GENERATION OF MAGNETIC FIELD IN THE COUETTE-TAYLOR SYSTEM

P. LAURE AND P. CHOSSAT
Institut Non-Linéaire de Nice, UMR 6618 - CNRS & UNSA
1361 route des Lucioles, 06560 Valbonne, France

AND

F. DAVIAUD
SPEC, CEA Saclay
91191 Gif sur Yvette cedex, France

1. Introduction

The governing equation for the magnetic field $\mathbf{B}$ in an electrically conducting fluid with conductivity σ and velocity $\mathbf{v}$ is the so-called induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \Delta \mathbf{B} \tag{1}$$

which follows from Maxwell equations and Ohm's law. The solution $\mathbf{B} = \mathbf{0}$ may become unstable for some critical value $\mathrm{Re}_{\mathrm{mc}}$ of the magnetic Reynolds number,

$$\mathrm{Re}_{\mathrm{m}} = \mu_0 \sigma \mathrm{LV}, \tag{2}$$

L and V being respectively typical length and velocity scales.

The induction equation can be coupled with the full Navier-Stokes equation, but the the problem becomes numerically very arduous (see [5]). Another approach consists to impose an arbitrary velocity field $\mathbf{v}$ and looking at if the solution $\mathbf{B}$ is amplified or not.

In the sequel, we assume that the velocity field is the Taylor vortex flow coming from the bifurcation of the Couette flow.

2. The Taylor vortex flow

The Couette-Taylor apparatus consists of two rotating cylinders moving with different angular velocities Ω_i and Ω_o [4].

P. Chossat et al. (eds.), Dynamo and Dynamics, a Mathematical Challenge, 17–24.

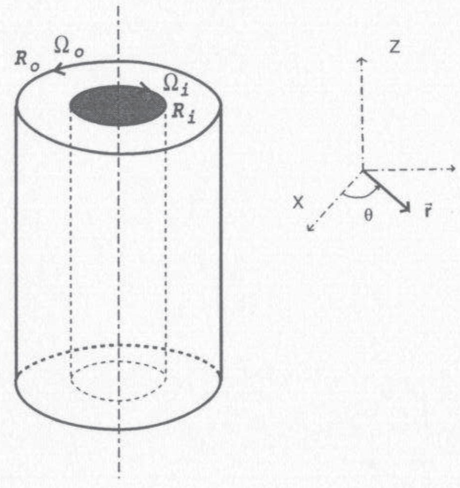

Figure 1. Geometry of the Couette-Taylor problem.

The parameters of this problem are the ratio of the inner and outer cylinder $\eta = R_i/R_o$, the inner and outer Reynolds numbers

$$Re_{i,o} = R_{i,o}\,\Omega_{i,o}(R_o - R_i)/\nu.$$

For small Re_i, the flow motion is purely azimuthal $V^c = (0, v_\theta^c(r), 0)$ (namely the Couette flow). A transition to axisymmetric motion (Taylor vortex) is observed for a value of the inner Reynolds number denoted Re_c if the outer Reynolds number Re_o is not too negative. This flow is stationary and periodic along the z-axis (α_c is its wavenumber).

In the following, we assume that the outer cylinder is at rest ($Re_o = 0$) and that the radii ratio is equal to .5. Indeed for theses parameter values [6], the second transition towards wavy vortices occurs at a large inner Reynolds number ($\epsilon_R = (Re_i - Re_c)/Re_c > 3$).

The asymptotic expansion of the Taylor flow in the neighborhood of the bifurcation point has the form

$$U = A\, v^t(r)\mathrm{e}^{\mathrm{i}\alpha_c \mathrm{z}} + \bar{A}\, \bar{v}^t(\mathrm{r})\mathrm{e}^{-\mathrm{i}\alpha_c \mathrm{z}} + \;.... \tag{3}$$

where A is the amplitude of the bifurcated solution. A can be evaluated thanks to amplitude equation [7]. Finally, the flow motion for $Re_i > Re_c$ is of the form

$$V = V^c + \rho(Re_i)V^t + ... \tag{4}$$

assuming that $V^t = 2\ \mathrm{Real}(v^t(\mathrm{r})\ \mathrm{e}^{\mathrm{i}\alpha_c \mathrm{z}})$ with the scaling $||v^t((R_i + R_o)/2)|| = 1$.

3. Equation for the Dynamo problem

We assume that the dynamo effect comes in as a secondary bifurcation after the Taylor vortex and the problem is now to determine the Reynolds number Re_i such that the linear operator in (5) has eigenvalues belonging to the imaginary axis. The dynamo problem reads for a velocity field V defined by relation (4) :

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{Re_m}\ \mathrm{curl}(\mathbf{V} \times \mathbf{B}) + \mathbf{\Delta B} \tag{5}$$

$$\mathrm{div}(\mathbf{B}) = \mathbf{0} \tag{6}$$

with conductor boundary condition for $r = R_i, R_o$

$$\mathbf{B.n} \;=\; 0 \tag{7}$$

$$\mathrm{curl}(\mathbf{B}) \wedge \mathbf{n} \;=\; 0 \tag{8}$$

and periodic boundary conditions in the z direction. Moreover, we also assume conductor boundary condition on the top and the bottom of cylinders. In this way, we fixe the representation of the symmetry, $z \to -z$, which is now "natural" [2, 1] :

$$(r, \theta, z) \to (r, \theta, -z)\ ,\ (v_r, v_\theta, v_z) \to (v_r, v_\theta, -v_z). \tag{9}$$

Therefore, a Fourier decomposition will be done in this direction and $\mathrm{Re_m}$ is defined with the same scaling as the Reynolds number and then

$$\mathrm{Re_m} = \mathrm{P_m}\ Re_\mathrm{i} \tag{10}$$

where $\mathrm{P_m} = \mu_o \sigma \nu$ is the magnetic Prandtl number. We change the scale in axial direction so that z is defined between 0 and 2 π and n_V is now the Fourier mode of V^t in the z-direction (namely the number of Taylor vortices).

At this stage, the number of parameters is rather important and it is very tedious to determine directly the critical Re_i. First, we seek if the first terms in the expansion (4) can give rise to dynamo effect. In fact, with the two first terms V^c and V^t, the velocity field has non-zero helicity density, $h = \mathbf{V}.\mathrm{curl}\mathbf{V}$ and therefore could exhibit dynamo action. More precisely, the helicity density is split in two terms

$$h(r,z) = \rho \sin(n_V\ z)\ h^c(r) + \rho^2 \sin(n_V\ z)\ \cos(n_V\ z)\ h^t(r) \tag{11}$$

$$h^c = v_\theta^c \left[Dv_z^t - \alpha_c\ v_r^t \right] + Dv_\theta^c\, v_z \text{ and } h^t = v_z^t\ Dv_\theta\ -\ v_\theta^t\ Dv_z^t.$$

The first one, h^c, comes from the coupling between the Couette and the Taylor flows and the other one, h^t, is due to the Taylor vortex. As shown in Fig. 2, the maximum of the first term is located close to the moving wall whereas the maximum of second term is close to the middle of the cylinder.

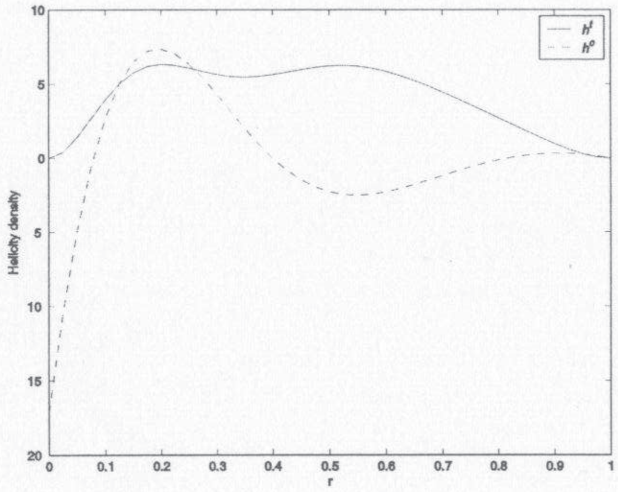

Figure 2. Helicity density of the basic flow versus the radial component : $Re_o = 0$ and $\eta = .5$

Finally, the methodology applied is the following : one fixes ρ and n_V and determines if there exists a critical magnetic Reynolds number.

3.1. NUMERICAL PROCEDURE

A Fourier analysis is made in the axial and azimuthal directions :

$$B(r,\theta,z) = \sum_{m,n} b^{m,n}(r)\ \mathrm{e}^{\mathrm{i}\ (\mathrm{m}\theta+\mathrm{nz})} \text{ with } \mathrm{b}^{-\mathrm{n},-\mathrm{m}} = \bar{\mathrm{b}}^{\mathrm{n,m}} \tag{12}$$

Each component $b^{m,n}(r)$ is discretized on N collocation points [3]. The condition of zero divergence is used to eliminate the components $b_z^{m,n}$ or $b_\theta^{m,n}$ according to the value of (m,n) :

$$\frac{1}{r} b_r^{m,n} + D b_r^{m,n} + \frac{\mathrm{i\ m}}{r}\ b_\theta^{m,n} + \mathrm{in\ b_z^{n,m}} = 0 \tag{13}$$

Due to reflection symmetry, $z \to -z$, we restrict our expansion to positive Fourier modes. The other modes can be deduced by the relation

$$b_r^{-n,m} = b_r^{n,m}\ ;\ b_\theta^{-n,m} = b_\theta^{n,m}\ ;\ b_z^{-n,m} = -b_z^{n,m} \tag{14}$$

Finally, one gets for each azimuthal mode m a generalized eigenvalue problem in a matrix form,

$$M(m, \mathrm{Re_m}, \rho, \mathrm{n_V})[\ \mathrm{b}\ ] = \mathrm{s\ N}[\ \mathrm{b}\ ] \tag{15}$$

where M is a real sparse matrix, N is a noninvertible diagonal matrix and the real part of the eigenvalue s is the dynamo growth rate. The aim is to compute the eigenvalue s with the largest real part and look at if this real part becomes positive as the magnetic Reynolds number increases. This algebraic problem is solved by an Arnoldi method coupled with an inverse iterative method [10, 11]. As the size of matrix M becomes rapidly important ($\sim 4\ N * (N_z + 1)$ for N_z Fourier modes), the GMRES algorithm with a suitable preconditioner is used to make the Arnoldi decomposition (instead of the usual LU factorization).

4. Numerical results : influence of n_V

The Fig. 3 shows the velocity field used in the computation. It consists of two counter-rotating vortices in the (r, z)-plane. It is displayed on one axial period and this pattern is repeated n_V times if the number of Taylor vortices is n_V.

The computations are made for $n_V = 1, 2, 3, 4, 6, 7$; $\rho = 1.$ and various azimuthal modes (see Table 1). First for $n_V > 1$ and $m = 1$, there always exists a critical Magnetic Reynolds number from which the dynamo growth rate, Real(s), is positive. Moreover, the amplified magnetic field is time-dependent (the imaginary part of s is always non zero). For higher values of the azimuthal modes m, additional computations show that the real part of eigenvalues s are always negative for the critical magnetic Reynolds numbers obtained for the mode 1. If $n_V = 1$, the perturbation are never amplified for magnetic Reynolds number up to 800. We have stopped our computation to this value as higher resolution is required [8] for large values of $\mathrm{Re_m}$. This is due to the concentration of magnetic field in sheets of thickness $O(\mathrm{Re_m}^{1/2})$ [9].

For $n_V = 2$, the two sets of two contra-rotating vortex (cf. Fig. 3) are located respectively between 0-π and π-2π (there are a total amount of four rolls). Fig 4a shows that the magnetic field is concentrated along the inner cylinder and along the outflow boundaries ($z = 0, \pi$ and 2π). As usual the magnetic field is more important near the stagnation points [9].

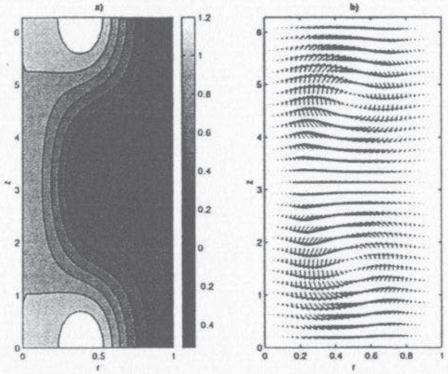

Figure 3. Velocity field $V^c + \rho V^t$ for $\rho = 1.$ and $Re_o = 0$ (the flow does not depend on Re_i if $Re_o = 0$: (a) isoline of V_θ in the (r, z)- plane; (b) (V_r, V_θ) in the (r, z)-plane.

n_V	Re_{mc}	Im(s)	N	N_Z	First non null Fourier modes
2	134.9	-72.56	31	20	1, 3, 5, 7, ...
3	180.	-24.51	41	50	1, 2, 4, 5, 7 ....
4	135.7	-76.10	41	50	2, 6, 10 ...
5	142.1	-80.06	41	70	2, 3, 7 ....
6	134.9	-75.57	41	90	3, 9, 15 ..
7	137.9	-77.40	41	110	3, 4, 10 ...

TABLE 1. Results : $\eta = .5$, $Re_o = 0$, $\rho = 1.$

For $n_V = 3$, the critical Reynolds number is higher than for $n_V = 2$ whereas the magnetic fields is rather concentrated along the inflow boundaries.

For $n_V > 3$, the characteristic length of the magnetic field seems to be always twice the height occupied by the basic Taylor vortex (c.f. Fig. 3). For odd n_V, this is only approximated and the system tends to neglect the contribution due to the rolls located around $z = \pi$, in order to get the magnetic field generated for an even n_V. This point is depicted in Figs. 4c-d and Figs. 4e-f for the case $n_V = 5$ and $n_V = 7$.

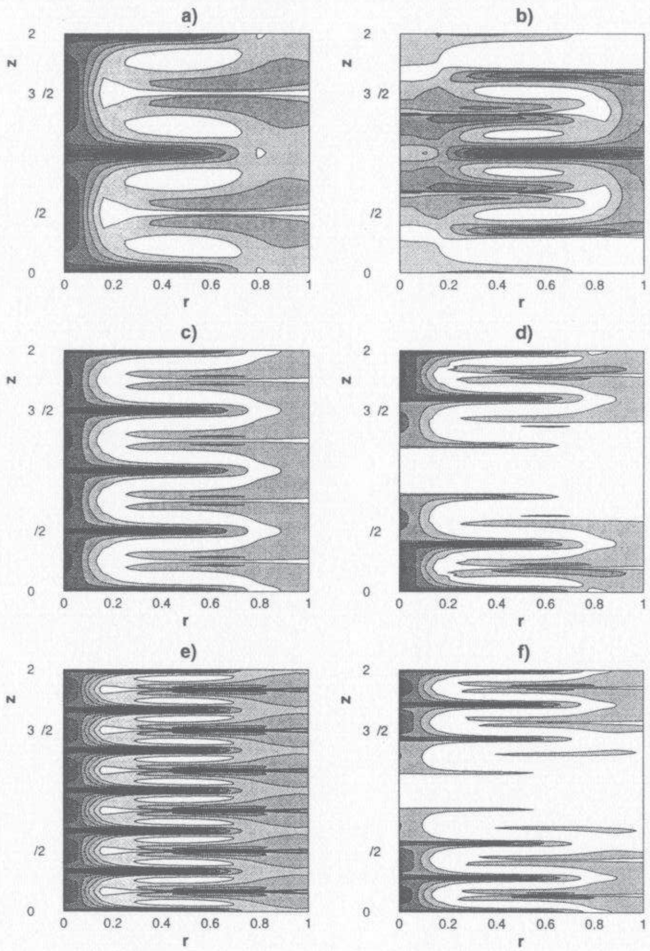

Figure 4. Isolines of modulus $|B|$ of the critical magnetic field in the (r, z)- plane for $\theta = 0$ and $t = 0$. The other parameters are $m = 1$, $\rho = 1.$, $Re_o = 0$: (a) $n_V = 2$, (b) $n_V = 3$, (c) $n_V = 4$, (d) $n_V = 5$, (e) $n_V = 6$, (f) $n_V = 7$. The maximum value corresponds to the black color.

5. Conclusion

Only very preliminary results of the dynamo effect on the Couette-Taylor system are presented. The influence of positive outer Reynolds number Re_o, the radii ratio η and the deviation ρ from Couette flow would be still studied. However, on can see that the characteristic length of dynamo field generated by Taylor cells are in a same order than the velocity field. Then, one cannot decrease the critical magnetic Reynolds number by increasing the size of the system (namely the number of Taylor vortices).

Acknowledgments

The authors thank S. Fauve for his advice and his constant encouragements.

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