Discontinuous Galerkin methods

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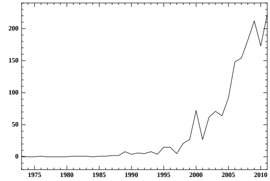
Introduction

- dG methods were introduced more than 40 years ago
- They have sparked extensive interest in the scientific computing and applied math communities



dG-related publications/year (Mathscinet)

A brief historical perspective

- Elliptic PDEs
  - boundary penalty methods [Nitsche 71]
  - interior penalty methods [Babuška 73, Douglas & Dupont 75, Baker 77, Wheeler 78, Arnold 82]
- First-order PDEs
  - neutron transport simulation [Reed & Hill 73] (steady, linear)
  - CV analysis [Lesaint & Raviart 74, Johnson & Pitkäranta 86]
  - time-dependent conservation laws [Cockburn & Shu 89-]
- Friedrichs systems
  - linear PDE systems with symmetry and L^2-positivity properties
  - unify mixed elliptic and first-order PDEs [AE & Guermond, 06-]

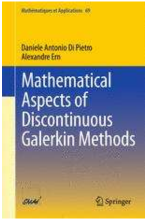
Motivations

- Discontinuous Galerkin (dG) methods can be viewed as
  - finite element methods with discontinuous discrete functions
  - finite volume methods with more than one DOF per mesh cell
- Possible motivations to consider dG methods
  - flexibility in the choice of basis functions
  - general meshes: non-matching interfaces, polyhedral cells
  - local discrete formulation using fluxes and local test functions (in particular, for **strongly-contrasted** material properties)
  - block-diagonal mass matrices for time-stepping
  - easily amenable to variable polynomial order, local time-stepping

Outline

- Part I: Diffusion
- Part II: Diffusion-advection-reaction

Main reference for this lecture **Di Pietro & AE, Mathematical aspects of DG methods, Springer 2012**



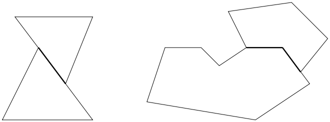
See also forthcoming book on FEM [AE & Guermond 16]

Diffusion

- Discrete setting
- Laplacian
- Variable diffusion

Discrete setting

- dG methods accommodate fairly general meshes
  - polyhedral cells (with various shapes)
  - nonmatching contact between adjacent cells (hanging nodes)



- Mesh indexed by h (e.g., maximal meshsize); CV analysis as $h \to 0$
- Given a mesh $\mathcal{T}_h$ of a domain Ω, examples of discrete spaces are the broken polynomial spaces ($k \geq 0$)

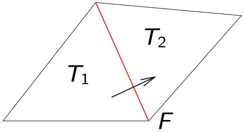
$$\mathbb{P}_d^k(\mathcal{T}_h) = \{v_h \in L^\infty(\Omega) \mid v_{h|T} \in \mathbb{P}_d^k(T),\ \forall T \in \mathcal{T}_h\}$$

Faces, mean-values, and jumps

- Interface $\mathcal{F}_h^i \ni F = \partial T_1 \cap \partial T_2$
  - oriented by unit normal n_F from T_1 to T_2 (fixed once and for all)
  - mean-values and jumps at interfaces ($v_i := v_{|T_i}$, $i \in \{1, 2\}$)

$$\{\!\!\{v\}\!\!\} = \frac{1}{2}(v_1 + v_2)$$

$$[\![v]\!] = v_1 - v_2$$



- Boundary face $\mathcal{F}_h^b \ni F = \partial T \cap \partial\Omega$, n_F pointing outward Ω

$$\{\!\!\{v\}\!\!\} = [\![v]\!] = v_{|T}$$

- Mesh faces are collected in the set $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$

Important algebraic identity

- Crucial when integrating by parts cellwise
- For pcw. smooth functions a (vector-valued) and b (scalar-valued)

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(ab) &= \sum_{T\in\mathcal{T}_h}\int_{\partial T}(ab)\cdot\mathrm{n}_T \quad \text{(outward unit normal to } T\text{)}\\ &= \sum_{F\in\mathcal{F}_h^i}\int_F [\![ab]\!]\cdot\mathrm{n}_F + \sum_{F\in\mathcal{F}_h^b}\int_F (ab)\cdot\mathrm{n}_F \quad (\mathrm{n}_F=\mathrm{n}_{T_1}=-\mathrm{n}_{T_2})\\ &= \sum_{F\in\mathcal{F}_h^i}\int_F (\{\!\{a\}\!\}[\![b]\!]+[\![a]\!]\{\!\{b\}\!\})\cdot\mathrm{n}_F + \sum_{F\in\mathcal{F}_h^b}\int_F (\{\!\{a\}\!\}[\![b]\!])\cdot\mathrm{n}_F\\ &= \sum_{F\in\mathcal{F}_h}\int_F \{\!\{a\}\!\}[\![b]\!]\cdot\mathrm{n}_F + \sum_{F\in\mathcal{F}_h^i}\int_F [\![a]\!]\{\!\{b\}\!\}\cdot\mathrm{n}_F\end{aligned}$$

Some basic facts from functional analysis

- Broken Sobolev spaces, e.g.,

$$H^1(\mathcal{T}_h) := \{v \in L^2(\Omega) \mid v_{|T} \in H^1(T),\ \forall T \in \mathcal{T}_h\}$$

- Broken gradient (defined cellwise) $\nabla_h : H^1(\mathcal{T}_h) \to [L^2(\Omega)]^d$

$$(\nabla_h v)_{|T} = \nabla(v_{|T}) \qquad \forall T \in \mathcal{T}_h$$

  We have $\nabla_h v = \nabla v$ if $v \in H^1(\Omega)$

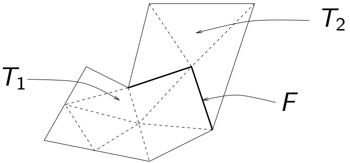
- A function $v \in H^1(\mathcal{T}_h)$ belongs to $H^1(\Omega)$ if and only if

$$[\![v]\!] = 0 \qquad \forall F \in \mathcal{F}_h^i$$

  (distributional argument)

Regularity of a mesh sequence $\{\mathcal{T}_h\}_{h>0}$

- Described by means of a matching simplicial submesh
  - shape-regular in the usual sense of Ciarlet
  - local meshsize comparable to that of $\mathcal{T}_h$



- Geometric properties resulting from mesh regularity
  - #(subsimplices) of $T \in \mathcal{T}_h$ is **uniformly bounded**
  - #(faces) of $T \in \mathcal{T}_h$ is **uniformly bounded**
  - $h_{T_1} \sim h_F \sim h_{T_2}$

Analysis tools

- **Local inverse inequality** $\forall v_h \in \mathbb{P}_d^k(T)$, $\forall T \in \mathcal{T}_h$,

$$\|\nabla v_h\|_{[L^2(T)]^d} \le C_{\mathrm{inv}} h_T^{-1} \|v_h\|_{L^2(T)}$$

  - Markov brothers' inequality in $L^\infty(-1,1)$ (1890)
  - $C_{\mathrm{inv}} \sim k^2$ [Schwab 98]; C_{inv} computable from eigenvalue pb.

- **Multiplicative trace inequality** $\forall v \in H^1(T)$, $\forall T \in \mathcal{T}_h$,

$$\|v\|_{L^2(\partial T)} \le C_{\mathrm{mtr}} \left(h_T^{-\frac{1}{2}} \|v\|_{L^2(T)} + \|v\|_{L^2(T)}^{\frac{1}{2}} \|\nabla v\|_{[L^2(T)]^d}^{\frac{1}{2}} \right)$$

  - lowest-order Raviart–Thomas functions and divergence formula [Carstensen & Funken 00; Stephansen 07; Di Pietro & AE 12]
  - in a polyhedral cell, carve a sub-simplex from each triangular sub-face with height $\sim h_T$ (allows for some face degeneration)

- **Discrete trace inequality** $\forall v_h \in \mathbb{P}_d^k(T)$, $\forall T \in \mathcal{T}_h$,

$$\|v_h\|_{L^2(\partial T)} \le C_{\mathrm{dtr}} h_T^{-\frac{1}{2}} \|v_h\|_{L^2(T)}$$

  - follows from LI and MT inequalities; $C_{\mathrm{dtr}} \sim k$

Polynomial approximation in polyhedral cells

- L^2-orthogonal projection $\pi_T^k : L^2(T) \to \mathbb{P}_d^k(T)$

$$(\pi_T^k(v) - v, q)_{L^2(T)} = 0 \qquad \forall q \in \mathbb{P}_d^k(T)$$

- **Poincaré–Steklov inequality** $\forall v \in H^1(T)$, $\forall T \in \mathcal{T}_h$,

$$\|v - \pi_h^0(v)\|_{L^2(T)} \leq C_{\mathrm{PS}} h_T \|\nabla v\|_{[L^2(T)]^d}$$

  - $\pi_h^0(v)$ is the mean-value of v over T
  - $C_{\mathrm{PS}} = \pi^{-1}$ for convex T (Poincaré (1894) [eigenvalue pb], Steklov (1897) [$d = 1$], Payne & Weinberger (60) [$d = 2$], Bebendorf (03) [$d \geq 3$])
  - For non-convex T, uniform bound on C_{PS} using simplicial sub-cells and MT inequality [AE & Guermond 16]

- PS inequality can be bootstrapped using Bramble–Hilbert polynomial to $|v - \pi_T^k(v)|_{H^m(T)} \leq C_{\mathrm{app}} h_T^{k+1-m} |v|_{H^{k+1}(T)}$ for all $0 \leq m \leq k+1$

- See also [Dupont & Scott 80] for alternate proof using averaged Taylor polynomials

Most useful properties

- $\forall v \in H^{k+1}(T)$, $\forall T \in \mathcal{T}_h$,

$$\|v - \pi_T^k v\|_{L^2(T)} \le C_{\mathrm{app}} h_T^{k+1} |v|_{H^{k+1}(T)}$$
$$\|\nabla(v - \pi_T^k v)\|_{[L^2(T)]^d} \le C_{\mathrm{app}} h_T^{k} |v|_{H^{k+1}(T)}$$
$$\|v - \pi_T^k v\|_{L^2(\partial T)} \le C_{\mathrm{app}} h_T^{k+\frac{1}{2}} |v|_{H^{k+1}(T)}$$

  - bounds extend to fractional Sobolev regularity [AE & Guermond 16]

- Global L^2-orth. projection $\pi_h^k : L^2(\Omega) \to \mathbb{P}_d^k(\mathcal{T}_h)$ is assembled cellwise

$$\pi_h^k(v)_{|T} = \pi_T^k(v_{|T}) \qquad \forall T \in \mathcal{T}_h$$

(global mass matrix is block-diagonal)

The Laplacian

- Let $f \in L^2(\Omega)$; seek $u : \Omega \to \mathbb{R}$ s.t. $-\triangle u = f$ in Ω and $u_{|\partial\Omega} = 0$
- Weak formulation: $u \in V := H^1_0(\Omega)$ s.t.

$$a(u, w) := \int_\Omega \nabla u \cdot \nabla w = \int_\Omega f w =: \ell(w) \qquad \forall w \in V$$

- The exact solution satisfies

$$[\![u]\!] = 0 \qquad \forall F \in \mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$$

- Other BC's (Neumann, Robin) can be considered as well

Normal flux

- Physically, the normal component of the diffusive flux $\sigma := -\nabla u$ is continuous across interfaces
- What is the mathematical meaning of $[\![\sigma]\!]\cdot \mathrm{n}_F = 0$ for $F \in \mathcal{F}_h^i$?
- If $\sigma \in [L^p(\Omega)]^d$, $p > 2$, and $\nabla\cdot\sigma \in L^2(\Omega)$ then

$$\sigma_{|T}\cdot \mathrm{n}_F \in W^{-\frac{1}{p},p}(F) \qquad \forall T \in \mathcal{T}_h,\ \forall F \subset \partial T$$

  - this holds provided $u \in H^{1+s}(\Omega)$, $s > 0$, and $\triangle u \in L^2(\Omega)$
- If $\sigma \in [H^s(\Omega)]^d$, $s > \frac{1}{2}$, then $\sigma_{|\partial T} \in [L^2(\partial T)]^d$
  - this holds provided $u \in H^{1+s}(\Omega)$, $s > \frac{1}{2}$
- Elliptic regularity theory shows that on a polyhedron, $u \in H^{1+s}(\Omega)$, $s > \frac{1}{2}$, and $s = 1$ if Ω is convex

Symmetric Interior Penalty

- Discrete space $V_h := \mathbb{P}_d^k(\mathcal{T}_h)$, $k \geq 1$
- Seek $u_h \in V_h$ s.t. $a_h(u_h, w_h) = \ell(w_h)$, $\forall w_h \in V_h$, with

$$a_h(v_h, w_h) := \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \underbrace{\sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h v_h\}\!\} \cdot \mathrm{n}_F [\![w_h]\!]}_{\text{consistency}}$$

$$- \underbrace{\sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \{\!\{\nabla_h w_h\}\!\} \cdot \mathrm{n}_F}_{\text{symmetry}} + \underbrace{\sum_{F \in \mathcal{F}_h} \frac{\eta}{h_F} \int_F [\![v_h]\!][\![w_h]\!]}_{\text{penalty}}$$

- Main properties of a_h
  - strong consistency: $a_h(u, w_h) = \ell(w_h)$, $\forall w_h \in V_h$
  - coercivity on V_h if η is large enough

Step-by-step derivation

- Starting point: Use broken gradient in exact bilinear form

$$a_h^{(0)}(v_h, w_h) := \int_\Omega \nabla_h v_h \cdot \nabla_h w_h$$

- Restore consistency ($a_h^{(0)}(u, w_h) = \ell(w_h) + \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla u\}\!\}\cdot \mathrm{n}_F [\![w_h]\!]$)

$$a_h^{(1)}(v_h, w_h) := \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h v_h\}\!\}\cdot \mathrm{n}_F [\![w_h]\!]$$

- Restore symmetry in a consistent way

$$a_h^{(2)}(v_h, w_h) := \int_\Omega \nabla_h v_h \cdot \nabla_h w_h - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h v_h\}\!\}\cdot \mathrm{n}_F [\![w_h]\!] - \sum_{F\in\mathcal{F}_h} \int_F [\![v_h]\!] \{\!\{\nabla_h w_h\}\!\}\cdot \mathrm{n}_F$$

- Achieve coercivity by penalizing jumps [Arnold 82]

Stability

- dG norm: broken gradient plus jump seminorm

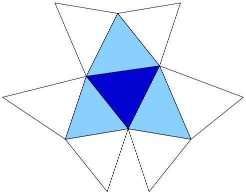
$$\|v_h\|_{\mathrm{dG}}^2 := \|\nabla_h v_h\|_{[L^2(\Omega)]^d}^2 + |v_h|_{\mathrm{J}}^2, \qquad |v_h|_{\mathrm{J}}^2 = \sum_{F\in\mathcal{F}_h} \frac{1}{h_F}\|[\![v_h]\!]\|_{L^2(F)}^2$$

- $\|\cdot\|_{\mathrm{dG}}$ is a norm on V_h (direct verification)
  - discrete Sobolev inequality $\|v_h\|_{L^q(\Omega)} \le \sigma_q \|v_h\|_{\mathrm{dG}}$, $\forall v_h \in V_h$, with $q \in [1, \frac{2d}{d-2}]$ if $d \ge 3$ and $q \in [1, \infty)$ if $d = 2$
  - see [Brenner 03] (for $q = 2$), [Eymard, Gallouët & Herbin 10] (for FV and general q), [Di Pietro & AE 10] (for general q, k)
- If $\eta > C_{\mathrm{dtr}}^2 N_\partial$, where
  - C_{dtr} results from discrete trace inequality (recall $C_{\mathrm{dtr}} \sim k$)
  - N_∂ is the maximum number of faces a mesh cell can have

  then $\exists C_{\mathrm{sta}} > 0$ s.t. $a_h(v_h, v_h) \ge C_{\mathrm{sta}} \|v_h\|_{\mathrm{dG}}^2$, $\forall v_h \in V_h$

Algebraic realization

- SPD stiffness matrix
- Compact stencil (only neighbors in the sense of faces)



Error analysis: Boundedness

- Approximation error $(u - u_h)$ is in $V_\flat = (H^{1+s}(\Omega) \cap V) + V_h$, $s > \frac{1}{2}$
- Boundedness: $a_h(v, w_h) \le C_{\mathrm{bnd}} \|v\|_{\mathrm{dG},\sharp} \|w_h\|_{\mathrm{dG}}$, $\forall (v, w_h) \in V_\flat \times V_h$

$$\|v\|_{\mathrm{dG},\sharp}^2 := \|v\|_{\mathrm{dG}}^2 + \sum_{T \in \mathcal{T}_h} h_T \|\nabla v \cdot \mathrm{n}_T\|_{L^2(\partial T)}^2$$

- The two norms are equivalent on V_h

$$\|v_h\|_{\mathrm{dG}} \le \|v_h\|_{\mathrm{dG},\sharp} \le C_\sharp \|v_h\|_{\mathrm{dG}} \qquad \forall v_h \in V_h$$

Error analysis: Second Strang's Lemma

- Optimal error estimate in $\|\cdot\|_{\mathrm{dG},\sharp}$-norm

$$\|u-u_h\|_{\mathrm{dG},\sharp} \le C \inf_{y_h\in V_h} \|u-y_h\|_{\mathrm{dG},\sharp}$$

- Let $y_h \in V_h$; coercivity, consistency, and boundedness imply

$$\begin{aligned}
\|u_h-y_h\|_{\mathrm{dG},\sharp} &\le C_\sharp \|u_h-y_h\|_{\mathrm{dG}} \\
&\le C_\sharp C_{\mathrm{sta}}^{-1} \sup_{w_h\in V_h\setminus\{0\}} \frac{a_h(u_h-y_h,w_h)}{\|w_h\|_{\mathrm{dG}}} \\
&= C_\sharp C_{\mathrm{sta}}^{-1} \sup_{w_h\in V_h\setminus\{0\}} \frac{a_h(u-y_h,w_h)}{\|w_h\|_{\mathrm{dG}}} \\
&\le C_\sharp C_{\mathrm{sta}}^{-1} C_{\mathrm{bnd}} \|u-y_h\|_{\mathrm{dG},\sharp}
\end{aligned}$$

and use the triangle inequality

$$\|u-u_h\|_{\mathrm{dG},\sharp} \le (1+C_\sharp C_{\mathrm{sta}}^{-1} C_{\mathrm{bnd}}) \|u-y_h\|_{\mathrm{dG},\sharp}$$

Convergence rates

- Assume exact solution u is smooth enough
- Using polynomial approximation properties in dG spaces yields

$$\|u - u_h\|_{\mathrm{dG},\sharp} \leq C \left(\sum_{T \in \mathcal{T}_h} h_T^{2k} |u|^2_{H^{k+1}(T)} \right)^{1/2}$$

- Assuming full elliptic regularity pickup, Aubin–Nitsche's duality argument leads to

$$\|u - u_h\|_{L^2(\Omega)} \leq C h \left(\sum_{T \in \mathcal{T}_h} h_T^{2k} |u|^2_{H^{k+1}(T)} \right)^{1/2}$$

Two side-excursions

- Lifting the jumps
- Mixed dG methods

Lifting the jumps I

- **Local lifting** Let $l \geq 0$, $F \in \mathcal{F}_h$; $\mathrm{r}_F^l : L^1(F) \to [\mathbb{P}_d^l(\mathcal{T}_h)]^d$ is s.t.

$$\int_\Omega \mathrm{r}_F^l(\varphi)\cdot\tau_h = \int_F \{\!\{\tau_h\}\!\}\cdot\mathrm{n}_F\varphi \qquad \forall\tau_h \in [\mathbb{P}_d^l(\mathcal{T}_h)]^d$$

  - r_F^l is vector-valued, collinear to n_F
  - the support of r_F^l reduces to the (one or two) mesh cells sharing F
  - r_F^l is easy to compute (invert local mass matrix)
  - see [Bassi, Rebay et al 97], [Brezzi et al 00]

- Penalizing local liftings of jumps instead of jumps yields coercivity for $\eta > N_\partial$ with the same stencil, $l \in \{k-1, k\}$

$$a_h(v_h, w_h) := \int_\Omega \nabla_h v_h\cdot\nabla_h w_h - \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\nabla_h v_h\}\!\}\cdot\mathrm{n}_F[\![w_h]\!]$$
$$- \sum_{F\in\mathcal{F}_h}\int_F [\![v_h]\!]\{\!\{\nabla_h w_h\}\!\}\cdot\mathrm{n}_F + \sum_{F\in\mathcal{F}_h}\eta\int_\Omega \mathrm{r}_F^l([\![v_h]\!])\cdot\mathrm{r}_F^l([\![w_h]\!])$$

Lifting the jumps II

- **Global lifting of jumps**: For all $v \in H^1(\mathcal{T}_h)$,

$$\mathrm{R}_h^l([\![v]\!]) := \sum_{F\in\mathcal{F}_h} \mathrm{r}_F^l([\![v]\!]) \in [\mathbb{P}_d^l(\mathcal{T}_h)]^d$$

- **Discrete gradient** $G_h^l : H^1(\mathcal{T}_h) \to [L^2(\Omega)]^d$ s.t.

$$G_h^l(v) := \nabla_h v - \mathrm{R}_h^l([\![v]\!])$$

- Discrete gradients are important tools in nonlinear problems
  - nonlinear elasticity [Lew et al. '04], Leray–Lions [Burman & AE '08, Buffa & Ortner '09], Navier–Stokes [Di Pietro & AE '10]
  - asymptotic consistency: Let $(v_h)_{h>0}$ be a sequence in $(V_h)_{h>0}$ bounded in the $\|\cdot\|_{\mathrm{dG}}$-norm. Then, $\exists v \in H_0^1(\Omega)$ s.t. as $h \to 0$, up to subseq., $v_h \to v$ strongly in $L^2(\Omega)$ and for all $l \geq 0$, $G_h^l(v_h) \rightharpoonup \nabla v$ weakly in $[L^2(\Omega)]^d$ [Di Pietro & AE '10]

Lifting the jumps III

- Local formulation with numerical fluxes (FV viewpoint)
- Let $T \in \mathcal{T}_h$ with faces collected in $\mathcal{F}_T$, let $\xi \in \mathbb{P}_d^k(T)$
- For the exact solution

$$\int_T \nabla u \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \Phi_F(u) \xi = \int_T f \xi$$

with $\epsilon_{T,F} = \mathrm{n}_T \cdot \mathrm{n}_F$ and exact flux $\Phi_F(u) = -\nabla u \cdot \mathrm{n}_F$

- For the discrete solution ($l \in \{k-1, k\}$)

$$\int_T (\nabla u_h - \mathrm{R}_h^l([\![u_h]\!])) \cdot \nabla \xi + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h) \xi = \int_T f \xi$$

with numerical flux $\phi_F(u_h) = -\{\!\{\nabla_h u_h\}\!\} \cdot \mathrm{n}_F + \frac{\eta}{h_F} [\![u_h]\!]$

Mixed dG methods I

- Mixed formulation: $\sigma + \nabla u = 0$ and $\nabla{\cdot}\sigma = f$ in Ω
- Mixed dG method: Find $u_h \in \mathbb{P}_d^k(\mathcal{T}_h)$, $\sigma_h \in [\mathbb{P}_d^k(\mathcal{T}_h)]^d$ (equal-order) s.t.

$$\int_T \sigma_h{\cdot}\zeta - \int_T u_h \nabla{\cdot}\zeta + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F \hat{u}_F(\zeta{\cdot}\mathrm{n}_F) = 0 \qquad \forall \zeta \in [\mathbb{P}_d^k(T)]^d$$

$$-\int_T \sigma_h{\cdot}\nabla\xi + \sum_{F\in\mathcal{F}_T} \epsilon_{T,F} \int_F (\hat{\sigma}_F{\cdot}\mathrm{n}_F)\xi = \int_T f\xi \qquad \forall \xi \in \mathbb{P}_d^k(T)$$

for all $T \in \mathcal{T}_h$, with numerical fluxes $\hat{u}_F$ and $\hat{\sigma}_F$

- σ_h can be eliminated locally whenever $\hat{u}_F$ does not depend on σ_h
- See [Arnold, Brezzi, Cockburn, and Marini 02] for a unified analysis of dG methods based on numerical fluxes

Mixed dG methods II

- Numerical fluxes for SIP

$$\hat{u}_F = \begin{cases} \{\!\{u_h\}\!\} & \forall F \in \mathcal{F}_h^i \\ 0 & \forall F \in \mathcal{F}_h^b \end{cases}$$

$$\hat{\sigma}_F = -\{\!\{\nabla_h u_h\}\!\} + \eta h_F^{-1} [\![u_h]\!] \mathrm{n}_F \quad \forall F \in \mathcal{F}_h$$

- Numerical fluxes for LDG (Local dG [Cockburn & Shu 98])
  $\hat{u}_F$ as for SIP and

$$\hat{\sigma}_F = \{\!\{\sigma_h\}\!\} + \eta h_F^{-1} [\![u_h]\!] \mathrm{n}_F$$

  - σ_h can be eliminated locally
  - main advantage: discrete coercivity for $\eta > 0$ (e.g. $\eta = 1$)
  - drawback: **larger stencil** (neighbors of neighbors)
  - stencil reduction [Castillo, Cockburn, Perugia, and Schötzau 00]

Mixed dG methods III

- Two-field approach [AE & Guermond 06]

$$\hat{u}_F = \begin{cases} \{\!\{u_h\}\!\} + \eta_\sigma [\![\sigma_h]\!]\cdot \mathrm{n}_F & \forall F \in \mathcal{F}_h^i \\ 0 & \forall F \in \mathcal{F}_h^b \end{cases}$$

$$\hat{\sigma}_F = \{\!\{\sigma_h\}\!\} + \eta_u [\![u_h]\!] \mathrm{n}_F \quad \forall F \in \mathcal{F}_h$$

- Drawback σ_h cannot be eliminated
- Advantages
  - a simple choice for penalty is $\eta_u = \eta_\sigma = 1$
  - the choice $k = 0$ is possible
  - quasi-optimal estimate on the diffusive flux

Mixed dG methods IV

- Hybridizable dG (HDG) methods introduce interface DOFs
  - [Cockburn, Gopalakrishnan, and Lazarov 09]
  - see also [Causin and Sacco 05], [Droniou and Eymard 06]
- Skeletal discrete space $\Lambda_h := \bigoplus_{F\in\mathcal{F}_h^i} \mathbb{P}_{d-1}^k(F)$
- Discrete unknowns $(\sigma_h, u_h, \lambda_h) \in \Sigma_h \times U_h \times \Lambda_h$
  - σ_h and u_h can be **eliminated locally**
  - global problem in $\lambda_h \in \Lambda_h$ with **compact stencil**
- A new viewpoint emerged recently: Hybrid High-Order (HHO) methods
  - introduced in [Di Pietro & AE 15], [Di Pietro, AE & Lemaire 14]
  - see tomorrow's lecture!

Variable diffusion

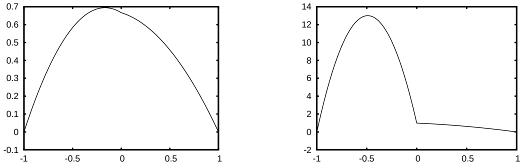
- Seek $u : \Omega \to \mathbb{R}$ s.t. $-\nabla\cdot(\kappa\nabla u) = f$ in Ω and $u_{|\partial\Omega} = 0$
- Weak formulation: For $f \in L^2(\Omega)$, seek $u \in V := H^1_0(\Omega)$ s.t.

$$a(u, v) := \int_\Omega \kappa\nabla u\cdot\nabla v = \int_\Omega fv \qquad \forall v \in V$$

  - κ is scalar-valued, bounded, and uniformly positive in Ω
  - the model problem is well-posed
- Application to groundwater flows
  - u: hydraulic head, $\sigma = -\kappa\nabla u$: Darcy velocity
  - κ: **highly-contrasted** hydraulic conductivity

Numerical illustration of high contrasts

- $\sigma = -\kappa \nabla u \in H(\text{div};\Omega)$
  - the normal component of σ is continuous across any interface
  - the normal component of ∇u is discontinuous if κ jumps
- $\Omega = (-1,1)$ partitioned into $\Omega_1 = (-1,0)$ and $\Omega_2 = (0,1)$, $\kappa_{|\Omega_1} = \alpha$ ($\alpha = 0.5$ on left; $\alpha = 0.01$ on right) and $\kappa_{|\Omega_2} = 1$



Discrete setting

- κ pcw. constant on a polyhedral partition $P_\Omega = \{\Omega_i\}_{1\le i\le N_\Omega}$ of Ω
  - $\mathcal{T}_h$ compatible with P_Ω (κ pcw. constant on $\mathcal{T}_h$)
- Discrete space $V_h := \mathbb{P}_d^k(\mathcal{T}_h)$, $k \ge 1$
- SIP bilinear form

$$a_h(v_h, w_h) = \int_\Omega \kappa\nabla_h v_h \cdot \nabla_h w_h - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\kappa\nabla_h v_h\}\!\}\cdot \mathrm{n}_F [\![w_h]\!]$$

$$- \sum_{F\in\mathcal{F}_h} \int_F [\![v_h]\!]\{\!\{\kappa\nabla_h w_h\}\!\}\cdot \mathrm{n}_F + \sum_{F\in\mathcal{F}_h} \eta \frac{\gamma_{\kappa,F}}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$

Diffusion-dependent penalty

- To achieve coercivity, penalty coefficient must depend on κ
  - $\gamma_{\kappa,F} = \{\!\{\kappa\}\!\}$ [Houston, Schwab & Süli 02]
  - for high contrasts, $\gamma_{\kappa,F}$ is controlled by the highest value of κ (the most permeable layer)
  - ... $\gamma_{\kappa,F}$ should be controlled by the lowest value (the least permeable layer) (as in Mixed FE and FV)
- One simple choice is harmonic averaging

$$\gamma_{\kappa,F}^{-1} := \{\!\{\kappa^{-1}\}\!\}$$

We need to modify the consistency and symmetry terms to maintain coercivity

Symmetric Weighted Interior Penalty (SWIP)

- Weighted average $\{\!\{v\}\!\}_{\omega,F} := \omega_{T_1,F} v_{|T_1} + \omega_{T_2,F} v_{|T_2}$
  - $\omega_{T_1,F} = \omega_{T_2,F} = \frac{1}{2}$ recovers usual arithmetic averages
  - diffusion-dependent weights $\omega_{T_1,F} := \frac{\kappa_2}{\kappa_1+\kappa_2}$, $\omega_{T_2,F} := \frac{\kappa_1}{\kappa_1+\kappa_2}$ (homogeneous diffusion yields back arithmetic averages)
  - see [Dryja 03] for idea, [Burman & Zunino 06] for mortaring, [AE, Stephansen & Zunino 09], [Di Pietro, AE & Guermond 08] for advection-diffusion with locally small or zero diffusion
- SWIP bilinear form

$$a_h(v_h, w_h) = \int_\Omega \kappa \nabla_h v_h \cdot \nabla_h w_h - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\kappa \nabla_h v_h\}\!\}_\omega \cdot \mathrm{n}_F [\![w_h]\!]$$
$$- \sum_{F \in \mathcal{F}_h} \int_F [\![v_h]\!] \{\!\{\kappa \nabla_h w_h\}\!\}_\omega \cdot \mathrm{n}_F + \sum_{F \in \mathcal{F}_h} \eta \frac{\gamma_{\kappa,F}}{h_F} \int_F [\![v_h]\!][\![w_h]\!]$$

- Strong consistency still holds

Error analysis

- C denotes a generic constant uniform w.r.t. h and κ
- Coercivity: Assuming $\eta > C_{\mathrm{tr}}^2 N_\partial$, $a_h(v_h, v_h) \geq C_{\mathrm{sta}} \|v_h\|_{\mathrm{dG}}^2$ with

$$\|v_h\|_{\mathrm{dG}}^2 := \|\kappa^{\frac{1}{2}} \nabla_h v_h\|_{[L^2(\Omega)]^d}^2 + \sum_{F \in \mathcal{F}_h} \frac{\gamma_{\kappa,F}}{h_F} \|[\![v_h]\!]\|_{L^2(F)}^2$$

- Boundedness: $a_h(v, w_h) \leq C_{\mathrm{bnd}} \|v\|_{\mathrm{dG},\sharp} \|w_h\|_{\mathrm{dG}}$ with $\|v\|_{\mathrm{dG},\sharp}^2 := \|v\|_{\mathrm{dG}}^2 + \sum_{T \in \mathcal{T}_h} h_T \|\kappa^{\frac{1}{2}} \nabla v \cdot \mathrm{n}_T\|_{L^2(\partial T)}^2$
- Error estimate: $\|u - u_h\|_{\mathrm{dG},\sharp} \leq C \inf_{y_h \in V_h} \|u - y_h\|_{\mathrm{dG},\sharp}$

$$\|u - u_h\|_{\mathrm{dG},\sharp} \leq C \left(\sum_{T \in \mathcal{T}_h} \kappa_T h_T^{2k} |u|_{H^{k+1}(T)}^2 \right)^{1/2}$$

- Extension to anisotropic κ: use normal component for penalty and averages (error estimate mildly depends on anisotropy ratio $\sim \rho^{\frac{1}{2}}$)

Outline

- Advection-reaction
- Péclet-robust diffusion-advection-reaction

Model problem

- Let Ω be a domain in $\mathbb{R}^d$ (open, bounded, connected, strongly Lipschitz set)
- Let $\beta \in [W^{1,\infty}(\Omega)]^d$ and $\mu \in L^\infty(\Omega)$ be s.t.

$$\mu - \frac{1}{2}\nabla\cdot\beta \geq \mu_0 > 0 \qquad \text{a.e. in } \Omega$$

- Inflow and outflow parts of boundary $\partial\Omega$

$$\partial\Omega^\pm = \{x \in \partial\Omega \mid \pm\beta(x)\cdot\mathrm{n}(x) > 0\}$$

- Let $f \in L^2(\Omega)$; the model problem is

$$\begin{cases} \mu u + \beta\cdot\nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega^- \end{cases}$$

Functional framework

- Graph space $W = \{v \in L^2(\Omega) \mid \beta\cdot\nabla v \in L^2(\Omega)\}$
  - Hilbert space with norm $\|v\|_W^2 = \|v\|_{L^2(\Omega)}^2 + \|\beta\cdot\nabla v\|_{L^2(\Omega)}^2$
- If $\partial\Omega^\pm$ are well-separated, there is a bounded trace map $\gamma : W \to L^2(|\beta\cdot\mathrm{n}|; \partial\Omega)$ s.t. for all $(v, w) \in W \times W$,

$$\int_\Omega (\nabla\cdot\beta) vw + \int_\Omega (\beta\cdot\nabla v) w + \int_\Omega v(\beta\cdot\nabla w) = \int_{\partial\Omega} (\beta\cdot\mathrm{n})\gamma(v)\gamma(w)$$

  - see [AE & Guermond 06]
  - the separation assumption cannot be circumvented for traces in $L^2(|\beta\cdot\mathrm{n}|; \partial\Omega)$

Weak formulation

- Define on $W \times W$ the bilinear form

$$a(v,w) := \int_\Omega \mu vw + (\beta\cdot\nabla v)w + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^\ominus vw$$

where for $x \in \mathbb{R}$, $x^\oplus = \frac{1}{2}(|x| + x)$ and $x^\ominus = \frac{1}{2}(|x| - x)$

- Define on W the linear form $\ell(w) := \int_\Omega fw$
- Seek $u \in W$ s.t. $a(u,w) = \ell(w)$, $\forall w \in W$
- BCs are weakly enforced

Well-posedness

- a is L^2-coercive on W: integrating by parts, we infer that

$$a(v,v) = \int_\Omega \left(\mu - \frac{1}{2}\nabla\cdot\beta\right) v^2 + \frac{1}{2}\int_{\partial\Omega} (\beta\cdot\mathrm{n})\gamma(v)^2 + \int_{\partial\Omega} (\beta\cdot\mathrm{n})^\ominus \gamma(v)^2$$
$$\geq \mu_0 \|v\|^2_{L^2(\Omega)} + \frac{1}{2}\int_{\partial\Omega} |\beta\cdot\mathrm{n}|\gamma(v)^2$$

- The weak problem is well-posed
  - L^2-coercivity implies uniqueness
  - existence by inf-sup argument [Ern & Guermond 06]

Discrete setting

- Discrete space $V_h := \mathbb{P}_d^k(\mathcal{T}_h)$, $k \geq 0$
- Discrete problem: Seek $u_h \in V_h$ s.t. $a_h(u_h, w_h) = \ell(w_h)$, $\forall w_h \in V_h$
- Main properties of a_h: strong consistency and L^2-coercivity on V_h
- We assume that $u \in H^s(\Omega)$, $s > \frac{1}{2}$; then,

$$(\beta \cdot \mathrm{n}_F)[\![u]\!] = 0 \qquad \forall F \in \mathcal{F}_h^i$$

(distributional argument)

Centered fluxes

- Use broken gradient in exact bilinear form
- Recover L^2-coercivity in a consistent way by setting

$$a_h^{\mathrm{cf}}(v_h, w_h) := \int_\Omega \mu v_h w_h + (\beta\cdot\nabla_h v_h) w_h + \sum_{F\in\mathcal{F}_h^b} \int_F (\beta\cdot\mathrm{n})^{\ominus} v_h w_h$$
$$- \sum_{F\in\mathcal{F}_h^i} \int_F (\beta\cdot\mathrm{n}_F)[\![v_h]\!]\{\!\{w_h\}\!\}$$

- $a_h^{\mathrm{cf}}(v_h, v_h) \geq \mu_0 \|v_h\|_{L^2(\Omega)}^2 + \sum_{F\in\mathcal{F}_h^b} \int_F \frac{1}{2}|\beta\cdot\mathrm{n}| v_h^2$
- Error estimate for smooth solution: $\|u - u_h\|_{L^2(\Omega)} \leq Ch^k |u|_{H^{k+1}(\Omega)}$
  - convergence for $k \geq 1$ only, and with suboptimal rate

Local formulation and stencil

- Let $T \in \mathcal{T}_h$, let $\xi \in \mathbb{P}_d^k(T)$ (FV viewpoint)

$$\int_T (\mu - \nabla{\cdot}\beta) u_h \xi - u_h(\beta{\cdot}\nabla\xi) + \sum_{F \in \mathcal{F}_T} \epsilon_{T,F} \int_F \phi_F(u_h)\xi = \int_T f\xi$$

with $\epsilon_{T,F} := \mathrm{n}_T{\cdot}\mathrm{n}_F = \pm 1$ and numerical fluxes

$$\phi_F(u_h) = \begin{cases} (\beta{\cdot}\mathrm{n}_F)\{\!\{u_h\}\!\} & \forall F \in \mathcal{F}_h^i \\ (\beta{\cdot}\mathrm{n})^{\oplus} u_h & \forall F \in \mathcal{F}_h^b \end{cases}$$

- Standard dG stencil (neighbors in the sense of faces)

Upwind fluxes

- Strengthen discrete stability by penalizing interface jumps in a least-squares sense [Brezzi, Marini & Süli 04]

$$a_h(v_h, w_h) := a_h^{\mathrm{cf}}(v_h, w_h) + s_h(v_h, w_h)$$

with stabilization bilinear form

$$s_h(v_h, w_h) = \sum_{F \in \mathcal{F}_h^i} \int_F \tfrac{1}{2} |\beta \cdot \mathrm{n}_F| [\![v_h]\!] [\![w_h]\!]$$

- Strong consistency is preserved

Stability

- Stability norm ($\beta_T := \|\beta\|_{[L^\infty(T)]^d}$)

$$\|v_h\|_{\mathrm{dG}}^2 := \mu_0 \|v_h\|_{L^2(\Omega)}^2 + \sum_{F\in\mathcal{F}_h^b} \int_F \frac{1}{2}|\beta\cdot\mathrm{n}| v_h^2 + \sum_{F\in\mathcal{F}_h^i} \int_F \frac{1}{2}|\beta\cdot\mathrm{n}_F| [\![v_h]\!]^2$$
$$+ \sum_{T\in\mathcal{T}_h} \beta_T^{-1} h_T \|\beta\cdot\nabla v\|_{L^2(T)}^2$$

- Assume for simplicity $h_T\mu_0 \le c_{\mu,\beta}\beta_T$, $L_{\beta,T} + \|\mu\|_{L^\infty(T)} \le c_{\mu,\beta}\mu_0$
  - we hide $c_{\mu,\beta}$ in the generic constants
  - general weight on adv. derivative: time-scale $\tau_T = \min(\mu_0^{-1}, \beta_T^{-1}h_T)$
- Discrete inf-sup condition [Johnson & Pitkäranta 86]

$$C_{\mathrm{sta}}\|v_h\|_{\mathrm{dG}} \le \sup_{w_h\in V_h\setminus\{0\}} \frac{a_h(v_h, w_h)}{\|w_h\|_{\mathrm{dG}}}$$

  - first three terms controlled by coercivity
  - bound on advective derivative: test with $w_{h|T} = \beta_T^{-1}h_T\langle\beta\rangle_T\cdot\nabla v_h$

Error analysis

- Boundedness: $a_h(v, w_h) \leq C_{\mathrm{bnd}} \|v\|_{\mathrm{dG},\sharp} \|w_h\|_{\mathrm{dG}}$ with

$$\|v\|_{\mathrm{dG},\sharp}^2 := \|v\|_{\mathrm{dG}}^2 + \sum_{T \in \mathcal{T}_h} \beta_T \left(h_T^{-1} \|v\|_{L^2(T)}^2 + \|v\|_{L^2(\partial T)}^2 \right)$$

- Error estimate: $\|u - u_h\|_{\mathrm{dG}} \leq C \inf_{y_h \in V_h} \|u - y_h\|_{\mathrm{dG},\sharp}$
  - $\|\cdot\|_{\mathrm{dG},\sharp}$ and $\|\cdot\|_{\mathrm{dG}}$ may not be equivalent on V_h, but they lead to the same decay rates of best-approximation errors on smooth functions
  - $\|u - u_h\|_{\mathrm{dG}} \leq C(\sum_{T \in \mathcal{T}_h} \beta_T h_T^{2k+1} |u|_{H^{k+1}(T)^2})^{1/2}$
  - quasi-optimal L^2-error estimate $O(h^{k+\frac{1}{2}})$
  - optimal error estimate on advective derivative

Local formulation and stencil

- Let $T \in \mathcal{T}_h$, let $\xi \in \mathbb{P}_d^k(T)$
- New numerical fluxes

$$\phi_F(u_h) = \begin{cases} (\beta\cdot\mathrm{n}_F)\{\!\{u_h\}\!\} + \frac{1}{2}|\beta\cdot\mathrm{n}_F|[\![u_h]\!] & \forall F \in \mathcal{F}_h^i \\ (\beta\cdot\mathrm{n})^{\oplus} u_h & \forall F \in \mathcal{F}_h^b \end{cases}$$

- Example: $F = \partial T_1 \cap \partial T_2$, β flows from T_1 to T_2 so that $\beta\cdot\mathrm{n}_F \geq 0$

$$\begin{aligned} \phi_F(u_h) &= (\beta\cdot\mathrm{n}_F)(\{\!\{u_h\}\!\} + \tfrac{1}{2}[\![u_h]\!]) \\ &= (\beta\cdot\mathrm{n}_F)\tfrac{1}{2}(u_{h|T_1} + u_{h|T_2} + u_{h|T_1} - u_{h|T_2}) \\ &= (\beta\cdot\mathrm{n}_F)u_{h|T_1} \end{aligned}$$

- Standard dG stencil (neighbors in the sense of faces)

Further comments

- L^2-coercivity can be relaxed to $\mu - \frac{1}{2}\nabla{\cdot}\beta \geq 0$
  - assume that there is $\zeta \in W^{1,\infty}(\Omega)$ s.t. $-\beta{\cdot}\nabla\zeta \geq \theta_0 > 0$
  - reasonable if β has no stationary points or closed curves [Devinatz, Ellis & Friedman 74]
- Localized error estimate to avoid global high-order Sobolev norm
  - cut-off functions, exponential decay away from singular layers
  - see [Johnson, Schatz & Wahlbin 87; Guzmán 06]
- Nonlinear conservation laws
  - upwinding promotes Gibbs phenomenon [AE & Guermond 13]
  - needs to add nonlinear stabilization mechanism to temper it

Diffusion-advection-reaction

- Model problem

$$\begin{cases} \mu u + \beta\cdot\nabla u - \nabla\cdot(\kappa\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- Assumptions on the data
  - $f \in L^2(\Omega)$
  - $\beta \in [W^{1,\infty}(\Omega)]^d$, $\mu \in L^\infty(\Omega)$, $\mu - \frac{1}{2}\nabla\cdot\beta \geq \mu_0 > 0$
  - κ scalar-valued, bounded, uniformly positive
- Local Péclet number $\mathrm{Pe}_T = \frac{\beta_T h_T}{\kappa_T}$ for all $T \in \mathcal{T}_h$
  - $\mathrm{Pe}_T \leq 1$: diffusion-dominated regime
  - $\mathrm{Pe}_T \geq 1$: advection-dominated regime
  - more generally, $\mathrm{Pe}_T = \frac{h_T^2}{\tau_T \kappa_T}$ with $\tau_T = \min(\mu_0^{-1}, \beta_T^{-1} h_T)$

Discrete setting

- Discrete space $V_h := \mathbb{P}_d^k(\mathcal{T}_h)$, $k \geq 1$
- Combine SWIP with upwind fluxes
  - centered fluxes can be used in diffusion-dominated regime
  - Scharfetter–Gummel-type weights can be used as well
- Discrete bilinear form (we drop the symmetry term and integrate by parts the advective derivative)

$$
\begin{aligned}
a_h(v_h, w_h) = & \int_\Omega (\mu - \nabla\cdot\beta) v_h w_h - v_h(\beta\cdot\nabla_h w_h) + \kappa \nabla_h v_h \cdot \nabla_h w_h \\
& - \sum_{F \in \mathcal{F}_h} \int_F (\{\!\{\kappa \nabla_h v_h\}\!\}_\omega + \beta \{\!\{v_h\}\!\}) \cdot \mathrm{n}_F [\![w_h]\!] \\
& + \sum_{F \in \mathcal{F}_h} \int_F \gamma_{\kappa,\beta,F} [\![v_h]\!] [\![w_h]\!]
\end{aligned}
$$

with $\gamma_{\kappa,\beta,F} = \eta \frac{\gamma_{\kappa,F}}{h_F} + \frac{1}{2}|\beta\cdot\mathrm{n}_F|$ if $F \in \mathcal{F}_h^i$ (or $\gamma_{\kappa,\beta,F} = \ldots + (\beta\cdot\mathrm{n}_F)^\ominus$ if $F \in \mathcal{F}_h^b$)

Error analysis

- Stability norm

$$\|v_h\|_{\mathrm{dG}}^2 := \sum_{T\in\mathcal{T}_h}\left(\mu_0\|v_h\|_{L^2(T)}^2 + \beta_T^{-1}h_T\|\beta\cdot\nabla v_h\|_{L^2(T)}^2 + \kappa_T\|\nabla v_h\|_{[L^2(T)]^d}^2\right) + \sum_{F\in\mathcal{F}_h}\gamma_{\kappa,\beta,F}\|[\![v_h]\!]\|_{L^2(F)}^2$$

- Main steps of error analysis
  - strong consistency
  - discrete inf-sup stability [technical difficulty for anisotropic κ]
  - boundedness in suitable $\|\cdot\|_{\mathrm{dG},\sharp}$-norm

- Error estimate for smooth solution

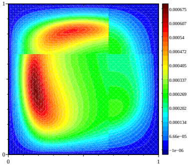
$$\|u-u_h\|_{\mathrm{dG}} \le C\left(\sum_{T\in\mathcal{T}_h}(\mu_0 h_T^2 + \beta_T h_T + \kappa_T)h_T^{2k}|u|_{H^{k+1}(T)}^2\right)^{1/2}$$

  - expected decay in both diffusion- and advection-dominated regimes

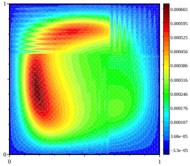
Numerical illustrations

- Rotating advective field [AE, Stephansen & Zunino 09]
  - strong x- or y-diffusion, anisotropy ratio 10^6
  - SIP+upw enforces zero jumps in under-resolved layers

SWIP+upw



SIP+upw



- Constant advective field with locally zero anisotropic diffusion [Di Pietro, AE & Guermond 08]

$$\beta = (-1, 0)$$

$$\kappa_{|\Omega_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$\kappa_{|\Omega_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$