 5.(e), we defined the projection $\widetilde{\pi}_\varepsilon$ on a subspace supplementary to the space spanned by $\underline{\zeta}_\varepsilon$, $\overline{\underline{\zeta}}_\varepsilon$, ξ_0 and ξ_1 :

$$\widetilde{\pi}_\varepsilon U = \pi_\varepsilon U - p_0^*(\pi_\varepsilon U)\xi_0 - p_1^*(\pi_\varepsilon U)\xi_1,$$

where we observe that

$$\zeta_\varepsilon^*(\widetilde{\pi}_\varepsilon U) = \overline{\zeta}_\varepsilon^*(\widetilde{\pi}_\varepsilon U) = p_0^*(\widetilde{\pi}_\varepsilon U) = p_1^*(\widetilde{\pi}_\varepsilon U) = 0,$$
$$||\widetilde{\pi}_\varepsilon||_{\mathcal{L}(\mathbb{H})} \leq c, \quad ||\widetilde{\pi}_\varepsilon||_{\mathcal{L}(\mathbb{D})} \leq c/\varepsilon.$$

In this section we prove the following (the notation W is not the one used at section 6).

Lemma 7.1. *For $|A| + |u| + |v| + ||W||_{\mathbb{H}} < M$, $0 < \varepsilon < \varepsilon_0$ there is an analytic mapping $\Phi_\varepsilon : \mathbb{C}^2 \times \mathbb{R}^2 \to \mathbb{E}_\varepsilon$, smoothly depending on ε, such that the change of variables*

$$\underline{U} = A\underline{\zeta}_\varepsilon + \overline{A\underline{\zeta}}_\varepsilon + u\xi_0 + v\xi_1 + W + \Phi_\varepsilon(A, \overline{A}, u, v),$$
$$\zeta_\varepsilon^*(W) = \overline{\zeta}_\varepsilon^*(W) = p_0^*(W) = p_1^*(W) = 0,$$

with (see theorem 6.3)

$$\Phi_\varepsilon(A, \overline{A}, u, v) = \sum_{\substack{p+q\geq 1 \\ 2\leq n+m+p+q\leq r+1}} \varepsilon^r u^n v^m A^p \overline{A}^q Y_{nmpqr}$$

transforms the system (4.1) into the following new reversible system in $\mathbb{C}\times\mathbb{R}^2\times\widetilde{\pi}_\varepsilon\mathbb{H}$

$$\frac{dA}{d\underline{x}} = iA\left[\frac{\lambda}{\varepsilon} + \gamma_1(u, v, |A|^2, \varepsilon)\right] + R_A(A, \overline{A}, u, v, W), \tag{7.1}$$
$$\frac{dv}{d\underline{x}} = p_1^*\mathcal{L}_\varepsilon W + R_v(A, \overline{A}, u, v, W), \tag{7.2}$$
$$\frac{du}{d\underline{x}} = p_0^*\mathcal{L}_\varepsilon W + R_u(A, \overline{A}, u, v, W), \tag{7.3}$$
$$\frac{dW}{d\underline{x}} = \widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon W + R_W(A, \overline{A}, u, v, W), \tag{7.4}$$

where $\gamma_1 = \gamma/\varepsilon$ (see theorem 6.3), R_A, R_u, R_v, R_W are analytic in their arguments, and

$$|R_A| + |R_v| + |R_u| + ||R_W||_{\widetilde{\pi}_\varepsilon\mathbb{H}} = O\left\{||\mathcal{P}W||_{\mathbb{D}}(|A| + |u| + |v| + ||\mathcal{P}W||_{\mathbb{D}})\right\},$$
$$||R_v\xi_1 + R_W||_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon} = O\left\{||\mathcal{P}W||_{\mathbb{D}}(|A| + |u| + |v| + ||\mathcal{P}W||_{\mathbb{D}})\right\},$$

where $\mathbb{F}_\varepsilon$ *is defined in (5.10). The system (7.1,7.2,7.3,7.4) is reversible under the symmetry* S, *acting in* $\mathbb{C}^2\times\mathbb{R}^2\times\widetilde{\pi}_\varepsilon\mathbb{H}$

$$(A,\overline{A},u,v,W)\rightarrow(\overline{A},A,u,v,SW).$$

Remark 1. The family of periodic solutions appears explicitly in the above normal form, in taking $W=0$, $u=u_0=const$, $v=v_0=const$ and $A=A_0e^{is}$, $|A_0|=const$. This is a very remarkable property of the above system, which simplifies a lot many of forthcoming calculations. Indeed, we *flatten the four-dimensional manifold of periodic solutions.*

Remark 2. In the equation for W, the 3rd and 4th components of the nonlinear part are y-dependent. Indeed, the very good fact is that the *nonlinear terms* (all terms except $\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon W$), have a fast exponential decay as $\underline{y}\rightarrow-\infty$, at least as $e^{\lambda\underline{y}/\varepsilon}$. Moreover these components are continuously differentiable in $\underline{y}$ such that $R_v\xi_1+R_W\in\mathbb{F}_\varepsilon$.

Remark 3. If we collect the equations for v and W, in making

$$\underline{W}=v\xi_1+W\in\underline{\pi}_\varepsilon\mathbb{D},$$

then we obtain

$$\frac{d\underline{W}}{d\underline{x}}=\underline{\pi}_\varepsilon\mathcal{L}_\varepsilon W+R_{\underline{W}}(A,\overline{A},u,v,W),\tag{7.5}$$

where we observe that $\mathcal{L}_\varepsilon$ operates only on the component W of $\underline{W}$, and the following estimate

$$||R_{\underline{W}}(A,\overline{A},u,v,W)||_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon}=O\left\{||\mathcal{P}W||_{\mathbb{D}}(|A|+|u|+|v|+||\mathcal{P}W||_{\mathbb{D}})\right\}$$

holds. *The important fact is that* $R_{\underline{W}}$, *as well as* R_A *and* R_u, *cancels if* PW *cancels.* This will allow us to improve this system (see next lemma).

Proof of lemma 7.1 From the form of the periodic solutions $\widehat{p}_{A_0,u_0,v_0,\varepsilon}(s)$, and the identities

$$\begin{aligned}&(\frac{\lambda}{\varepsilon}+\gamma_1)\frac{d}{ds}\widehat{p}=\mathcal{L}_\varepsilon\widehat{p}+\mathcal{N}(\varepsilon;\widehat{p}),\\&\widehat{p}_{A_0,u,v,\varepsilon}(s)=u\xi_0+v\xi_1+A\underline{\zeta}_\varepsilon+\overline{A\underline{\zeta}}_\varepsilon+\Phi_\varepsilon(A,\overline{A},u,v),\quad A=A_0e^{is},\end{aligned}$$

and defining

$$N_A=iA\gamma_1(u,v,|A|^2,\varepsilon),$$

we obtain the new identity in $\mathbb{K}_\varepsilon$

$$\begin{aligned}&N_A\underline{\zeta}_\varepsilon+\overline{N_A\underline{\zeta}}_\varepsilon+\left(\frac{i\lambda A}{\varepsilon}+N_A\right)\partial_A\Phi_\varepsilon+\left(-\frac{i\lambda\overline{A}}{\varepsilon}+\overline{N_A}\right)\partial_{\overline{A}}\Phi_\varepsilon\\&=\mathcal{L}_\varepsilon\Phi_\varepsilon+\mathcal{N}(\varepsilon;A\underline{\zeta}_\varepsilon+\overline{A\underline{\zeta}}_\varepsilon+u\xi_0+v\xi_1+\Phi_\varepsilon),\end{aligned}\tag{7.6}$$

which is valid for any u,v,A,ε such that $|A|+|u|+|v|<M,\ 0<\varepsilon<\varepsilon_0$. It results that

$$R_A(\underline{\zeta}_\varepsilon+\partial_A\Phi_\varepsilon)+\overline{R_A}(\overline{\underline{\zeta}}_\varepsilon+\partial_{\overline{A}}\Phi_\varepsilon)+R_W+R_u(\xi_0+\partial_u\Phi_\varepsilon)+R_v(\xi_1+\partial_v\Phi_\varepsilon)=$$

$$\begin{aligned}
&= \mathcal{N}(\varepsilon; A\underline{\zeta}_\varepsilon + \overline{A\underline{\zeta}}_\varepsilon + u\xi_0 + v\xi_1 + W + \Phi_\varepsilon) + \\
&- \mathcal{N}(\varepsilon; A\underline{\zeta}_\varepsilon + \overline{A\underline{\zeta}}_\varepsilon + u\xi_0 + v\xi_1 + \Phi_\varepsilon) + \\
&- p_0^*(\mathcal{L}_\varepsilon W)\partial_u \Phi_\varepsilon - p_1^*(\mathcal{L}_\varepsilon W)\partial_v \Phi_\varepsilon
\end{aligned} \tag{7.7}$$

holds, and then $R_v\xi_1 + R_W$ lies in fact in $\mathbb{F}_\varepsilon$, because ξ_0, ζ_ε, Φ_ε, $\partial_A\Phi_\varepsilon$, $\partial_u\Phi_\varepsilon$, $\partial_v\Phi_\varepsilon \in \mathbb{F}_\varepsilon$, and $\mathcal{N}$ has no 3rd and 4th components. We now use the estimates

$$||\partial_A\Phi_\varepsilon||_{\mathbb{F}_\varepsilon} + ||\partial_u\Phi_\varepsilon||_{\mathbb{F}_\varepsilon} + ||\partial_v\Phi_\varepsilon||_{\mathbb{F}_\varepsilon} \leq c\varepsilon(|A| + |u| + |v|),$$

for being able to solve equation (7.7) with respect to $(R_A, R_u, R_v, R_W) \in \mathbb{C} \times \mathbb{R}^2 \times \widetilde{\pi}_\varepsilon\mathbb{H}$, in projecting this equation on $\underline{\zeta}_\varepsilon, \xi_0, \xi_1, \widetilde{\pi}_\varepsilon\mathbb{H}$. We observe that

$$\begin{aligned}
||p_0^*(\mathcal{L}_\varepsilon W)\partial_u\Phi_\varepsilon||_{\mathbb{F}_\varepsilon} &\leq c|A|||\mathcal{P}W||_{\widetilde{\pi}_\varepsilon\mathbb{D}}, \\
||p_1^*(\mathcal{L}_\varepsilon W)\partial_v\Phi_\varepsilon||_{\mathbb{F}_\varepsilon} &\leq c|A|||\mathcal{P}W||_{\widetilde{\pi}_\varepsilon\mathbb{D}},
\end{aligned}$$

holds, and $\mathcal{N}(\varepsilon; W)$ only depends on $\mathcal{P}W$, hence the lemma is proved, after straightforward estimates.

Now, we can improve lemma 7.1 in treating, by a normal form technique, all terms which are linear in W, with coefficients depending on A. This is useful for eliminating oscillating terms having a not as good decay rate when $|x| \to \infty$ as higher order nonlinear terms in W. Below we will be only interested in solutions such that u v, and W tend towards 0 as $|\underline{x}| \to \infty$. We might do the same analysis for solutions such that the limiting u and v are not 0. We have the following

Lemma 7.2. *There exists $\delta > 0$, such that for $|A| < \delta$, $|u| + |v| + ||W||_{\mathbb{H}} < M$, $0 < \varepsilon < \varepsilon_0$, there is a change of variables of the form*

$$\begin{aligned}
A &= A' + \mu_\varepsilon^*(A', \overline{A}')[W'], \\
u &= u' + \nu_\varepsilon^*(A', \overline{A}')[W'], \\
\underline{W} &= \underline{W}' + \Gamma_\varepsilon(A', \overline{A}')[W'],
\end{aligned} \tag{7.8}$$

where $\mu_\varepsilon^*(A, \overline{A})$, $\nu_\varepsilon^*(A, \overline{A})$, $\Gamma_\varepsilon(A, \overline{A})$ have respectively their values in

$$\mathcal{L}(\widetilde{\pi}_\varepsilon\mathbb{H}, \mathbb{C}), \mathcal{L}(\widetilde{\pi}_\varepsilon\mathbb{H}, \mathbb{R}), \mathcal{L}(\widetilde{\pi}_\varepsilon\mathbb{H}, \underline{\pi}_\varepsilon\mathbb{E}_\varepsilon),$$

they are analytic in their arguments and

$$\mu_\varepsilon^* = O(|A|), \nu_\varepsilon^* = O(|A|), \Gamma_\varepsilon = O(\varepsilon|A|),$$

and they are such that the system satisfied by $(A', u', \underline{W}' = v'\xi_1 + W')$ takes the form (dropping the primes)

$$\frac{dA}{d\underline{x}} = iA\left[\frac{\lambda}{\varepsilon} + \gamma_1(u, v, |A|^2, \varepsilon)\right] + R_A(A, \overline{A}, u, \underline{W}),$$

$$\frac{du}{d\underline{x}} = p_0^*(\mathcal{L}_\varepsilon W) + R_u(A, \overline{A}, u, \underline{W}),$$

$$\frac{d\underline{W}}{d\underline{x}} = \underline{\pi}_\varepsilon\mathcal{L}_\varepsilon W + \Delta_\varepsilon(A, \overline{A})[W] + R_{\underline{W}}(A, \overline{A}, u, \underline{W}),$$

where $\Delta_\varepsilon(A,\overline{A})$ *takes its values in* $\mathcal{L}(\widetilde{\pi}_\varepsilon\mathbb{D},\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon)$ *, and is analytic in its arguments and satisfies*

$$\Delta_\varepsilon(SW)=-S\Delta_\varepsilon(W),\ ||\Delta_\varepsilon||_{\mathcal{L}(\widetilde{\pi}_\varepsilon\mathbb{D},\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon)}=O(|A|),$$

$$\eta_\varepsilon^*\{\Delta_\varepsilon(A,\overline{A})[W]\}=0,$$

where η_ε^* *is defined in (5.7), and the following estimate*

$$|R_A|+|R_u|+||R_{\underline{W}}||_{\mathbb{F}_\varepsilon}=O\left[||W||_{\mathbb{D}}(|u|+||\underline{W}||_{\mathbb{D}})\right]$$

holds.

We observe in the above formulation, that the linear terms in W are simplified in the two first equations. In addition, in the equation for $\underline{W}$, the term $\Delta_\varepsilon(A,\overline{A})[W]$ satisfies a compatibility condition, which will be very helpful later.

Remark. We notice, in the above lemma, that if we suppress the equation for A, and make $A=0$ in the two other equations, we recover a system similar to the original one, but in the subspace $\pi_\varepsilon\mathbb{H}$. This is helpful for *recognizing the homoclinic given at main order by Benjamin-Ono equation* (see next section).

Proof of lemma 7.2. This proof is given in Appendix Normal form. The fact that this reduction is possible is not a surprise, except for the a priori resonant terms of the form $A|A|^{2n}l^*(W)|$ and $|A|^{2n}n^*(W)$ respectively in the equation for A and the equation for u. The elimination of these terms results from a particular property of our system, where we can extend the basic space to functions having a growth in $\ln|\underline{y}|$ as $\underline{y}\rightarrow-\infty$, and observe that in such space the linear operator $\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon$ has a bounded inverse, while this is not the case in $\widetilde{\pi}_\varepsilon\mathbb{H}$. Another new fact is the *ability to impose the compatibility condition*

$$\eta_\varepsilon^*\{\Delta_\varepsilon(A,\overline{A})[W]\}=0,$$

which will be essential for being able to invert a certain linear operator, this operator being basic in our proof of existence of homoclinics to periodic solutions. This condition allows to treat the term $\Delta_\varepsilon(A,\overline{A})[W]$ which has an insufficient decay rate in x (see Appendix Resolvent 0).

8. New working system

(a) Rescaling and Bernoulli first integral

For the rest of the paper, the good scaling is $\underline{W}=\varepsilon Y$, i.e.

$$\begin{aligned}\underline{U}=&\left\{A+\varepsilon\mu_\varepsilon^*(A,\overline{A})[\widetilde{\pi}_\varepsilon Y]\right\}\underline{\zeta_\varepsilon}+\left\{\overline{A}+\varepsilon\overline{\mu_\varepsilon^*}(A,\overline{A})[\widetilde{\pi}_\varepsilon Y]\right\}\underline{\overline{\zeta}_\varepsilon}+V+\\&+\Phi_\varepsilon\left\{A+\varepsilon\mu_\varepsilon^*(A,\overline{A})[\widetilde{\pi}_\varepsilon Y],\overline{A}+\varepsilon\overline{\mu_\varepsilon^*}(A,\overline{A})[\widetilde{\pi}_\varepsilon Y],\right.\\&\left.u+\varepsilon\nu_\varepsilon^*(A,\overline{A})[\widetilde{\pi}_\varepsilon Y],\varepsilon p_1^*\left(Y+\Gamma_\varepsilon(A,\overline{A})[\widetilde{\pi}_\varepsilon Y]\right)\right\},\\V=&\left\{u+\varepsilon\nu_\varepsilon^*(A,\overline{A})[\widetilde{\pi}_\varepsilon Y]\right\}\xi_0+\varepsilon Y+\varepsilon\Gamma_\varepsilon(A,\overline{A})[\widetilde{\pi}_\varepsilon Y],\\p_0^*(Y)=&\ \zeta_\varepsilon^*(Y)=\overline{\zeta}_\varepsilon^*(Y)=0,\quad Y\in\underline{\pi}_\varepsilon\mathbb{D}\end{aligned}\tag{8.1}$$

then the system satisfied by $(A, \overline{A}, u, Y)$ reads (new R_A, R_u and R_Y here)

$$\begin{aligned} \frac{dA}{d\underline{x}} &= iA\left[\frac{\lambda}{\varepsilon} + \gamma_1(u, \varepsilon p_1^*(Y), |A|^2, \varepsilon)\right] + R_A(A, \overline{A}, u, Y), \\ \frac{du}{d\underline{x}} &= p_0^*(\varepsilon \mathcal{L}_\varepsilon Y) + 2\varepsilon u p_0^*\left[\pi_\varepsilon N_\varepsilon^{(2)}(\xi_0, Y)\right] + R_u(A, \overline{A}, u, Y), \\ \frac{dY}{d\underline{x}} &= \underline{\pi}_\varepsilon \mathcal{L}_\varepsilon Y + 2u\underline{\pi}_\varepsilon N_\varepsilon^{(2)}(\xi_0, Y) + \Delta_\varepsilon(A, \overline{A})[\widetilde{\pi}_\varepsilon Y] + R_Y(A, \overline{A}, u, Y), \end{aligned} \tag{8.2}$$

with

$$\eta_\varepsilon^*\{\Delta_\varepsilon(A, \overline{A})[\widetilde{\pi}_\varepsilon Y]\} = 0,$$

and

$$\begin{aligned} |R_A| &= O\left\{\varepsilon ||Y||_{\mathbb{D}}[|u| + \varepsilon ||Y||_{\mathbb{D}}]\right\}, \\ |R_u| &= O\left\{\varepsilon^2 ||Y||_{\mathbb{D}}[(|A| + |u|)|u| + ||Y||_{\mathbb{D}}]\right\}, \\ ||R_Y||_{\mathbb{F}_\varepsilon} &= O\left\{\varepsilon ||Y||_{\mathbb{D}}[(|A| + |u|)|u| + ||Y||_{\mathbb{D}}]\right\}. \end{aligned} \tag{8.3}$$

Remark. We observe that

$$\xi_\varepsilon^*(\underline{\pi}_\varepsilon \mathcal{L}_\varepsilon Y) \equiv p_0^*(\varepsilon \mathcal{L}_\varepsilon Y).$$

The aim of this section is to replace the system (8.2), by another equivalent one where the oscillating part in A is kept, and where the (u, Y) part is transformed into one equation expressing directly Y in terms of u and of the nonlinear terms, and another equation which is a perturbation of the Benjamin-Ono equation for u. This new system has the following form [see (8.20)]

$$\begin{aligned} \frac{dA}{d\underline{x}} &= iA\left[\frac{\lambda}{\varepsilon} + \gamma_1(u, |A|^2, \varepsilon)\right] + R_A(A, \overline{A}, u, Y), \\ Y &= \underline{\pi}_\varepsilon \mathcal{T}_0 u + \mathcal{R}_Y(A, \overline{A}, u, Y), \\ \rho \mathcal{H}&\left(\frac{du}{d\underline{x}}\right) + u + \frac{3}{2}u^2 = \mathcal{B}_\varepsilon(A, \overline{A}, u, Y), \end{aligned}$$

where the operators $\mathcal{T}_0$, $\mathcal{R}_Y$, $\mathcal{B}_\varepsilon$ are non local in $\underline{x}$, and $\mathcal{R}_Y$, $\mathcal{B}_\varepsilon$ are small in a suitable norm. In the above system $\mathcal{T}_0$ is linear and $\mathcal{H}$ is the Hilbert transform, occurring in the Benjamin-Ono equation (Benjamin 1967; Davis & Acrivos 1967; Ono 1975). This equation plays a deep role here, and will be derived with the help of the Bernoulli first integral $h(\varepsilon; \underline{U})$ as defined in (4.9).

Indeed, we now consider the *Bernoulli first integral*

$$h[\varepsilon; \underline{U}(\underline{x})] = const,$$

where we know that for $\underline{U} \in \mathbb{H}$, one has

$$\begin{aligned} &h[\varepsilon; \underline{U}] - \xi_\varepsilon^*(\underline{U}) = \\ &\qquad \varepsilon\left[\beta_{10}^2 - \rho\beta_{20}^2 - (1-\rho)\beta_{21}^2 - \tfrac{1-\varepsilon}{2}\int_0^1 (\beta_2^2 - \alpha_2^2)dy\right] + O(\varepsilon^2 ||\mathcal{P}\underline{U}||_{\mathbb{H}}^3). \end{aligned}$$

We already computed the Bernoulli integral for the family of periodic solutions (see lemma 6.5), which leads to the identity

$$h[\varepsilon; A\underline{\zeta}_{\varepsilon} + \overline{A\underline{\zeta}_{\varepsilon}} + u\xi_0 + v\xi_1 + \Phi_{\varepsilon}(A, \overline{A}, u)] = v - \varepsilon[u + h_{\varepsilon}(|A|^2, u, v)].$$

Finally, we have

Lemma 8.1. *The Bernoulli first integral of the system (4.1), written with the new variables defined in (8.1), takes the form*

$$u = \xi_{\varepsilon}^{*}(Y) - \frac{3}{2}u^2 + \widetilde{h}_{\varepsilon}[|A|^2, u, \varepsilon p_1^{*}(Y)] + \widetilde{R}_u(A, \overline{A}, u, Y) + c_0, \tag{8.4}$$

$$|\widetilde{R}_u| = O[\varepsilon(|A| + |u|)||Y||_{\mathbb{H}} + \varepsilon^2||Y||_{\mathbb{H}}^2),$$

$$|\widetilde{h}_{\varepsilon}| = O(|A|^2 + \varepsilon u^2 + \varepsilon|A|^2(|u| + \varepsilon|p_1^{*}(Y)|) + \varepsilon^2|p_1^{*}(Y)|^2).$$

where $\widetilde{h}_{\varepsilon}$ *and* $\widetilde{R}_u$ *are analytic in their arguments, and* c_0 *a constant.*

From now on, we really need to specify the required behavior in $\underline{x}$ near infinity, specially for (u, Y). So we introduce new basic spaces.

(b) Basic spaces for the $\underline{x}$ *- dependence*

Let us introduce the following (Hölder) spaces for the $\underline{x}$ dependence, where $\mathbb{E}$ is a Banach space:

$$B_p^{\alpha}(\mathbb{E}) = \{f \in C^{\alpha}(\mathbb{E}); ||f||_{\mathbb{E},p}^{\alpha} < \infty\}, \quad 0 < \alpha < 1,$$

$$||f||_{\mathbb{E},p}^{\alpha} = \sup_{x\in\mathbb{R}}(1 + |x|^p)||f(x)||_{\mathbb{E}} + \sup_{x\in\mathbb{R},\ |\delta|\leq 1}(1 + |x|^p)\frac{||f(x+\delta) - f(x)||_{\mathbb{E}}}{|\delta|^{\alpha}},$$

hence, we use for example in the following $B_2^{\alpha}(\mathbb{R})$, $B_3^{\alpha}(\widetilde{\pi}_{\varepsilon}\mathbb{F}_{\varepsilon})$... and also $B_2^{1,\alpha}(\mathbb{R})$ defined by

$$B_2^{1,\alpha}(\mathbb{R}) = \left\{ f \in B_2^{\alpha}(\mathbb{R}); \frac{df}{d\underline{x}} \in B_2^{\alpha}(\mathbb{R}) \right\},$$

and we denote for instance $||f||_{\mathbb{R},2}^{1,\alpha}$ the corresponding norm. We also introduce the spaces $B_{\mathbb{H},w}^{\alpha}$ and $B_{\mathbb{D},w}^{\alpha}$ defined by

$$B_{\mathbb{H},w}^{\alpha} = \{V = (a, b, f_1, g_1, f_2, g_2)^t; V(x) \in \mathbb{H},$$
$$(a, b) \in B_2^{\alpha}(\mathbb{R}^2), (f_1, g_1) \in (B_w^{-})^2, (f_2, g_2) \in (B_w^{+})^2\},$$

where

$$B_w^{-} = \{f(x, y);\ (x, y) \in \mathbb{R} \times \mathbb{R}^{-},\ f \text{ is } C^{\alpha} \text{ in } x,\ C^0 \text{ in } y,\ ||f||_{B_w^{-}} < \infty\},$$
$$B_w^{+} = B_2^{\alpha}[C^0(0,1)],\ B_w^{1,+} = B_2^{\alpha}[C^1(0,1)],$$

$$||f||_{B_w^{-}} = \sup_{x\in\mathbb{R}, y<0}\frac{(1 + |x|^2 + |y|^2)}{1 + |y|}|f(x, y)| +$$
$$+ \sup_{x\in\mathbb{R}, y<0,\ |\delta|\leq 1}\frac{(1 + |x|^2 + |y|^2)}{1 + |y|}\frac{|f(x+\delta, y) - f(x, y)|}{|\delta|^{\alpha}},$$

$$B^{\alpha}_{\mathbb{D},w} = \{U = (\beta_{10}, \beta_{21}, \alpha_1, \beta_1, \alpha_2, \beta_2)^t; \beta_{10} = \beta_1|_{y=0}, \alpha_{20} = \alpha_{10}, \beta_{21} = \beta_2|_{y=1}$$
$$(\alpha_1, \beta_1) \in (B^-_w)^2, (\alpha_2, \beta_2) \in (B^{1,+}_w)^2\},$$
$$B^{1,\alpha}_{\mathbb{H},w} = \left\{V \in B^{\alpha}_{\mathbb{H},w}; \frac{dV}{d\underline{x}} \in B^{\alpha}_{\mathbb{H},w}\right\},$$

and we denote by $||U||^{\alpha}_{\mathbb{D},w}$, and $||V||^{1,\alpha}_{\mathbb{H},w}$ the corresponding norms.

(c) *A new linear lemma*

In this section, we consider the following linear system appearing as a part of system (8.2):

$$\frac{du}{d\underline{x}} = p_0^* \varepsilon \mathcal{L}_\varepsilon Y + \varepsilon T_u, \tag{8.5}$$
$$\frac{dY}{d\underline{x}} = \underline{\pi}_\varepsilon \mathcal{L}_\varepsilon Y + T_Y, \tag{8.6}$$

where (T_u, T_Y) satisfies

$$T_u \in B_3^{\alpha}(\mathbb{R}), \text{ odd}$$
$$T_Y \in [B_2^{\alpha}(\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon)_{\eta^*} + B_3^{\alpha}(\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon)], \text{ antireversible,}$$

where we denote by $[B_2^{\alpha}(\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon)_{\eta^*} + B_3^{\alpha}(\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon)]$ the set of $T_Y = T_Y^{(1)} + T_Y^{(2)}$, such that

$$T_Y^{(1)} \in B_2^{\alpha}(\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon) \cap \ker \eta_\varepsilon^*,$$
$$T_Y^{(2)} \in B_3^{\alpha}(\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon).$$

Taking the Fourier transform of the system (8.5,8.6), we are lead to use the results of lemma 5.4 for the resolvent near $k = 0$. We now introduce a splitting of unity, with smooth *even* functions φ_0 and φ_1 such that

$$\varphi_0(\varepsilon k) = \begin{cases} 1, & \varepsilon|k| < \delta/2 \\ 0, & \varepsilon|k| > \delta \end{cases},$$
$$\varphi_1(\varepsilon k) = \begin{cases} 0, & \varepsilon|k| < \delta/2 \\ 1, & \varepsilon|k| > \delta \end{cases}.$$
$$\varphi_0(\varepsilon k) + \varphi_1(\varepsilon k) = 1.$$

Then we have first the following results

Lemma 8.2. *For any $u \in B_2^{\alpha}(\mathbb{R})$, we define the function*

$$u_0 = \psi_0 * u,$$
$$\psi_0 = \mathcal{F}^{-1}[\varphi_0(\varepsilon k)].$$

Then u_0 and all derivatives $u_0^{(n)} \in B_2^{\alpha}(\mathbb{R})$, and

$$||u_0||^{\alpha}_{\mathbb{R},2} \leq c||u||^{\alpha}_{\mathbb{R},2},$$
$$||u_0^{(n)}||^{\alpha}_{\mathbb{R},2} \leq \frac{c_n}{\varepsilon^n}||u||^{\alpha}_{\mathbb{R},2}$$

holds.

Lemma 8.3. *For $u \in B_2^{1,\alpha}(\mathbb{R})$, and u even, let define the operator $\mathcal{T}_0 : B_2^{1,\alpha}(\mathbb{R}) \rightarrow B_{\mathbb{D},w}^{\alpha}$ by*

$$\begin{aligned}
&\mathcal{T}_0 u = (\beta_{10}, 0, \alpha_1, \beta_1, \alpha_2, 0)^t, \\
&\beta_{10} = -\widetilde{\rho}\mathcal{H}\left(\frac{du_0}{d\underline{x}}\right), \ \alpha_2 = (\widetilde{\rho} - y)\left(\frac{du_0}{d\underline{x}}\right), \\
&\alpha_1 = \frac{\widetilde{\rho}}{\pi} u_0 * \left[\frac{2\underline{x}\underline{y}}{(\underline{x}^2 + \underline{y}^2)^2}\right], \ \beta_1 = \frac{\widetilde{\rho}}{\pi} u_0 * \left[\frac{\underline{x}^2 - \underline{y}^2}{(\underline{x}^2 + \underline{y}^2)^2}\right],
\end{aligned}$$

where

$$u_0 = \psi_0 * u.$$

Then, we have

$$\xi_\varepsilon^*(\mathcal{T}_0 u) = \beta_{10} = -\widetilde{\rho}\mathcal{H}\left(\frac{du_0}{d\underline{x}}\right),$$

$$||\mathcal{T}_0 u||_{\mathbb{D},w}^{\alpha} \leq c||u||_{\mathbb{R},2}^{1,\alpha}.$$

The main result of this section is then the following

Lemma 8.4. *The solution $(u, Y) \in B_2^{1,\alpha}(\mathbb{R}) \times B_{\underline{\pi}_\varepsilon \mathbb{D},w}^{\alpha}$ of system (8.5,8.6) satisfies*

$$Y = \mathcal{T}_0 u + \mathcal{T}_1(T_u) + \mathcal{T}_2(T_Y), \tag{8.7}$$

$$\xi_\varepsilon^*[Y] + \widetilde{\rho}\mathcal{H}\left(\frac{du}{d\underline{x}}\right) = \mathcal{C}_\varepsilon^{(1)}(T_u) + \mathcal{C}_\varepsilon^{(2)}(T_Y^{(1)}) + \mathcal{C}_\varepsilon^{(3)}(T_Y^{(2)}), \tag{8.8}$$

where $\mathcal{T}_0 u$ is defined at previous lemma, and the following estimates

$$\begin{aligned}
&||\mathcal{T}_1(T_u)||_{\mathbb{D},w}^{\alpha} \leq c\varepsilon||T_u||_{\mathbb{R},3}^{\alpha}, \\
&||\mathcal{T}_2(T_Y)||_{\mathbb{D},w}^{\alpha} \leq c\varepsilon(||T_Y^{(1)}||_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,2}^{\alpha} + ||T_Y^{(2)}||_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,3}^{\alpha}),
\end{aligned}$$

$$\begin{aligned}
&||\mathcal{C}_\varepsilon^{(1)}(T_u)||_{\mathbb{R},2}^{\alpha} \leq c\varepsilon||T_u||_{\mathbb{R},3}^{\alpha}, \\
&||\mathcal{C}_\varepsilon^{(2)}(T_Y^{(1)})||_{\mathbb{R},2}^{\alpha} \leq c\varepsilon||T_Y^{(1)}||_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,2}^{\alpha}, \\
&||\mathcal{C}_\varepsilon^{(3)}(T_Y^{(2)})||_{\mathbb{R},2}^{\alpha} \leq c\varepsilon||T_Y^{(2)}||_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,3}^{\alpha}
\end{aligned}$$

hold.

Proof of lemma 8.2. We have ψ_0 even, indefinitely differentiable, and decaying fast at infinity. Moreover, for any fixed n

$$|\psi_0(x)| \leq c \min\{\varepsilon^{-1}, \varepsilon/x^2, ...\varepsilon^{2n-1}/x^{2n})$$

holds, as can be deduced from the identity

$$\psi_0(x) = \frac{(-1)^n \varepsilon^{2n}}{\pi x^{2n}} \int_0^\infty \cos(kx)\varphi_0^{(2n)}(\varepsilon k)dk$$

which is valid for any $n \geq 0$. This implies

$$||(1+x^2)\psi_0||_{L^\infty} \leq c,$$

hence, the identity

$$u_0 = \psi_0 * u$$

leads to the result of the lemma 8.2, using the property (see for instance (Iooss & Kirrmann 1996))

$$\int_{\mathbb{R}} \frac{1+x^2}{(1+t^2)[1+(x-t)^2]} dt < 2\pi.$$

Proof of lemma 8.3. The fact that $\beta_{10} \in B_2^\alpha(\mathbb{R})$ results from the property that u_0 and $u_0' \in B_2^\alpha(\mathbb{R})$ and a result of Corollary 12.2 of Appendix A, where we show that $\mathcal{H}(u_0') \in B_2^\alpha(\mathbb{R})$ (see also (Amick & Toland 1991)). Moreover, the result for α_2 is straightforward. For α_1 and β_1 we prove in Appendix A that

$$\alpha_1(\underline{x}, \underline{y}) = -\frac{\partial}{\partial x}\left[\frac{\widetilde{\rho}}{\pi} u_0 * \left(\frac{\underline{y}}{\underline{x}^2 + \underline{y}^2}\right)\right],$$
$$\beta_1(\underline{x}, \underline{y}) = \frac{\partial}{\partial y}\left[\frac{\widetilde{\rho}}{\pi} u_0 * \left(\frac{\underline{y}}{\underline{x}^2 + \underline{y}^2}\right)\right],$$

satisfy

$$||\alpha_1||_{B_w^-} + ||\beta_1||_{B_w^-} \leq c||u_0||_{\mathbb{R},2}^{1,\alpha}.$$

In addition, we notice that

$$\widehat{\mathcal{T}_0 u}(k) = ik\widehat{u}_0(k)(\widetilde{\rho}\chi_k - y\xi_2)$$

holds, which shows how we extract this operator $\mathcal{T}_0$ from our previous computation of the resolvent near 0 (see section 5.(d)).

Proof of lemma 8.4. We have by Fourier transform

$$(ik\mathbb{I} - \mathcal{L}_\varepsilon)(\widehat{u}\xi_0 + \varepsilon\widehat{Y}) = \varepsilon\left(\widehat{T_u}\xi_0 + \widehat{T_Y}\right),$$

then, using the splitting of unity, we now solve the system

$$(ik\mathbb{I} - \mathcal{L}_\varepsilon)(\widehat{u}_0\xi_0 + \varepsilon\widehat{Y}_0) = \varepsilon\varphi_0\left(\widehat{T_u}\xi_0 + \widehat{T_Y}\right), \tag{8.9}$$
$$(ik\mathbb{I} - \mathcal{L}_\varepsilon)(\widehat{u}_1\xi_0 + \varepsilon\widehat{Y}_1) = \varepsilon\varphi_1\left(\widehat{T_u}\xi_0 + \widehat{T_Y}\right) \tag{8.10}$$

where

$$\widehat{u}_0 = \varphi_0(\varepsilon k)\widehat{u},\ \widehat{u}_1 = \varphi_1(\varepsilon k)\widehat{u},$$
$$\widehat{Y}_0 = \varphi_0(\varepsilon k)\widehat{Y},\ \widehat{Y}_1 = \varphi_1(\varepsilon k)\widehat{Y},$$
$$u = u_0 + u_1,\ Y = Y_0 + Y_1.$$

Step 1. For solving the first equation (8.9), we use lemma 5.4 (since the right hand side cancels for $|k| > \delta/\varepsilon$), and then find, in fact for any $k \in \mathbb{R}$ (below is the definition of operators $S_u^{(0)}$ and $S_Y^{(0)}$, detailed in Appendix Resolvent 0)

$$\widehat{u}_0(k) = -\frac{1}{ik(1+\widetilde{\rho}|k|)}\varphi_0\xi_\varepsilon^*\left(\widehat{T_Y}\right) + \varphi_0 S_u^{(0)}\left(\widehat{T_Y}\right) + \frac{\varepsilon}{ik}\varphi_0\widehat{T_u},$$
$$\widehat{Y}_0(k) = \frac{1}{1+\widetilde{\rho}|k|}\varphi_0\xi_\varepsilon^*\left(\widehat{T_Y}\right)(y\xi_2 - \widetilde{\rho}\chi_k) + \varphi_0 S_Y^{(0)}\left(\widehat{T_Y}\right),$$

with

$$\varphi_0 S_u^{(0)}\left(\widehat{T_Y}\right)(k) = O\left(\frac{\varepsilon}{1+|k|}||\varphi_0\widehat{T_Y}||_{\mathbb{F}_\varepsilon}\right),$$
$$\varphi_0 S_Y^{(0)}\left(\widehat{T_Y}\right)(k) = [\Phi(f_1, g_1)](k) + O\left(\varepsilon||\varphi_0\widehat{T_Y}||_{\mathbb{H}}\right),$$

where f_1 and g_1 are the 3rd and 4th components of $\varphi_0\widehat{T_Y}$, and where the estimate of $\widehat{Y}_0(k)$ is in $\mathbb{D}$. We show at Appendix Resolvent 0, that

$$\mathcal{F}^{-1}\left[\varphi_0 S_u^{(0)}\left(\widehat{T_Y}\right)\right] \in B_2^{1,\alpha}(\mathbb{R}), \tag{8.11}$$

with

$$||\mathcal{F}^{-1}\left[\varphi_0 S_u^{(0)}\left(\widehat{T_Y}\right)\right]||_{\mathbb{R},2}^{1,\alpha} \leq c\varepsilon(||T_Y^{(1)}||^\alpha_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,2} + ||T_Y^{(2)}||^\alpha_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,3}).$$

We observe now that the fast exponential decay in y of the 3rd and 4th components of T_Y implies

$$||\left[\Phi(f_1, g_1)\right](k)||_{\mathbb{D}} = O\left(\varepsilon||\varphi_0(\widehat{T_Y})||_{\mathbb{F}_\varepsilon}\right).$$

Finally we have, for any $k \in \mathbb{R}$

$$\varphi_0 S_Y^{(0)}\left(\widehat{T_Y}\right)(k) = O\left(\varepsilon||\varphi_0\left(\widehat{T_Y}\right)||_{\mathbb{F}_\varepsilon}\right).$$

Now we can compute

$$\widehat{Y}_0(k) = \widehat{\mathcal{T}_0 u}(k) + \varphi_0(\varepsilon k)S_Y^{(0)}\left(\widehat{T_Y}\right)(k) + \\ + (y\xi_2 - \widetilde{\rho}\chi_k)\varphi_0(\varepsilon k)\left[ikS_u^{(0)}\left(\widehat{T_Y}\right)(k) + \varepsilon\widehat{T_u}(k)\right], \tag{8.12}$$

which allows to define

$$\mathcal{T}_{10}(T_u) = \mathcal{F}^{-1}\left\{\varepsilon\varphi_0(\varepsilon k)\widehat{T_u}(k)(y\xi_2 - \widetilde{\rho}\chi_k)\right\},$$
$$\mathcal{T}_{20}(T_Y) = \mathcal{F}^{-1}\left\{\varphi_0(\varepsilon k)\left[S_Y^{(0)}\left(\widehat{T_Y}\right)(k) + ik(y\xi_2 - \widetilde{\rho}\chi_k)S_u^{(0)}\left(\widehat{T_Y}\right)(k)\right]\right\},$$

and we obtain in addition

$$\xi_\varepsilon^*[Y_0] + \widetilde{\rho}\mathcal{H}(\frac{du_0}{d\underline{x}}) = \xi_\varepsilon^*[\mathcal{T}_{10}(T_u) + \mathcal{T}_{20}(T_Y)]. \tag{8.13}$$

We observe that $\mathcal{T}_{10}(T_u)$ is easily estimated, since it is the same estimate as at lemma 8.3, once we notice that $\int_{-\infty}^{\underline{x}} T_u(\tau)d\tau \in B_2^{\alpha}(\mathbb{R})$ is an even primitive of T_u. Hence we have directly

$$||\mathcal{T}_{10}(T_u)||^{\alpha}_{\mathbb{D},w} \leq c\varepsilon ||T_u||^{\alpha}_{\mathbb{R},3}. \tag{8.14}$$

We prove at the Appendix Resolvent 0, that we have the estimate

$$||\mathcal{T}_{20}(T_Y)||^{\alpha}_{\mathbb{D},w} \leq c\varepsilon(||T_Y^{(1)}||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,2} + ||T_Y^{(2)}||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,3}), \tag{8.15}$$

where the above estimate (8.11) on $\mathcal{F}^{-1}\left[\varphi_0 S_u^{(0)}\left(\widehat{T_Y}\right)\right]$ solves half of this estimate, because of lemma 8.3. Hence we finally obtain

$$||\xi_\varepsilon^*[Y_0] + \widetilde{\rho}\mathcal{H}(\frac{du_0}{d\underline{x}})||^{\alpha}_{\mathbb{R},2} \leq c\varepsilon(||T_u||^{\alpha}_{\mathbb{R},3} + ||T_Y^{(1)}||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,2} + ||T_Y^{(2)}||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,3}). \tag{8.16}$$

Step 2. For solving the equation (8.10), we use lemma 5.3. Indeed we have

$$\begin{aligned} ||(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1}\pi_\varepsilon||_{\mathcal{L}(\mathbb{H})} &\leq c/|k|, \\ ||\mathcal{P}(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1}\pi_\varepsilon||_{\mathcal{L}(\mathbb{H},\mathbb{D}^{\mathcal{P}})} &\leq c\varepsilon, \end{aligned}$$

and noticing that $\widehat{u}_1(k)$ is the β_{21} component of $\varepsilon\varphi_1(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1}\left(\widehat{T_u}\xi_0 + \widehat{T_Y}\right)$, this leads to (below is the definition of the operators $S_u^{(1)}, S_Y^{(1)}$)

$$\begin{aligned} \widehat{u}_1(k) &= \frac{\varepsilon}{ik}\varphi_1\widehat{T_u} + \varphi_1 S_u^{(1)}\left(\widehat{T_Y}\right) = O\left(\frac{\varepsilon}{|k|}||\varphi_1\left(\widehat{T_u}\xi_0 + \widehat{T_Y}\right)||_{\mathbb{H}}\right), \\ \widehat{Y}_1(k) &= \varphi_1 S_Y^{(1)}\left(\widehat{T_Y}\right) = O(\varepsilon||\varphi_1\left(\widehat{T_Y}\right)||_{\mathbb{H}}), \end{aligned}$$

hence we can define

$$\begin{aligned} \mathcal{T}_{11}(T_u) &= 0, \\ \mathcal{T}_{21}(T_Y) &= \mathcal{F}^{-1}\left[\varphi_1(\varepsilon k)S_Y^{(1)}\left(\widehat{T_Y}\right)(k)\right], \end{aligned}$$

with the following estimate proved at Appendix Resolvent ∞

$$||\mathcal{T}_{21}(T_Y)||^{\alpha}_{\mathbb{D},w} \leq c\varepsilon(||T_Y^{(1)}||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,2} + ||T_Y^{(2)}||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,3}), \tag{8.17}$$

hence the lemma for (8.7) is proved. Moreover it is also proved in Appendix Resolvent ∞ that

$$\mathcal{F}^{-1}\left[\varphi_1 S_u^{(1)}\left(\widehat{T_Y}\right)\right] \in B_2^{1,\alpha}(\mathbb{R}) \tag{8.18}$$

holds, with

$$||\mathcal{F}^{-1}\left[\varphi_1 S_u^{(1)}\left(\widehat{T_Y}\right)\right]||^{1,\alpha}_{\mathbb{R},2} \leq c\varepsilon(||T_Y^{(1)}||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,2} + ||T_Y^{(2)}||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,3}),$$

so, we obtain in addition

$$\begin{aligned} \xi_\varepsilon^*[Y_1] = &-\widetilde{\rho}\mathcal{H}(\frac{du_1}{d\underline{x}}) + \mathcal{F}^{-1}\left[\widetilde{\rho}|k|\varphi_1(\varepsilon k)S_u^{(1)}\left(\widehat{T_Y}\right)\right] + \xi_\varepsilon^*[\mathcal{T}_{21}(T_Y)] + \\ &- \mathcal{F}^{-1}\left[isgn(k)\widetilde{\rho}\varepsilon\varphi_1(\varepsilon k)\widehat{T_u}(k)\right], \end{aligned} \tag{8.19}$$

with an estimate (see corollary 12.2 in Appendix A, and use the fact that $\int_{-\infty}^{\underline{x}} T_u d\tau$ and $\psi_0 * \int_{-\infty}^{\underline{x}} T_u d\tau \in B_2^{1,\alpha}(\mathbb{R})$)

$$||\xi_\varepsilon^*[Y_1] + \widetilde{\rho}\mathcal{H}(\frac{du_1}{d\underline{x}})||_{\mathbb{R},2}^\alpha \leq c\varepsilon(||T_u||_{\mathbb{R},3}^\alpha + ||T_Y^{(1)}||_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,2}^\alpha + ||T_Y^{(2)}||_{\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon,3}^\alpha).$$

Step 3. We now collect the results (8.13,8.19), and finally obtain (8.8)

$$\xi_\varepsilon^*[Y] + \widetilde{\rho}\mathcal{H}\left(\frac{du}{d\underline{x}}\right) = \mathcal{C}_\varepsilon^{(1)}(T_u) + \mathcal{C}_\varepsilon^{(2)}(T_Y^{(1)}) + \mathcal{C}_\varepsilon^{(3)}(T_Y^{(2)}),$$

with

$$\mathcal{C}_\varepsilon^{(1)}(T_u) = \xi_\varepsilon^*[\mathcal{T}_{10}(T_u)] - \mathcal{F}^{-1}\left[isgn(k)\widetilde{\rho}\varepsilon\varphi_1(\varepsilon k)\widehat{T_u}(k)\right],$$

$$\begin{aligned}\mathcal{C}_\varepsilon^{(2)}(T_Y^{(1)}) + \mathcal{C}_\varepsilon^{(3)}(T_Y^{(2)}) = \mathcal{F}^{-1}\left[\widetilde{\rho}|k|\varphi_1(\varepsilon k)S_u^{(1)}\left(\widehat{T_Y}\right)\right] + \\ + \xi_\varepsilon^*[\mathcal{T}_{21}(T_Y) + \mathcal{T}_{20}(T_Y)],\end{aligned}$$

and with estimates announced at the lemma.

(d) New system

We now replace the two last equations in (8.2) by two equivalent new equations. The first equation will be (8.7) where T_u and T_Y are expressed in terms of the nonlinear terms on the right hand side of (8.2). The second one is a combination of the Bernoulli first integral (which results from the full system, see lemma 8.1) and the identity (8.8) which comes from the two last equations, and where T_u and T_Y are expressed in terms of the nonlinear terms as above. Let us express our new system in the following lemma

Lemma 8.5. *There exists $\delta > 0$, such that for any $M > 0$, $|A| < \delta$, $|u| + \varepsilon||Y||_{\underline{\pi}_\varepsilon \mathbb{D}} < M$, $0 < \varepsilon < \varepsilon_0$ the system satisfied by A, u, Y as defined by (8.1) takes the following equivalent form*

$$\begin{aligned}\frac{dA}{d\underline{x}} &= iA\left[\frac{\lambda}{\varepsilon} + \gamma_1(u, \varepsilon p_1^*(Y), |A|^2, \varepsilon)\right] + R_A(A, \overline{A}, u, Y), \\ Y &= \underline{\pi}_\varepsilon[\mathcal{T}_0 u + \mathcal{T}_1(T_u) + \mathcal{T}_2(T_Y)], \\ \rho\mathcal{H}&\left(\frac{du}{d\underline{x}}\right) + u + \frac{3}{2}u^2 = \mathcal{B}_\varepsilon(A, \overline{A}, u, Y),\end{aligned} \tag{8.20}$$

provided the nonlocal operators are defined, and where

$$\begin{aligned}T_u &= 2up_0^*\pi_\varepsilon N_\varepsilon^{(2)}(\xi_0, Y) + \varepsilon^{-1}R_u(A, \overline{A}, u, Y), \\ T_Y &= T_Y^{(1)} + T_Y^{(2)}, \\ T_Y^{(1)} &= \Delta_\varepsilon(A, \overline{A})[Y], \\ T_Y^{(2)} &= 2u\underline{\pi}_\varepsilon N_\varepsilon^{(2)}(\xi_0, Y) + R_Y(A, \overline{A}, u, Y),\end{aligned}$$

$$\mathcal{B}_\varepsilon(A,\overline{A},u,Y) = \frac{\varepsilon}{\lambda}\mathcal{H}\left(\frac{du}{d\underline{x}}\right) + \widetilde{h}_\varepsilon[|A|^2,u,\varepsilon p_1^*(Y)] + \mathcal{C}_\varepsilon^{(1)}(T_u) + \\ + \mathcal{C}_\varepsilon^{(2)}(T_Y^{(1)}) + \mathcal{C}_\varepsilon^{(3)}(T_Y^{(2)}) + \widetilde{R}_u(A,\overline{A},u,Y) + c_0.$$

The right hand side of this system is well defined for $A \in C^\alpha(\mathbb{R},\mathbb{C})$, $u \in B_2^{1,\alpha}(\mathbb{R})$, $Y \in B^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}$ *and the operators* $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{C}_\varepsilon^{(1)}$, $\mathcal{C}_\varepsilon^{(2)}$, $\mathcal{C}_\varepsilon^{(3)}$ *satisfy the estimates of lemma 8.4.*

Remark. The operators $\mathcal{H}$, $\mathcal{B}_\varepsilon$, $\mathcal{T}_i$, $\mathcal{C}^{(i)}$ are nonlocal with respect to the $\underline{x}$ coordinate.

Proof. We put the projection $\underline{\pi}_\varepsilon$ in equation $(8.20)_2$ since this does not change the result, $Y(\underline{x})$ lying in $\underline{\pi}_\varepsilon\mathbb{D}$. Equation $(8.20)_3$ is obtained in replacing $\xi_\varepsilon^*(Y)$ in (8.4) by its expression from (8.8) in lemma 8.4. The remarkable fact here is that the left hand side of $(8.20)_3$ is the Benjamin-Ono equation! (see (Benjamin 1967; Davis & Acrivos 1967; Ono 1975)), and that its right hand side will play the role of a nice perturbation.

The *equivalence* between system (8.2) and (8.20) can be seen in realizing that (8.2) is equivalent to a system with the same equation for A, and where we deduce (u, Y) from the inverse Fourier transform of the resolvent, as done in the proof of lemma 8.4. The combination $Y - \mathcal{T}_0 u$ gives one of the final equations, and the result for u is precisely identified with the Benjamin-Ono equation differentiated with respect to $\underline{x}$, as might be seen in the expression found at step 1 of the proof of lemma 8.4:

$$ik(1+\widetilde{\rho}|k|)\widehat{u}_0(k) = -\varphi_0\xi_\varepsilon^*\left(\widehat{T_Y}\right) + ik(1+\widetilde{\rho}|k|)\varphi_0 S_u^{(0)}\left(\widehat{T_Y}\right) + \varepsilon(1+\widetilde{\rho}|k|)\varphi_0\widehat{T_u}.$$

It can be seen that the term $-3\widehat{u'u}$ which corresponds to the main next term in Benjamin-Ono equation comes from the term

$$2u\underline{\pi}_\varepsilon N_0^{(2)}(\xi_0, Y)$$

belonging to T_Y, inserted in $-\xi_\varepsilon^*\left(\widehat{T_Y}\right)$, where Y is replaced by $\mathcal{T}_0 u$ (or $\underline{\pi}_\varepsilon\mathcal{T}_0 u$, which gives the same result). We did not use this way for deriving the Benjamin-Ono equation, because it is simpler to use the Bernoulli first integral. So the equivalence of (8.2) and (8.20) results from the resolution of the linear part of equations for (u, Y), thanks to a double combination made on their Fourier transforms.

Remark. It should be clear that our new system (8.20) is non local, which implies the necessity to define a priori an acceptable behavior in $\underline{x}$, for the solution. Before doing this, we give in the next section an approximate solution of the full system, under the form of an homoclinic, whose principal part is solution of the Benjamin-Ono equation.

9. Asymptotic expansion of a solitary wave

In this section we give an asymptotic result without proof, since it is not used in our further proofs, but which seems interesting by itself. It gives the asymptotic expansion of a solitary wave, corresponding to a formal solution of our system (4.1), this expansion is expected to diverge. In the later proofs, we only use its principal part. We have the following

Theorem 9.1. *The system (4.1) has a formal reversible solution in the form of a power series in* ε :

$$\underline{U} = u_0^h \xi_0 + \sum_{n\geq 1} \varepsilon^n [u_n \xi_0 + Z_{n-1}],$$

where coefficients $u_n \in B_2^\alpha(\mathbb{R})$, Z_n *satisfies* $p_0^*(Z_n) = 0$ *and* $Z_n \in B_{\mathbb{D},w}^\alpha$ *with a decay of 3rd and 4th components at least in* $1/\underline{y}^2$ *as* $\underline{y} \to -\infty$. *Coefficient* u_0^h *is given by*

$$u_0^h(\underline{x}) = \frac{-4\rho^2}{3(\rho^2 + \underline{x}^2)}, \tag{9.1}$$

and Z_0 *is given by*

$$Z_0(\underline{x}, y) = \left(\frac{4\rho^2(\rho^2 - \underline{x}^2)}{3(\rho^2 + \underline{x}^2)^2}, 0, \frac{8\rho^2 \underline{x}(\rho - \underline{y})}{3[(\rho - \underline{y})^2 + \underline{x}^2]^2}, \frac{4\rho^2[(\rho - \underline{y})^2 - \underline{x}^2]}{3[(\rho - \underline{y})^2 + \underline{x}^2]^2}, \right.$$
$$\left. \frac{8\rho^2 \underline{x}(\rho - y)}{3(\rho^2 + \underline{x}^2)^2}, 0 \right)^t .$$

Moreover, we have the following

Lemma 9.2. *Let us define for a fixed integer* $p \geq 0$

$$H_p = (u_0^h + \sum_{1\leq k\leq p+1} \varepsilon^k u_k)\xi_0 + \sum_{1\leq k\leq p+2} \varepsilon^k Z_{k-1},$$

then, by construction

$$\frac{dH_p}{d\underline{x}} - \mathcal{L}_\varepsilon H_p - \mathcal{N}(\varepsilon; H_p) = \varepsilon^{p+2} \mathcal{K}_p$$

holds, with $\mathcal{K}_p = (a_p, b_p, 0, 0, f_{2,p}, g_{2,p})^t \in \mathcal{P}B_{\mathbb{D},w}^\alpha$, *and as* $|x| \to \infty$, *we have*

$$a_p = O(1/\underline{x}^3), \quad b_p = O(1/\underline{x}^3), \quad f_{2,p} = O(1/\underline{x}^4), \quad g_{2,p} = O(1/\underline{x}^3).$$

The proof of the theorem and lemma above is based on an identification of powers of ε in system (4.4,4.5), where we recover at main order the Benjamin-Ono equation

$$u_0^h + \rho \mathcal{H}\left(\frac{du_0^h}{d\underline{x}}\right) + \tfrac{3}{2}(u_0^h)^2 = 0,$$

whose *unique even solution, tending towards 0 at infinity,* is given by (9.1) (as it is proved in (Amick & Toland 1991)). We observe that this equation also appears on the new system (8.20) in lemma 8.5, hence we can state the following

Lemma 9.3. *There is a* reversible homoclinic, *approximate solution of the system (8.20) of the form*

$$A = 0,$$
$$u = u_0^h,$$
$$Y = Y_0^h = \underline{\pi}_\varepsilon \mathcal{T}_0 u_0^h,$$

where $u_0^h \in B_2^{1,\alpha}(\mathbb{R})$ *is even in* $\underline{x}$*, and* $Y_0^h \in B^\alpha_{\frac{\pi}{\varepsilon}\mathbb{D},w}$ *and is reversible, and* u_0^h *satisfies*

$$u_0^h + \rho\mathcal{H}\left(\frac{du_0^h}{d\underline{x}}\right) + \tfrac{3}{2}(u_0^h)^2 = 0.$$

Remark (Physical shape of the approximate homoclinic). From the expressions of the interface and free surface, we have

$$\frac{dZ_I}{dx}(x) \sim \alpha_{10}(x) = \alpha_{20}(x),$$
$$\frac{dZ}{dx}(x) \sim \alpha_{21}(x).$$

Hence, after reestablishing the unscaled variables, and from the expressions of components of $Z_0(\underline{x}, y)$, we obtain for the approximate homoclinic

$$Z_I(x) \sim -\frac{4\varepsilon^2\rho^3}{3(\rho^2+\varepsilon^2x^2)},$$
$$Z(x) \sim \frac{4\varepsilon^2\rho^2(1-\rho)}{3(\rho^2+\varepsilon^2x^2)},$$

which gives at finite distance, the physical shape indicated at figure 8, once we notice that $x \sim \xi$. For the shape of the solutions near infinity, this corresponds to the periodic waves whose principal part is based on the eigenvectors belonging to eigenvalues $\pm i\lambda/\varepsilon$ of $\mathcal{L}_\varepsilon$. For the computation of $Z_I(x)$ and $Z(x)$, it is better to use the formulas

$$Z_I(x) \sim -\int_{-\infty}^0 \beta_1(x,y)dy,$$
$$Z(x) \sim -\int_{-\infty}^0 \beta_1(x,y)dy - \int_0^1 \beta_2(x,y)dy,$$

and it is shown in (Iooss 1999) that the periodic waves at the free surface and at the interface are in phase, hence this will be the case for our generalized solitary waves here, as $|x| \to \infty$ (this is proved at next section).

10. Homoclinics to periodic solutions

The purpose of the paper is to find solutions homoclinic to each of the periodic solutions $p_{A_0,0,0,\varepsilon}$ found at section 6, such that $u_0 = v_0 = 0$ (we might generalize our results for $u_0.v_0 \neq 0$). In this section we prove

Theorem 10.1. *For any* $0 < \alpha \leq 1/2$*, there exist* δ*,* δ_0*,* $\varepsilon_0 > 0$*, such that for* $0 < \varepsilon < \varepsilon_0$*, and* $\delta_0\varepsilon^{2-\alpha} < A_0 < \delta$*, the following statement holds: there exist two distinct reversible solutions* $\underline{U}^{(j)}_{A_0,\varepsilon}$ $(j = 1, 2)$ *of the scaled system (4.1), Hölder continuous in* $\mathbb{D}$*, homoclinic to each periodic solution* $p_{A_0,0,0,\varepsilon}$ *found at theorem 6.3 which satisfy*

$$U^{(j)}_{A_0,\varepsilon}(\underline{x}) = p_{A_0,0,0,\varepsilon}\Big(x + \phi_j\rho\arctan(\underline{x}/\rho)\Big) + u_0^h(\underline{x})\xi_0 + \mathcal{O}\left(\frac{\varepsilon^{1-\alpha}+A_0}{1+|\underline{x}|}\right).$$

where u_0^h *is the Benjamin-Ono homoclinic, of order 1, decaying at infinity as* $1/|\underline{x}|^2$.

As shown in the last remark of section 9, provided that the integrals invoked in the remark are convergent, these solutions give generalized solitary waves with a shape indicated at figure 8, where they all look the same at finite distance, and where they fit with one of the periodic travelling waves at infinity, with opposite phase shifts at $\pm\infty$. Moreover, the form of periodic solutions found at theorem 6.3 shows that these ones have principal parts given by $A_0(\underline{\zeta}_\varepsilon e^{is} + \overline{\underline{\zeta}}_\varepsilon e^{-is})$, and then the form of $\underline{\zeta}_\varepsilon$ shows that the free surface and the interface are in phase.

Remark 1. A natural question is how can we improve the above result in allowing a smaller size for A_0? A first improvement may be made in improving the normal form in lemma 7.2. This can be done in treating terms of order $|u|^m||W||$ in the A equation. Indeed they can be suppressed up to an arbitrary order, by a technique analogous to the one showed at Appendix Normal Form. A second improvement may be made if we use a better approximation of the approximate homoclinic (see lemma 9.2). These two actions may arrive to a lower bound for $|A_0|$ in ε^m with $m > 2$. Now, the major improvement would be to obtain an exponentially small lower bound for $|A_0|$. This needs to work on analytic functions in a strip of the complex plane containing the real axis, using methods developed for example in (Lombardi 2000). However, in the context here, where an essential spectrum passes through 0, this needs additional work which is provided in the forthcoming paper (Lombardi & Iooss 2001).

Remark 2. In the above theorem the limiting periodic solutions $p_{A_0,0,0,\varepsilon}$ are those with parameters $u_0 = v_0 = 0$. We might produce the same result for $|u_0| + |v_0| < \delta$, in adapting the normal form of lemma 7.2, as indicated in the previous remark (modulo additional complications). These new homoclinics would correspond to *non zero mean horizontal flows* in both layers.

The rest of this section is devoted to the proof of this theorem.

(*a*) *Shifted system*

For proving theorem 10.1, we use a fixed point technique, in starting from an approximate homoclinic connection computed in section 9, and coming from the Benjamin-Ono equation.

After the changes of coordinates made in sections 7 and 8, the system (4.1) is equivalent to (8.20), and the family of periodic orbits $p_{A_0,0,0,\varepsilon}$ found at section 6 reads now

$$\begin{aligned}\mathsf{p}_{A_0,\varepsilon}(s) &= (\mathsf{A}_{A_0,\varepsilon}, \mathsf{Y}_{A_0,\varepsilon}, \mathsf{u}_{A_0,\varepsilon}) \\ &= (A_0 e^{is}, 0, 0)\end{aligned}$$

with $A_0 \in \mathbb{R}$ (reversible solutions), and

$$s = \left[\frac{\lambda}{\varepsilon} + \gamma_1(0, 0, A_0^2, \varepsilon)\right] \underline{x}.$$

We look for homoclinic connections to the periodic orbits $\mathsf{p}_{A_0,\varepsilon}$ under the form

$$\mathsf{H}_{A_0,\varepsilon} = \mathsf{p}_{A_0,\varepsilon} \circ \psi_\phi(\underline{x}) + \mathsf{h}_\varepsilon(\underline{x}) \tag{10.1}$$

where

$$\psi_\phi(\underline{x}) \stackrel{def}{=} \left[\frac{\lambda}{\varepsilon} + \gamma_1(0, 0, A_0^2, \varepsilon)\right] \left[\underline{x} + \varepsilon\rho\phi\theta(\underline{x}/\rho)\right],$$

$$\theta(\underline{x}) = \arctan(\underline{x}).$$

The unknown are $\phi \in \mathbb{R}$, which is proportional to the phase shift at infinity, and h_ε which is required to be a *reversible homoclinic connection to 0.* We look for $\mathsf{h}_\varepsilon(\underline{x})$ under the form

$$\mathsf{h}_\varepsilon(\underline{x}) = \left((iq_1 + q_2)e^{i\psi_\phi(\underline{x})}, Y, u\right).$$

So, we are looking for $\phi \in \mathbb{R}$ and (q_1, q_2, Y, u) tending towards 0 as $\underline{x} \to \pm\infty$ with q_1 odd, (q_2, u) even, and Y satisfying $SY(\underline{x}) = Y(-\underline{x})$. Notice that we have a freedom in the choice of the odd function θ; the important request is that θ' is positive and decays at least as $1/\underline{x}^2$.

The new system satisfied by (q_1, q_2, Y, u, ϕ) reads

$$\begin{aligned}
\frac{dq_1}{d\underline{x}} &= (A_0 + q_2)\left[\gamma_1(u, 0, 0, \varepsilon) - \phi\rho_0\right] + R_{q_1}, \\
\frac{dq_2}{d\underline{x}} &= -q_1\left[\gamma_1(u, 0, 0, \varepsilon) - \phi\rho_0\right] + R_{q_2}, \\
Y &= \underline{\pi}_\varepsilon[\mathcal{T}_0 u] + \mathcal{R}'_Y, \\
&\rho\mathcal{H}\left(\frac{du}{d\underline{x}}\right) + u + \frac{3}{2}u^2 = \mathcal{B}'_\varepsilon,
\end{aligned} \tag{10.2}$$

where

$$\rho_0(\underline{x}) = \left[\lambda + \gamma(0, 0, A_0^2, \varepsilon)\right]\frac{\rho^2}{\underline{x}^2 + \rho^2},$$

$$\begin{aligned}
R_{q_2} + iR_{q_1} &= R_A e^{-i\psi_\phi(\underline{x})} + i(A_0 + q_2 + iq_1)\left\{\gamma_1(u, \varepsilon p_1^*(Y), |A|^2, \varepsilon) + \right. \\
&\quad \left. -\gamma_1(u, 0, 0, \varepsilon) - \gamma_1(0, 0, A_0^2, \varepsilon)\right\}, \\
\mathcal{R}'_Y &= \underline{\pi}_\varepsilon[\mathcal{T}_1(T'_u) + \mathcal{T}_2(T'_Y)],
\end{aligned}$$

where we put a prime when we need to replace A by $(A_0 + q_2 + iq_1)e^{i\psi_\phi}$, and where we choose the constant c_0 in $\mathcal{B}_\varepsilon$ such that $(q_1, q_2, Y, u) = 0$ cancels it.

(*b*) *Decay rates*

Let us consider the expected decay rates as $\underline{x} \to \infty$ for (q_1, q_2, Y, u). We have an approximate homoclinic computed at section 9, which decays as $1/\underline{x}^2$ for the components (Y_0^h, u_0^h). Moreover, if we make $R_{q_1} = R_{q_2} = 0$, and replace u by u_0^h in the two first equations of (10.2), we can compute explicitly a solution tending towards 0 at infinity, with non zero q_1 and q_2 functions:

$$q_1 = A_0 \sin\int_{-\infty}^{\underline{x}}\left[\gamma_1(u_0^h(\tau), 0, 0, \varepsilon) - \phi\rho_0(\tau)\right]d\tau \tag{10.3}$$

$$q_2 = A_0\left\{\cos\int_{-\infty}^{\underline{x}}\left[\gamma_1(u_0^h(\tau), 0, 0, \varepsilon) - \phi\rho_0(\tau)\right]d\tau - 1\right\}, \tag{10.4}$$

where ϕ is determined by

$$\int_{-\infty}^{+\infty}\left[\gamma_1(u_0^h(\tau), 0, 0, \varepsilon) - \phi\rho_0(\tau)\right]d\tau = 0. \tag{10.5}$$

The decay rate of $\gamma_1(u_0^h(\tau), 0, 0, \varepsilon)$ and $\rho_0(\tau)$ is in $1/\tau^2$, hence q_1 decays as $1/\underline{x}$ while q_2 decays as $1/\underline{x}^2$. It results that we choose to work in spaces such that q_1 decays as $1/\underline{x}$, while q_2, u and Y decay as $1/\underline{x}^2$.

We now observe, because of reversibility of (8.2) under the symmetry

$$(A, \overline{A}, Y, u) \mapsto (\overline{A}, A, SY, u),$$

that the equations (10.2) are invariant under the symmetry $\widehat{S}$:

$$[q_1(\underline{x}), q_2(\underline{x}), Y(\underline{x}), u(\underline{x})] \mapsto [-q_1(-\underline{x}), q_2(-\underline{x}), SY(-\underline{x}), u(-\underline{x})]$$

and that if we assume $(q_1, q_2, Y, u) \in B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D}, w} \times B_2^{1,\alpha}(\mathbb{R})$ then the right hand side of the system (10.2) lies in $B_2^\alpha(\mathbb{R}) \times B_3^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D}, w} \times B_2^\alpha(\mathbb{R})$.

These decay rates are shown below to be sufficient for our proof. We just observe that this constitutes the big benefits of the normal form reduction made at section 7, allowing to kill all not sufficiently decaying terms, linear in Y. In addition we observe that, since the right hand side of equation for q_1 is even, we have to write a compatibility condition for insuring the limit to 0 at both infinities [notice this on (10.3,10.4)]. Precisely, this compatibility condition will allow to determine ϕ, as in (10.5).

(*c*) *Strategy for the resolution of the full equation*

We look for a reversible homoclinic connection $\mathfrak{h} = (q_1, q_2, Y, u)$ to 0 of the full equation (10.2) under the form

$$\mathfrak{h} = \mathfrak{h}_{0,\varepsilon} + \mathfrak{h}_1$$

with

$$\begin{aligned} \mathfrak{h}_{0,\varepsilon} &= (0, 0, Y_0^h, u_0^h), \\ \mathfrak{h}_1 &= (q_1, q_2, Z, w), \end{aligned}$$

where (u_0^h, Y_0^h) are the components of the approximate homoclinic defined in lemma 9.3 and where

$$\mathfrak{h}_1 = (q_1, q_2, Z, w) = O\left(\frac{1}{\underline{x}}, \frac{1}{\underline{x}^2}, \frac{1}{\underline{x}^2}, \frac{1}{\underline{x}^2}\right) \text{ as } x \to \pm\infty.$$

More precisely, we look for $\phi \in \mathbb{R}$ and

$$\mathfrak{h}_1 \in B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D}, w} \times B_2^{1,\alpha}(\mathbb{R}),$$

which satisfies

$$\mathbf{L}_\phi(\underline{x})\mathfrak{h}_1 = \mathbf{G}(\mathfrak{h}_1, \varepsilon, A_0, \phi) \tag{10.6}$$

where

$$\mathbf{L}_\phi(\underline{x})\mathfrak{h}_1 = \left| \begin{array}{l} \frac{dq_1}{d\underline{x}} - \left[\gamma_1(u_0^h, 0, 0, \varepsilon) - \phi\rho_0\right] q_2 \\ \frac{dq_2}{d\underline{x}} + \left[\gamma_1(u_0^h, 0, 0, \varepsilon) - \phi\rho_0\right] q_1 \\ Z - \underline{\pi}_\varepsilon[\mathcal{T}_0 w] \\ \rho\mathcal{H}\left(\frac{dw}{d\underline{x}}\right) + w + 3u_0^h w \end{array} \right.$$

and

$$\mathbf{G} = (G_{q_1}, G_{q_2}, \mathcal{G}_Z, \mathcal{G}_w)$$

with

$$\begin{aligned} G_{q_1} &= A_0[\gamma_1(u_0^h + w, 0, 0, \varepsilon) - \phi\rho_0] + \\ &\quad + q_2[\gamma_1(u_0^h + w, 0, 0, \varepsilon) - \gamma_1(u_0^h, 0, 0, \varepsilon)] + R_{q_1}, \\ G_{q_2} &= -q_1[\gamma_1(u_0^h + w, 0, 0, \varepsilon) - \gamma_1(u_0^h, 0, 0, \varepsilon)] + R_{q_2}, \\ \mathcal{G}_Z &= \mathcal{R}'_Y, \\ \mathcal{G}_w &= \mathcal{B}'_\varepsilon - \frac{3}{2}w^2. \end{aligned}$$

Here the reversibility comes from the invariance of the system under the symmetry

$$(\underline{x}, q_1, q_2, Z, w) \mapsto (-\underline{x}, -q_1, q_2, SZ, w).$$

Moreover the map

$$(\phi, q_1, q_2, Z, w) \mapsto (G_{q_1}, G_{q_2}, \mathcal{G}_Z, \mathcal{G}_w)$$

is analytic from a fixed ball

$$|\phi| < M, \quad ||q_1||^\alpha_{\mathbb{R},1} + ||q_2||^\alpha_{\mathbb{R},2} + ||w||^{1,\alpha}_{\mathbb{R},2} + ||Z||^\alpha_{\mathbb{D},w} < \delta,$$

of

$$\mathbb{R} \times B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^{1,\alpha}(\mathbb{R})$$

to

$$B_2^\alpha(\mathbb{R}) \times B_3^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^\alpha(\mathbb{R}).$$

For finding homoclinic connection to 0 of (10.6) we proceed in several steps:

Step 1. In subsection 10.(d) we consider the affine equation

$$\mathbf{L}_\phi(\underline{x})\mathfrak{h} = \mathbf{F}.$$

More precisely we prove that for any antireversible $\mathbf{F} \in B_2^\alpha(\mathbb{R}) \times B_3^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^\alpha(\mathbb{R})$ there exists a reversible solution $\mathfrak{h}$ in $B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^{1,\alpha}(\mathbb{R})$ if and only if $\mathbf{F}$ satisfies the solvability condition

$$\int_0^\infty < r_-(x), \mathbf{F}(x) > dx = 0 \tag{10.7}$$

where r_- is given by

$$r_- = (\cos\Gamma(\underline{x}), -\sin\Gamma(\underline{x}), 0, 0),$$

with

$$\Gamma(\underline{x}) = \int_0^{\underline{x}} \left[\gamma_1(u_0^h(\tau), 0, 0, \varepsilon) - \phi\rho_0(\tau)\right] d\tau.$$

In other words the range $R(\mathbf{L}_\phi)$ of $\mathbf{L}_\phi$ is the subset of $B_2^\alpha(\mathbb{R}) \times B_3^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^\alpha(\mathbb{R})$ spanned by the functions which satisfy (10.7). So, a necessary condition for the existence of a solution $\mathfrak{h}_1$ of (10.6) in $B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^{1,\alpha}(\mathbb{R})$ is that

$$J(\mathfrak{h}_1, \phi, A_0, \varepsilon) = 0,$$

where

$$J = \int_0^\infty < r_-, \mathbf{G}(\mathfrak{h}_1, \varepsilon, A_0, \phi) > dx.$$

Step 2. For studying J and (10.7), we need precise estimates of $R_{q_1}, R_{q_2}, \mathcal{R}'_Y, \mathcal{B}'_\varepsilon$. They are given in subsection 10.(e).

Step 3. In subsection 10.(f), we study the solvability function J and we compute its principal part.

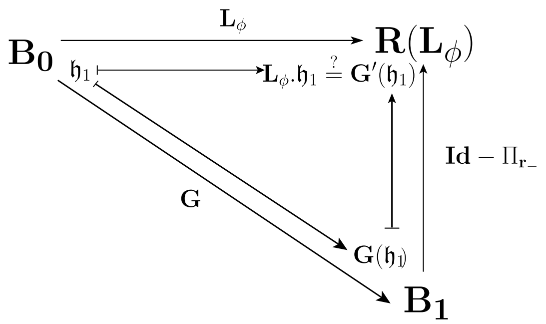

Figure 9. Diagram of the modified equation where $R(\mathbf{L}_\phi)$ is the range of $\mathbf{L}_\phi$; where $\mathbf{B}_0 = B_2^\alpha(\mathbb{R}) \times B_3^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^\alpha(\mathbb{R})$ and $\mathbf{B}_1 = B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^{1,\alpha}(\mathbb{R})$; and where $\mathrm{Id} - \Pi_{r_-}$ is a projection onto the range of $\mathbf{L}_\phi$ with $\Pi_{r_-}(\mathbf{G}) = \frac{2}{\sqrt{\pi}} J(\mathbf{G}) e^{-\underline{x}^2} r_-(\underline{x})$ and $J(\mathbf{G}) = \int_0^\infty < r_-(\tau), \mathbf{G}(\tau) > dx$

Step 4. In subsection 10.(g), we introduce the modified equation

$$\mathbf{L}_\phi(\underline{x})\mathfrak{h}_1 = \mathbf{G}'(\mathfrak{h}_1, \varepsilon, A_0, \phi) \tag{10.8}$$

where

$$\mathbf{G}' = \mathbf{G} - \frac{2}{\sqrt{\pi}} J e^{-\underline{x}^2} r_-(\underline{x}).$$

The term $\mathbf{G}'$ has been designed so that it lies in the range $R(\mathbf{L}_\phi)$ of $\mathbf{L}_\phi$, for every $\varepsilon, A_0, \phi, \mathfrak{h}_1$, i.e.

$$\int_0^\infty < r_-, \mathbf{G}'(\mathfrak{h}_1, \varepsilon, A_0, \phi) > dx = 0.$$

Then, using the implicit function theorem, we prove that for any ϕ and any sufficiently small $|A_0|, \varepsilon$ the system (10.8) admits a solution $\mathfrak{h}_{1,\varepsilon,A_0,\phi}$ in $B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^{1,\alpha}(\mathbb{R})$.

Step 5. Finally, in subsection 10.(g), using the study of J made in 10.(f), we prove that for $0 < \alpha \leq 1/2$, there exist δ, δ_0, ε_0 such that for every $0 < \varepsilon < \varepsilon_0$, $\delta_0 \varepsilon^{2-\alpha} < A_0 < \delta$, there exists $\phi(\varepsilon, A_0)$ such that

$$J[\mathfrak{h}_{1,\varepsilon,A_0,\phi(\varepsilon,A_0)}, \phi(\varepsilon, A_0), A_0, \varepsilon] = 0.$$

Hence, $\mathfrak{h}_{1,\varepsilon,A_0,\phi(\varepsilon,A_0)}$ is a reversible solution of (10.6) in $B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^{1,\alpha}(\mathbb{R})$ which gives the existence of an homoclinic connection to 0 for (10.2) under the form

$$\mathfrak{h} = \mathfrak{h}_{0,\varepsilon} + \mathfrak{h}_{1,\varepsilon,A_0,\phi(\varepsilon,A_0)}.$$

(d) Linearized system around the approximate homoclinic

This subsection is devoted to the study of the affine equation

$$\mathbf{L}_\phi(\underline{x})\mathfrak{h} = \mathbf{F}, \tag{10.9}$$

for any given $\mathbf{F} = (F_{q_1}, F_{q_2}, F_Z, F_w) \in B_2^\alpha(\mathbb{R}) \times B_3^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^\alpha(\mathbb{R})$ which is antireversible, i.e. such that F_{q_1} and F_w are *even*, F_{q_2} is *odd*, while F_Z is *reversible* (i.e. $\widehat{S}F_Z = F_Z$). Equation (10.9) reads

$$\begin{aligned}
\frac{dq_1}{d\underline{x}} &= q_2\left[\gamma_1(u_0^h, 0, 0, \varepsilon) - \phi\rho_0\right] + F_{q_1},\\
\frac{dq_2}{d\underline{x}} &= -q_1\left[\gamma_1(u_0^h, 0, 0, \varepsilon) - \phi\rho_0\right] + F_{q_2},\\
Z &= \underline{\pi}_\varepsilon[\mathcal{T}_0 w] + F_Z,
\end{aligned} \tag{10.10}$$

$$\rho\mathcal{H}\left(\frac{dw}{d\underline{x}}\right) + w + 3u_0^h w = F_w.$$

Let us first show the inversion for the two first coordinates. Let us consider a basis of solutions of the homogeneous system in (q_1, q_2)

$$\begin{aligned}
r_+ &= (\sin\Gamma(\underline{x}), \cos\Gamma(\underline{x}), 0, 0)\,,\\
r_- &= (\cos\Gamma(\underline{x}), -\sin\Gamma(\underline{x}), 0, 0),\\
\Gamma(\underline{x}) &= \int_0^{\underline{x}} \left[\gamma_1(u_0^h(\tau), 0, 0, \varepsilon) - \phi\rho_0(\tau)\right] d\tau
\end{aligned} \tag{10.11}$$

then r_+ is reversible, while r_- is antireversible, and Γ is odd and may be also written as

$$\Gamma(\underline{x}) = \int_0^{\underline{x}} \gamma_1(u_0^h(\tau), 0, 0, \varepsilon)d\tau - \phi\rho[\lambda + \gamma(0, 0, A_0^2, \varepsilon)]\arctan(\underline{x}/\rho).$$

We show the following

Lemma 10.2. *Let consider the affine system*

$$\begin{aligned}
\frac{dq_1}{d\underline{x}} &= q_2\left[\gamma_1(u_0^h, 0, 0, \varepsilon) - \phi\rho_0\right] + F_{q_1},\\
\frac{dq_2}{d\underline{x}} &= -q_1\left[\gamma_1(u_0^h, 0, 0, \varepsilon) - \phi\rho_0\right] + F_{q_2},
\end{aligned}$$

with $F_q = (F_{q_1}, F_{q_2}) \in B_2^\alpha(\mathbb{R}) \times B_3^\alpha(\mathbb{R})$, antireversible ($F_{q_1}$ even, F_{q_2} odd). This system has a unique reversible, continuously differentiable solution $(q_1, q_2) = \mathcal{F}_q(F_q)$,

(q_1 odd, q_2 even) tending towards 0 at infinity, if and only if (we identify r_- with its two first components)

$$\int_0^\infty < r_-(x), F_q(x) > dx = 0. \tag{10.12}$$

We have

$$\mathcal{F}_q(F_q)(\underline{x}) = -r_+(\underline{x}) \int_{\underline{x}}^\infty < r_+(\tau), F_q(\tau) > d\tau - r_-(\underline{x}) \int_{\underline{x}}^\infty < r_-(\tau), F_q(\tau) > d\tau,$$

and $\mathcal{F}_{q_1}(F_q) \in B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R})$, *with*

$$||\mathcal{F}_{q_1}(F_q)||^\alpha_{\mathbb{R},1} + ||\mathcal{F}_{q_2}(F_q)||^\alpha_{\mathbb{R},2} \leq c||F_{q_1}||^\alpha_{\mathbb{R},2} + ||F_{q_2}||^\alpha_{\mathbb{R},3}.$$

Proof. Variation of constants method leads to

$$\begin{aligned}\mathcal{F}_q(F_q)(\underline{x}) = &\left(c_+ + \int_0^{\underline{x}} < r_+(\tau), F_q(\tau) > d\tau\right) r_+(\underline{x}) + \\ &+ \left(c_- + \int_0^{\underline{x}} < r_-(\tau), F_q(\tau) > d\tau\right) r_-(\underline{x}).\end{aligned}$$

Reversibility of $\mathcal{F}_q(F_q)$ leads to $c_- = 0$. The imposed decay towards 0 at infinity implies the conditions

$$\begin{aligned} c_+ &= -\int_0^\infty < r_+(\tau), F_q(\tau) > d\tau, \\ c_- &= -\int_0^\infty < r_-(\tau), F_q(\tau) > d\tau. \end{aligned}$$

We deduce the compatibility condition (10.12), and the explicit form of $\mathcal{F}_q(F_q)$. The sufficiency of the compatibility condition follows easily. About the decay rate at infinity, we first observe that the decay of order $1/\underline{x}^2$ of F_q and the reversibility of the solution, give immediately a decay rate in $1/\underline{x}$ for $\mathcal{F}_q(F_q)$. It remains to prove the decay rate in $1/\underline{x}^2$ of the component $\mathcal{F}_{q_2}(F_q)$, which is easy in using the differential equation, since $dq_2/d\underline{x}$ decays as $1/\underline{x}^3$. Hence the lemma is proved.

It remains to invert the second part of system (10.10) with respect to (w, Z). This is given by the following

Lemma 10.3. *Let consider the affine system in* $B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^\alpha(\mathbb{R})$

$$\begin{aligned} Z &= \underline{\pi}_\varepsilon[\mathcal{T}_0 w] + F_Z, \\ \rho\mathcal{H}\left(\frac{dw}{d\underline{x}}\right) &+ w + 3u_0^h w = F_w, \end{aligned}$$

where F_Z is reversible, and F_w is even. Then, there is a unique reversible solution (Z, w) such that $(F_Z, F_w) \mapsto (Z, w)$ is a bounded linear map:

$$B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^\alpha(\mathbb{R}) \to B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^{1,\alpha}(\mathbb{R})$$

with an estimate

$$||w||^{1,\alpha}_{\mathbb{R},2} + ||Z||^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \leq c(||F_w||^\alpha_{\mathbb{R},2} + ||F_Z||^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w}).$$

Proof. From the result of lemma 8.3, it is sufficient to solve the equation for w, which is the linearized Benjamin-Ono equation. It is shown in (Amick 1994) that if $F_w \in B_2^\alpha(\mathbb{R})$, then the solution w of the linearized B-O equation lies in $B_2^{1,\alpha}(\mathbb{R})$, with the above estimate.

(e) Estimates of the rests

In this subsection we give estimates on R_{q_1}, R_{q_2}, $\mathcal{R}'_Y$, $\mathcal{B}'_\varepsilon$ which occur in system (10.2). We observe that

$$\gamma_1(u,\varepsilon v,|A|^2,\varepsilon)-\gamma_1(0,0,A_0^2,\varepsilon)=\gamma_1(u,0,0,\varepsilon)+\varepsilon O\left[|A_0||q_2|+ +|v|+q_1^2+q_2^2+\varepsilon|u|A_0^2\right],$$

and that we have a "bad" Hölder norm of $e^{i\psi_\phi}$

$$||e^{i\psi_\phi}||_{C^\alpha}\leq c\varepsilon^{-\alpha}.$$

Now, for

$$\begin{aligned}|A_0|+||q_1||^\alpha_{\mathbb{R},1}+||q_2||^\alpha_{\mathbb{R},2}<\delta,\\ ||u||^\alpha_{\mathbb{R},2}+\varepsilon||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}<M,\end{aligned}$$

we have from lemma 8.5

$$\begin{aligned}||T_u||^\alpha_{\mathbb{R},3}&\leq c||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}(||u||^\alpha_{\mathbb{R},2}+\varepsilon||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}),\\ ||T_Y^{(1)}||^\alpha_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,2}&\leq c\varepsilon^{-\alpha}(|A_0|+||q_1||^\alpha_{\mathbb{R},1}+||q_2||^\alpha_{\mathbb{R},2})||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w},\\ ||T_Y^{(2)}||^\alpha_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,3}&\leq c||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}(||u||^\alpha_{\mathbb{R},2}+\varepsilon||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}).\end{aligned}$$

Hence it results the following estimates

$$\begin{aligned}||R_{q_1}||^\alpha_{\mathbb{R},2}&\leq c\varepsilon(|A_0|+||q_2||^\alpha_{\mathbb{R},2})\left[\varepsilon A_0^2||u||^\alpha_{\mathbb{R},2}+(||q_1||^\alpha_{\mathbb{R},1})^2\right]\\ &\quad+c\varepsilon^{1-\alpha}||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}(||u||^\alpha_{\mathbb{R},2}+\varepsilon^\alpha(|A_0|+||q_2||^\alpha_{\mathbb{R},2})+\varepsilon||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w})\\ &\quad+c\varepsilon||q_2||^\alpha_{\mathbb{R},2}(|A_0|+||q_2||^\alpha_{\mathbb{R},2})^2,\\ ||R_{q_2}||^\alpha_{\mathbb{R},3}&\leq c\varepsilon||q_1||^\alpha_{\mathbb{R},1}\left[|A_0|||q_2||^\alpha_{\mathbb{R},2}+\varepsilon A_0^2||u||^\alpha_{\mathbb{R},2}+(||q_1||^\alpha_{\mathbb{R},1})^2+(||q_2||^\alpha_{\mathbb{R},2})^2\right]\\ &\quad+c\varepsilon^{1-\alpha}||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}(||u||^\alpha_{\mathbb{R},2}+\varepsilon^\alpha||q_1||^\alpha_{\mathbb{R},1}+\varepsilon||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}),\\ ||\mathcal{R}'_Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}&\leq c\left[\varepsilon^{1-\alpha}(|A_0|+||q_1||^\alpha_{\mathbb{R},1}+||q_2||^\alpha_{\mathbb{R},2})+\varepsilon(||u||^\alpha_{\mathbb{R},2}+\varepsilon||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w})\right]||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w},\\ ||\mathcal{B}'_\varepsilon||^\alpha_{\mathbb{R},2}&\leq c\left[\varepsilon^{1-\alpha}(|A_0|+||q_1||^\alpha_{\mathbb{R},1}+||q_2||^\alpha_{\mathbb{R},2})+\varepsilon(||u||^\alpha_{\mathbb{R},2}+\varepsilon||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w})\right]||Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w}\\ &\quad+c\left[(||q_1||^\alpha_{\mathbb{R},1})^2+||q_2||^\alpha_{\mathbb{R},2}(|A_0|+||q_2||^\alpha_{\mathbb{R},2})\right]\\ &\quad+c\varepsilon[||u||^{1,\alpha}_{\mathbb{R},2}+(||u||^\alpha_{\mathbb{R},2})^2],\end{aligned}\tag{10.13}$$

which comes from the estimates given at section 8 on system (8.2), and where we observe that R_{q_1}, R_{q_2}, $\mathcal{R}'_Y$, $\mathcal{B}'_\varepsilon$ are analytic in (q_1,q_2,u,Y,A_0). Notice that $s=\psi_\phi(\underline{x})$ implies a loss of $\varepsilon^{-\alpha}$ in the Holder constant, each time q_1,q_2,A_0 occur, except when A appears as $|A|^2$.

(*f*) *Principal part of J*

This subsection is devoted to the computation of the principal part of J. More precisely we prove

Lemma 10.4. *For every sufficiently small* $\delta, \varepsilon, |A_0|$, *every* $\phi \in \mathbb{R}$, *and every*

$$\mathfrak{h} = (q_1, q_2, Z, w) \in B_1^{\alpha}(\mathbb{R}) \times B_2^{\alpha}(\mathbb{R}) \times B^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^{1,\alpha}(\mathbb{R})$$

with

$$||q_1||^{\alpha}_{\mathbb{R},1} + ||q_2||^{\alpha}_{\mathbb{R},2} + ||w||^{1,\alpha}_{\mathbb{R},2} + ||Z||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{D},w} \leq \delta,$$

$J(\mathfrak{h}, \phi, A_0, \varepsilon)$ *reads*

$$J(\mathfrak{h}, \phi, A_0, \varepsilon) = A_0 \sin\Gamma(\infty) + J'(\mathfrak{h}, \phi, A_0, \varepsilon)$$

where

$$\begin{aligned} J' = O\Big\{&\varepsilon^m + \varepsilon(||w||^{\alpha}_{\mathbb{R},2} + ||Z||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{D},w}) + \\ &+ (||q_1||^{\alpha}_{\mathbb{R},1} + ||q_2||^{\alpha}_{\mathbb{R},2})[\varepsilon^2 + ||w||^{\alpha}_{\mathbb{R},2} + \varepsilon(||q_1||^{\alpha}_{\mathbb{R},1})^2 + \varepsilon(||q_2||^{\alpha}_{\mathbb{R},2})^2] \\ &+ |A_0|[\varepsilon^2 + ||w||^{\alpha}_{\mathbb{R},2} + \varepsilon|A_0|||q_2||^{\alpha}_{\mathbb{R},2} + \varepsilon(||q_1||^{\alpha}_{\mathbb{R},1})^2 + \varepsilon(||q_2||^{\alpha}_{\mathbb{R},2})^2]\Big\}, \end{aligned} \tag{10.14}$$

and ($\Gamma(\underline{x})$ *is given by (10.11)*

$$\Gamma(\infty) = \int_0^{\infty} \gamma_1[u_0^h(\tau), 0, 0, \varepsilon] d\tau - \phi \frac{\rho\pi}{2}[\lambda + \gamma(0, 0, A_0^2, \varepsilon)].$$

Proof. Assume that

$$||q_1||^{\alpha}_{\mathbb{R},1} + ||q_2||^{\alpha}_{\mathbb{R},2} + ||w||^{1,\alpha}_{\mathbb{R},2} + ||Z||^{\alpha}_{\underline{\pi}_\varepsilon \mathbb{D},w} \leq \delta,$$

holds, where δ is small enough, J may be written more precisely

$$\begin{aligned} J = &\int_0^{\infty} < r_-(x), R_q(x) > dx + \int_0^{\infty} A_0 \left[\gamma_1(u_0^h, 0, 0, \varepsilon) - \phi\rho_0\right] \cos\Gamma dx + \\ &+ \int_0^{\infty} [(A_0 + q_2)\cos\Gamma + q_1 \sin\Gamma][\gamma_1(u_0^h + w, 0, 0, \varepsilon) - \gamma_1(u_0^h, 0, 0, \varepsilon)] dx, \end{aligned} \tag{10.15}$$

where we notice already that the last integral is bounded by

$$O[(|A_0| + ||q_1||^{\alpha}_{\mathbb{R},1} + ||q_2||^{\alpha}_{\mathbb{R},2})||w||^{\alpha}_{\mathbb{R},2}].$$

We notice also that the second integral reads

$$\int_0^{\infty} A_0 \frac{d\Gamma}{dx} \cos\Gamma dx = A_0 \sin\Gamma(\infty).$$

Now, from the expression of R_q [see (10.2)], we have

$$\begin{aligned} \int_0^{\infty} < r_-(x), R_q(x) > dx = \operatorname{Im}\left(\int_0^{\infty} R_A e^{-i(\Gamma+\psi_\phi)} dx\right) + \\ + O[\varepsilon(|A_0| + ||q_1||^{\alpha}_{\mathbb{R},1} + ||q_2||^{\alpha}_{\mathbb{R},2})\left[\varepsilon + |A_0|||q_2||^{\alpha}_{\mathbb{R},2} + (||q_1||^{\alpha}_{\mathbb{R},1})^2 + (||q_2||^{\alpha}_{\mathbb{R},2})^2\right]]. \end{aligned}$$

Since we look for reversible solutions, we have in the reversible system (8.2)

$$\begin{aligned}-\overline{R}_A[A(x),\overline{A}(x),u(x),Y(x)] &= R_A[\overline{A}(x),A(x),u(x),SY(x)]\\ &= R_A[A(-x),\overline{A}(-x),u(-x),Y(-x)],\end{aligned}$$

hence

$$\mathrm{Im}\left(\int_0^\infty R_A e^{-i(\Gamma+\psi_\phi)}dx\right)=\frac{1}{2i}\int_{-\infty}^\infty R_A e^{-i(\Gamma+\psi_\phi)}dx.$$

Now we can write from (8.2)

$$\begin{aligned}R_A &= R_A^{(0)}+R_A^{(1)},\\ R_A^{(0)} &= R_A(0,0,u_0^h,Y_0^h),\\ ||R_A^{(1)}||_{L^1} &= O[\varepsilon(||w||^\alpha_{\mathbb{R},2}+||Z||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w})],\end{aligned}$$

hence

$$J=A_0\sin\Gamma(\infty)+J_1+J_2$$

with

$$J_1=\frac{1}{2i}\int_{-\infty}^\infty R_A^{(0)}e^{-i(\Gamma+\psi_\phi)}dx, \tag{10.16}$$

$$\begin{aligned}J_2 = &\, O[\varepsilon(||w||^\alpha_{\mathbb{R},2}+||Z||^\alpha_{\underline{\pi}_\varepsilon\mathbb{D},w})+||w||^\alpha_{\mathbb{R},2}(|A_0|+||q_1||^\alpha_{\mathbb{R},1}+||q_2||^\alpha_{\mathbb{R},2})]+\\ &+O[\varepsilon(|A_0|+||q_1||^\alpha_{\mathbb{R},1}+||q_2||^\alpha_{\mathbb{R},2})\left[\varepsilon+|A_0|||q_2||^\alpha_{\mathbb{R},2}+(||q_1||^\alpha_{\mathbb{R},1})^2+(||q_2||^\alpha_{\mathbb{R},2})^2\right]].\end{aligned}$$

We observe that J_1 is an *oscillating integral*, since $R_A^{(0)}$ decays at least in $1/\underline{x}^2$, is indefinitely differentiable with good decays of its derivatives, and $\Gamma+\psi_\phi=(\frac{\lambda}{\varepsilon}+\gamma_{10})\underline{x}+$ smooth function tending towards a constant at infinity. It results that

$$|J_1|=O(\varepsilon^m),\ \text{for any fixed } m>0$$

holds (this can be improved in using analyticity of $R_A^{(0)}$ in a strip near real axis). We only need $m=2$ in the proof. The estimate of the lemma follows.

(*g*) *Proof of theorem 10.1*

(i) *Homoclinics of the modified equation (10.8)*

As already explained in subsection 10.(c), for finding homoclinic connections of (10.6) we first study the modified equation (10.8)

$$\mathbf{L}_\phi(\underline{x})\mathfrak{h}_1=\mathbf{G}'(\mathfrak{h}_1,\varepsilon,A_0,\phi)$$

where

$$\mathbf{G}'=\mathbf{G}-\frac{2}{\sqrt{\pi}}Je^{-\underline{x}^2}r_-(\underline{x})=(G'_{q_1},G'_{q_2},\mathcal{G}_Z,\mathcal{G}_w).$$

We first prove

Proposition 10.5. *For every $0 < \alpha < 1$, and every $T > 0$, there exist δ, ε_0, $c > 0$ such that for every $\phi \in [-T, T]$, $\varepsilon \in (0, \varepsilon_0)$, $|A_0| \leq \delta$, equation (10.8) admits an homoclinic connection $\mathfrak{h}_{1,\varepsilon,A_0,\phi}$ to 0, satisfying*

$$\mathfrak{h}_{1,\varepsilon,A_0,\phi} = (q_1, q_2, Z, w) \in B_1^\alpha(\mathbb{R}) \times B_2^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^{1,\alpha}(\mathbb{R})$$

and

$$||q_1||^\alpha_{\mathbb{R},1} + ||q_2||^\alpha_{\mathbb{R},2} \leq c(|A_0| + \varepsilon^{1-\alpha}),$$
$$||w||^\alpha_{\mathbb{R},2} + ||Z||^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \leq c\left[\varepsilon + (|A_0| + \varepsilon^{1-\alpha})^2\right].$$

Proof. Our aim is to solve (10.8) by using the analytic implicit function theorem. For this purpose we need estimates on $(G'_{q_1}, G'_{q_2}, \mathcal{G}_Z, \mathcal{G}_w)$. Indeed, in the ball

$$||q_1||^\alpha_{\mathbb{R},1} + ||q_2||^\alpha_{\mathbb{R},2} + ||w||^{1,\alpha}_{\mathbb{R},2} + ||Z||^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \leq \delta,$$

and taking into account that the Hölder norm of $e^{i\psi_\phi(\underline{x})}$ is bounded by $c\varepsilon^{-\alpha}$, we obtain, due to (10.13) and (10.14)

$$\begin{aligned}
||G'_q||_{B_2^\alpha(\mathbb{R})\times B_3^\alpha(\mathbb{R})} &\leq c\left[|A_0| + \varepsilon^{1-\alpha} + ||w||^\alpha_{\mathbb{R},2}(||q_1||^\alpha_{\mathbb{R},1} + ||q_2||^\alpha_{\mathbb{R},2})\right],\\
||\mathcal{G}_Z||^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} &\leq c\left[\varepsilon + \varepsilon^{1-\alpha}(|A_0| + ||q_1||^\alpha_{\mathbb{R},1} + ||q_2||^\alpha_{\mathbb{R},2})\right],\\
||\mathcal{G}_w||^\alpha_{\mathbb{R},2} &\leq c\left[\varepsilon + \varepsilon^{1-\alpha}(|A_0| + ||q_1||^\alpha_{\mathbb{R},1} + ||q_2||^\alpha_{\mathbb{R},2}) + |A_0|||q_2||^\alpha_{\mathbb{R},2}\right] +\\
&\quad + c\left(||w||^\alpha_{\mathbb{R},2} + ||q_1||^\alpha_{\mathbb{R},1} + ||q_2||^\alpha_{\mathbb{R},2}\right)^2.
\end{aligned}$$

We also need estimates on the derivatives of $G'_q, \mathcal{G}_Z, \mathcal{G}_w$ with respect to (q_1, q_2, Z, w). The corresponding estimates are not mentioned below, since they are in the same spirit as above, and often simpler. In all this process the differentials at the origin are close to the invertible operators defined at lemmas 10.2 and 10.3, hence have a bounded inverse. In fact we need a slight adaptation of the implicit function theorem, since we fix ε small enough, but non zero here. We replace $\mathbf{G}'(\mathfrak{h}_1, \varepsilon, A_0, \phi)$ by

$$\mathbf{G}'(\mathfrak{h}_1, \varepsilon, A_0, \phi) - (1 - \mu\varepsilon^{\alpha-1})\mathbf{G}'(0, \varepsilon, 0, \phi),$$

and consider the analytic implicit function theorem for $(\mathfrak{h}_1, A_0, \mu)$ near 0, observing that $\varepsilon^{\alpha-1}\mathbf{G}'(0, \varepsilon, 0, \phi)$ is bounded in $B_2^\alpha(\mathbb{R}) \times B_3^\alpha(\mathbb{R}) \times B^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \times B_2^\alpha(\mathbb{R})$. For $\mu = 0$, we have the trivial solution $(\mathfrak{h}_1, A_0) = 0$, while our system (10.8) corresponds to $\mu = \varepsilon^{1-\alpha}$, which lies in the domain of existence of the solution, for ε and $|A_0|$ small enough.

We first solve the two first equations with respect to (q_1, q_2), using lemma 10.2 and implicit function theorem. We then obtain (q_1, q_2) as an analytic function of (ϕ, Z, v) for δ, A_0, ε small enough, and

$$||q_1||^\alpha_{\mathbb{R},1} + ||q_2||^\alpha_{\mathbb{R},2} \leq c(|A_0| + \varepsilon^{1-\alpha}).$$

Now solving the two last equations with respect to (Z, w) in using lemma 10.3 in the analytic implicit function theorem, we finally obtain (w, Z) analytic in ϕ, which satisfies

$$||w||^\alpha_{\mathbb{R},2} + ||Z||^\alpha_{\underline{\pi}_\varepsilon \mathbb{D},w} \leq c\left[\varepsilon + (|A_0| + \varepsilon^{1-\alpha})^2\right].$$

(ii) *Compatibility condition*

The previously found homoclinic connections $\mathfrak{h}_{1,\varepsilon,A_0,\phi}$ of the modified equation (10.8) are solutions of the full equation (10.6) if and only if

$$J(\mathfrak{h}_{1,\varepsilon,A_0,\phi},\phi,A_0,\varepsilon)=0.$$

The study of J, made in subsection 10.(f) ensures that for every ϕ, $\alpha\le 1/2$, $0\le|A_0|\le\delta$, $\varepsilon\in(0,\varepsilon_0)$, J reads

$$J(\mathfrak{h}_{1,\varepsilon,A_0,\phi},\phi,A_0,\varepsilon)=A_0\sin\Gamma(\infty)+J'(\mathfrak{h}_{1,\varepsilon,A_0,\phi},\phi,A_0,\varepsilon)$$

with

$$J'(\mathfrak{h}_{1,\varepsilon,A_0,\phi},\phi,A_0,\varepsilon)=O(\varepsilon^{2-\alpha}+\varepsilon|A_0|+\varepsilon^{1-\alpha}|A_0|^2+|A_0|^3),$$

and

$$\Gamma(\infty)=\int_0^\infty\gamma_1[u_0^h(\tau),0,0,\varepsilon]d\tau-\phi\frac{\rho\pi}{2}[\lambda+\gamma(0,0,A_0^2,\varepsilon)].$$

Equation $J=0$ can be solved with respect to ϕ for fixed values of A_0 and ε, provided that $\delta_0\varepsilon^{2-\alpha}<|A_0|$, $\varepsilon<\varepsilon_1$. Indeed this allows to have an acceptable value of $\sin\Gamma(\infty)\in(-1,1)$ giving two angles $\Gamma(\infty)$ modulo 2π. If $|\sin\Gamma(\infty)|<1$ strictly, an implicit function theorem argument provides two corresponding solutions for ϕ. Indeed, we consider the solutions $\phi_j^{(0)}\in[0,\pi[$ (or $[\pi,2\pi[$ depending of the signs of A_0 and J') of the equation

$$A_0\sin\Gamma(\infty)+J'(\mathfrak{h}_{1,\varepsilon,0,0},0,0,\varepsilon)=0,$$

then a rescaling of the form $\varepsilon^{2-\alpha}=A_0\varepsilon'$, and the implicit function theorem allows to find two solutions ϕ_j near $\phi_j^{(0)}$ for ε' small enough.

However, because of the modulo 2π indeterminacy, we find infinitely many values of the phase shift ϕ. For instance when $|A_0|>>\varepsilon^{2-\alpha}$, they are near the values

$$\phi_k=\frac{2}{\rho\pi[\lambda+\gamma(0,0,A_0^2,\varepsilon)]}\left[\int_0^\infty\gamma_1[u_0^h(\tau),0,0,\varepsilon]d\tau-k\pi\right].$$

More generally, we observe, that changing ϕ_k into ϕ_{k+1} is equivalent to changing A_0 into $-A_0$ in the principal part of the equation. We observe also that our homoclinics to periodic solutions are not really well defined by (10.1) since a phase shift in $\underline{x}$ of $n\pi\varepsilon/[\lambda+\gamma(0,0,A_0^2,\varepsilon)]$ at $+\infty$, and $-n\pi\varepsilon/[\lambda+\gamma(0,0,A_0^2,\varepsilon)]$ at $-\infty$ in (10.1) would lead to the same solution, with ϕ_k changed into ϕ_{k+n}. So, observing that a fixed value of $\sin\Gamma(\infty)$ gives only two solutions for $\Gamma(\infty)$ in an interval of length 2π, we deduce that there are only two different solutions of our problem for a given value of A_0, provided $|A_0|>\delta_0\varepsilon^2$. We also observe that the two solutions for $-A_0$ correspond to the previous one in changing the phase ($n=1$ in the above phase shift). Finally theorem 10.1 is proved.

11. Appendix Normal Form

In this appendix, we prove lemma 7.2. The change of variables (7.8) is determined by its property to manage all linear terms in W only depending on $(A,\overline{A})$ (not depending on u and v) in (7.1,7.3,7.5). Indeed we need to solve with respect to $\mu_\varepsilon^*,\nu_\varepsilon^*,\Gamma_\varepsilon$ [unknown functions of $(A,\overline{A})$], and $\Delta_\varepsilon(A,\overline{A})$ such that $\eta_\varepsilon^*\{\Delta_\varepsilon(A,\overline{A})[W]\}=0$ (η_ε^* is defined in (5.7)), the following system

$$\mu_\varepsilon^*\left[(i\frac{\lambda}{\varepsilon}+i\gamma_{10}-\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon)W\right]=(\frac{i\lambda}{\varepsilon}+i\gamma_{10})\left[A\frac{\partial\mu_\varepsilon^*}{\partial A}(W)-\overline{A}\frac{\partial\mu_\varepsilon^*}{\partial\overline{A}}(W)\right]+ \\ +\mathcal{M}_A^*(\mu_\varepsilon^*,\nu_\varepsilon^*,\Gamma_\varepsilon)(W)-\mathcal{R}_A(A,\overline{A})[W], \tag{11.1}$$

$$-\nu_\varepsilon^*\left[\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon W\right]=(\frac{i\lambda}{\varepsilon}+i\gamma_{10})\left[A\frac{\partial\nu_\varepsilon^*}{\partial A}(W)-\overline{A}\frac{\partial\nu_\varepsilon^*}{\partial\overline{A}}(W)\right]+ \\ +\mathcal{M}_u^*(\nu_\varepsilon^*,\Gamma_\varepsilon)(W)-\mathcal{R}_u(A,\overline{A})[W]-p_0^*(\mathcal{L}_\varepsilon\Gamma_\varepsilon W), \tag{11.2}$$

$$\underline{\pi}_\varepsilon\mathcal{L}_\varepsilon\Gamma_\varepsilon W=\Gamma_\varepsilon\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon W+(\frac{i\lambda}{\varepsilon}+i\gamma_{10})\left[A\frac{\partial\Gamma_\varepsilon}{\partial A}(W)-\overline{A}\frac{\partial\Gamma_\varepsilon}{\partial\overline{A}}(W)\right]+ \\ +\mathcal{M}_{\underline{W}}(\Gamma_\varepsilon)(W)+\Delta_\varepsilon(A,\overline{A})[W]-\mathcal{R}_{\underline{W}}(A,\overline{A})[W], \tag{11.3}$$

with

$$\mathcal{M}_A^*(\mu_\varepsilon^*,\nu_\varepsilon^*,\Gamma_\varepsilon)(W)=\mu_\varepsilon^*[\widetilde{\pi}_\varepsilon\Delta_\varepsilon(A,\overline{A})](W)-\mathcal{R}_A(A,\overline{A})[\Gamma_\varepsilon W]+ \\ -iA\gamma_{20}(|A|^2)\nu_\varepsilon^*(W)-iA\gamma_{30}(|A|^2)p_1^*[\Gamma_\varepsilon(W)]+ \\ -i\gamma_{10}'(|A|^2)[|A|^2\mu_\varepsilon^*(W)+A^2\overline{\mu_\varepsilon^*}(W)],$$

$$\mathcal{M}_u^*(\nu_\varepsilon^*,\Gamma_\varepsilon)(W)=\nu_\varepsilon^*[\widetilde{\pi}_\varepsilon\Delta_\varepsilon(A,\overline{A})](W)-\mathcal{R}_u(A,\overline{A})[\Gamma_\varepsilon W],$$

$$\mathcal{M}_{\underline{W}}(\Gamma_\varepsilon)(W)=\Gamma_\varepsilon\widetilde{\pi}_\varepsilon\Delta_\varepsilon(A,\overline{A})[W]-\mathcal{R}_{\underline{W}}(A,\overline{A})[\Gamma_\varepsilon W],$$

where [from (7.1,7.3,7.4)]

$$\gamma_{10}(|A|^2)=\gamma_1(0,0,|A|^2,\varepsilon)=O(|A|^2),$$
$$\gamma_{20}(|A|^2)=\frac{\partial\gamma_1}{\partial u}(0,0,|A|^2,\varepsilon)=O(1),$$
$$\gamma_{30}(|A|^2)=\frac{\partial\gamma_1}{\partial v}(0,0,|A|^2,\varepsilon)=O(1),$$
$$\mathcal{R}_A(A,\overline{A})=D_WR_A(A,\overline{A},0,0,0)=O(|A|),$$
$$\mathcal{R}_u(A,\overline{A})=D_WR_u(A,\overline{A},0,0,0)=O(|A|),$$
$$\mathcal{R}_{\underline{W}}(A,\overline{A})=D_WR_{\underline{W}}(A,\overline{A},0,0,0)=O(|A|),$$

and it is clear that $\mathcal{R}_A(A,\overline{A}),\mathcal{R}_u(A,\overline{A}),\mathcal{R}_{\underline{W}}(A,\overline{A})$ are analytic in their arguments, in a ball $|A|<M$, with coefficients of order ε^{p+q-1} for $A^p\overline{A}^q$, and that they operate linearly only on the component $\mathcal{P}W$ of W. In (11.3) we notice that both projections $\widetilde{\pi}_\varepsilon$ and $\underline{\pi}_\varepsilon$ occur, since this equation stays in $\underline{\pi}_\varepsilon\mathbb{H}$ ($\supset\widetilde{\pi}_\varepsilon\mathbb{H}$).

Step 1. Let us first solve (11.3) with respect to Γ_ε, and find Δ_ε such that $\eta_\varepsilon^*[\Delta_\varepsilon(W)] = 0$. Let consider the converging Taylor series in powers of $A, \overline{A}$ of $\Gamma_\varepsilon(A,\overline{A})$, $\Delta_\varepsilon(A,\overline{A})$, $\mathcal{R}_{\underline{W}}(A,\overline{A})$, coefficients of $A^p\overline{A}^q$ being denoted respectively by Γ_{pq}, Δ_{pq}, $\mathcal{R}_{\underline{W},pq}$. For solving this system, let us first consider a more basic problem, where we look for a linear form Γ^* such that

$$\Gamma^*\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon W + (\frac{i\lambda}{\varepsilon} + i\gamma_{10})\left[A\frac{\partial\Gamma^*}{\partial A}(W) - \overline{A}\frac{\partial\Gamma^*}{\partial\overline{A}}(W)\right] = f^*(W) \tag{11.4}$$

where we assume that

$$f^* \text{ analytic in } (A,\overline{A}) \text{ taking values in } (\widetilde{\pi}_\varepsilon\mathbb{D})^*$$

and we look for

$$\Gamma^* \text{ analytic in } (A,\overline{A}) \text{ taking values in } (\widetilde{\pi}_\varepsilon\mathbb{H})^*.$$

Remark. Notice that we incorporate $\gamma_{10}(|A|^2)$ inside the equation (11.4). If we chose to treat this term in the process of identification of powers of $A, \overline{A}$, this would lead to serious difficulties, related with the unboundedness of the perturbation terms in any reasonable Banach norm of analytic functions.

For $p \neq q$, we can find for any $Z \in \widetilde{\pi}_\varepsilon\mathbb{H}$

$$\Gamma^*_{pq}(Z) = f^*_{pq}\left\{[\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon + (\frac{i\lambda}{\varepsilon} + i\gamma_{10})(p-q)]^{-1}Z\right\}$$

and we have (see lemma 5.3)

$$|\Gamma^*_{pq}(Z)| \leq c||f^*_{pq}||_{(\widetilde{\pi}_\varepsilon\mathbb{D})^*}||Z||_{\widetilde{\pi}_\varepsilon\mathbb{H}}$$

where c is independent of p,q, and where Γ^*_{pq} is analytic in $|A|^2$ [the inverse operator is analytic in $(\lambda + \varepsilon\gamma_{10})$]. Now, for $p = q$ we are faced with the problem of non invertibility of $\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon$ (see lemma 5.5), since we have to solve

$$\Gamma^*_{pp}(\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon W) = f^*_{pp}(W).$$

We now observe that in all equations we need to solve, the linear form f^* can be extended to a space larger than $\widetilde{\pi}_\varepsilon\mathbb{D}$ in the sense that the components α_1, β_1 may grow as $\underline{y} \to -\infty$. This is mainly due to the property of $\mathcal{R}_A, \mathcal{R}_u, \mathcal{R}_W$ which do not depend on 3rd and 4th components of their argument.

Let us introduce the following Banach space

$$\begin{aligned}\widetilde{\mathbb{D}} = \{U = (\beta_{10}, \beta_{21}, \alpha_1, \beta_1, \alpha_2, \beta_2)^t,\\ (\alpha_1', \beta_1') \in [C^0_1(\mathbb{R}^-)]^2, (\alpha_2, \beta_2) \in [C^1(0,1)]^2,\\ \beta_{10} = \beta_1|_{\underline{y}=0}, \beta_{21} = \beta_2|_{y=1},\ \alpha_{10} = \alpha_{20}\},\end{aligned}$$

with the norm

$$||U||_{\widetilde{\mathbb{D}}} = ||\alpha_1'||_{1,\infty} + ||\beta_1'||_{1,\infty} + ||\mathcal{P}U||_{\mathbb{D}^{\mathcal{P}}}.$$

We observe that α_1 and β_1 may grow as $\underline{y} \to -\infty$, like $|\ln|\underline{y}||$.

Now, we can prove the following

Lemma 11.1. *For any given linear form f^* analytic in $(A, \overline{A})$ for $|A| < \delta$, taking values in $(\widetilde{\pi}_\varepsilon \widetilde{\mathbb{D}})^*$, there exists $\delta' > 0$, $\delta' \leq \delta$ such that equation (11.4) has a solution Γ^*, analytic in $(A, \overline{A})$ for $|A| < \delta'$, and taking its values in $(\widetilde{\pi}_\varepsilon \mathbb{H})^*$, denoted by*

$$\Gamma^* = \mathcal{Q}(A, \overline{A}) f^*$$

and we have

$$||\mathcal{Q}(A, \overline{A}) f^*||_{(\widetilde{\pi}_\varepsilon \mathbb{H})^*} \leq c||f^*||_{(\widetilde{\pi}_\varepsilon \widetilde{\mathbb{D}})^*}.$$

Notice that in this lemma there is no need to have uniqueness of the solution Γ^*; in fact this solution is indeed unique, but this needs a little more work. Before starting the proof of this lemma, let us examine the computation of the resolvent, made at section 5.(c), and show the following

Lemma 11.2. *The operator $\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon + (ik/\varepsilon)\mathbb{I}$ which acts from $\widetilde{\pi}_\varepsilon \widetilde{\mathbb{D}}$ onto $\widetilde{\pi}_\varepsilon \mathbb{H}$ has a bounded inverse for $|k| > 2\varepsilon$, which satisfies*

$$||(\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon + (ik/\varepsilon)\mathbb{I})^{-1}||_{\mathcal{L}(\widetilde{\pi}_\varepsilon \mathbb{H}, \widetilde{\pi}_\varepsilon \widetilde{\mathbb{D}})} \leq c,$$
$$||\mathcal{P}(\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon + (ik/\varepsilon)\mathbb{I})^{-1}||_{\mathcal{L}(\widetilde{\pi}_\varepsilon \mathbb{H}, \widetilde{\pi}_\varepsilon \mathbb{D})} \leq c\varepsilon,$$

where c is independent of k and ε.

Proof: this lemma follows directly from lemma 5.3, where the operator $\widetilde{\pi}_\varepsilon$ does not perturb the computation, more than when we obtained lemma 5.3 with π_ε. Notice that the integrals in the expression of the linear form ζ_ε^* are all convergent, which allows to define the space $\widetilde{\pi}_\varepsilon \widetilde{\mathbb{D}}$. The estimate follows from the continuous linear embedding

$$\mathbb{D} \hookrightarrow \widetilde{\mathbb{D}}.$$

Now a nice property is that $\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon$ is invertible in the above space! This is the following

Lemma 11.3. *The operator $\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon$ which acts from $\widetilde{\pi}_\varepsilon \widetilde{\mathbb{D}}$ onto $\widetilde{\pi}_\varepsilon \mathbb{H}$ has a bounded inverse $(\widetilde{\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon})^{-1}$, which satisfies*

$$||(\widetilde{\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon})^{-1}||_{\mathcal{L}(\widetilde{\pi}_\varepsilon \mathbb{H}, \widetilde{\pi}_\varepsilon \widetilde{\mathbb{D}})} \leq c,$$
$$||\mathcal{P}(\widetilde{\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon})^{-1}||_{\mathcal{L}(\widetilde{\pi}_\varepsilon \mathbb{H}, \widetilde{\pi}_\varepsilon \mathbb{D})} \leq c.$$

Proof. Let us consider the linear equation

$$\widetilde{\pi}_\varepsilon \mathcal{L}_\varepsilon W = Z \in \widetilde{\pi}_\varepsilon \mathbb{H}.$$

For $Z=(a,b,f_1,g_1,f_2,g_2)^t$ given in $\widetilde{\pi}_\varepsilon\mathbb{H}$, this equation determines a unique $W=(0,0,\alpha_1,\beta_1,\alpha_2,\beta_2)\in\widetilde{\pi}_\varepsilon\widetilde{\mathbb{D}}$ of the form

$$\begin{aligned}
\alpha_1 &= \int_{\underline{y}}^{0}[g_1(\tau)-\lim g_1]d\tau+\widetilde{\rho}[\xi_\varepsilon^*(Z)-\lim g_1]+\\
&\quad +\varepsilon\left(\int_0^1 g_2(\tau)d\tau-b/\lambda\right),\\
\beta_1 &= -\int_{\underline{y}}^{0} f_1(\tau)d\tau,\\
\alpha_2 &= \varepsilon\left(\int_y^1 g_2(\tau)d\tau-b/\lambda\right)+(\widetilde{\rho}-y)[\xi_\varepsilon^*(Z)-\lim g_1],\\
\beta_2 &= -\varepsilon\int_y^1 f_2(\tau)d\tau.
\end{aligned}$$

This W is denoted $(\widetilde{\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon})^{-1}Z$ and is such that $\beta_{10}=0$, $p_0^*(W)=p_1^*(W)=\zeta_\varepsilon^*(W)=\overline{\zeta}_\varepsilon^*(W)=0$, so it is clear that $W\in\widetilde{\pi}_\varepsilon\widetilde{\mathbb{D}}$. We observe that $\mathcal{P}W\in\widetilde{\pi}_\varepsilon\mathbb{D}$, and the estimates of the lemma are straightforward.

Proof of lemma 11.1. Coming back to (11.4), we now assume the linear form $f^*\in(\widetilde{\pi}_\varepsilon\widetilde{\mathbb{D}})^*$. Then the solution Γ^* may be explicitly written, for any $Z\in\widetilde{\pi}_\varepsilon\mathbb{H}$, as

$$\begin{aligned}
\Gamma^*(Z) &= \sum_{p\neq q} f_{pq}^*\left\{[\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon+(\frac{i\lambda}{\varepsilon}+i\gamma_{10})(p-q)]^{-1}Z\right\}+\\
&\quad+\sum_p f_{pp}^*\left\{(\widetilde{\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon})^{-1}Z\right\},
\end{aligned}$$

which is denoted by

$$\Gamma^*=\mathcal{Q}(A,\overline{A})f^*$$

and $(A,\overline{A})\to\mathcal{Q}(A,\overline{A})f^*$ is analytic for $|A|<\delta'$, for some $\delta'\leq\delta$. Moreover, we have

$$||\mathcal{Q}(A,\overline{A})f^*||_{(\widetilde{\pi}_\varepsilon\mathbb{H})^*}\leq c||f^*||_{(\widetilde{\pi}_\varepsilon\widetilde{\mathbb{D}})^*}.$$

End of step 1. We now come to equation (11.3), which is solved first, since we observe that it is uncoupled from equations for μ_ε^*, and ν_ε^*. Let us introduce η_ε such that

$$\eta_\varepsilon\in\underline{\pi}_\varepsilon\mathbb{E}_\varepsilon,\ \eta_\varepsilon^*(\eta_\varepsilon)=1,\quad ||\eta_\varepsilon||_{\mathbb{E}_\varepsilon}=O(1),\ ||\mathcal{L}_\varepsilon\eta_\varepsilon||_{\mathbb{F}_\varepsilon}=O(1/\varepsilon),\ \eta_\varepsilon^*(\underline{\pi}_\varepsilon\mathcal{L}_\varepsilon\eta_\varepsilon)=0.$$

Indeed, we can take

$$\eta_\varepsilon=(\beta_{10},0,0,\beta_1,0,\beta_2)^t$$

with

$$\beta_1=b_1e^{\frac{\lambda\underline{y}}{2\varepsilon}},\quad \beta_2=b_2(y-1),$$

and $\eta_\varepsilon \in \underline{\pi}_\varepsilon \mathbb{E}_\varepsilon$ as soon as

$$\frac{b_1}{3} - \frac{\rho b_2}{\lambda}(2\lambda + 1 - e^\lambda) = 0,$$

and

$$||\eta_\varepsilon||_{\mathbb{E}_\varepsilon} = (2+\lambda/2)|b_1| + 2|b_2|,$$
$$||\mathcal{L}_\varepsilon \eta_\varepsilon||_{\mathbb{F}_\varepsilon} = \frac{1}{\varepsilon}\left(|b_2| + \frac{\lambda}{2}(1+\lambda/2)|b_1|\right).$$

The condition $\eta_\varepsilon^*(\underline{\pi}_\varepsilon \mathcal{L}_\varepsilon \eta_\varepsilon) = 0$ is satisfied, since

$$\underline{\pi}_\varepsilon \mathcal{L}_\varepsilon \eta_\varepsilon = \mathcal{L}_\varepsilon \eta_\varepsilon, \text{ and } \eta_\varepsilon^*(\mathcal{L}_\varepsilon \eta_\varepsilon) = 0,$$

and the condition $\eta_\varepsilon^*(\eta_\varepsilon) = 1$ is satisfied for

$$\frac{2}{\lambda} b_1 - \frac{1}{2} b_2 = 1.$$

Solving the two equations for b_1 and b_2 then gives a suitable η_ε (if ρ and λ are such that this system cannot be solved, then we might choose another η_ε with $\beta_2 = b_2'(y-1)^2$).

Now, we choose Γ_ε of the form

$$\Gamma_\varepsilon(W) = \eta_\varepsilon \Gamma_\varepsilon^*(W),$$

where we look for $\Gamma_\varepsilon^* \in (\widetilde{\pi}_\varepsilon \mathbb{H})^*$. Then (11.3) reads (we omit the variable A)

$$\begin{aligned} \Gamma_\varepsilon^* &= \mathcal{Q}\left\{\chi_\varepsilon^*(\mathcal{R}_{\underline{W}}[\cdot]) - \Gamma_\varepsilon^*(\widetilde{\pi}_\varepsilon \Delta_\varepsilon[\cdot]) + \eta_\varepsilon^*(\mathcal{R}_{\underline{W}}[\eta_\varepsilon])\Gamma_\varepsilon^*[\cdot]\right\}, \\ \Delta_\varepsilon[W] &= (\underline{\pi}_\varepsilon \mathcal{L}_\varepsilon \eta_\varepsilon)\Gamma_\varepsilon^* W + \{\mathcal{R}_{\underline{W}}[W] - \eta_\varepsilon^*(\mathcal{R}_{\underline{W}}[W])\eta_\varepsilon\} + \\ &\quad + \left\{\mathcal{R}_{\underline{W}}[\eta_\varepsilon]\Gamma_\varepsilon^*(W) - \eta_\varepsilon^*(\mathcal{R}_{\underline{W}}[\eta_\varepsilon])\Gamma_\varepsilon^*(W)\eta_\varepsilon\right\}, \end{aligned} \tag{11.5}$$

where we may replace $\Delta_\varepsilon[\cdot]$ by its expression in the first equation, which is then polynomial of degree two in Γ_ε^*. We indeed observe that $\eta_\varepsilon^*(\Delta_\varepsilon[\cdot]) = 0$ as required. We can solve (11.5) in using the implicit function theorem in the Banach space of analytic functions of $(A, \overline{A})$ in the ball of radius $\delta' \le \delta$, taking values in $(\widetilde{\pi}_\varepsilon \mathbb{H})^*$ (with the sup norm with respect to A). Notice that the extension of the operator to space $\widetilde{\pi}_\varepsilon \mathbb{H}$ allows to have an equation above, which is well posed in the required spaces. Separating now (for obtaining better estimates) the sum with $p \ne q$, and the sum for $p = q$, we decompose $\mathcal{R}_{\underline{W}}[\cdot] = \mathcal{R}'_{\underline{W}}[\cdot] + \mathcal{R}^{(0)}_{\underline{W}}[\cdot]$ which satisfies, because of the independence of $\mathcal{R}_{\underline{W}}[\cdot]$ into the 3rd and 4th components of its argument

$$||\mathcal{Q}\chi_\varepsilon^*(\mathcal{R}'_{\underline{W}}[\cdot])||_{(\widetilde{\pi}_\varepsilon \mathbb{H})^*} = O(\varepsilon|A|),$$
$$||\mathcal{Q}\chi_\varepsilon^*(\mathcal{R}^{(0)}_{\underline{W}}[\cdot])||_{(\widetilde{\pi}_\varepsilon \mathbb{H})^*} = O(\varepsilon|A|^2),$$

as it comes from lemma 11.2 (where we win a factor ε).

This leads to $\Gamma_\varepsilon^* \in (\widetilde{\pi}_\varepsilon \mathbb{H})^*$ for $|A| < \delta'$, δ' being small enough, independent of ε,

$$||\Gamma_\varepsilon^*(A, \overline{A})||_{(\widetilde{\pi}_\varepsilon \mathbb{H})^*} \le c\varepsilon|A|,$$
$$||\Delta_\varepsilon(A, \overline{A})[\cdot]||_{\mathcal{L}(\widetilde{\pi}_\varepsilon \widetilde{\mathbb{D}}, \underline{\pi}_\varepsilon \mathbb{F}_\varepsilon)} \le c|A|.$$

Important remark. Notice that this solves (11.3) also for $\Gamma_\varepsilon^*(A,\overline{A})$ in $(\widetilde{\pi}_\varepsilon\mathbb{H})^*$ and $\Delta_\varepsilon(A,\overline{A})[\cdot]$ in $\mathcal{L}(\widetilde{\pi}_\varepsilon\mathbb{D},\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon)$, since we showed that the definitions of these operators are extended to larger spaces!

Step 2. Now we can invert (11.2), after replacing Γ_ε by the result given by step 1 above. This equation may be written in $(\widetilde{\pi}_\varepsilon\mathbb{H})^*$ as

$$\nu_\varepsilon^* = \mathcal{Q}\left\{\mathcal{R}_u[\cdot] + p_0^*(\mathcal{L}_\varepsilon\eta_\varepsilon)\Gamma_\varepsilon^*[\cdot] + \mathcal{R}_u[\eta_\varepsilon]\Gamma_\varepsilon^*[\cdot] - \nu_\varepsilon^*(\widetilde{\pi}_\varepsilon\Delta_\varepsilon[\cdot])\right\},$$

which is linear in ν_ε^*. It is clear that for $|A| < \delta''$ $(\leq \delta')$ small enough, we obtain a unique $\nu_\varepsilon^*(A,\overline{A}) \in (\widetilde{\pi}_\varepsilon\mathbb{H})^*$, analytic in $(A,\overline{A})$, such that

$$||\nu_\varepsilon^*(A,\overline{A})||_{(\widetilde{\pi}_\varepsilon\mathbb{H})^*} \leq c|A|.$$

Step 3. We now invert (11.1) after replacing ν_ε^* and Γ_ε by their expressions, obtained above at steps 1 and 2. We define an operator $\mathcal{Q}_1$ of the same type as $\mathcal{Q}$ in solving

$$\Gamma_1^*\left[\widetilde{\pi}_\varepsilon\mathcal{L}_\varepsilon - (\frac{i\lambda}{\varepsilon} + i\gamma_{10})\right]W + (\frac{i\lambda}{\varepsilon} + i\gamma_{10})\left[A\frac{\partial\Gamma_1^*}{\partial A}(W) - \overline{A}\frac{\partial\Gamma_1^*}{\partial\overline{A}}(W)\right] = f^*(W)$$

for any $f^* \in (\widetilde{\pi}_\varepsilon\widetilde{\mathbb{D}})^*$, by

$$\Gamma_1^* = \mathcal{Q}_1(A,\overline{A})f^* \in (\widetilde{\pi}_\varepsilon\mathbb{H})^*,$$

analytic in $(A,\overline{A})$, and satisfying

$$||\mathcal{Q}_1(A,\overline{A})f^*||_{(\widetilde{\pi}_\varepsilon\mathbb{H})^*} \leq c||f^*||_{(\widetilde{\pi}_\varepsilon\widetilde{\mathbb{D}})^*}.$$

Equation (11.1) may be written as follows (we omit A)

$$\begin{aligned}\mu_\varepsilon^* = {} & \mathcal{Q}_1\left\{\mathcal{R}_A[\cdot] + \mathcal{R}_A[\eta_\varepsilon]\Gamma_\varepsilon^*[\cdot] + iA\gamma_{20}\nu_\varepsilon^*[\cdot] + iA\gamma_{30}p_1^*(\eta_\varepsilon)\Gamma_\varepsilon^*[\cdot]\right\} + \\ & + \mathcal{Q}_1\{i\gamma_{10}'\left(|A|^2\mu_\varepsilon^*[\cdot] + A^2\overline{\mu}_\varepsilon^*[\cdot]\right) - \mu_\varepsilon^*(\widetilde{\pi}_\varepsilon\Delta_\varepsilon[\cdot])\},\end{aligned}$$

which is linear in μ_ε^*. It is clear that for $|A| < \delta'''$ $(\leq \delta'')$ small enough, we obtain a unique $\mu_\varepsilon^*(A,\overline{A}) \in (\widetilde{\pi}_\varepsilon\mathbb{H})^*$, analytic in $(A,\overline{A})$. Now we have

$$||\mathcal{Q}_1\mathcal{R}_A[\cdot]||_{(\widetilde{\pi}_\varepsilon\mathbb{H})^*} = O(|A|)$$

hence the estimate

$$||\mu_\varepsilon^*(A,\overline{A})||_{(\widetilde{\pi}_\varepsilon\mathbb{H})^*} \leq c|A|$$

holds. This ends the resolution of system (11.1,11.2,11.3).

For ending the proof of lemma 7.2, we just need to check the new estimates of the new rests $R_A, R_u, R_{\underline{W}}$, which is straightforward.

12. Appendix A

In this appendix, we prove a technical lemma (lemma 12.1) which is useful here and in the next appendices, We also give a corollary 12.2 useful at various places, in particular for lemma 8.3 and we provide the rest of the proof of lemma 8.3 (corollary 12.3).

Lemma 12.1. *(a) Assume K is a real function which is C^1 on $\mathbb{R}\backslash\{0\}$, such that*
i) $|K(x)| \leq C_0/|x|, \;\; |K'(x)| \leq C_0/|x|^2$ *for* $|x| \leq 1$,
ii) $|K(x)| \leq C_1/|x|^2$ *for* $|x| \geq 1$, *and* $p.v. \int_{-1}^{1} K(x)dx < \infty$.
Then, the linear map $\mathcal{K}$ defined by

$$f \mapsto \mathcal{K}f = p.v. \int_{\mathbb{R}} K(s)f(\cdot - s)ds$$

is bounded from $B_2^\alpha(\mathbb{R})$ into itself.

(b) Let $\mathbb{E}$ be a Banach space and $\mathcal{L}(\mathbb{E})$ be the space of bounded linear operators in $\mathbb{E}$. Assume that $K : \mathbb{R}\backslash\{0\} \to \mathcal{L}(\mathbb{E})$ is C^1 such that
i) $||K(x)||_{\mathcal{L}(\mathbb{E})} \leq C_0/|x|, \;\; ||K'(x)||_{\mathcal{L}(\mathbb{E})} \leq C_0/|x|^2$ *for* $|x| \leq 1$,
ii) $||K(x)||_{\mathcal{L}(\mathbb{E})} \leq C_1/|x|^2$ *for* $|x| \geq 1$, *and* $p.v. \int_{-1}^{1} K(x)dx \in \mathcal{L}(\mathbb{E})$.
Then, the linear map $\mathcal{K}$ defined by

$$f \mapsto \mathcal{K}f = p.v. \int_{\mathbb{R}} K(s)f(\cdot - s)ds$$

is bounded from $B_2^\alpha(\mathbb{E})$ into itself.

Proof. We write

$$\mathcal{K}f(x) = \left(\int_{-\infty}^{-1} + p.v. \int_{-1}^{1} + \int_{1}^{\infty} \right) K(s)f(x-s)ds = I_{-1} + I_0 + I_1.$$

First consider I_{-1} and I_1. Using the estimate

$$\int_{\mathbb{R}} \frac{1+x^2}{(1+t^2)[1+(x-t)^2]} dt < 2\pi,$$

we already see that there exists $c > 0$ such that I_1 and $I_{-1} \in B_2^\alpha(\mathbb{R})$ and

$$||I_{-1} + I_1||_{\mathbb{R},2}^\alpha \leq c||f||_{\mathbb{R},2}^\alpha.$$

We consider now I_0. We can write

$$I_0(x) = \int_{-1}^{1} K(s)\left[f(x-s) - f(x)\right] ds + f(x) \left(p.v. \int_{-1}^{1} K(s)ds \right),$$

hence

$$\begin{aligned} |I_0(x)| &\leq C_0||f||_{\mathbb{R},2}^\alpha \int_{-1}^{1} \left(\frac{1}{(1+x^2)|s|^{1-\alpha}} + \frac{c}{1+x^2} \right) ds \\ &\leq \frac{C}{1+x^2}||f||_{\mathbb{R},2}^\alpha. \end{aligned}$$

Now, for the Hölder estimate, we have, for δ small enough

$$\begin{aligned} \widetilde{I_0} &= I_0(x+\delta) - I_0(x) = p.v. \int_{-1}^{1} K(s)\left[f(x+\delta-s) - f(x-s)\right] ds \\ &= p.v. \int_{-1}^{1} [K(s+\delta) - K(s)]f(x-s)ds + \\ &+ \left(\int_{1}^{1-\delta} + \int_{-1-\delta}^{-1} \right) K(s+\delta)f(x-s)ds \\ &= \widetilde{II}_0 + \widetilde{II}_{00}. \end{aligned}$$

The estimate for $\widetilde{II}_{00}$ is straightforward:

$$|\widetilde{II}_{00}| \leq \frac{c|\delta|}{1+x^2}||f||^{\alpha}_{\mathbb{R},2}.$$

Now, we notice that we may rewrite $\widetilde{II}_0$ as ($\delta > 0$)

$$\begin{aligned}
\widetilde{II}_0 &= \int_{-1}^{1} K(s+\delta)[f(x-s) - f(x+\delta)]ds - I_0(x) + \\
&+ p.v.\int_{-1}^{1} K(s+\delta) f(x+\delta) ds \\
&= \left(\int_{-1}^{-2\delta} + \int_{-2\delta}^{\delta} + \int_{\delta}^{1}\right) K(s+\delta)[f(x-s) - f(x+\delta)]ds + \\
&- \left(\int_{-1}^{-2\delta} + \int_{-2\delta}^{\delta} + \int_{\delta}^{1}\right) K(s)[f(x-s) - f(x)]ds + \\
&+ p.v.\int_{-1}^{1} K(s+\delta) f(x+\delta) ds - p.v.\int_{-1}^{1} K(s) f(x) ds.
\end{aligned}$$

We first see that

$$\begin{aligned}
&p.v.\int_{-1}^{1} K(s+\delta) f(x+\delta) ds - p.v.\int_{-1}^{1} K(s) f(x) ds \\
&= [f(x+\delta) - f(x)]\left(p.v.\int_{-1}^{1} K(s) ds\right) + f(x+\delta)\left(\int_{-1+\delta}^{-1} + \int_{1}^{1+\delta}\right) K(s) ds,
\end{aligned}$$

is bounded by

$$\frac{c|\delta|^{\alpha}}{1+x^2}||f||^{\alpha}_{\mathbb{R},2}.$$

Now consider

$$\widetilde{II}_1 = \int_{-2\delta}^{\delta} K(s+\delta)[f(x-s) - f(x+\delta)]ds,$$

then we have

$$|\widetilde{II}_1| \leq \frac{c||f||^{\alpha}_{\mathbb{R},2}}{1+x^2}\int_{-2\delta}^{\delta} \frac{|s+\delta|^{\alpha}}{|s+\delta|} ds = c\alpha^{-1}(1+2^{\alpha})\frac{\delta^{\alpha}||f||^{\alpha}_{\mathbb{R},2}}{1+x^2},$$

and a similar estimate holds for

$$\int_{-2\delta}^{\delta} K(s)[f(x-s) - f(x)]ds.$$

Finally, the rest of $\widetilde{II}_0$ can be written as

$$\begin{aligned}
\widetilde{II}_2 &= \left(\int_{-1}^{-2\delta} + \int_{\delta}^{1}\right) [K(s+\delta) - K(s)]\,[f(x-s) - f(x)]\,ds \\
&+ \left(\int_{-1}^{-2\delta} + \int_{\delta}^{1}\right) K(s+\delta)\,[f(x) - f(x+\delta)]\,ds \\
&= \widetilde{II}_3 + \widetilde{II}_4,
\end{aligned}$$

where the last integral $\widetilde{II}_4$ may be estimated in using the fact that, for δ small enough

$$\left|\left(\int_{-1+\delta}^{-\delta}+\int_{2\delta}^{1+\delta}\right)K(s)ds - p.v.\int_{-1}^{1}K(s)ds\right| \leq c,$$

since $|K(s)| \leq C_0|s|^{-1}$ for $|s| \leq 1$; hence

$$|\widetilde{II}_4| \leq c\frac{\delta^\alpha ||f||^\alpha_{\mathbb{R},2}}{1+x^2}.$$

Now consider

$$\widetilde{II}_5 = \int_{-1}^{-2\delta}[K(s+\delta)-K(s)]\,[f(x-s)-f(x)]\,ds,$$

and use the assumption on $K'(x)$ giving ($\delta > 0$)

$$|K(s+\delta)-K(s)| \leq C_0\delta|s+\delta|^{-2},$$

for $s \in (-1,-2\delta)$, then $\widetilde{II}_5$ satisfies

$$\begin{aligned}|\widetilde{II}_5| &\leq c\frac{\delta||f||^\alpha_{\mathbb{R},2}}{1+x^2}\int_{-1}^{-2\delta}\frac{|s|^\alpha}{|s+\delta|^2}ds\\ &\leq 2^\alpha c\frac{\delta||f||^\alpha_{\mathbb{R},2}}{1+x^2}\int_{\delta}^{1-\delta}\frac{s^\alpha+\delta^\alpha}{s^2}ds\\ &\leq c'\frac{\delta^\alpha||f||^\alpha_{\mathbb{R},2}}{1+x^2},\end{aligned}$$

and the same holds for the part $\int_\delta^1$ of $\widetilde{II}_2$. All these estimates, for $|\delta|$ small enough lead to the required property in the lemma.

Part (b) of the lemma may be proved in the same way.

Corollary 12.2. *Let $u \in B^{1,\alpha}_{\mathbb{R},2}$, then $\mathcal{H}(u') \in B^\alpha_{\mathbb{R},2}$ and*

$$||\mathcal{H}(u')||^\alpha_{\mathbb{R},2} \leq c||u||^{1,\alpha}_{\mathbb{R},2}.$$

Proof. We can write

$$\mathcal{H}(u') = p.v.\frac{1}{\pi}\int_{\mathbb{R}}K(s)u'(x-s)ds$$

with $K(s) = 1/s$. We introduce $\phi \in C^\infty(\mathbb{R})$ such that $\phi(s) = 1$ for $|s| \leq 1$, and $\phi(s) = 0$ for $|s| \geq 2$, then we have

$$\begin{aligned}\mathcal{H}(u') &= p.v.\frac{1}{\pi}\int_{\mathbb{R}}\phi(s)K(s)u'(x-s)ds+\\ &+\frac{1}{\pi}\int_{\mathbb{R}}\left([1-\phi(s)]K(s)\right)'_s u(x-s)ds.\end{aligned}$$

We now observe that both ϕK and $\left([1-\phi(s)]K(s)\right)'_s$ satisfy the hypothesis on the kernel K in the above lemma, hence the result of the corollary follows.

Corollary 12.3. *The functions* α_1 *and* β_1 *defined in lemma 8.3, satisfy*

$$||\alpha_1||_{B_w^-} \leq c||u_0'||^{\alpha}_{\mathbb{R},2},$$
$$||\beta_1||_{B_w^-} \leq c||u_0||^{1,\alpha}_{\mathbb{R},2}.$$

Moreover $\beta_1(x,\cdot)$ *is integrable in* y *on* $(-\infty,0)$.

Proof. First consider

$$I(x,y) = \int_{\mathbb{R}} u_0'(s)\frac{y}{(x-s)^2+y^2}ds,$$

and, use the identity

$$\int_{\mathbb{R}} \frac{ds}{(1+s^2)[(x-s)^2+y^2]} = \frac{\pi(1+|y|)}{|y|[(x^2+(1+|y|)^2]}. \tag{12.1}$$

It results that

$$|I(x,y)| \leq \frac{\pi(1+|y|)}{x^2+(1+|y|)^2}||u_0'||^{\alpha}_{\mathbb{R},2},$$

hence

$$\sup_{x\in\mathbb{R},y<0}\frac{1+x^2+y^2}{1+|y|}|I(x,y)| \leq \pi||u_0'||^{\alpha}_{\mathbb{R},2}.$$

It remains to proceed similarly for the Hölder estimate

$$I(x+\delta,y)-I(x,y) = \int_{\mathbb{R}}[u_0'(s+\delta)-u_0'(s)]\frac{y}{(x-s)^2+y^2}ds$$

in using

$$|u_0'(s+\delta)-u_0'(s)| \leq ||u_0'||^{\alpha}_{\mathbb{R},2}\frac{\delta^{\alpha}}{1+s^2}.$$

We finally obtain the desired result for $\alpha_1(\underline{x},\underline{y}) = -\frac{\widetilde{\rho}}{\pi}I(\underline{x},\underline{y})$

$$||\alpha_1||_{B_w^-} \leq c||u_0'||^{\alpha}_{\mathbb{R},2}.$$

For β_1 we first observe that

$$\begin{aligned}\beta_1 &= \frac{\partial}{\partial y}\left[\frac{\widetilde{\rho}}{2\pi}u_0 * \frac{\partial}{\partial y}\ln(x^2+y^2)\right]\\ &= -\frac{\partial}{\partial x}\left[\frac{\widetilde{\rho}}{2\pi}u_0 * \frac{\partial}{\partial x}\ln(x^2+y^2)\right]\\ &= -\frac{\widetilde{\rho}}{\pi}J(x,y)\end{aligned}$$

holds, with

$$\begin{aligned}J(x,y) &= \int_{\mathbb{R}} u_0'(x-s)\frac{s}{s^2+y^2}ds\\ &= \int_{\mathbb{R}} u_0(x-s)\left(\frac{s}{s^2+y^2}\right)'_s ds.\end{aligned}$$

Since

$$\left| \left(\frac{s}{s^2+y^2} \right)'_s \right| \leq \frac{1}{s^2+y^2},$$

and in using the above method, we already have the required estimate for $|y| \geq 1$, i.e. more precisely

$$\frac{1+x^2+y^2}{1+|y|} \left(|J(x,y)| + \delta^{-\alpha} |J(x+\delta,y) - J(x,y)| \right) \leq \frac{c}{|y|} ||u_0||^{\alpha}_{\mathbb{R},2}.$$

Now, for $|y| \leq 1$, we can write

$$J(x,y) = \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) u_0(x-s) \left(\frac{s[1-\phi(s)]}{s^2+y^2} \right)'_s ds +$$
$$+ \int_{-2}^{2} u_0'(x-s) \frac{s\phi(s)}{s^2+y^2} ds$$

and the kernels $\left(\frac{s[1-\phi(s)]}{s^2+y^2} \right)'_s$ and $\frac{s\phi(s)}{s^2+y^2}$ satisfy the conditions of the lemma above, uniformly in $|y| \leq 1$, hence

$$||J(\cdot,y)||^{\alpha}_{\mathbb{R},2} \leq c ||u_0||^{1,\alpha}_{\mathbb{R},2},$$

where c is independent of y such that $|y| \leq 1$. This estimate with the preceding one ends the proof of

$$||\beta_1||_{B_w^-} \leq c ||u_0||^{1,\alpha}_{\mathbb{R},2}.$$

Moreover we observe that $\beta_1(x,y)$ *is integrable in* y *on* $(-\infty,0)$, which may be used for justifying the convergence of integrals in the Bernoulli first integral.

13. Appendix Resolvent ∞

In this appendix we prove estimates (8.18) and (8.17).

First, let us introduce some notations. We denote by $a(x)$ a function of $x \in \mathbb{R}$, $f_1(x,y)$ a function with $x \in \mathbb{R}, y \in (-\infty,0]$, $f_2(x,y)$ a function with $x \in \mathbb{R}, y \in [0,1]$. For proving (8.18) and (8.17) we work on the Fourier transform (in x) of $(\alpha_1(x,y), \beta_1(x,y), \alpha_2(x,y), \beta_2(x,y))$ given by the formulas of subsections 5.(a) and 5.(b). These formulas give the components of $(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1}\widehat{V}$, and here we need to take

$$\widehat{V} = \varepsilon \varphi_1 \widehat{T}_Y,$$

where

$$T_Y \in B_2^\alpha(\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon)_{\eta^*} + B_3^\alpha(\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon),$$

and where functions a, f_1, f_2 are typical components of $\widehat{V}$ (which cancel for k near 0). What we denote by $\varphi_1 S_u^{(1)}(\widehat{T}_Y)$ is the component β_{21}, i.e. $\beta_2|_{y=1}$, and we have by construction

$$(ik\mathbb{I} - \mathcal{L}_\varepsilon)^{-1} \varepsilon \varphi_1 \widehat{T}_Y = \varphi_1 S_u^{(1)}(\widehat{T}_Y)\xi_0 + \varepsilon \varphi_1 S_Y^{(1)}(\widehat{T}_Y).$$

Now looking at formulas of section 5.(a) giving α_1 and β_1 we claim that they are sums of typical terms as

$$
\begin{aligned}
K_{1,0}(k;\varepsilon)\widehat{a} &= \varepsilon A(\varepsilon k)e^{ky}\widehat{a},\\
K_{1,10}(k,y;\varepsilon)\widehat{f_1} &= \int_{-\infty}^{0} sgn(y-\tau)\widehat{f_1}(k,\tau)e^{-|k||y-\tau|}d\tau,\\
K_{1,11}(k,y;\varepsilon)\widehat{f_1} &= \int_{-\infty}^{0} sgn(k)\widehat{f_1}(k,\tau)e^{-|k||y-\tau|}d\tau,\\
K_{1,12}(k,y;\varepsilon)\widehat{f_1} &= \int_{-\infty}^{0} \widehat{f_1}(k,\tau)e^{-|k||y+\tau|}d\tau,
\end{aligned}
$$

$$
\begin{aligned}
K_{1,13}(k,y;\varepsilon)\widehat{f_1} &= \int_{-\infty}^{0} sgn(k)\widehat{f_1}(k,\tau)e^{-|k||y+\tau|}d\tau,\\
K_{1,14}(k,y;\varepsilon)\widehat{f_1} &= \varepsilon k A(\varepsilon k)\int_{-\infty}^{0} \widehat{f_1}(k,\tau)e^{-|k||y+\tau|}d\tau,\\
K_{1,2}(k,y;\varepsilon)\widehat{f_2} &= \varepsilon e^{|k|y}\int_{0}^{1} B(\varepsilon k,\tau)\widehat{f_2}(k,\tau)d\tau,
\end{aligned}
$$

where A and B are analytic functions, except at 0 for the first argument lying in a sector of the complex plane centered on the real axis, and of angle $O(1)$, either even or odd and such that for $|\widetilde{k}|$ large we have the estimates

$$
\begin{aligned}
A(\widetilde{k}) &= \frac{c}{\widetilde{k}} + A_0(\widetilde{k}),\\
B(\widetilde{k},\tau) &= \widetilde{c}_1 e^{-c_1(\tau)\widetilde{k}} + \frac{\widetilde{c}_2}{\widetilde{k}}e^{-c_2(\tau)\widetilde{k}} + B_0(\widetilde{k},\tau),
\end{aligned}
$$

with $c_1(\cdot)$ and $c_2(\cdot) \geq 0$ linear functions of their argument, and

$$
|A_0| + |B_0| \leq c|\widetilde{k}|^{-2}, \quad \text{uniformly in } \tau \in (0,1).
$$

We can also see for α_2 and β_2 that they are sums of typical terms as

$$
\begin{aligned}
K_{2,0}(\varepsilon k,y;\varepsilon)\widehat{a} &= \varepsilon E(\varepsilon k,y)\widehat{a},\\
K_{2,1}(k,y;\varepsilon)\widehat{f_1} &= C(\varepsilon k,y)\int_{-\infty}^{0} \widehat{f_1}(k,\tau)e^{|k|\tau}d\tau,\\
K_{2,20}(\varepsilon k,y;\varepsilon)\widehat{f_2} &= \varepsilon\int_{0}^{1} D_0(\varepsilon k,y-\tau;\varepsilon)\widehat{f_2}(k,\tau)d\tau,\\
K_{2,21}(\varepsilon k,y;\varepsilon)\widehat{f_2} &= \varepsilon\int_{0}^{1} D_1(\varepsilon k,y,\tau;\varepsilon)\widehat{f_2}(k,\tau)d\tau,
\end{aligned}
$$

where C, D and E are analytic in $\widetilde{k} = \varepsilon k$ except at 0 for the first argument lying in a sector of the complex plane centered on the real axis, and of angle $O(1)$, either even or odd, and satisfy the estimates for $|\widetilde{k}|$ large

$$\begin{aligned} E(\widetilde{k},y) &= \frac{c}{\widetilde{k}} e^{-c_0(y)|\widetilde{k}|} + E_0(\widetilde{k},y), \\ C(\widetilde{k},y) &= \widetilde{c}_1 e^{-c_1(y)|\widetilde{k}|} + \frac{\widetilde{c}_2}{\widetilde{k}} e^{-c_2(y)|\widetilde{k}|} + C_0(\widetilde{k},y), \\ D_0(\widetilde{k},y-\tau;\varepsilon) &= c e^{-|y-\tau||\widetilde{k}|} + D_{00}(\widetilde{k},y-\tau;\varepsilon), \\ D_1(\widetilde{k},y,\tau;\varepsilon) &= \widetilde{c}_1 e^{-|\widetilde{k}|c_1(y+\tau)} + \frac{\widetilde{c}_2}{\widetilde{k}} e^{-|\widetilde{k}|c_2(y+\tau)} + D_{10}(\widetilde{k},y,\tau;\varepsilon), \end{aligned}$$

where $c, \widetilde{c}_j$ are generic constants, and $c_0(\cdot)$, $c_1(\cdot)$, $c_2(\cdot) \geq 0$ are linear functions of their argument, and

$$|E_0| + |C_0| + |D_{00}| + |D_{10}| \leq c|\widetilde{k}|^{-2}, \tag{13.1}$$

$$|\partial_y E_0| + |\partial_y C_0| + |\partial_y D_{00}| + |\partial_y D_{10}| \leq c|\widetilde{k}|^{-2}, \tag{13.2}$$

holds uniformly in $(y,\tau) \in (0,1)^2$. We should notice in these estimates that we took account of the elimination of the poles at $\varepsilon k = \pm\lambda$, thanks to the projection π_ε.

Let us denote by

$$\begin{aligned} K^{(1)}_{i,j}\widehat{f} &= \varphi_1 K_{i,j}\widehat{f} \\ \mathcal{L}^{(1)}_{i,j} f &= \mathcal{F}^{-1}[K^{(1)}_{i,j}\widehat{f}] = \mathcal{K}^{(1)}_{i,j} * f \end{aligned}$$

where $*$ means convolution in x and $\mathcal{K}^{(1)}_{i,j} = \mathcal{F}^{-1}K^{(1)}_{i,j}$, then we have the following lemma

Lemma 13.1. *For any given $a \in B_2^\alpha(\mathbb{R})$, $f_1 \in B_2^\alpha(C_\varepsilon^{0,\exp})$, and $f_2 \in B_w^+$, the following holds*

(i) $\mathcal{L}^{(1)}_{1,0} a \in B_w^-$ *and* $||\mathcal{L}^{(1)}_{1,0} a||_{B_w^-} \leq c\varepsilon ||a||^\alpha_{\mathbb{R},2}$,

(ii) $\mathcal{L}^{(1)}_{1,j} f_1 \in B_w^-$ *and* $||\mathcal{L}^{(1)}_{1,j} f_1||_{B_w^-} \leq c\varepsilon ||f_1||_{B_2^\alpha(C_\varepsilon^{0,\exp})}$, $j = 10, 11, 12, 13, 14$,

(iii) $\mathcal{L}^{(1)}_{1,2} f_2 \in B_w^-$ *and* $||\mathcal{L}^{(1)}_{1,2} f_2||_{B_w^-} \leq c\varepsilon ||f_2||_{B_w^+}$,

(iv) $\mathcal{L}^{(1)}_{2,0} a \in B_w^{1,+}$ *and* $||\mathcal{L}^{(1)}_{2,0} a||_{B_w^+} + ||\frac{\partial}{\partial y}\mathcal{L}^{(1)}_{2,0} a||_{B_w^+} \leq c\varepsilon ||a||^\alpha_{\mathbb{R},2}$,

(v) $\mathcal{L}^{(1)}_{2,1} f_1 \in B_w^{1,+}$ *and* $||\mathcal{L}^{(1)}_{2,1} f_1||_{B_w^+} + ||\frac{\partial}{\partial y}\mathcal{L}^{(1)}_{2,1} f_1||_{B_w^+} \leq c\varepsilon ||f_1||_{B_2^\alpha(C_\varepsilon^{0,\exp})}$,

(vi) $\mathcal{L}^{(1)}_{2,j} f_2 \in B_w^{1,+}$ *and* $||\mathcal{L}^{(1)}_{2,j} f_2||_{B_w^+} + ||\frac{\partial}{\partial y}\mathcal{L}^{(1)}_{2,j} f_2||_{B_w^+} \leq c\varepsilon ||f_2||_{B_w^+}$, $j = 20, 21$.

Proof of estimates (8.17)**,** (8.18)**.** This lemma, for $\widehat{V} = \varepsilon\varphi_1\widehat{T}_Y$, gives directly the estimate (8.17), by construction of

$$\mathcal{T}_{21}(T_Y) = \mathcal{F}^{-1}\left[\varphi_1 S_Y^{(1)}(\widehat{T}_Y)\right].$$

Moreover, we have the following estimate in $B_2^\alpha(\mathbb{R})$ of $\mathcal{F}^{-1}\left[\varphi_1 S_u^{(1)}(\widehat{T}_Y)\right] = \beta_{21}$:

$$||\mathcal{F}^{-1}\left[\varphi_1 S_u^{(1)}(\widehat{T}_Y)\right]||^\alpha_{\mathbb{R},2} \leq c\varepsilon^2(||T_Y^{(1)}||^\alpha_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,2} + ||T_Y^{(2)}||^\alpha_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,3})$$

which, with the relationship

$$ik\widehat{\beta}_2 + \varepsilon^{-1}\widehat{\alpha}_2' = \widehat{g}_2,$$

leads to

$$\begin{aligned}||(d/dx)\mathcal{F}^{-1}\left[\varphi_1 S_u^{(1)}(\widehat{T}_Y)\right]||_{\mathbb{R},2}^{\alpha} &\leq \varepsilon^{-1}||\frac{\partial}{\partial y}\mathcal{F}^{-1}\widehat{\alpha}_2|_{y=1}||_{\mathbb{R},2}^{\alpha} + ||g_{21}||_{\mathbb{R},2}^{\alpha} \\ &\leq c\varepsilon(||T_Y^{(1)}||_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,2}^{\alpha} + ||T_Y^{(2)}||_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,3}^{\alpha}).\end{aligned}$$

Hence the estimate (8.18) holds.

Proof of lemma 13.1. We only prove the lemma for $\mathcal{L}_{1,10}^{(1)}, \mathcal{L}_{2,0}^{(1)}, \mathcal{L}_{2,20}^{(1)}$. The rest of the proof would be similar.

Step 1. Let first consider $\mathcal{L}_{1,10}^{(1)}f_1$, and introduce, for $y-\tau \neq 0$

$$I(x, y-\tau) = \int_{\mathbb{R}} sgn(y-\tau)\varphi_1(\varepsilon k)e^{ixk-|k||y-\tau|}dk.$$

Then we have

$$\left(\mathcal{L}_{1,10}^{(1)}f_1\right)(x,y) = \frac{1}{2\pi}\int_{-\infty}^{0}\int_{\mathbb{R}} I(x-s, y-\tau)f_1(s,\tau)dsd\tau.$$

We can work on I and obtain first (since $\varphi_1' = 0$ outside $(\delta/2\varepsilon, \delta/\varepsilon)$)

$$I(x, y-\tau) = -2\,\mathrm{Re}\left(\frac{\varepsilon}{ix - |y-\tau|}\int_{\delta/2\varepsilon}^{\delta/\varepsilon} sgn(y-\tau)\varphi_1'(\varepsilon k)e^{ixk-k|y-\tau|}dk\right),$$

which (after several integrations by parts for the second and last estimates), leads to

$$\begin{aligned}|I(x,y-\tau)| &\leq \frac{c}{(x^2+|y-\tau|^2)^{1/2}}e^{-(\delta/2\varepsilon)|y-\tau|}, \\ |I(x,y-\tau)| &\leq \frac{c\varepsilon^n}{|x|^{n+1}}e^{-(\delta/2\varepsilon)|y-\tau|}, \\ |I_x'(x,y-\tau)| &\leq \frac{c}{(x^2+|y-\tau|^2)}e^{-(\delta/2\varepsilon)|y-\tau|} \leq \frac{c}{x^2}e^{-(\delta/2\varepsilon)|y-\tau|},\end{aligned}$$

where c only depends on $n \geq 0$. Moreover, we have

$$\int_{-1}^{1} I(x,y-\tau)dx = e^{-(\delta/2\varepsilon)|y-\tau|}\int_{\delta/2\varepsilon}^{\infty} 4\frac{\sin k}{k} sgn(y-\tau)\varphi_1(\varepsilon k)e^{-[k-(\delta/2\varepsilon)]|y-\tau|}dk,$$

hence

$$|\int_{-1}^{1} I(x,y-\tau)dx| \leq ce^{-(\delta/2\varepsilon)|y-\tau|}$$

holds. We can then apply lemma 12.1 for any fixed τ, and obtain

$$\left\|\int_{\mathbb{R}} I(x-s,y-\tau)f_1(s,\tau)ds\right\|_{\mathbb{R},2}^{\alpha} \leq ce^{-(\delta/2\varepsilon)|y-\tau|}||f_1(\cdot,\tau)||_{\mathbb{R},2}^{\alpha}.$$

Let us choose $\delta = \lambda$, then we have the identity

$$\int_{-\infty}^{0} e^{-(\lambda/2\varepsilon)|y-\tau|} e^{-(\lambda/2\varepsilon)|\tau|} d\tau = \frac{2\varepsilon}{\lambda}\left(2e^{-(\lambda/2\varepsilon)|y|} - e^{-2(\lambda/2\varepsilon)|y|}\right) \leq c\varepsilon e^{-(\lambda/2\varepsilon)|y|},$$

and thanks to the inequality (valid for $\varepsilon < e\lambda/2$)

$$(1+x^2)^{-1} e^{-(\lambda/2\varepsilon)|y|} \leq \frac{1+|y|}{1+x^2+y^2}$$

we can deduce the required estimate in B_w^- :

$$\left\| \mathcal{L}_{1,10}^{(1)} f_1 \right\|_{B_w^-} \leq c\varepsilon ||f_1||_{B_2^\alpha(C_\varepsilon^{0,\exp})}.$$

Step 2. Let consider now $\mathcal{L}_{2,0}^{(1)}$ and introduce

$$\mathcal{K}_{2,0}^{(1)}(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx} \varepsilon E(\varepsilon k, y)\varphi_1(\varepsilon k) dk,$$

such that

$$\left(\mathcal{L}_{2,0}^{(1)} a\right)(x,y) = \int_{\mathbb{R}} \mathcal{K}_{2,0}^{(1)}(x-s,y) a(s) ds.$$

We intend to use the lemma 12.1 again, uniformly in y. Splitting the kernel

$$\mathcal{K}_{2,0}^{(1)}(x,y) = \mathcal{K}_{2,0}^{(1)+}(x,y) + \mathcal{K}_{2,0}^{(1)-}(x,y)$$

in separating the integral on $\mathbb{R}^+$ and $\mathbb{R}^-$, one integration by parts leads to ($\varphi_1(\varepsilon k) = 0$ for $|k| < \delta/2\varepsilon$)

$$\mathcal{K}_{2,0}^{(1)+}(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}^+} \frac{\varepsilon}{ix - \varepsilon c_0(y)} e^{[ixk - \varepsilon c_0(y)|k|]} \left(\frac{c}{\varepsilon k}\varphi_1(\varepsilon k)\right)'_k dk + \\ -\frac{1}{2\pi}\int_{\mathbb{R}^+} \frac{\varepsilon}{ix} e^{ikx} \left(E_0(\varepsilon k, y)\varphi_1(\varepsilon k)\right)'_k dk,$$

which gives the estimate

$$|\mathcal{K}_{2,0}^{(1)}(x,y)| \leq c\frac{\varepsilon}{|x|}, \quad \text{uniformly in } y,$$

and a second integration by parts gives

$$|\mathcal{K}_{2,0}^{(1)}(x,y)| \leq c\frac{\varepsilon^2}{|x|^2}, \quad \text{uniformly in } y.$$

We can also write

$$\int_{-1}^{1} \mathcal{K}_{2,0}^{(1)}(x,y) dx = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\sin k}{k} \varepsilon E(\varepsilon k, y)\varphi_1(\varepsilon k) dk = O(\varepsilon)$$

where the integral is absolutely convergent. Now, after one integration by parts for half of the terms, we have

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{K}^{(1)+}_{2,0}(x,y) = &\frac{i}{2\pi}\int_{\mathbb{R}+}\frac{\varepsilon}{[ix-\varepsilon c_0(y)]^2}e^{[ix-\varepsilon c_0(y)]k}\left(\frac{c}{\varepsilon k}\varphi_1(\varepsilon k)\right)'_k dk+\\ &+\frac{i}{2\pi}\int_{\mathbb{R}+}\frac{\varepsilon c}{[ix-\varepsilon c_0(y)]^2}e^{[ix-\varepsilon c_0(y)]k}\left(\varphi_1'(\varepsilon k)-\frac{\varphi_1(\varepsilon k)}{\varepsilon k}\right)'_k dk+\\ &+\frac{1}{2\pi}\int_{\mathbb{R}+}\frac{\varepsilon}{ix^2}e^{ikx}\left(E_0(\varepsilon k,y)\varphi_1(\varepsilon k)\right)'_k dk+\\ &+\frac{1}{2\pi}\int_{\mathbb{R}+}\frac{\varepsilon}{ix^2}e^{ikx}\left[k\left(E_0(\varepsilon k,y)\varphi_1(\varepsilon k)\right)'_k\right]'_k dk\end{aligned}$$

and a similar formula holds for $\frac{\partial}{\partial x}\mathcal{K}^{(1)-}_{2,0}(x,y)$. Since $\frac{\partial^2}{\partial \widetilde{k}^2}E_0(\widetilde{k},y)=O(1/\widetilde{k}^4)$ uniformly in y (thanks to analyticity in $\widetilde{k}$, to the uniformity of (13.1), and to the Cauchy formula), all terms give the estimate

$$|\frac{\partial}{\partial x}\mathcal{K}^{(1)}_{2,0}(x,y)|\leq c\frac{\varepsilon}{|x|^2},\quad \text{uniformly in } y.$$

Then using lemma 12.1, we have directly

$$||\mathcal{L}^{(1)}_{2,0}a||_{B_w^+}\leq c\varepsilon||a||^{\alpha}_{\mathbb{R},2}.$$

We proceed in the same way with

$$\begin{aligned}\frac{\partial}{\partial y}\mathcal{K}^{(1)+}_{2,0}(x,y) = &-\frac{1}{2\pi}\int_{\mathbb{R}+}\frac{\varepsilon c c_0'}{ix-\varepsilon c_0(y)}e^{[ix-\varepsilon c_0(y)]k}\varepsilon\varphi_1'(\varepsilon k)dk+\\ &-\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\varepsilon}{ix}e^{ikx}\left(\partial_y E_0(\varepsilon k,y)\varphi_1(\varepsilon k)\right)'_k dk,\end{aligned}$$

where we observe that c_0' is constant, and we obtain easily, uniformly in y :

$$\begin{aligned}|\frac{\partial}{\partial y}\mathcal{K}^{(1)}_{2,0}(x,y)|&\leq c\frac{\varepsilon}{|x|},\\ |\frac{\partial}{\partial y}\mathcal{K}^{(1)}_{2,0}(x,y)|&\leq c\frac{\varepsilon^2}{|x|^2},\\ \int_{-1}^{1}\frac{\partial}{\partial y}\mathcal{K}^{(1)}_{2,0}(x,y)dx&=O(\varepsilon),\\ \left|\frac{\partial^2}{\partial x\partial y}\mathcal{K}^{(1)}_{2,0}(x,y)\right|&\leq c\frac{\varepsilon}{|x|^2}.\end{aligned}$$

By lemma 12.1, it then results that

$$||\frac{\partial}{\partial y}\mathcal{L}^{(1)}_{2,0}a||_{B_w^+}\leq c\varepsilon||a||^{\alpha}_{\mathbb{R},2},$$

hence this part of the lemma is proved.

Step 3. Let consider now $\mathcal{L}^{(1)}_{2,20}$ and introduce

$$\mathcal{K}^{(1)}_{2,20}(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}\varepsilon D_0(\varepsilon k, y)\varphi_1(\varepsilon k)dk,$$

such that

$$\left(\mathcal{L}^{(1)}_{2,20}f_2\right)(x,y) = \int_{\mathbb{R}}\int_0^1 \mathcal{K}^{(1)}_{2,20}(x-s,y-\tau)f_2(s,\tau)d\tau ds.$$

We intend to use the lemma 12.1 again, uniformly in y. We notice that we have

$$\mathcal{K}^{(1)}_{2,20}(x,y-\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} \varepsilon c e^{ixk-\varepsilon|y-\tau||k|}\varphi_1(\varepsilon k)dk + \\ +\frac{1}{2\pi}\int_{\mathbb{R}} e^{ikx}\varepsilon D_{00}(\varepsilon k, y-\tau)\varphi_1(\varepsilon k)dk,$$

hence the second integral may be treated exactly as above with the integral in E_0. Let us split the first integral on $\mathbb{R}^+$ and $\mathbb{R}^-$:

$$\mathcal{K}^{(1)}_{2,201}(x,y-\tau) = \mathcal{K}^{(1)+}_{2,201}(x,y-\tau) + \mathcal{K}^{(1)-}_{2,201}(x,y-\tau), \\ \mathcal{K}^{(1)-}_{2,201}(x,y-\tau) = \mathcal{K}^{(1)+}_{2,201}(-x,y-\tau),$$

it then remains to study

$$\mathcal{K}^{(1)+}_{2,201}(x,y-\tau) = \frac{1}{2\pi}\int_{\delta/2\varepsilon}^{\infty} \varepsilon c e^{(ix-\varepsilon|y-\tau|)k}\varphi_1(\varepsilon k)dk \\ = -\frac{1}{2\pi}\int_{\delta/2\varepsilon}^{\delta/\varepsilon} \frac{\varepsilon c}{ix-\varepsilon|y-\tau|}e^{(ix-\varepsilon|y-\tau|)k}\varepsilon\varphi_1'(\varepsilon k)dk,$$

where these integrals are convergent for $|y-\tau| \neq 0$. It is then clear that we have, with an integration by parts for the second estimate

$$|\mathcal{K}^{(1)+}_{2,201}(x,y-\tau)| \leq c\frac{\varepsilon}{|x|}, \\ |\mathcal{K}^{(1)+}_{2,201}(x,y-\tau)| \leq c\frac{\varepsilon^2}{|x|^2},$$

and

$$\int_{-1}^{1} \mathcal{K}^{(1)+}_{2,201}(x,y-\tau)dx = \frac{\varepsilon c}{\pi}\int_{\delta/2\varepsilon}^{\infty} \frac{\sin k}{k}e^{-\varepsilon|y-\tau|k}\varphi_1(\varepsilon k)dk \\ = \frac{\varepsilon c}{2\pi}\int_{\delta/2\varepsilon}^{\infty} \mathrm{Re}\left(\frac{ie^{ik-\varepsilon|y-\tau|k}}{i-\varepsilon|y-\tau|}\right)\left(\frac{\varphi_1(\varepsilon k)}{k}\right)'_k dk = O(\varepsilon^2)$$

uniformly in $|y-\tau| \in (0,1)$. In addition we have

$$\frac{\partial}{\partial x}\mathcal{K}^{(1)+}_{2,201}(x,y-\tau) = \frac{i}{2\pi}\int_{\delta/2\varepsilon}^{\delta/\varepsilon} \frac{\varepsilon c}{(ix-\varepsilon|y-\tau|)^2}e^{(ix-\varepsilon|y-\tau|)k}\varepsilon\varphi_1'(\varepsilon k)dk + \\ + \frac{i\varepsilon c}{2\pi}\int_{\delta/2\varepsilon}^{\delta/\varepsilon} \frac{1}{(ix-\varepsilon|y-\tau|)^2}e^{(ix-\varepsilon|y-\tau|)k}\left[\varepsilon k\varphi_1'(\varepsilon k)\right]'_k dk$$

hence

$$\left|\frac{\partial}{\partial x}\mathcal{K}^{(1)+}_{2,201}(x,y-\tau)\right| \leq c\frac{\varepsilon}{|x|^2}$$

holds uniformly in $|y-\tau| \in (0,1)$. These estimates allow to use lemma 12.1 for obtaining the estimate

$$||\mathcal{L}^{(1)}_{2,20}f_2||_{B^+_w} \leq c\varepsilon||f_2||_{B^+_w}.$$

We need to do the same with $\frac{\partial}{\partial y}\mathcal{K}^{(1)+}_{2,201}$. Integrating by parts and splitting the resulting integral into $\int_{\delta/2\varepsilon}^{\delta/\varepsilon} + \int_{\delta/\varepsilon}^{\infty}$ we have the following decomposition for $|y-\tau| \neq 0$,

$$\begin{aligned}
\frac{\partial}{\partial y}\mathcal{K}^{(1)+}_{2,201}(x,y-\tau) &= -\frac{1}{2\pi}\int_{\delta/2\varepsilon}^{\infty}[sgn(y-\tau)]\varepsilon^2 ce^{(ix-\varepsilon|y-\tau|)k}k\varphi_1(\varepsilon k)dk \\
&= \widetilde{\partial}\mathcal{K}^{(1)+}_{2,201}(x,y-\tau) + \widetilde{\mathcal{K}}^{(1)+}_{2,201}(x,y-\tau), \\
\widetilde{\mathcal{K}}^{(1)+}_{2,201}(x,y-\tau) &= -\frac{1}{2\pi}\frac{\varepsilon^2 c[sgn(y-\tau)]}{(ix-\varepsilon|y-\tau|)^2}e^{(i\frac{\delta}{\varepsilon}x-\delta|y-\tau|)}, \\
\widetilde{\partial}\mathcal{K}^{(1)+}_{2,201}(x,y-\tau) &= \frac{1}{2\pi}\int_{\delta/2\varepsilon}^{\delta/\varepsilon}\frac{\varepsilon c[sgn(y-\tau)]}{ix-\varepsilon|y-\tau|}e^{(ix-\varepsilon|y-\tau|)k}\left(\varepsilon k\varphi_1(\varepsilon k)\right)'_k dk \\
&= -\frac{1}{2\pi}\int_{\delta/2\varepsilon}^{\delta/\varepsilon}\frac{\varepsilon c[sgn(y-\tau)]}{(ix-\varepsilon|y-\tau|)^2}e^{(ix-\varepsilon|y-\tau|)k}\left(\varepsilon k\varphi_1(\varepsilon k)\right)''_{kk} dk,
\end{aligned}$$

hence

$$|\widetilde{\partial}\mathcal{K}^{(1)+}_{2,201}(x,y-\tau)| \leq c\min\left(\frac{\varepsilon}{|x|},\frac{\varepsilon^2}{|x|^2}\right),$$

and we treat the kernel $\widetilde{\mathcal{K}}^{(1)+}_{2,201}(x,y-\tau)$ later. In addition we have

$$\begin{aligned}
\int_{-1}^{1}\frac{\partial}{\partial y}\mathcal{K}^{(1)+}_{2,201}(x,y-\tau)dx &= -\frac{\varepsilon^2 c[sgn(y-\tau)]}{\pi}\int_{\delta/2\varepsilon}^{\infty}\sin ke^{-\varepsilon|y-\tau|k}\varphi_1(\varepsilon k)dk \\
&= \frac{\varepsilon^2 c[sgn(y-\tau)]}{2\pi}\int_{\delta/2\varepsilon}^{\delta/\varepsilon}\mathrm{Re}\left(\frac{ie^{(i-\varepsilon|y-\tau|)k}}{i-\varepsilon|y-\tau|}\right)\varepsilon\varphi_1'(\varepsilon k)dk \\
&= O(\varepsilon^2)
\end{aligned}$$

uniformly in $|y-\tau| \neq 0$, and

$$\begin{aligned}
\frac{\partial}{\partial x}\widetilde{\partial}\mathcal{K}^{(1)+}_{2,201}(x,y) &= \frac{1}{2\pi}\int_{\delta/2\varepsilon}^{\delta/\varepsilon}\frac{-i\varepsilon c[sgn(y-\tau)]}{(ix-\varepsilon|y-\tau|)^2}e^{(ix-\varepsilon|y-\tau|)k}\left(\varepsilon k\varphi_1(\varepsilon k)\right)'_k dk + \\
&- \frac{1}{2\pi}\int_{\delta/2\varepsilon}^{\delta/\varepsilon}\frac{i\varepsilon c[sgn(y-\tau)]}{(ix-\varepsilon|y-\tau|)^2}e^{(ix-\varepsilon|y-\tau|)k}\left[k\left(\varepsilon k\varphi_1(\varepsilon k)\right)'_k\right]'_k dk,
\end{aligned}$$

which gives

$$\left|\frac{\partial}{\partial x}\widetilde{\partial}\mathcal{K}^{(1)+}_{2,201}(x,y-\tau)\right| \leq c\frac{\varepsilon}{|x|^2},$$

uniformly in $|y-\tau| \neq 0$. Now we can check that

$$\int_{-1}^{1}\widetilde{\partial}\mathcal{K}^{(1)+}_{2,201}(x,y-\tau)dx = O(\varepsilon^2),$$

uniformly in $|y-\tau| \neq 0$. Indeed we have

$$\begin{aligned}\int_{-1}^{1} \widetilde{\mathcal{K}}_{2,201}^{(1)+}(x, y-\tau)dx &= -\frac{c\varepsilon^2}{2\pi} \int_{-1}^{1} \frac{[sgn(y-\tau)]}{(ix-\varepsilon|y-\tau|)^2} e^{(ix-\varepsilon|y-\tau|)\delta/\varepsilon} dx \\ &= -\frac{c\varepsilon}{2\pi} \int_{-1/\varepsilon}^{1/\varepsilon} \frac{[sgn(y-\tau)]}{(ix-|y-\tau|)^2} e^{(ix-|y-\tau|)\delta} dx \\ &= -\frac{c\varepsilon}{2\pi} \int_{-\infty}^{\infty} \frac{[sgn(y-\tau)]}{(ix-|y-\tau|)^2} e^{(ix-|y-\tau|)\delta} dx + O(\varepsilon^2)\end{aligned}$$

uniformly in $|y-\tau| \neq 0$, since we have uniformly

$$\int_{1/\varepsilon}^{\infty} \frac{dx}{x^2+y^2} < \varepsilon,$$

and by the residue formula, the last integral on $\mathbb{R}$ is zero, hence the estimate follows. It results that we can apply the lemma 12.1(a) for the part $\widetilde{\partial}\mathcal{K}_{2,201}^{(1)+}$ of the kernel, and it remains to study the action of the explicit kernel $\widetilde{\mathcal{K}}_{2,201}^{(1)+}$.

In fact we want to control the function

$$\int_{\mathbb{R}} \int_{0}^{1} \widetilde{\mathcal{K}}_{2,201}^{(1)+}(s, y-\tau) f_2(x-s, \tau) d\tau ds = \int_{\mathbb{R}} \widetilde{\widetilde{\mathcal{K}}}(s) f_2(x-s, \cdot) ds$$

in $B_w^+ = B_2^\alpha[C^0(0,1)]$, where the kernel $\widetilde{\widetilde{\mathcal{K}}}$ is defined for $g \in C^0(0,1)$, by

$$\left(\widetilde{\widetilde{\mathcal{K}}}(x) g\right)(y) = \int_0^1 \widetilde{\mathcal{K}}_{2,201}^{(1)+}(x, y-\tau) g(\tau) d\tau.$$

The idea is to use lemma 12.1(b), in noticing that $x \mapsto \widetilde{\widetilde{\mathcal{K}}}(x)$ is C^1 in $\mathcal{L}[C^0(0,1)]$ for $x \neq 0$. Thanks to the explicit expression

$$\widetilde{\mathcal{K}}_{2,201}^{(1)+}(x, y-\tau) = -\frac{1}{2\pi} \frac{\varepsilon^2 c[sgn(y-\tau)]}{(ix-\varepsilon|y-\tau|)^2} e^{(i\frac{\delta}{\varepsilon}x - \delta|y-\tau|)},$$

we already have

$$\begin{aligned}\left| \int_0^1 \widetilde{\mathcal{K}}_{2,201}^{(1)+}(x, y-\tau) g(\tau) d\tau \right| &\leq c \frac{\varepsilon^2}{|x|^2} \sup_{\tau \in (0,1)} |g(\tau)|, \\ \left| \int_0^1 \widetilde{\mathcal{K}}_{2,201}^{(1)+}(x, y-\tau) g(\tau) d\tau \right| &\leq c \left(\int_0^1 \frac{\varepsilon^2}{x^2+\varepsilon^2|y-\tau|^2} d\tau \right) \sup_{\tau \in (0,1)} |g(\tau)| \\ &\leq 2c \frac{\varepsilon}{|x|} \left(\int_0^{\frac{\varepsilon}{|x|}} \frac{1}{1+u^2} du \right) \sup_{\tau \in (0,1)} |g(\tau)| \\ &\leq c' \frac{\varepsilon}{|x|} \sup_{\tau \in (0,1)} |g(\tau)|.\end{aligned}$$

Moreover, the calculation made above, shows that, for $\mu > 0$ (Fubini's theorem applies)

$$\begin{aligned}&\int_{(-1,-\mu)\cup(\mu,1)}\int_0^1 \widetilde{\mathcal{K}}_{2,201}^{(1)+}(x,y-\tau)g(\tau)d\tau dx\\&=\int_0^1 d\tau\int_{(-1,-\mu)\cup(\mu,1)}\widetilde{\mathcal{K}}_{2,201}^{(1)+}(x,y-\tau)g(\tau)dx\\&=O(\varepsilon^2)\sup_{\tau\in(0,1)}|g(\tau)|,\end{aligned}$$

with a uniform limit for $\mu\to 0$. It remains to check the $x-$ derivative

$$\begin{aligned}&\frac{\partial}{\partial x}\int_0^1 \widetilde{\mathcal{K}}_{2,201}^{(1)+}(x,y-\tau)g(\tau)d\tau\\&=\frac{1}{\pi}\int_0^1\frac{i\varepsilon^2 c[sgn(y-\tau)]}{(ix-\varepsilon|y-\tau|)^3}e^{(i\frac{\delta}{\varepsilon}x-\delta|y-\tau|)}g(\tau)d\tau+\\&+\frac{1}{2\pi}\int_0^1\frac{i\varepsilon\delta c[sgn(y-\tau)]}{(ix-\varepsilon|y-\tau|)^2}e^{(i\frac{\delta}{\varepsilon}x-\delta|y-\tau|)}g(\tau)d\tau\end{aligned}$$

and by the same type of estimate as above, we obtain

$$\left|\frac{\partial}{\partial x}\int_0^1 \widetilde{\mathcal{K}}_{2,201}^{(1)+}(x,y-\tau)g(\tau)d\tau\right|\leq c\frac{\varepsilon}{|x|^2}\sup_{\tau\in(0,1)}|g(\tau)|,$$

and lemma 12.1(b) applies for getting the required estimate

$$||\frac{\partial}{\partial y}\mathcal{L}_{2,20}^{(1)}f_2||_{B_w^+}\leq c\varepsilon||f_2||_{B_w^+}.$$

14. Appendix Resolvent 0

In this appendix we prove estimates (8.11) and (8.15). First let us give a more detailed form for the operators $S_u^{(0)}$ and $S_Y^{(0)}$. If we denote $\widehat{T}_Y=(\widehat{a},\widehat{b},\widehat{f}_1,\widehat{g}_1,\widehat{f}_2,\widehat{g}_2)^t$ then a straightforward computation from lemma 5.4 leads to

$$\begin{aligned}S_u^{(0)}(\widehat{T}_Y)&=-\varepsilon\frac{isgn(k)}{1+\widetilde{\rho}|k|}\eta_\varepsilon^*(\widehat{T}_Y)+\widetilde{S}_u(\widehat{T}_Y)(k),\\S_Y^{(0)}(\widehat{T}_Y)&=-\frac{\varepsilon}{1+\widetilde{\rho}|k|}\eta_\varepsilon^*(\widehat{T}_Y)\chi_k+\varepsilon\widetilde{\Phi}(\widehat{f}_2,\widehat{g}_2)+\widetilde{\widetilde{\Phi}}(\widehat{f}_1,\widehat{g}_1)+\widetilde{S}_Y(\widehat{T}_Y)(k),\end{aligned}$$

where

$$\begin{aligned}|\eta_\varepsilon^*(\widehat{T}_Y)|&\leq c||\widehat{T}_Y||_{\mathbb{F}_\varepsilon},\\\widetilde{S}_u(\widehat{T}_Y)(k)&=O\left(\frac{\varepsilon}{1+|k|}||\widehat{T}_Y||_{\mathbb{F}_\varepsilon}\right),\\\widetilde{S}_Y(\widehat{T}_Y)(k)&=O\left(\frac{\varepsilon|k|}{1+|k|}||\widehat{T}_Y||_{\mathbb{F}_\varepsilon}\right),\end{aligned}$$

the $O(..)$ terms are now continuous in k, and for $f_2, g_2 \in C^0(0,1)$ the term $\widetilde{\Phi}(f_2, g_2)$ is independent of k, defined in lemma 5.4, and for $f_1, g_1 \in C^{\exp}_{0,\varepsilon}$ the term $\widetilde{\widetilde{\Phi}}(f_1, g_1)$ is continuous in k, and is defined by

$$\widetilde{\widetilde{\Phi}}(f_1,g_1)(k) = \Big(\widetilde{\widetilde{K}}_{10}(f_1,g_1)(k), 0, \widetilde{\widetilde{H}}_1(f_1,g_1)(k,\underline{y}),$$

$$\widetilde{\widetilde{K}}_1(f_1,g_1)(k,\underline{y}), \widetilde{\widetilde{H}}_2(f_1,g_1)(k,y), 0\Big)^t,$$

$$\begin{aligned}
\widetilde{\widetilde{H}}_1[f_1,g_1](k,\underline{y}) &= H_1[f_1,g_1](k,\underline{y}) + \frac{1}{1+\widetilde{\rho}|k|}\int_{-\infty}^0 g_1(\tau)e^{|k|\underline{y}}(1+\widetilde{\rho}|k|e^{|k|\tau})d\tau + \\
&- \frac{i\widetilde{\rho}k}{1+\widetilde{\rho}|k|}\int_{-\infty}^0 f_1(\tau)e^{|k|(\tau+\underline{y})}d\tau, \\
\widetilde{\widetilde{K}}_1[f_1,g_1](k,\underline{y}) &= -\frac{1}{2}\int_{-\infty}^0 f_1(\tau)\left[\frac{1-\widetilde{\rho}|k|}{1+\widetilde{\rho}|k|}e^{|k|(\tau+\underline{y})} + sgn(y-\tau)e^{-|k||\tau-\underline{y}|}\right]d\tau + \\
&- \frac{isgn(k)}{2}\int_{-\infty}^0 g_1(\tau)\left[\frac{1-\widetilde{\rho}|k|}{1+\widetilde{\rho}|k|}e^{|k|(\tau+\underline{y})} + e^{-|k||\tau-\underline{y}|} - \frac{2e^{|k|\underline{y}}}{1+\widetilde{\rho}|k|}\right]d\tau, \\
\widetilde{\widetilde{K}}_{10}[f_1,g_1](k) &= -\int_{-\infty}^0 \left[f_1(\tau)\frac{e^{|k|\tau}}{1+\widetilde{\rho}|k|} + isgn(k)g_1(\tau)\left(\frac{e^{|k|\tau}-1}{1+\widetilde{\rho}|k|}\right)\right]d\tau, \\
\widetilde{\widetilde{H}}_2[f_1,g_1](k,y) &= \frac{ik(y-\widetilde{\rho})}{1+\widetilde{\rho}|k|}\int_{-\infty}^0 [f_1(\tau)+i(sgnk)g_1(\tau)]e^{|k|\tau}d\tau + \frac{1}{1+\widetilde{\rho}|k|}\int_{-\infty}^0 g_1(\tau)d\tau.
\end{aligned}$$

We notice that the terms with $\eta^*_\varepsilon(\widehat{T}_Y)$ only occur for $T^{(2)}_Y$ since $T^{(1)}_Y \in \ker \eta^*_\varepsilon$. Now, $T^{(2)}_Y \in B^\alpha_3(\underline{\pi}_\varepsilon \mathbb{F}_\varepsilon)$ is antireversible, hence $\eta^*_\varepsilon(T_Y)$ is odd and $\int_{-\infty}^{x} \eta^*_\varepsilon(T_Y)ds \in B^\alpha_2(\mathbb{R})$ is even. We can then apply lemma 8.3 to $\int_{-\infty}^{x} \eta^*_\varepsilon(T_Y)ds \in B^{1,\alpha}_2(\mathbb{R})$ which shows that

$$\mathcal{F}^{-1}\left(\varphi_0\eta^*_\varepsilon(\widehat{T}_Y)\chi_k\right) \in B^\alpha_{\mathbb{D},w},$$

$$\mathcal{F}^{-1}\left(isgn(k)\varphi_0\eta^*_\varepsilon(\widehat{T}_Y)\right) \in B^\alpha_2(\mathbb{R}),$$

with

$$||\mathcal{F}^{-1}\left(\varphi_0\eta^*_\varepsilon(\widehat{T}_Y)\chi_k\right)||^\alpha_{\mathbb{D},w} \le c||T^{(2)}_Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,3},$$

$$||\mathcal{F}^{-1}\left(isgn(k)\varphi_0\eta^*_\varepsilon(\widehat{T}_Y)\right)||^\alpha_{\mathbb{R},2} \le c||T^{(2)}_Y||^\alpha_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,3}.$$

Now, we observe that the function

$$K(x) = \mathcal{F}^{-1}\left(\frac{\varepsilon\varphi_0(\varepsilon k)}{1+\widetilde{\rho}|k|}\right)$$

satisfies the conditions of lemma 12.1(a) (we denoted here $\varphi_0^2(\varepsilon k)$ the function previously noted $\varphi_0(\varepsilon k)$ which does not change its properties). It results by lemma 12.1(a) that

$$\mathcal{F}^{-1}\left(\varepsilon\frac{isgn(k)}{1+\widetilde{\rho}|k|}\varphi_0(\varepsilon k)\eta^*_\varepsilon(\widehat{T}_Y)\right) \in B^\alpha_2(\mathbb{R})$$

with

$$\left\| \mathcal{F}^{-1}\left(\varepsilon\frac{isgn(k)}{1+\widetilde{\rho}|k|}\varphi_0(\varepsilon k)\eta^*_\varepsilon(\widehat{T}_Y)\right)\right\|^{\alpha}_{\mathbb{R},2} \leq c\varepsilon||T^{(2)}_Y||^{\alpha}_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,3}.$$

Moreover, we have $\frac{|k|}{1+\widetilde{\rho}|k|} = \frac{1}{\widetilde{\rho}} - \frac{1}{\widetilde{\rho}(1+\widetilde{\rho}|k|)}$ so we can apply lemma 8.2 and lemma 12.1(a) to show that

$$\mathcal{F}^{-1}\left(\varepsilon\frac{|k|}{1+\widetilde{\rho}|k|}\varphi_0(\varepsilon k)\eta^*_\varepsilon(\widehat{T}_Y)\right) \in B^{\alpha}_2(\mathbb{R})$$

and finally

$$\left\| \mathcal{F}^{-1}\left(\varepsilon\frac{isgn(k)}{1+\widetilde{\rho}|k|}\varphi_0(\varepsilon k)\eta^*_\varepsilon(\widehat{T}_Y)\right)\right\|^{1,\alpha}_{\mathbb{R},2} \leq c\varepsilon||T^{(2)}_Y||^{\alpha}_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,3}. \tag{14.1}$$

It now results directly from lemma 12.1(b) for components $(\beta_{10}, \alpha_2, \beta_2)$ and for the components α_1 and β_1 restricted to $\underline{y} \in (-1, 0)$, that these components of $\mathcal{F}^{-1}\left(\frac{\varepsilon}{1+\widetilde{\rho}|k|}\varphi_0(\varepsilon k)\eta^*_\varepsilon(\widehat{T}_Y)\chi_k\right)$ may be estimated in B^{α}_2 of the corresponding spaces. It remains to study the α_1 and β_1 components for $\underline{y} \in (-\infty, -1)$ (i.e. the decay rate in $\underline{y}$) for finally showing that

$$\mathcal{F}^{-1}\left(\frac{\varepsilon}{1+\widetilde{\rho}|k|}\varphi_0(\varepsilon k)\eta^*_\varepsilon(\widehat{T}_Y)\chi_k\right) \in B^{\alpha}_{\mathbb{D},w},$$

with

$$\left\| \mathcal{F}^{-1}\left(\frac{\varepsilon}{1+\widetilde{\rho}|k|}\varphi_0(\varepsilon k)\eta^*_\varepsilon(\widehat{T}_Y)\chi_k\right)\right\|^{\alpha}_{\mathbb{D},w} \leq c\varepsilon||T^{(2)}_Y||^{\alpha}_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,3}. \tag{14.2}$$

This last part is proved as soon as we show that

$$\left\| \mathcal{F}^{-1}\left(\frac{|k|}{1+\widetilde{\rho}|k|}\varphi_0(\varepsilon k)e^{|k|y}\widehat{f}(k)\right)\right\|_{B^-_w} \leq c||f||_{B^{\alpha}_2(\mathbb{R})}$$

for any $f \in B^{\alpha}_2(\mathbb{R})$, where it is only needed to show the estimate for $|y| > 1$. We may introduce

$$\begin{aligned}\widetilde{I}(x,y) &= \mathcal{F}^{-1}\left(\frac{|k|}{1+\widetilde{\rho}|k|}\varphi_0(\varepsilon k)e^{|k|y}\right)\\ &= \frac{y}{\pi(x^2+y^2)} - \mathrm{Re}\left\{\frac{1}{\pi(y+ix)}\int_0^{\delta/\varepsilon}\left(\frac{k}{1+\widetilde{\rho}k}\varphi_0(\varepsilon k)\right)'_k e^{k(y+ix)}dk\right\}\end{aligned}$$

where the first part is treated at Corollary 12.3 for α_1. The second part may be split in two, where the part

$$\mathrm{Re}\,\frac{1}{\pi(y+ix)}\int_{\delta/2\varepsilon}^{\delta/\varepsilon}\frac{\varepsilon k}{1+\widetilde{\rho}k}\varphi_0'(\varepsilon k)e^{k(y+ix)}dk$$

may be treated exactly as $I(x, y)$ at step 1 of the proof of lemma 13.1. It then still remains to estimate for $|y| > 1$ the convolution by the kernel

$$\begin{aligned}&\frac{1}{\pi(y+ix)}\int_0^{\delta/\varepsilon}\frac{1}{(1+\widetilde{\rho}k)^2}\varphi_0(\varepsilon k)e^{k(y+ix)}dk\\ &= -\frac{1}{\pi(y+ix)^2} - \frac{1}{\pi(y+ix)^2}\int_0^{\delta/\varepsilon}\left(\frac{\varphi_0(\varepsilon k)}{(1+\widetilde{\rho}k)^2}\right)'_k e^{k(y+ix)}dk.\end{aligned}$$

This kernel is no longer singular (since $|y|>1$), moreover it is regular in (x,y) and $O[1/(x^2+y^2)]$ near infinity. Then, the identity (12.1) gives

$$\int_{\mathbb{R}} \frac{ds}{(1+s^2)[(x-s)^2+y^2]} \leq \frac{c}{1+x^2+y^2} \quad \text{for } |y|>1,$$

and we obtain the required estimate (14.2).

For the term $\varepsilon\varphi_0(\varepsilon k)\widetilde{\Phi}(\widehat{f}_2,\widehat{g}_2)$ we have easily by lemma 8.2 (see the form of $\widetilde{\Phi}(\widehat{f}_2,\widehat{g}_2)$ in lemma 5.4)

$$||\mathcal{F}^{-1}\left(\varepsilon\varphi_0(\varepsilon k)\widetilde{\Phi}(\widehat{f}_2,\widehat{g}_2)\right)||^{\alpha}_{\mathbb{D},w} \leq c\varepsilon||T_Y||^{\alpha}_{\underline{\pi}_\varepsilon\mathbb{F}_\varepsilon,2}. \tag{14.3}$$

The three estimates (14.1), (14.2) and (14.3) are parts of (8.11) and (8.15).

Now looking at formulas of section 5.(d) giving $\alpha_1,\beta_1,\alpha_2,\beta_2$ and $\widetilde{\widetilde{\Phi}}(\widehat{f}_1,\widehat{g}_1)$, $\widetilde{S}_u(\widehat{T}_Y)$, $\widetilde{S}_Y(\widehat{T}_Y)$ defined above, we claim that it remains to study sums of typical terms as

$$\begin{aligned}
K_{1,0}(k;\varepsilon)\widehat{a} &= \varepsilon|k|(1+|k|)^{-1}A(\varepsilon k)e^{|k|\underline{y}}\widehat{a},\\
K_{1,10}(k,\underline{y};\varepsilon)\widehat{f}_1 &= \int_{-\infty}^{0}\widehat{f}_1(k,\tau)e^{-|k||\underline{y}-\tau|}d\tau,\\
K_{1,11}(k,\underline{y};\varepsilon)\widehat{f}_1 &= \int_{-\infty}^{0} sgn(\underline{y}-\tau)\widehat{f}_1(k,\tau)e^{-|k||\underline{y}-\tau|}d\tau,\\
K_{1,12}(k,\underline{y};\varepsilon)\widehat{f}_1 &= \int_{-\infty}^{0}\widehat{f}_1(k,\tau)e^{-|k||\underline{y}+\tau|}d\tau,
\end{aligned}$$

$$\begin{aligned}
K_{1,13}(k,\underline{y};\varepsilon)\widehat{f}_1 &= \int_{-\infty}^{0} sgn(k)\widehat{f}_1(k,\tau)(e^{-|k||\underline{y}-\tau|}-e^{-|k||\underline{y}+\tau|})d\tau,\\
K_{1,14}(k,\underline{y};\varepsilon)\widehat{f}_1 &= (1+|k|)^{-1}\int_{-\infty}^{0} sgn(k)\widehat{f}_1(k,\tau)(e^{|k|\underline{y}}-e^{|k|(\underline{y}+\tau)})d\tau,\\
K_{1,15}(k,\underline{y};\varepsilon)\widehat{f}_1 &= |k|(1+|k|)^{-1}A(\varepsilon k)\int_{-\infty}^{0}\widehat{f}_1(k,\tau)e^{-|k||\underline{y}+\tau|}d\tau,\\
K_{1,2}(k,\underline{y};\varepsilon)\widehat{f}_2 &= \varepsilon|k|(1+|k|)^{-1}e^{|k|\underline{y}}\int_{0}^{1}\widehat{f}_2(k,\tau)B(\varepsilon k,\varepsilon k\tau)d\tau,
\end{aligned}$$

$$\begin{aligned}
K_{2,0}(\varepsilon k,y;\varepsilon)\widehat{a} &= \varepsilon|k|(1+|k|)^{-1}C_0(\varepsilon k,\varepsilon ky)\widehat{a},\\
K_{2,1}(k,y;\varepsilon)\widehat{f}_1 &= |k|(1+|k|)^{-1}C_1(\varepsilon k,y)\int_{-\infty}^{0}\widehat{f}_1(k,\tau)e^{|k|\tau}d\tau,\\
K_{2,2}(\varepsilon k,y;\varepsilon)\widehat{f}_2 &= \varepsilon|k|(1+|k|)^{-1}\int_{0}^{1}\widehat{f}_2(k,\tau)D(\varepsilon k,y,\tau;\varepsilon)d\tau,
\end{aligned}$$

where A, B, C_0, C_1, D are uniformly bounded, as well as their derivatives in their arguments, for $\varepsilon(1+|k|)\leq\delta$, analytic for $k\neq 0$, continuous for $k=0$.

Let us denote by

$$K_{i,j}^{(0)}\widehat{f} = \varphi_0 K_{i,j}\widehat{f}$$
$$\mathcal{L}_{i,j}^{(0)} f = \mathcal{F}^{-1}[K_{i,j}^{(0)}\widehat{f}] = \mathcal{K}_{i,j}^{(0)} * f$$

where $*$ means convolution in x and $\mathcal{K}_{i,j}^{(0)} = \mathcal{F}^{-1}K_{i,j}^{(0)}$, then we have the following lemma

Lemma 14.1. *For any given $a \in B_2^\alpha(\mathbb{R})$, $f_1 \in B_2^\alpha(C_\varepsilon^{0,\exp})$, and $f_2 \in B_w^+ = B_2^\alpha[C^0(0,1)]$, the following holds*

(i) $\mathcal{L}_{1,0}^{(0)}a \in B_w^-$ and $||\mathcal{L}_{1,0}^{(0)}a||_{B_w^-} \leq c\varepsilon||a||_{\mathbb{R},2}^\alpha$,

(ii) $\mathcal{L}_{1,j}^{(0)}f_1 \in B_w^-$ and $||\mathcal{L}_{1,j}^{(0)}f_1||_{B_w^-} \leq c\varepsilon||f_1||_{B_2^\alpha(C_\varepsilon^{0,\exp})}$, $j = 10, 11, 12, 13, 14, 15$

(iii) $\mathcal{L}_{1,2}^{(0)}f_2 \in B_w^-$ and $||\mathcal{L}_{1,2}^{(0)}f_2||_{B_w^-} \leq c\varepsilon||f_2||_{B_w^+}$,

(iv) $\mathcal{L}_{2,0}^{(0)}a \in B_w^{1,+}$ and $||\mathcal{L}_{2,0}^{(0)}a||_{B_w^+} + ||\frac{\partial}{\partial y}\mathcal{L}_{2,0}^{(0)}a||_{B_w^+} \leq c\varepsilon||a||_{\mathbb{R},2}^\alpha$,

(v) $\mathcal{L}_{2,1}^{(0)}f_1 \in B_w^{1,+}$ and $||\mathcal{L}_{2,1}^{(0)}f_1||_{B_w^+} + ||\frac{\partial}{\partial y}\mathcal{L}_{2,1}^{(0)}f_1||_{B_w^+} \leq c\varepsilon||f_1||_{B_2^\alpha(C_\varepsilon^{0,\exp})}$,

(vi) $\mathcal{L}_{2,2}^{(0)}f_2 \in B_w^{1,+}$ and $||\mathcal{L}_{2,2}^{(0)}f_2||_{B_w^+} + ||\frac{\partial}{\partial y}\mathcal{L}_{2,2}^{(0)}f_2||_{B_w^+} \leq c\varepsilon||f_2||_{B_w^+}$.

Proof of lemma 14.1. To prove this lemma we only consider $\mathcal{L}_{2,0}^{(0)}$ and $\mathcal{L}_{1,10}^{(0)}$, the rest of the proof being similar. For $\mathcal{L}_{2,0}^{(0)}$, $\mathcal{L}_{2,1}^{(0)}$, $\mathcal{L}_{2,2}^{(0)}$ we use lemma 12.1(b). Let us take for example the case when $K_{2,0}(\varepsilon k, y; \varepsilon)$ is even in k, then

$$\mathcal{K}_{2,0}^{(0)}(x, y; \varepsilon) = \frac{1}{\pi}\int_0^{\delta/\varepsilon} \varphi_0(\varepsilon k)K_{2,0}(\varepsilon k, y; \varepsilon)\cos(kx)dk.$$

It is straightforward to show (by simple integration by parts) that

$$|\mathcal{K}_{2,0}^{(0)}(x, y; \varepsilon)| \leq c\varepsilon \min\left(\frac{1}{|x|}, \frac{1}{|x|^2}\right),$$
$$\left|\frac{\partial\mathcal{K}_{2,0}^{(0)}(x, y; \varepsilon)}{\partial x}\right| \leq \frac{c\varepsilon}{|x|^2},$$
$$\int_{-1}^{1} \mathcal{K}_{2,0}^{(0)}(x, y; \varepsilon)dx = O(\varepsilon)$$

uniformly in $y \in [0, 1]$. Similarly, the derivative with respect to y, $\frac{\partial\mathcal{K}_{2,0}^{(0)}(x,y;\varepsilon)}{\partial y}$ satisfies the same estimates. Hence by lemma 12.1(b), part (iv) of lemma 14.1 is proved. Exactly the same argument applies for parts (vi), and (v) just noticing for the last case that the factor ε comes from the fact that the integral in τ for $K_{2,1}$ is $O(\varepsilon)$ because $f_1(x, \cdot) \in C_\varepsilon^{0,\exp}$ (space $\mathbb{E}$ in lemma 12.1(b)).

Let consider now the operator $\mathcal{L}_{1,10}^{(0)}$, which can be written as

$$\left(\mathcal{L}_{1,10}^{(0)}f_1\right)(x, y) = \int_{-\infty}^{0}\int_{\mathbb{R}} \widetilde{\mathcal{K}}_{1,10}^{(0)}(s, y - \tau)f_1(x - s, \tau)dsd\tau$$

where $f_1 \in B_2^\alpha(C_\varepsilon^{0,\exp})$ and with

$$\widetilde{\mathcal{K}}_{1,10}^{(0)}(x, y) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi_0(\varepsilon k)e^{ikx - |k||y|}dk.$$

After one integration by parts we obtain

$$\widetilde{\mathcal{K}}_{1,10}^{(0)}(x,y) = \frac{|y|}{\pi(x^2+y^2)} + \widetilde{\widetilde{\mathcal{K}}}_{1,10}^{(0)}(x,y)$$

with

$$\widetilde{\widetilde{\mathcal{K}}}_{1,10}^{(0)}(x,y) = \frac{\varepsilon}{2\pi(|y|-ix)}\int_{\delta/2\varepsilon}^{\delta/\varepsilon}\varphi_0'(\varepsilon k)e^{k(ix-|y|)}dk + \\ + \frac{\varepsilon}{2\pi(|y|+ix)}\int_{\delta/2\varepsilon}^{\delta/\varepsilon}\varphi_0'(\varepsilon k)e^{-k(ix+|y|)}dk.$$

Now the first part with $\frac{|y|}{\pi(x^2+y^2)}$ may be treated as we did for α_1 in Corollary 12.3 in Appendix A. The estimate in B_w^- is straightforward, once we replace $u_0'(s)$ by $f_1(s,\tau)$ and we notice that

$$||f_1(\cdot,\tau)||_{\mathbb{R},2}^{\alpha} \le e^{\lambda\tau/2\varepsilon}||f_1||_{B_2^\alpha(C_\varepsilon^{0,\exp})},$$

and

$$\int_{-\infty}^{0}\left(\frac{1+x^2+y^2}{1+|y|}\right)\left(\frac{1+|y-\tau|}{1+x^2+|y-\tau|^2}\right)e^{\lambda\tau/2\varepsilon}d\tau \le c\varepsilon \tag{14.4}$$

with c independent of $(x,y)\in\mathbb{R}\times\mathbb{R}^-$. The inequality (14.4) can be obtained simply in splitting the integral into the part where $|y-\tau|>|y|/2$, and the complementary part, noticing that

$$\int_{-\infty}^{0}(1+|\tau|)e^{\lambda\tau/2\varepsilon}d\tau = O(\varepsilon),$$
$$(1+y^2)e^{\lambda y/\varepsilon} = O(1) \text{ uniformly in } y.$$

The second part $\widetilde{\widetilde{\mathcal{K}}}_{1,10}^{(0)}(x,y)$ of the kernel is of the same form as $I(x,y)$ at step 1 of the proof of lemma 13.1, hence the estimate (ii) of lemma 14.1 holds for $\mathcal{L}_{1,10}^{(0)}f_1$. This ends the proof of lemma 14.1, hence estimate (8.15) is proved, as well as the part in B_2^α of (8.11).

It remains to estimate in $B_2^{1,\alpha}$ the function $\mathcal{F}^{-1}(\varphi_0\widetilde{S}_u(\widehat{T}_Y))$ (in particular the $x-$derivative). The principal part of this term is such that

$$\widetilde{S}_u(\widehat{T}_Y)(k) = -\frac{1}{1+\widetilde{\rho}|k|}\int_{-\infty}^{0}[\widehat{f}_1(k,\tau)e^{|k|\tau} + isgn(k)(e^{|k|\tau}-1)\widehat{g}_1(k,\tau)d\tau + \\ + \frac{\varepsilon}{1+\widetilde{\rho}|k|}\int_0^1[\tau(1-\varepsilon)-\rho]\widehat{f}_2(k,\tau)d\tau$$

where f_1 and $g_1\in B_2^\alpha(C_\varepsilon^{0,\exp})$, $f_2\in B_2^\alpha[C^0(0,1)]$ are components of T_Y. It can be checked that the kernels

$$\mathcal{F}^{-1}\left(\frac{\varphi_0}{1+\widetilde{\rho}|k|}\right),\quad \mathcal{F}^{-1}\left(\frac{ik\varphi_0}{1+\widetilde{\rho}|k|}\right)$$

satisfy the conditions of lemma 12.1, with constants of order $O(1)$, hence, by lemma 12.1(b), the part of $\mathcal{F}^{-1}(\varphi_0 \widetilde{S}_u(\widehat{T}_Y))$ depending on f_2 has the good estimate in $B_2^{1,\alpha}(\mathbb{R})$ bounded by $c\varepsilon ||f_2||_{B_w^+}$. Now we observe that

$$\widetilde{J}(x,\tau) = \mathcal{F}^{-1}\left(\frac{\varphi_0 e^{|k|\tau}}{1+\widetilde{\rho}|k|}\right)$$
$$= \frac{|\tau|}{\pi(x^2+\tau^2)} + \mathrm{Re}\left\{\frac{1}{\pi(\tau+ix)}\int_0^{\delta/\varepsilon} e^{k(\tau+ix)}\left(\frac{\varphi_0(\varepsilon k)}{1+\widetilde{\rho}k}\right)'_k dk\right\},$$

and the first part leads to an estimate in $B_2^\alpha(\mathbb{R})$ of the integral over $(-\infty, 0)$ of its convolution product with f_1, thanks to the identity (12.1), and due to the fact that

$$\int_{-\infty}^0 (1+|\tau|)e^{\lambda\tau/2\varepsilon} d\tau \leq c\varepsilon \text{ } (c \text{ independent of } \varepsilon).$$

The second part of the kernel satisfies all assumptions of lemma 12.1, with constants of order 1, uniformly in $\tau \in \mathbb{R}^-$, hence thanks to lemma 12.1(b), this leads to

$$\left\|\int_{-\infty}^0 \widetilde{J}(\cdot,\tau) * f_1(\cdot,\tau)d\tau\right\|_{\mathbb{R},2}^\alpha \leq c\varepsilon ||f_1||_{B_2^\alpha(C_\varepsilon^{\varepsilon,\exp})}.$$

Now we also have

$$\frac{\partial}{\partial x}\widetilde{J}(x,\tau) = \mathrm{Re}\left(\frac{1}{\pi}\int_0^{\delta/\varepsilon} \frac{ik\varphi_0(\varepsilon k)}{1+\widetilde{\rho}k} e^{k(\tau+ix)} dk\right)$$

which satisfies all assumptions of lemma 12.1, with constants of order 1, uniformly in $\tau \in \mathbb{R}^-$, so in using lemma 12.1(b), the part of $\mathcal{F}^{-1}(\varphi_0 \widetilde{S}_u(\widehat{T}_Y))$ depending on f_1 has the good estimate in $B_2^{1,\alpha}(\mathbb{R})$ bounded by $c\varepsilon ||f_1||_{B_2^\alpha(C_\varepsilon^{\varepsilon,\exp})}$. For the part depending on g_1 the method is the same. Higher order terms have the same form, multiplied by extra $\varepsilon k\psi(k)$ where $\psi(k)$ is smooth and bounded for $\varepsilon(1+|k|) \leq \delta$, hence the same method applies, and (8.11) is proved.

References

Amick, C. 1994 On the theory of internal waves of permanent form in fluids of great depth. Trans.Amer.Math.Soc. **346**, 399-419.

Amick, C. & Kirchgässner, K. 1989 A theory of solitary water-waves in the presence of surface tension. Arch. Rat. Mech. Anal. **105**, 1-49.

Amick, C. & Toland, J. 1991 Uniqueness and related analytic properties for the Benjamin-Ono equation - a nonlinear Neumann problem in the plane. Acta Math. **167**, 107-126.

Benjamin, T.B. 1967 Internal waves of permanent form in fluids of great depth. J.Fluid Mech. **29**, 559-592.

Davis, R.E. & Acrivos, A. 1967 Solitary internal waves in deep water. J.Fluid Mech. **29**, 593-607.

Dias, F. & Iooss, G. 2001 Water-waves as a spatial dynamical system. *Handbook of Mathematical Fluid Dynamics. To appear.*

Iooss, G. 1999 Gravity and Capillary-gravity periodic travelling waves for two superposed fluid layers, one being of infinite depth. J.Math. Fluid Mech. **1**, 24-61.

Iooss, G. & Kirchgässner, K. 1990 Bifurcation d'ondes solitaires en présence d'une faible tension superficielle. C.R.Acad. Sci. Paris, **311**, I, 265-268.

Iooss, G. & Kirrmann, P. 1996 Capillary gravity waves on the free surface of an inviscid fluid of infinite depth. Existence of solitary waves. *Arch. Rat. Mech. Anal.* **136**, 1-19.

Iooss, G. & Pérouème, M.C. 1993 Perturbed homoclinic solutions in 1:1 resonance vector fields. J. Diff. Equ. **102**, 1, 62-88.

Kirchgässner, K. 1988 Nonlinearly resonant surface waves and homoclinic bifurcation. *Adv. Appl. Mech.* **26**, 135-181.

Levi-Civita, T. 1925 Détermination rigoureuse des ondes permanentes d'ampleur finie. *Math. Annalen*, **93**, 264-314.

Lombardi, E. 1997 Orbits homoclinic to exponentially small periodic orbits for a class of reversible systems. Application to water waves. *Arch. Rat. Mech. Anal.* **137**, 227-304 .

Lombardi, E. 2000 *Oscillatory integrals and phenomena beyond all algebraic orders, with applications to homoclinic orbits in reversible systems.* Lecture Notes in Maths vol. **1741**. Springer-Verlag.

Lombardi, E. & Iooss, G. 2001 Gravity solitary waves with polynomial decay to exponentially small ripples at infinity. Preprint INLN 2001/40. *to appear in Annales de l'IHP. Analyse non linéaire.*

Mielke, A. 1988 Reduction of quasilinear elliptic equation in cylindrical domains with applications. Math. Meth. Appl. Sci. **10**, 51-66.

Ono, H. 1975 Algebraic solitary waves in stratified fluids. *J.Phys. Soc. Japan*, **39**, 1082-1091.

Părău, E. & Dias, F. 2001 Interfacial periodic waves of permanent form with free-surface boundary conditions. *J.Fluid Mech*, **437**, 325-336.

Sun, S.M. & Shen, M.C. 1993 Exponentially small estimate for the amplitude of capillary ripples of a generalized solitary wave. *J. Math. Anal. Appl.* **172**, 533-566.

Sun, S.M. 1997 Existence of solitary internal waves in a two-layer fluid of infinite depth. *Nonlinear Analysis* **30**, 8, 5481-5490.

Sun, S.M. 1999 Non existence of truly solitary waves in water with small surface tension. *Proc. Roy. London A*, **455**, 2191-2228.