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Quasiperiodic drift flow in the Couette-Taylor problem

by

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1. Introduction

The Couette-Taylor problem deals with the flow of an incompressible, viscous fluid between two coaxial rotating cylinders. Depending on the angular velocities of the cylinders, different flow patterns are observed in experiments. Mathematically, transitions between different flow patterns can be described by instabilities and bifurcations of certain solutions of the Navier-Stokes equations for this problem. In this paper we consider the counterrotating case, i.e. the cylinders rotate in opposite directions. We describe a sequence of three successive instabilities and corresponding bifurcations which occur in a certain parameter regime when the Reynolds number is increased. The primary bifurcation is the classical bifurcation from Couette flow to Görtler-Taylor vortex flow (cf. Taylor [1923]), the secondary one leads to wavy vortex flow, and the tertiary one leads to what we call quasiperiodic drift flow. We also indicate how this result is linked to related work on the Couette-Taylor problem and give an outline of the method by which we have obtained it. A key step of our method is the reduction of the Navier-Stokes equations to a system of ordinary differential equations. This is achieved by invariant manifold theory and ideas from the theory of dynamical systems with symmetry.

The plan of the paper is as follows. In section 2 we introduce the basic equations. In section 3 we discuss stability of the Couette flow. Section 4 is devoted to the reduction procedure. Finally, in section 5 we analyse the reduced equation and state the main result.

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2. The basic equations and solutions

Let R_i and R_0 be the radii of the inner and outer cylinders, respectively, and denote their angular velocities by Ω_i and Ω_0. According to the geometry of the experimental apparatus, we choose cylindrical coordinates r, Θ, z and denote the velocity field in the fluid by $V = (V_r, V_\Theta, V_z)^T$ and the pressure field by p. The superscript T denotes the corresponding column vector. Both are functions of the spatial coordinates and time t. Also, we introduce the dimensionless parameters

$$\Omega = \Omega_0/\Omega_i, \quad \eta = R_i/R_0, \quad r_1 = \frac{\eta}{1-\eta}, \quad r_2 = \frac{1}{1-\eta} \tag{2.1}$$

and

$$R = \frac{R_i \Omega_i d}{\nu}, \tag{2.2}$$

where R is the Reynolds number, $d = R_0 - R_i$ is the width of the gap between the cylinders, and ν is the kinematic viscosity of the fluid. As usual, we impose non-slip boundary conditions at the cylinder walls and assume $2\pi/\alpha$-periodicity of V and p in the axial direction (infinitely long cylinders). Here the wave number α will be fixed eventually. Then the Navier-Stokes equations for V and p in dimensionless form read as follows:

$$\begin{array}{l} \left.\begin{array}{l} \dfrac{\partial V}{\partial t} = \Delta V - R(V\cdot\nabla)V - \nabla p \\ \nabla\cdot V = 0 \end{array}\right\} (r_1 \le r \le r_2, \Theta \in \mathbb{R}, z \in \mathbb{R}) \\ V_r = V_z = 0 \quad (r = r_1, r = r_2) \\ V_\Theta = 1 \quad (r = r_1) \\ V_\Theta = \Omega/\eta \quad (r = r_2) \\ V \text{ and } p \text{ are } 2\pi/\alpha - \text{periodic in } z \text{ and } 2\pi - \text{periodic in } \Theta. \end{array} \tag{2.3}$$

Here Δ is the Laplace operator, and ∇ the nabla operator.

These equations are covariant with respect to the symmetry group $\Gamma = SO(2) \times O(2)$, where $SO(2)$ acts by rotations $R_\varphi : \Theta \mapsto \Theta + \varphi$ with angle $\varphi \in [0, 2\pi)$ around the z-axis, and $O(2)$ acts by translations $T_a : z \mapsto z + a$ along the z-axis and through the flip $S : z \mapsto -z$ (see Golubitsky and Stewart [1986]).

One advantage of the symmetry is that one can describe the transitions between different flow patterns by symmetry breaking bifurcations and characterize different solutions (V, p) by their symmetry, or more precisely by the isotropy subgroup of V

$$\Sigma_V = \{\gamma \in \Gamma \mid \gamma V = V\}.$$

There is a stationary solution, namely the Couette flow (COU), for which one has the explicit analytic expression

$$V^0 = (0, V^0_\Theta, 0)^T, \quad p^0 = R\int (V^0_\Theta(r)^2/r)dr$$
(2.4)
$$V^0_\Theta(r) = Ar + B/r, \quad A = \frac{\Omega - \eta^2}{\eta(1+\eta)}, \quad B = \frac{\eta(1-\Omega)}{(1-\eta)(1-\eta^2)}.$$

It has the symmetry of the full group Γ, i.e. $\sum_{COU} = \Gamma$, and represents an azimuthal flow. Another type of stationary solution which is going to play a role in the discussion below is the Görtler-Taylor vortex flow (GTV). Its isotropy subgroup is $\Sigma_{GTV} = SO(2) \times \mathbb{Z}_2(S)$ where $\mathbb{Z}_2(S)$ is the subgroup of $O(2)$ generated by the flip S. Consequently, this flow is not invariant under the translations T_a. Here flat flow cells form in the fluid. In contrast to those flows, for the wavy vortex flow (WV) also the $SO(2)$-symmetry is broken. This flow is time periodic and for fixed time invariant under $\sum_{WV} = \mathbb{Z}_2(\mathbb{R}_\pi, S)$ only, i.e. under a rotation R_π followed by the flip S. Here wavy flow cells form in the fluid. Actually, the wavy vortex solution is a rotating wave or relative equilibrium with respect to the group $SO(2)$, i.e. its trajectory in (V,p)-space is also a group orbit. The corresponding flow cells rotate periodically around the axis of the cylinders. Since the translational symmetry is broken, these solutions occur in families of conjugate trajectories which just differ by a translation T_a.

3. Stability of the Couette flow

Next we discuss stability of the Couette flow depending on the parameters R, Ω and η. To this end we introduce relative variables U and q via

$$V = V^0 + U, \quad p = p^0 + q \tag{3.1}$$

and write the basic equations (2.3) as an evolution equation

$$\frac{dU}{dt} = L(R,\Omega,\eta)U + N(R,\Omega,\eta)(U) \qquad (U \in H) \tag{3.2}$$

for U in a suitable Hilbert space $H \subset [L_2((r_1,r_2) \times \mathbb{R} \times \mathbb{R})]^3$ of solenoidal vector fields which are periodic in Θ and z. It is well known that this can be achieved using the Weyl projection operator to eliminate the pressure field q (see Ladyshenskaya [1963], Iudovich [1965], Iooss [1971], and Témam [1977]). Here L is a closed linear operator with dense domain of definition D in H. The elements of D have zero trace at $r = r_1$ and $r = r_2$. By $\|\cdot\|_D$ we denote the graph norm in D with respect to L. The resolvent of L is compact, i.e. L has pure point spectrum, and L generates a holomorphic compact semigroup $\exp(Lt)_{t\geq 0}$ in H. The operator $N : D \to H$ is quadratic and continuous.

There is a local existence and uniqueness theorem for the initial value problem corresponding to (3.2):

<u>**Theorem.**</u> For all $T > 0$, there exists a δ such that (3.2) has an unique (classical) solution $U \in C^0([0,T], D) \cap C^1((0,T], H)$ with $U(0) = U_0$ for all $U_0 \in D$ with $\|U\|_D < \delta$.

Hence, the evolution equation (3.2) generates a local semiflow S_t in D. Moreover, it has been shown, e.g. by Sattinger [1969/70] and Kirchgässner and Kielhöfer [1973], that the <u>principle of linearized stability</u> holds true for the trivial solution $U = 0$ which represents the Couette flow. It says the following: if all eigenvalues of L have negative real parts, then $U = 0$ is asymptotically stable with respect to the semiflow S_t (in the sense of Liapunov). If there is at least one eigenvalue of L with positive real part, then $U = 0$ is unstable. Note, that here in Liapunov's notion of stability the norm $\|\cdot\|_D$ is used to measure initial values and the H-norm is used to measure $U(t)$ for $t > 0$.

Using this principle, it is not difficult to prove that for sufficiently small values of R the Couette flow is asymptotically stable. Indeed, in this case L is just a small perturbation of the Laplace operator which is negative definite. Hence, all eigenvalues of L have negative real parts. However, for all Ω and η, there exists a critical Reynolds number $R_c = R_c(\Omega, \eta)$ such that for $R > R_c(\Omega, \eta)$ the Couette flow is unstable. How many eigenvalues cross the imaginary axis right at $R = R_c$ and how the corresponding eigenfunctions which we call <u>critical modes</u>, look like depends on Ω and η. By definition of H, the general complex form of the critical modes is

$$U = U(r)e^{i(k\alpha z + m\Theta)} \qquad (k, m \in \mathbb{Z}).$$

Numerically one finds a curve in the rectangle $-1.2 \leq \Omega \leq -0.4$, $0.4 \leq \eta \leq 1.0$, along which critical modes with $k = 1$ and two different azimuthal wave numbers $m = 0$ and $m = 1$ occur simultaneously at $R = R_c(\Omega, \eta)$ (cf. Langford et al. [1988]). This is called a <u>bicritical instability</u> of the Couette flow. We now fix a point (Ω_c, η_c) on this curve, choose α appropriately, and consider (3.2) for $\eta = \eta_c$ and (R, Ω) near the corresponding critical point $P_c = (R_c(\Omega_c, \eta_c), \Omega_c)$ in the (R, Ω)-plane. There L has a real eigenvalue $\mu(R, \Omega)$ and a pair of complex conjugate eigenvalues $\gamma(R, \Omega) \pm i\omega(R, \Omega)$. All these eigenvalues have multiplicity two. For $R = R_c$ and $\Omega = \Omega_c$ they simultaneously sit on the imaginary axis. Hence, the corresponding <u>critical eigenspace</u> E is six-dimensional. The other eigenvalues of L are strictly bounded away from the imaginary axis for (R, Ω) near P_c.

4. Reduction

Because of the above properties of L and N, one can use center manifold theory (see e.g. Henry [1981]) to reduce (3.2) to a six-dimensional system of first order ordinary differential equations for (R,Ω) near P_c and $U \in D$ near 0

$$\frac{dx}{dt} = X(R,\Omega)(x) \qquad (x \text{ near } 0 \text{ in } \mathbb{R}^6), \tag{4.1}$$

where x is a vector of coordinates in the critical eigenspace E. The vector field $X = X(R,\Omega,\cdot)$ is equivariant with respect to the symmetry group Γ, i.e. it commutes with a certain representation of Γ on E. This system fully describes all solutions of (3.2) which exist and stay close to $U = 0$ (in the H-norm) for all t, including stability properties. It is obtained by restricting (3.2) to a six-dimensional invariant manifold $M \subset D$, the center manifold, which is represented as the graph of a smooth map $\Psi = \Psi(R,\Omega,\cdot)$. The latter is defined for x near 0 in E and has values in the complementary eigenspace of $L(R_c,\Omega_c,\eta_c)$. The vector field X and the map Ψ satisfy a so-called homological equation

$$[id + D\Psi]X = L(\cdot + \Psi) + N(\cdot + \Psi) \tag{4.2}$$

which can be used to compute Taylor expansions for both X and Ψ at $x = 0$. This leads to linear elliptic boundary value problems for the Taylor coefficients of Ψ. The results in this paper only depend on terms of order up through two of Ψ and up through three of X.

A further reduction can be achieved using the symmetry of X. For example, following Menck [1991], one can devide out the center manifold M by the action of the subgroup $\tilde{\Gamma} = SO(2) \times S^1$ of Γ, which leads to the orbit space $M_{|\tilde{\Gamma}}$. Here different points on the group orbits of $\tilde{\Gamma}$ are identified. Globally, this is not a manifold, rather it is an algebraic variety with cone-like singularities. But corresponding to a certain region on M, where the group orbits of $\tilde{\Gamma}$ are two-tori, $M_{|\tilde{\Gamma}}$ has a stratum which is a four-dimensional manifold locally. There the motion of (4.1) transverse to the group orbits of $\tilde{\Gamma}$ is described by a smooth four-dimensional system

$$\frac{d\xi}{dt} = Y(R,\Omega)(\xi) \qquad (\xi \text{ near } 0 \text{ in } \mathbb{R}^4), \tag{4.3}$$

where ξ is a vector of suitably choosen invariant coordinates ξ_1, ξ_2, ξ_3 and ξ_4. To describe the relation between ξ and the coordinates on the center manifold M we introduce complex coordinates ζ_j $(j = 0,1,2)$ in E defined by

$$x = \sum_{j=0}^{2} (\zeta_j V_j + \bar{\zeta}_j W_j), \tag{4.4}$$

where $V_0 = \hat{U}_0(r)e^{i\alpha z}, W_0 = SV_0$ and $V_1 = \hat{U}_1(r)e^{i(\alpha z+\theta)}, V_2 = SV_1, W_1 = \bar{V}_1, W_2 = \bar{V}_2$ denote critical modes corresponding to the zero eigenvalue and the pair of complex conjugate eigenvalues, respectively. Then we have

$$\text{(4.5)}\qquad \begin{aligned} \xi_1 &= |\zeta_0|^2, \quad \xi_2 = \frac{1}{2}(|\zeta_1|^2 + |\zeta_2|^2) \\ \xi_3 &= \frac{1}{2}(|\zeta_2|^2 - |\zeta_1|^2), \quad \xi_4 = Im(\bar{\zeta}_0^2\zeta_1\bar{\zeta}_2). \end{aligned}$$

It turns out, that the $\mathbb{Z}_2$-symmetry $\xi_1 \mapsto \xi_1$, $\xi_2 \mapsto \xi_2$, $\xi_3 \mapsto -\xi_3$ and $\xi_4 \mapsto -\xi_4$ still acts nontrivially on this system. Hence, the restriction of (4.3) to the fixed point subspace $Fix(\mathbb{Z}_2) = \{\xi \in \mathbb{R}^4 | \xi_3 = \xi_4 = 0\}$ finally leads to a two-dimensional system for ξ_1 and ξ_2. Setting

$$\text{(4.6)}\qquad \begin{aligned} \mu(R,\Omega) &= \lambda \\ \gamma(R,\Omega) &= \lambda - \sigma, \end{aligned}$$

this system has the form

$$\text{(4.7)}\qquad \begin{aligned} \dot{\xi}_1 &= \lambda\xi_1 + b_1\xi_1\xi_2 + b_2\xi_1^2 + \text{ h.o.t.} \\ \dot{\xi}_2 &= (\lambda-\sigma)\xi_2 + a_1\xi_1\xi_2 + a_2\xi_2^2 + \text{ h.o.t..} \end{aligned}$$

The coefficients depend on the parameters λ and σ. Note, that the terms which are quadratic in (ξ_1, ξ_2) involve third order terms of the vector field X. This is a consequence of the fact that the orbit space $M_{|\tilde{\Gamma}}$ is globally nonlinear.

5. Analysis of the reduced system

Numerical computations show that roughly speaking, we can think of λ and σ given by (4.6) as being

(5.1) $$\lambda \approx R - R_c(\Omega_c, \eta_c) \quad \text{and} \quad \sigma \approx \Omega - \Omega_c \quad (\text{as} \quad (R, \Omega) \to P_c).$$

To compute the relevant Taylor coefficients of X and Ψ from (4.2), we used a combination of symbolic computations and a numerical boundary value problem solver (cf. Laure and Demay [1988]).

Here is a table of results for X which we obtained for differnt values of $\eta_c \in [0.4, 0.85]$, $\alpha \sim 3.6, R = R_c$ and $\Omega = \Omega_c$:

Table

η_c	c_1^1	p_2^1	p_0^3	c_0^3	$p_1^1 - p_0^3$	$c_2^1 - c_0^3$	p_0^2	c_0^2	q_0^3	q_0^2
0.85	-7.918	-152.5	-109.5	-191.2	2.8	-88.2	-44.17	-182.5	-62.16	-117
0.825	-18.43	-188.7	-118.8	-208.1	-6.5	-122.5	-46.96	-204.8	-68.44	-130.4
0.8	-30.04	-228.2	-129	-227.1	-16.5	-160.1	-50.02	-227.9	-74.47	-144.2
0.775	-42.71	-270.9	-139.8	-248.2	-27.5	-201.4	-53.35	-252.2	-80.44	-158.5
0.75	-57.01	-318.4	-152	-272.4	-39.6	-247.4	-57.2	-277.7	-86.15	-173.4
0.725	-72.83	-370.4	-165.3	-299.4	-52.6	-298.8	-61.4	-305.3	-91.92	-189.3
0.7	-90.89	-428.7	-180.4	-330.7	-67.1	-357.1	-66.35	-334.9	-97.5	-206.4
0.675	-111.4	-494	-197.3	-366.5	-83	-423.3	-72	-367.6	-103.1	-224.9
0.65	-134.8	-567.2	-216.3	-407.4	-100.5	-499.2	-78.5	-404	-109	-245.4
0.625	-162.1	-651	-238.4	-455.6	-120.2	-587	-86.2	-444.7	-114.8	-268.1
0.6	-194.1	-747.2	-263.9	-512.1	-142.5	-689.1	-95.2	-491.2	-120.9	-294
0.575	-232.2	-858.7	-293.8	-579.2	-168	-809.2	-105.9	-545	-127.4	-323.7
0.55	-278.5	-989.9	-329.6	-660.5	-197.9	-951.9	-118.7	-608.2	-134.5	-358.6
0.525	-335.6	-1146	-373.2	-760.5	-233.3	-1124	-134.1	-684.4	-142.5	-400.7
0.5	-408.2	-1337	-427.8	-885.7	-276.9	-1334	-152.9	-777.8	-151.9	-452.5
0.475	-503.2	-1573	-498.6	-1047	-332.4	-1595	-176.4	-896.1	-163.6	-518.8
0.45	-633	-1877	-594.8	-1260	-405.2	-1932	-206.4	-1053	-180.2	-608.1
0.425	-822	-2288	-735.6	-1557	-509.4	-2383	-246.9	-1272	-206.6	-736.7
0.4	-1125	-2890	-965.9	-2001	-674.1	-3041	-306	-1610	-257.4	-942.5

This table refers to the following representation of X as a Γ-equivariant vector-field (cf. Golubitsky and Langford [1988]):

$$
\begin{aligned}
X \; &= (c^1 + i2\xi_4 c^2)\begin{pmatrix} \zeta_1 \\ 0 \\ 0 \end{pmatrix} \quad + (c^3 + i2\xi_4 c^4)\begin{pmatrix} \zeta_0\bar{\zeta}_1\zeta_2 \\ 0 \\ 0 \end{pmatrix} \\
&+ (p^1 + iq^1)\begin{pmatrix} 0 \\ \zeta_1 \\ \zeta_2 \end{pmatrix} \quad + 2\xi_4(p^2 + iq^2)\begin{pmatrix} 0 \\ \zeta_1 \\ -\zeta_2 \end{pmatrix} \qquad (5.2) \\
&+ (p^3 + iq^3)\begin{pmatrix} 0 \\ \zeta_0^2\zeta_2 \\ \bar{\zeta}_0^2\zeta_1 \end{pmatrix} \quad + 2\xi_4(p^4 + iq^4)\begin{pmatrix} 0 \\ \zeta_0^2\zeta_2 \\ -\bar{\zeta}_0^2\zeta_1 \end{pmatrix}
\end{aligned}
$$

Here c^j, p^j, q^j $(j = 1,2,3,4)$ denote functions of R, Ω and the Γ-invariants $\xi_1, \xi_2, \xi_3^2, \xi_3\xi_4$ and $\xi_5 = Re(\bar{\zeta}_0^2\zeta_1\bar{\zeta}_2)$. A subscript 0 denotes the value of these functions for $R = R_c, \Omega = \Omega_c$ and all the other arguments equal to zero. A subscript 1 or 2 denotes the corresponding value of the partial derivative with respect to ξ_1 and ξ_2, respectively.

The correspondence between the values of the coefficients in (4.7) at $\lambda = \sigma = 0$ and the entries of the table is as follows:

$$
\begin{aligned}
b_1 &= c_2^1 - c_0^3, \quad b_2 = c_1^1 \\
a_1 &= p_1^1 - p_0^3, \quad a_2 = p_2^1
\end{aligned} \qquad (5.3)
$$

Therefore, the following inequalities are satisfied:

$$
\begin{aligned}
b_1, b_2, b_2 - a_1 &< 0 \\
b_1 - a_2 &\lesseqgtr 0 \quad \text{for} \quad \eta_c \lesseqgtr 0.49 \\
d = a_2 b_2 - a_1 b_1 &> 0
\end{aligned} \qquad (5.4)
$$

We now analyse (4.7) under these conditions. Neglecting the higher order terms and choosing the values of the coefficients at $\lambda = \sigma = 0$, by simple algebraic computations one obtains the following approximations. The curve given by $\lambda = 0$ in the (λ, σ)-plane is a curve of primary bifurcations of the trivial solution $\xi_1 = \xi_2 = 0$. This corresponds to the classical primary instability of the Couette flow which has already been studied by Taylor [1923]. Early rigorous treatments of this instability are in Iudovich [1965], Velte [1966], and Kirchgässner and Sorger [1969]. The bifurcating Taylor vortex flow is represented by the equilibria

$$
\xi_1 = \pm\sqrt{-\lambda/b_2}, \quad \xi_2 = 0 \qquad (\lambda > 0, \quad \beta \ \text{arbitrary}) \qquad (5.5)
$$

of the truncated system (4.7). Furthermore, one finds the curve of secondary bifurcations $\lambda = \sigma b_2/(b_2 - a_1)$, along which another family of equilibria given by

$$
\begin{aligned}
&\xi_1 = \frac{(b_1 - a_2)\lambda - b_1\sigma}{d}, \quad \xi_2 = \frac{(a_1 - b_2)\lambda + b_2\sigma}{d} \\
&\left(\lambda > 0,\ \rho_2\lambda > \sigma > \rho_1\lambda, \quad \text{where} \quad \rho_1 = \frac{b_1 - a_2}{b_1}, \quad \rho_2 = \frac{b_2 - a_1}{b_2}\right)
\end{aligned} \qquad (5.6)
$$

branches off from the previous one. This corresponds to a curve of secondary instabilities in the Couette-Taylor problem where the Taylor vortex flow looses stability to wavy vortex flow. This happens actually through a Hopf bifurcation (see Davey, DiPrima and Stewart [1968], Chossat and Iooss [1985], Golubitsky and Stewart [1986] and Golubitsky and Langford [1988]). We point out, that the numerical value of ρ_1 in (5.6) is positive for $0.4 \leq \eta_c \leq 0.475$ and negative for $0.5 \leq \eta_c \leq 0.85$ according to our computations. Also note, that the stability of the approximate Couette and Görtler-Taylor vortex flows inside $Fix(\mathbb{Z}_2)$ is consistent with their actual stability. However, to really prove stability by the present method, one has to take into account the full orbit space $M_{|\tilde{\Gamma}}$. Equation (4.3) is not adequate to do this in case of the Couette and Görtler-Taylor vortex flows. Since those are $SO(2)$-symmetric, their $\tilde{\Gamma}$-orbits in M are not two-tori. Menck [1992] uses an extended system to overcome this difficulty.

To analyse the stability of the wavy vortex flow, we can use equation (4.3). Inside $Fix(\mathbb{Z}_2)$ the corresponding equilibria (5.6) are asymptotically stable for all values of λ and σ for which they exist. According to Menck [1991], their stability in $M_{|\tilde{\Gamma}}$ is therefore determined by the eigenvalues of the 2×2 matrix

$$B = \begin{pmatrix} -2p_0^2\xi_2 + p_0^3\xi_1 & -q_0^3\xi_1 \\ q_0^3\xi_1 + 2(q_0^2 - c_0^2)\xi_2 & c_0^3\xi_2 + p_0^3\xi_1 \end{pmatrix}. \tag{5.7}$$

In points of $Fix(\mathbb{Z}_2)$ the matrix of the linearization of (4.3) block-diagonalizes in a (ξ_1, ξ_2)- and a (ξ_3, ξ_4)-block. The matrix B is equivalent to the latter. If we evaluate B along the branch of equilibria (5.6), then trace $B(\lambda, \sigma)$ becomes a linear function of σ and det $B(\lambda, \sigma)$ becomes a quadratic function of σ. According to our computations,

$$\begin{aligned} &\text{trace} \quad B(\lambda,\sigma) = 2p_0^3\xi_1 < 0, \quad \det B(\lambda,\sigma) = \xi_1[(p_0^3)^2 + (q_0^3)^2] > 0 \quad \text{for} \quad \sigma = \lambda\rho_2 \\ &\text{trace} \quad B(\lambda,\sigma) = \xi_2(c_0^3 - 2p_0^2) < 0, \quad \det B(\lambda,\sigma) = \xi_2^2 p_0^2 c_0^3 < 0 \quad \text{for} \quad \sigma = \lambda\rho_1. \end{aligned} \tag{5.8}$$

Therefore, trace $B(\lambda, \sigma) < 0$ for all λ and σ as in (5.6) and the function det $B(\lambda, \cdot)$ has a unique simple zero $\sigma = \sigma_0(\lambda) \in (\lambda\rho_1, \lambda\rho_2)$ for all $\lambda > 0$,

$$\det B(\lambda, \rho_0(\lambda)) = 0. \tag{5.9}$$

This implies that $B(\lambda, \sigma)$ has a simple zero eigenvalue and a negative real eigenvalue along the curve $\sigma = \sigma_0(\lambda) \quad (\lambda > 0)$ in the (λ, σ)-plane. Hence, along such a curve the equilibria of (4.3) corresponding to (5.6) loose their stability through a bifurcation of still another family of equilibria when either σ is decreased or λ is increased. Because of the $\mathbb{Z}_2$-symmetry, this is actually a pitchfork bifurcation. Note, that the eigenvector belonging to the zero eigenvalue is antisymmetric with respect to this symmetry. This follows from the block structure of the corresponding matrix. Consequently, the bifurcating equilibria are not $\mathbb{Z}_2$-symmetric.

We also mention, that $\sigma_0(\lambda)$ turns out to be positive for $0.4 \leq \eta_c \leq 0.55$ and negative for $0.6 \leq \eta_c \leq 0.85$.

A more careful analysis shows that this tertiary bifurcation is subcritical, and the bifurcating equilibria are unstable near the bifurcation point. It is an interesting open question whether the bifurcating solution branch turns to the right and attains stability somewhere away from the bifurcation point. Correspondingly, in the Couette-Taylor problem there is a curve of tertiary instabilities starting at the origin in the (λ, σ)-plane. There the wavy vortex flow looses stability through a bifurcation of quasiperiodic drift solutions (QD). These do not have any obvious spatial symmetry and, therefore, from a generic point of view they fill the Γ-orbits, i.e. 2-tori, corresponding to the bifurcating equilibria of (4.3) densly. Actually, they are quasiperiodic rotating waves with two frequencies. In a still photograph the corresponding fluid flow almost looks like the wavy vortex flow. But the flow cells do not have any spatial symmetry. As time increases, they do not only rotate in the azimuthal direction, but also slowly drift in the axial direction of the cylinders.

We summarize our results in the following theorem.

Theorem. There is a parameter regime in the Couette-Taylor problem (2.3), in particular $0.4 \leq \eta \leq 0.55, \alpha \sim 3.6$, where the following sequence of successive bifurcations occurs when the Reynolds number R is increased quasistatically:

$$COU \longrightarrow GTV \longrightarrow WV \longrightarrow QD.$$

At the primary and secondary bifurcations asymptotic stability (in the sense of Liapunov) is exchanged to the bifurcating solutions. The tertiary bifurcation is a subcritical pitchfork bifurcation, through which the wavy vortex flow looses stability. Here the bifurcating quasiperiodic drift flow is unstable close to the bifurcation point.

Remark. The quasiperiodic drift flow is not to be confused with the modulated wavy vortex flow which usually is observed after the tertiary instability in the standard Couette-Taylor experiment, where the outer cylinder is held fixed. That flow is also quasiperiodic, but still has a spatial $\mathbb{Z}_2$-symmetry. In the parameter regime, which we have studied, there appear to be no such solutions. As far as the identification of the quasiperiodic solutions as drift states and a theoretical and numerical computation of the curve of bifurcation, the direction of bifurcation and stability of that drift state is concerned, our theorem provides a supplement to the general bifurcation picture for this parameter regime developed by Golubitsky and Stewart [1986] and Golubitsky and Langford [1988] (see also Chossat and Iooss [1992]). For an analysis of bifurcations to drift states in related contexts, cf. Chossat and Golubistky [1988] and Golubitsky, Krupa and Lin [1991].

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