# The Couette Taylor Problem in the small gap approximation.

Yves DEMAY Gérard IOOSS

Patrice LAURE

*Institut Non Linéaire de Nice, UMR CNRS 129, Université de Nice*

*Parc Valrose, 06034 NICE Cedex, France*

## 1 INTRODUCTION

We consider the classical Couette-Taylor problem in the limiting case when the radii ratio ($\eta = \frac{R_1}{R_2}$) is very close to 1 and the critical Reynolds number is very large. It is well know that the critical modes which destabilise the Couette flow are either stationary axisymmetric modes or oscillatory non axisymmetric modes with an integer azimutal wavenumber. Langford *and al.* [1] observe when $\eta$ tends to 1 that the most critical oscillatory modes come altogether (Figure 4 in [2]). In this paper, our main motivation is to analyse the transition between axisymmetric and non axisymmetric modes at this limit and to determine the selected azimuthal wavelength. By this way, we give some informations on the occurrence of oscillatory motion when the two cylinders move in the opposite direction. The governing equations considered here are deduced from the Navier-Stokes equations by a suitable choice of scale taken at the limit $\eta = 1$. The new dimensionless parameters are the inner and outer Taylor numbers $\mathcal{T}_i$, and, the basic flow is the planar Couette flow. The linear analysis points out the possibility of oscillatory instabilities for negative value of $\mathcal{T}_2$. The bifurcated structures are studied by means of Ginzburg-Landau type of equations. Thus, we can take into account the interaction between critical modes with spatial wave numbers close to the critical one. In addition, we give precisely the coefficients occuring these equations. At the critical point, these coefficients can be obtained in the same way as those of classical amplitude equations ([3]).

## 2 GOUVERNING EQUATION

Let us denote respectively by $R_1$, $R_2$ and $\Omega_1$, $\Omega_2$ the radii and rotation rates of the inner and outer cylinders. We consider the case when the inner and eventually the outer Reynolds numbers $\mathcal{R}_j = \frac{R_j\Omega_j d}{\nu}$, $j = 1$ and 2, ($\nu$ being the kinematic viscosity) are very large when $\eta$ is close to 1. We use the same scales as Tabeling [4] and the parameters are $\mathcal{T}_j$, $j = 1$ and 2, defined by

$$\mathcal{T}_j = \frac{R_j\Omega_j d}{\nu}\sqrt{2(1-\eta)}, \quad j = 1, 2 \tag{1}$$

# The Couette Taylor Problem in the small gap approximation.

Yves DEMAY Gérard IOOSS

Patrice LAURE

*Institut Non Linéaire de Nice, UMR CNRS 129, Université de Nice*

*Parc Valrose, 06034 NICE Cedex, France*

## 1 INTRODUCTION

We consider the classical Couette-Taylor problem in the limiting case when the radii ratio ($\eta = \frac{R_1}{R_2}$) is very close to 1 and the critical Reynolds number is very large. It is well know that the critical modes which destabilise the Couette flow are either stationary axisymmetric modes or oscillatory non axisymmetric modes with an integer azimutal wavenumber. Langford *and al.* [1] observe when $\eta$ tends to 1 that the most critical oscillatory modes come altogether (Figure 4 in [2]). In this paper, our main motivation is to analyse the transition between axisymmetric and non axisymmetric modes at this limit and to determine the selected azimuthal wavelength. By this way, we give some informations on the occurrence of oscillatory motion when the two cylinders move in the opposite direction. The governing equations considered here are deduced from the Navier-Stokes equations by a suitable choice of scale taken at the limit $\eta = 1$. The new dimensionless parameters are the inner and outer Taylor numbers $\mathcal{T}_i$, and, the basic flow is the planar Couette flow. The linear analysis points out the possibility of oscillatory instabilities for negative value of $\mathcal{T}_2$. The bifurcated structures are studied by means of Ginzburg-Landau type of equations. Thus, we can take into account the interaction between critical modes with spatial wave numbers close to the critical one. In addition, we give precisely the coefficients occuring these equations. At the critical point, these coefficients can be obtained in the same way as those of classical amplitude equations ([3]).

## 2 GOUVERNING EQUATION

Let us denote respectively by $R_1$, $R_2$ and $\Omega_1$, $\Omega_2$ the radii and rotation rates of the inner and outer cylinders. We consider the case when the inner and eventually the outer Reynolds numbers $\mathcal{R}_j = \frac{R_j \Omega_j d}{\nu}$, $j = 1$ and 2, ($\nu$ being the kinematic viscosity) are very large when $\eta$ is close to 1. We use the same scales as Tabeling [4] and the parameters are $\mathcal{T}_j$, $j = 1$ and 2, defined by

$$\mathcal{T}_j = \frac{R_j \Omega_j d}{\nu} \sqrt{2(1-\eta)}, \quad j = 1,2 \tag{1}$$

where $d = R_2 - R_1$ and $\mathcal{T}_1$ is positive while $\mathcal{T}_2$ may be negative.

More precisely, we choose a new set of dimensionless variables defined by the relation

$$x = \frac{r}{d} - \frac{R_1 + R_2}{2d} \ , \ y = \theta\sqrt{\frac{2R_1}{d}} \ , \ z = \frac{Z}{d} \tag{2}$$

where r is the distance from the axis of cylinders, $\theta$ the azimuthal coordinate and Z the axial coordinate. In addition, the scaling factors for velocity are taken in a such way that the incompressibility condition is recovered in the new cartesian coordinates :

$$v_x^* = v_z^* = \frac{\nu}{d} \ , \ v_y^* = \frac{\nu}{d}\sqrt{\frac{R_1}{2d}}. \tag{3}$$

After suppression of terms of order $O(1-\eta)$ in the Navier-Stokes equations written in cylindrical coordinates, we have a solution which corresponds to the Couette solution given by

$$V_0 = (0, v_0(x), 0), \text{ with } v_0(x) = (\mathcal{T}_2 - \mathcal{T}_1)x + \frac{\mathcal{T}_2 + \mathcal{T}_1}{2} \tag{4}$$

and the perturbation $U$ now satisfies ([5])

$$\left\{ \begin{array}{lcl} \frac{\partial U}{\partial t} & = & \Delta_{xz} U - v_0(x)\frac{\partial U}{\partial y} + \begin{pmatrix} v_0(x)u_y + \frac{u_y^2}{2} \\ -v_0'(x)u_x \\ 0 \end{pmatrix} - (U.\nabla)U - \begin{pmatrix} H_x(x,z) \\ H_y(y) \\ H_z(x,z) \end{pmatrix}, \\ \frac{\partial H_z}{\partial x} & = & \frac{\partial H_x}{\partial z} \ , \\ \nabla .\, U & = & 0 \ , \end{array} \right. \tag{5}$$

with the boundary conditions

$$U_{|x=\pm 1/2} = 0, \tag{6}$$

where $\Delta_{xz}$ is the usual Laplace operator in the two coordinates $x$ and $z$. System (5) differs from the system used by Tabeling [4] by the "pressure" term. In fact, it is shown at the limit $\eta \to 1$ that there is an additional variable $H_y$ on its second component which is only a function of $y$ (in [4] $H_y$ is 0, which is wrong for y dependent motion)

Let us finally observe that the system (5) is translation invariant in $y$ and $z$ directions, and is invariant under $z \mapsto -z$ symmetry.

## 3 LINEAR STABILITY ANALYSIS

We look for eigenmodes of the form

$$U = \hat{U}(x)e^{i(\alpha z + \beta y)} \tag{7}$$

belonging to some eigenvalue $\sigma$. Let us remark that the eigenvalues have the following properties due to the form of the linearized operator from equation (5) and the symmetry $z \mapsto -z$:

$$\sigma(-\alpha, \beta, \mathcal{T}_1, \mathcal{T}_2) = \sigma(\alpha, \beta, \mathcal{T}_1, \mathcal{T}_2) \text{ and } \sigma(\alpha, -\beta, \mathcal{T}_1, \mathcal{T}_2) = \overline{\sigma(\alpha, \beta, \mathcal{T}_1, \mathcal{T}_2)} \tag{8}$$

The most unstable mode is given by the eigenvalue $\sigma_0$ with largest real part, and the neutral stability curve in the $(\mathcal{T}_1, \mathcal{T}_2)$ plane, corresponds to the function

$$\mathcal{T}_{1c} = \min_{\alpha,\beta} \mathcal{T}_1(\alpha, \beta, \mathcal{T}_2) \tag{9}$$

with $\mathcal{T}_1$ solution of $\Re(\sigma_0(\alpha, \beta, \mathcal{T}_1, \mathcal{T}_2)) = 0$ for fixed values of $\mathcal{T}_2$,$\alpha$ and $\beta$.This gives in particular the critical wavelengths $\alpha_c$ (Table 1), $\beta_c$ (Figure 2) and the neutral curve $\mathcal{T}_{1c}(\mathcal{T}_2)$ is plotted in Figure 1. The critical curve is obtained for $\beta = 0$ and $\sigma_0 = 0$ up to a value $\mathcal{T}_2^*$ ($\sim -62.5$) of $\mathcal{T}_2$.

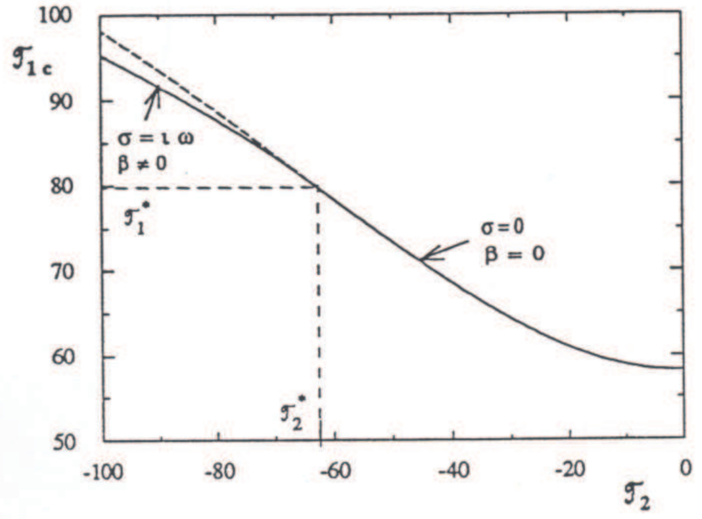


Figure 1: Neutral stability curve.

For $\mathcal{T}_2 < \mathcal{T}_2^*$, the minimum corresponds to positive value of $\beta$ and a complex eigenvalue. In order to explain this transition occuring at point $(\mathcal{T}_2^*, \mathcal{T}_1^*)$, we analyse more precisely the dependency of critical variables with respect to $\beta$. We solve numerically the equation $\Re(\sigma_0(\alpha, \beta, \mathcal{T}_1, \mathcal{T}_2)) = 0$ for various values of $\mathcal{T}_2$, and, the following functions

$$\mathcal{T}_1'(\beta, \mathcal{T}_2) = \min_{\alpha} \mathcal{T}_1(\alpha, \beta, \mathcal{T}_2), \tag{10}$$

are plotted in Figure 2. Using the properties (8), the Taylor expansion of $\sigma_0$ in the neighbourhood of $(\alpha_c,\ \beta = 0,\ \mathcal{T}_1 c)$ yields for $\mathcal{T}_2 > \mathcal{T}_2^*$

$$\sigma_0 = i a_1 \beta + a_2(\mathcal{T}_1 - \mathcal{T}_{1c}) - a_3(\alpha^2 - \alpha_c^2)^2 - a_4 \beta^2 - i a_5 \beta(\alpha^2 - \alpha_c^2) + \ \ldots \tag{11}$$

where the coefficients $a_i$ are real. In this way, we show that at leading order, $\mathcal{T}_1'$ verifies the relation

$$\mathcal{T}_1' = \mathcal{T}_{1c} + \frac{a_4}{a_2}\beta^2 + \ \ldots \tag{12}$$

Consequently, we can deduced from figure 2 that the coefficient $a_4$ changes its sign at $\mathcal{T}_2 = \mathcal{T}_2^*$, and for $\mathcal{T}_2 < \mathcal{T}_2^*$ the criticality is now reached for non zero azimuthal wavelength $\beta$. Moreover, due to coefficients $a_5$ and $a_1$, $\sigma_0$ becomes complex as soon as $\beta$ is non zero. Let us finally remark that for the monotonic transition the curves plotted in Figure 2 are very flat, so $\mathcal{T}_1'(\beta)$ is very close to $\mathcal{T}_{1c}$ for small values of $\beta$.

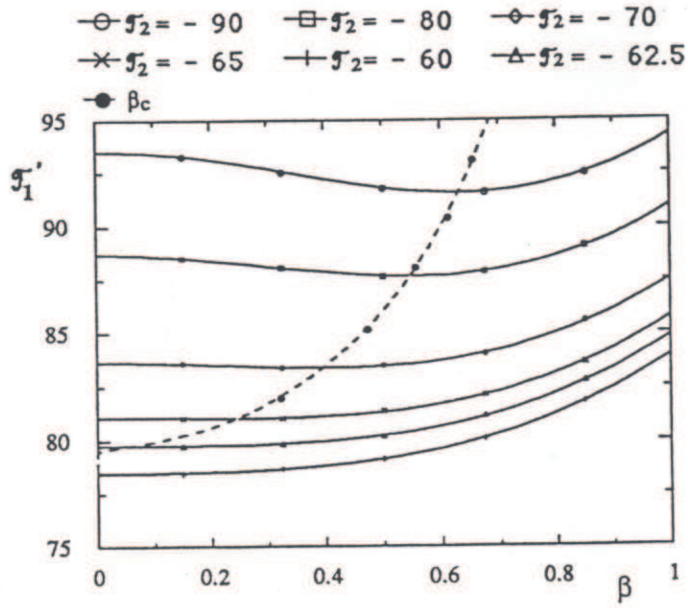


Figure 2: Graphs of $\mathcal{T}_1'(\alpha_c, \beta, \mathcal{T}_2)$ for fixed values of $\mathcal{T}_2$.

## 4 WEAKLY NON LINEAR ANALYSIS

We present the envelope (or Ginzburg-Landau) equations which describe the spatio-temporal evolution of the amplitude of the perturbation at criticality. The form of Taylor expansion of these equations is obtained by means of invariance properties, and in addition the numerical computation of main coefficients are been made. As only the sign and the weight of each coefficient with respect to others are important, we only present the relevant equations obtained after changes of variable and scales. In all the cases, they are given in a rigid frame moving in $y$ direction at velocity proportional to $a_1$ ($= i\partial\sigma_0/\partial\beta$). This allows us to eliminate the advection term ($\partial A/\partial y$) which means that any disturbance of the bifurcated solution propagates along the mean flow at constant speed. Let us note that this term comes from the anisotropy of the problem. The computations show that the bifurcation towards rolls are becoming subcritical at $\mathcal{T}_2 \sim -48.5$. Then, in the monotonic case we have two situations of high codimension which correspond to the annulation of third order nonlinear coefficient and the annulation of coefficient $a_4$ as noted before. We finally exhibit four different GL equations as $\mathcal{T}_2$ decreases.

### 4.1 Steady bifurcation.

Let first consider the case where the critical eigenvalue and the critical wavelength $\beta$ are null. The eigenmode is $U_0(x)\exp(\alpha_c z)$ and its complex conjugate $\bar{U}_0 \exp(-\alpha_c z)$. If we denote by $A$ the amplitude of the perturbation, we obtain when the first nonlinear term is negative ($\mathcal{T}_2 > -48.5$) the same type of equation as already presented by Tabeling [4]

$$\partial_t A = \mu A + \partial^2_{z^2} A + \partial^2_{y^2} A + ia\partial^2_{yz} A - A^2 \bar{A} \tag{13}$$

where the bifurcation parameter $\mu$ is proportional to $\mathcal{T}_1 - \mathcal{T}_{1c}$; values of coefficient $a$ are given in Table 1.

| $\mathcal{T}_2$ | $\alpha_c$ | $\mathcal{T}_{1c}$ | $a$ |
|---|---|---|---|
| 0. | 3.13 | 58.21 | .15 |
| -10. | 3.14 | 58.86 | .21 |
| -20. | 3.15 | 60.98 | .29 |
| -30. | 3.19 | 64.31 | .39 |
| -40. | 3.24 | 68.55 | .51 |
| -45. | 3.27 | 70.91 | .58 |

Table 1: Values of coefficient $a$ occuring in equation (13).

Equation (13) has non trivial solutions of the form :

$$A_0 = Q \exp i(qy + pz + \Omega t) \text{ with } Q^2 = \mu - p^2 - q^2 \text{ and } \Omega = -apq \tag{14}$$

where $p$ measures the dilatation (or the compression) of rolls and $q$ the modulation in azimuthal direction. Following [4] and [6], one obtains an equation describing the dynamics of the phase variable $\phi$ of a solution $A = A_0(qy + pz + \Omega t + \phi(z, y, t))$. A generalization of the Tabeling's approach yields ,

$$\frac{\partial \phi}{\partial t} + v_y \frac{\partial \phi}{\partial y} + v_z \frac{\partial \phi}{\partial z} = \alpha_{20} \frac{\partial^2 \phi}{\partial y^2} + \alpha_{11} \frac{\partial^2 \phi}{\partial y \partial z} + \alpha_{02} \frac{\partial^2 \phi}{\partial y^2} \tag{15}$$

with

$$v_y = ap \ , \ v_z = aq \ ,$$

$$\alpha_{20} = 1 - 2\frac{q^2}{Q^2} \ , \ \alpha_{11} = -\frac{4pq}{Q^2} \ , \ \alpha_{02} = 1 - \frac{2p^2}{Q^2} \ .$$

On the physical grounds, $v_y$ and $v_z$ mean that any disturbances along the y- (resp. z-) direction propagate in the z- (resp. y-) direction. In the moving frame at velocity $v = (0, v_y, v_z)$, the two conditions

$$\alpha_{20} > 0 \text{ and } 4\alpha_{20}\alpha_{02} - \alpha_{11}^2 > 0$$

giving the stability of the solution (14) yields

$$\mu \geq 3(q^2 + p^2) \tag{16}$$

which is the classical Eckhaus criterion.

The numerical computations show that for $\mathcal{T}_2 \sim -48.5$ the first nonlinear coefficient of the amplitude equation vanishes. Thus, it is necessary to expand the previous equation up to fifth order. As this coefficient is negative, we obtain

$$
\begin{aligned}
\partial_t A &= \mu A + \partial^2_{z^2} A + \partial^2_{y^2} A + ia\partial^2_{yz} A + \epsilon A^2 \bar{A} - A^3 \bar{A}^2 \\
&\quad + if_1 A^2 \partial_z \bar{A} + if_2 A\bar{A}\partial_z A + g_1 A^2 \partial_y \bar{A} + g_2 A\bar{A}\partial_y A + \ldots
\end{aligned}
\tag{17}
$$

where $\mu$ and $\epsilon$ are the two small parameters describing this codimension two bifurcation; other coefficients are given in Table 2.

| $\mathcal{T}_2$ | $\alpha_c$ | $\mathcal{T}_1$ | $a$ | $f_1$ | $f_2$ | $g_1$ | $g_2$ |
|---|---|---|---|---|---|---|---|
| -48.5 | 3.305 | 72.62 | .64 | -.54 | -.70 | 17.02 | 1.38 |

Table 2: values of coefficients occuring in equation (17).

If we neglect the spatial dependency with respect to y, we recover the analysis presented in Refs [7] and [8] for the degenerate case. Here, computations are more complicated, and the non trivial solution (14) verifies the relations

$$
Q^2_{\pm} = \frac{\epsilon + pf \pm \sqrt{\Delta}}{2} \;,\; \Omega = -(apq + Q^2 gq)
$$

with $f = f_1 - f_2$ , $g = g_1 - g_2$ and $\Delta = (\epsilon + pf)^2 + 4(\mu - p^2 - q^2)$. We show that the solution $Q^2_-$ may exist in a specified domain under the surface ($\mu_c = p^2 + q^2$), but it is always unstable. However, the coefficients of the phase equation (15) around the solution $Q^2_+$ are now

$$
\begin{aligned}
\alpha_{20} &= 1 - \tfrac{2q^2}{\delta}\left(\tfrac{1}{X} - \tfrac{gg_1}{\delta} + \tfrac{g^2q^2}{X\delta^2}\right) \\
\alpha_{11} &= -\tfrac{q}{\delta}\left[-ag + 2f_2 + 4\tfrac{p}{X} + g_1 g\tfrac{-2p+fX}{\delta} - 2g^2pq^2\tfrac{-2p+fX}{X\delta^2}\right] \\
\alpha_{02} &= 1 - \tfrac{-2p+fX}{2\delta}\left[-\tfrac{2p+X(f_1+f_2)}{X} + g^2q^2\tfrac{(-2p+fX)}{X\delta^2}\right]
\end{aligned}
$$

where $X = Q^2_+$ and $\delta = \sqrt{\Delta}$. It is difficult to exhibit a so nice relation as the Eckhaus criterion, but numerical trials show that the surface delimiting the fields where the solution $Q^2_+$ is stable is close to the paraboloid (16). It is under this paraboloid in three sectors ($p > 0$ and $q \sim 0$; $p \sim 0$ and either $q > 0$ or $q < 0$) starting from the point $(\mu, p, q) = (0, 0, 0)$, and above it in the other cases.

The next codimension two situation occuring at $\mathcal{T}_2 \sim -62.5$ is related to the following envelope equation

$$
\partial_t A = \mu A + \partial^2_{z^2} A - \nu\partial^2_{y^2} A - \partial^4_{y^4} A + ia\partial^2_{yz} A + \epsilon A^2 \bar{A} - A^3 \bar{A}^2
$$

$$+if_1 A^2\partial_z\bar{A} + if_2 A\bar{A}\partial_z A + g_1 A^2\partial_y\bar{A} + g_2 A\bar{A}\partial_y A + ... \quad (18)$$

where now the two small parameters are $\nu$ and $\mu$. As $\epsilon$ is rather important, the rolls are very subcritical and we have to take into account terms $\mu A^2$ and $\mu^2$ in the expansion (18). Thus, the complete study of this case has not been made.

| $\mathcal{T}_2$ | $\alpha_c$ | $\mathcal{T}_1$ | $\epsilon$ | $a$ | $f_1$ | $f_2$ | $g_1$ | $g_2$ |
|---|---|---|---|---|---|---|---|---|
| -62.5 | 3.462 | 79.77 | .50 | .64 | -.31 | -.56 | 19.36 | 1.02 |

Table 3: values of coefficients occuring in equation (18).

### 4.2 Hopf bifurcation

For $\mathcal{T}_2 < \mathcal{T}_2^*$, we have four critical modes $U_0(x)\exp i(\alpha_c z + \beta_c y)$, $U_1(x)\exp i(-\alpha_c z + \beta_c y)$ and complex conjugates. The Ginzburg-Landau equations have the following form :

$$\begin{aligned}
\partial_t A &= a_2\mu A + i\omega A + 2b_1\alpha_c\partial_z A + 4a_3\alpha_c^2\partial_{z^2}^2 A + a_4\partial_{y^2}^2 A + 2ia_5\partial_{yz}^2 A + bA^2\bar{A} + cAB\bar{B} \\
\partial_t B &= a_2\mu B + i\omega B - 2b_1\alpha_c\partial_z B + 4a_3\alpha_c^2\partial_{z^2}^2 B + a_4\partial_{y^2}^2 B + 2ia_5\partial_{yz}^2 B + bB^2\bar{B} + cA\bar{A}B
\end{aligned} \quad (19)$$

where $A$ and $B$ are the amplitudes of the perturbation associated to critical modes $U_0$ and $U_1$ respectively. Neglecting spatial dependencies, equation (19) possesses two types of non trivial solutions, the travelling waves ($A \neq 0$ and $B = 0$) or standing waves($|A| = |B|$). The former is stable with respect to spatially homogeneous perturbation when $\Re(b) < 0$ and $\Re(c) - \Re(b) < 0$, the latter when $\Re(b) < 0$ and $\Re(c) - \Re(b) > 0$. Unfortunately, the numerical computations show that neither of these solutions is stable.

## 5 CONCLUSION

The determination of the two points of high codimension for $\mathcal{T}_2$ equal to $-48.5$ or $-62.5$ can be deduced from previous works on the Couette-Taylor problem by a limit process as $\eta$ tends to 1. As shown on Figure 3, the former value corresponds to the outer Taylor number for which the Taylor vortices are becoming subcritical, and the latter to the interaction between the axisymmetric mode and the first non axisymmetric mode. But with the small gap formulation, we have now a smooth transition between steady axisymmetric critical modes and non axisymmetric oscillatory modes, and the azimuthal wavelength increases continuously from 0 as $\mathcal{T}_2$ decreases from $\mathcal{T}_2^*$. This fact was a priori not obvious with a formulation using outer and

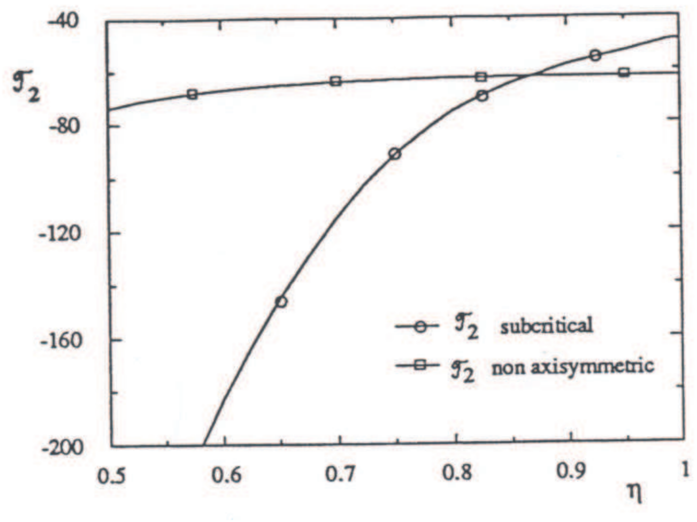


Figure 3: Evolution versus $\eta$ of $\mathcal{T}_2$ giving the subcriticality (O) of Taylor vortices and the interaction between axisymmetric and first non axisymmetric critical modes(□)

inner Reynolds numbers as parameters ([1]). Moreover, this work gives an example of bifurcations with anisotropic properties which are characterized by the presence of advection and cross derivative terms in the amplitude equations and the fact that any disturbance azimuthal direction involves a temporal modulation.

## References


[1] W. Langford, R. Tagg, E. Kostelich, H. Swinney and M. Golubitsky, Phys. Fluids, 31, 776 (1987).

[2] M. Golubitsky, W.F. Langford, Physica D 32, 362-392 (1988).

[3] Y. Demay and G. Iooss, J. Méca. Théo. Appli., $n^0$ spécial, 193 (1984) .

[4] P. Tabeling, J. Physique-lettres, 44, 16, 665 (1983).

[5] Y. Demay, G. Iooss and P. Laure, in preparation (1991).

[6] J. Lega, Thesis, Nice University (1989).

[7] W. Eckhaus and G. Iooss, Phys. D 39, 124 (1989).

[8] A. Doelman, Thesis, Utrecht University (1990).