Implementation of dG methods for diffusion problems: Interior penalty and BR2 schemes

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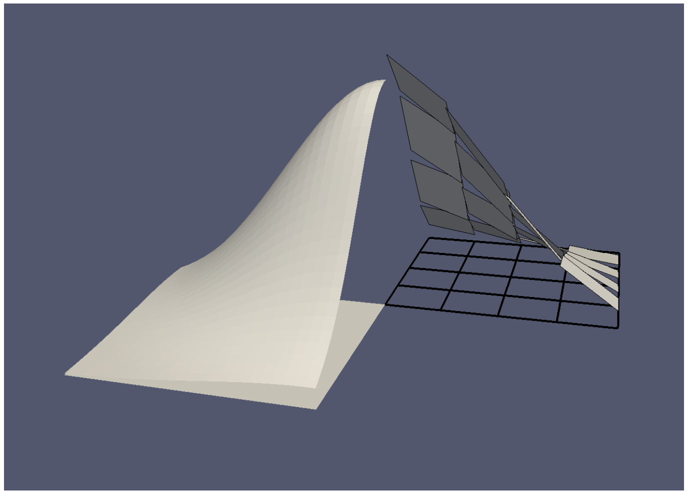
Broken polynomials.

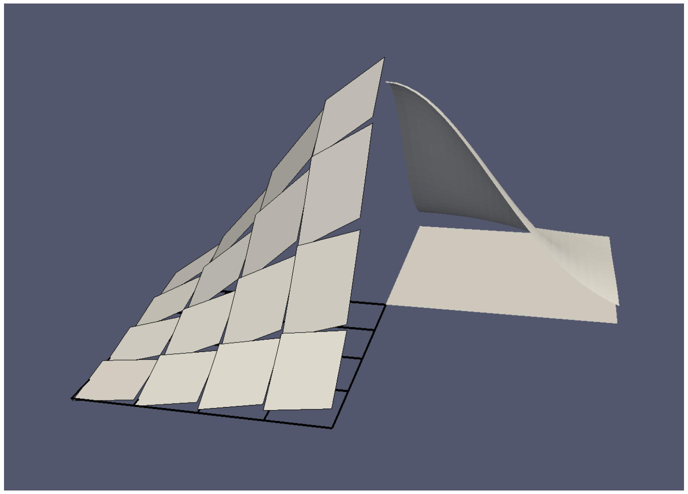
$\mathbb{P}^k_d(\mathcal{T}_h) = \{v \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h, v|_T \in \mathbb{P}^k_d(T)\}$

For example $\Omega = [0,1]^2$, 16 quad elems mesh $\mathcal{T}_h$.
First degree polynomials in each mesh element $\mathbb{P}^1_2(\mathcal{T}_h)$.

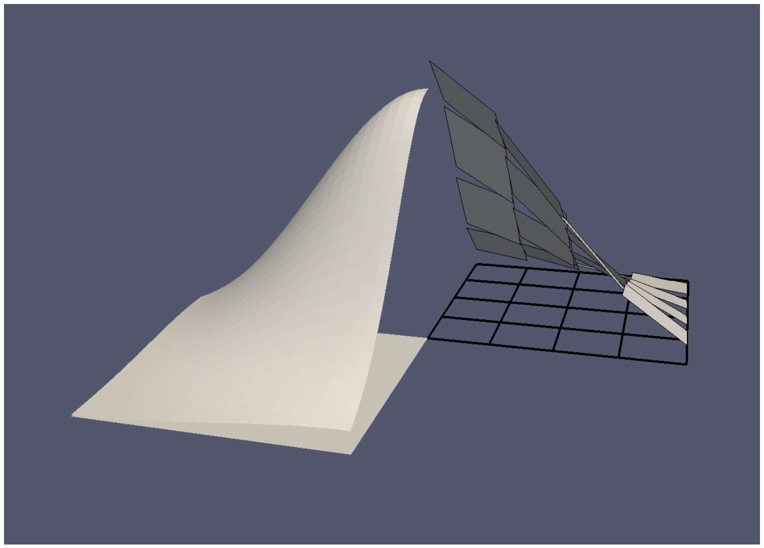
Poisson problem, exact solution $u = e^{-2.5((x-1)^2+(y-1)^2)}$

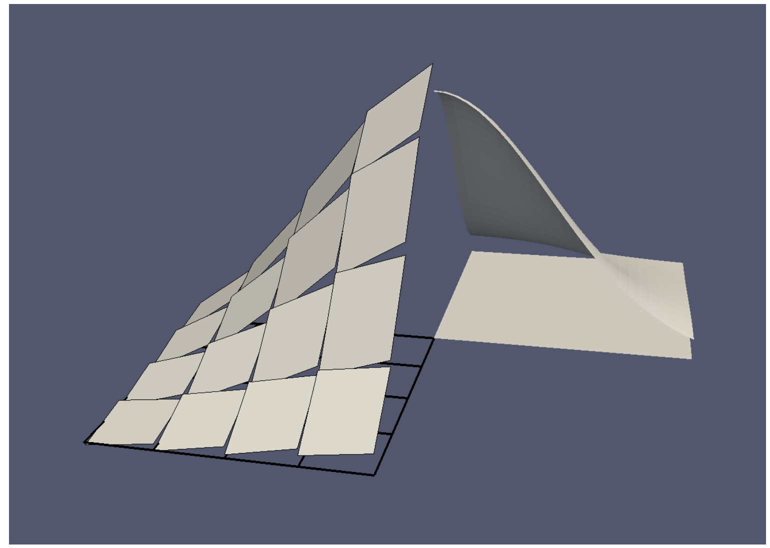
$$\begin{cases} -\triangle u &= f, \text{ in } \Omega, \\ u &= g, \text{ on } \partial\Omega. \end{cases}$$





Average and jump operators





v admits a two-valued trace at interelement boundaries.

We define the average and jump operators over a face $F \in \mathcal{F}_h^i$.

$$\{\!\{ v \}\!\}_F = \frac{1}{2}\left(v|_{T_1} + v|_{T_2} \right), \quad [\![v]\!]_F = \left(v|_{T_1} - v|_{T_2} \right).$$

On boundary faces $F \in \mathcal{F}_h^b$, $\{\!\{ v \}\!\}_F = [\![v]\!]_F = v|_T$.

IP: Interior penalty [Arnold 1982].

$$a_h^{\mathrm{IP}}(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h u_h\}\!\} \cdot \mathbf{n}_F [\![v_h]\!] + [\![u_h]\!] \{\!\{\nabla_h v_h\}\!\} \cdot \mathbf{n}_F + \sum_{F\in\mathcal{F}_h} \frac{\eta_F}{h_{T,F}} \int_F [\![u_h]\!] [\![v_h]\!]$$

It is not trivial to choose η_F on general meshes. We set $\eta_F = k^2 N_\partial$, where N_∂ is the maximum number of faces of the elements sharing F and k is the polynomial degree.

Moreover it is convenient to set $h_{T,F} = 0.5\frac{|T_1|_d + |T_2|_d}{|F|_{d-1}}$ where $|\cdot|_d$ is the d-dimensional measure and T_1, T_2 are the elements sharing F.

IP: Solving a Poisson problem. The discrete operator A^{IP} resulting from an IP dG discreization reads

$$(A_h^{\mathrm{IP}} u_h, v_h)_{L^2(\Omega)} = a_h^{\mathrm{IP}}(u_h, v_h), \quad \forall u_h, v_h \in \mathbb{P}^k(\mathcal{T}_h)$$

Solving a Poisson problem by means of the IP method requires to find $u_h \in \mathbb{P}^k(\mathcal{T}_h)$ such that

$$A_h^{\mathrm{IP}} u_h = \pi_{\mathcal{T}_h}^k f, \tag{1}$$

where $\pi_{\mathcal{T}_h}^k$ is the $L^2(\Omega)$-orthogonal projection operator.

Basis functions and the algebraic problem. Inside the generic element $T \in \mathcal{T}_h$ the unknown u_h can be expressed as a linear combination of the basis functions

$$\forall T \in \mathcal{T}_h \quad u_h|_T = \sum_{j=1}^{N_{\text{dof}}} \widehat{u}_j^T \varphi_j^T$$

where the coefficients

$$\widehat{\mathbf{u}} = \{\widehat{u}_j^T\}_{j=1,\dots,N_{\text{dof}}}$$

are the degrees of freedom of the polynomial expansion.

Having introduced the basis functions for the space $\mathbb{P}^k(\mathcal{T}_h)$, problem 1 reads: find $\widehat{\mathbf{u}}$ such that

$$\mathbf{A}^{\text{IP}}\widehat{\mathbf{u}} = \mathbf{f}, \tag{2}$$

where $\mathbf{A}^{\text{IP}}$ is the global matrix.

Global system matrix

1. The global system matrix $\mathbf{A}^{\mathrm{IP}}$ has a blocked structure.
2. Each (blocked)-row is associated to an element $T \in \mathcal{T}_h$.
3. The number of off-diagonal blocks is equal to the number of neighbors of T.

Global matrix assembly

The local contributions are build and transferred to the appropriate row of the global sistem matrix.

IP: Local contributions. Let us split the bilinear form in volumic, interface and boundary face contributions. Let $\mathcal{F}_h = \mathcal{F}_h^i \cup \mathcal{F}_h^b$, we have

$$a_h^{\text{IP}}(u_h, v_h) = a_h^{\text{IP},V}(u_h, v_h) + a_h^{\text{IP},IF}(u_h, v_h) + a_h^{\text{IP},BF}(u_h, v_h)$$

$$a_h^{\text{IP},V}(u_h, v_h) = \sum_{T \in \mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h$$

$$a_h^{\text{IP},IF}(u_h, v_h) = -\sum_{F \in \mathcal{F}_h^i} \int_F \{\{\nabla_h u_h\}\} \cdot \mathbf{n}_F \, [[v_h]] + [[u_h]] \, \{\{\nabla_h v_h\}\} \cdot \mathbf{n}_F$$

$$+ \sum_{F \in \mathcal{F}_h^i} \frac{\eta_F}{h_{T,F}} \int_F [[u_h]] \, [[v_h]]$$

$$a_h^{\text{IP},BF}(u_h, v_h) = \sum_{F \in \mathcal{F}_h^b} \int_F \left[\frac{\eta_F}{h_{T,F}} u_h \, v_h - (\nabla_h u_h \cdot \mathbf{n} \, v_h + u_h \, \nabla_h v_h \cdot \mathbf{n}) \right]$$

IP: Local contributions.

Each summation involves a loop over the corresponding mesh entities.

Each mesh entity provide its local contributions (dense matrices of size $N_{\text{dof}} \times N_{\text{dof}}$).

for $T \in \mathcal{T}_h$ **do**
  assemble $(\mathbf{A}^{V}_{T,T})_{i,j} = a_h^{\text{IP,V}}(\varphi_i^T, \varphi_j^T)$ {contribution to the diagonal blocks}
for $F \in \mathcal{F}_h^i$ **do**
  find T_1, T_2 such that $F = \partial T_1 \cap \partial T_2$
  assemble $(\mathbf{A}^{IF}_{T_1,T_1})_{i,j} = a_h^{\text{IP,IF}}(\varphi_i^{T_1}, \varphi_j^{T_1})$ {contribution to the diagonal blocks}
  assemble $(\mathbf{A}^{IF}_{T_2,T_2})_{i,j} = a_h^{\text{IP,IF}}(\varphi_i^{T_2}, \varphi_j^{T_2})$ {contribution to the diagonal blocks}
  assemble $(\mathbf{A}^{IF}_{T_1,T_2})_{i,j} = a_h^{\text{IP,IF}}(\varphi_i^{T_1}, \varphi_j^{T_2})$ {contribution to the off-diagonal blocks}
  assemble $(\mathbf{A}^{IF}_{T_2,T_1})_{i,j} = a_h^{\text{IP,IF}}(\varphi_i^{T_2}, \varphi_j^{T_1})$ {contribution to the off-diagonal blocks}
for $F \in \mathcal{F}_h^b$ **do**
  find T such that $F = \partial T \cap \partial\Omega$
  assemble $(\mathbf{A}^{BF}_{T,T})_{i,j} = a_h^{\text{IP,BF}}(\varphi_i^T, \varphi_j^T)$ {contribution to the diagonal blocks}

IP: Assembly of local contributions.

Consider the consistency term for a face $F \in \mathcal{F}_h^i$

$$\int_F \{\!\{\nabla u_h\}\!\} [\![v_h]\!] \cdot \mathbf{n}_F \to \sum_{i=1}^{N_{\rm dof}} \sum_{j=1}^{N_{\rm dof}} \int_F 0.5 \left(\nabla \varphi_j^{T_1} + \nabla \varphi_j^{T_2}\right) \cdot \mathbf{n}_F \; \left(\varphi_i^{T_1} - \varphi_i^{T_2}\right)$$

Its local contribution are

$$(\mathbf{A}_{T_1,T_1}^{IF,cnst})_{i,j} = +0.5 \int_F \varphi_i^{T_1} \, \nabla \varphi_j^{T_1} \cdot \mathbf{n}_F \quad \text{(diagonal block, row } T_1\text{)}$$

$$(\mathbf{A}_{T_2,T_2}^{IF,cnst})_{i,j} = -0.5 \int_F \varphi_i^{T_2} \, \nabla \varphi_j^{T_2} \cdot \mathbf{n}_F \quad \text{(diagonal block, row } T_2\text{)}$$

$$(\mathbf{A}_{T_1,T_2}^{IF,cnst})_{i,j} = +0.5 \int_F \varphi_i^{T_1} \, \nabla \varphi_j^{T_2} \cdot \mathbf{n}_F \quad \text{(off-diagonal block, row } T_1\text{, col } T_2\text{)}$$

$$(\mathbf{A}_{T_2,T_1}^{IF,cnst})_{i,j} = -0.5 \int_F \varphi_i^{T_2} \, \nabla \varphi_j^{T_1} \cdot \mathbf{n}_F \quad \text{(off-diagonal block, row } T_2\text{, col } T_1\text{)}$$

IP: Assembly of local contributions.

Consider the symmetry term for a face $F \in \mathcal{F}_h^i$

$$\int_F [\![u_h]\!] \{\!\{\nabla v_h\}\!\} \cdot \mathbf{n}_F \to \sum_{i=1}^{N_{\text{dof}}} \sum_{j=1}^{N_{\text{dof}}} \int_F \left(\varphi_j^{T_1} - \varphi_j^{T_2}\right) 0.5 \left(\nabla \varphi_i^{T_1} + \nabla \varphi_i^{T_2}\right) \cdot \mathbf{n}_F$$

Its local contribution are

$$(\mathbf{A}_{T_1,T_1}^{IF,symm})_{i,j} = +0.5 \int_F \nabla \varphi_i^{T_1} \cdot \mathbf{n}_F \, \varphi_j^{T_1} \quad \text{(diagonal block, row } T_1)$$

$$(\mathbf{A}_{T_2,T_2}^{IF,symm})_{i,j} = -0.5 \int_F \nabla \varphi_i^{T_2} \cdot \mathbf{n}_F \, \varphi_j^{T_2} \quad \text{(diagonal block, row } T_2)$$

$$(\mathbf{A}_{T_1,T_2}^{IF,symm})_{i,j} = -0.5 \int_F \nabla \varphi_i^{T_1} \cdot \mathbf{n}_F \, \varphi_j^{T_2} \quad \text{(off-diagonal block, row } T_1\text{, col } T_2)$$

$$(\mathbf{A}_{T_2,T_1}^{IF,symm})_{i,j} = +0.5 \int_F \nabla \varphi_i^{T_2} \cdot \mathbf{n}_F \, \varphi_j^{T_1} \quad \text{(off-diagonal block, row } T_2\text{, col } T_1)$$

IP: Assembly of local contributions.

Consider the penalty term for a face $F \in \mathcal{F}_h^i$

$$\frac{\eta_F}{h_{T,F}} \int_F [\![u_h]\!]\,[\![v_h]\!] \to \sum_{i=1}^{N_{\mathrm{dof}}} \sum_{j=1}^{N_{\mathrm{dof}}} \frac{\eta_F}{h_{T,F}} \int_F \left(\varphi_j^{T_1} - \varphi_j^{T_2}\right)\,\left(\varphi_i^{T_1} - \varphi_i^{T_2}\right)$$

Its local contribution are

$$(\mathbf{A}_{T_1,T_1}^{IF,stab})_{i,j} = +\frac{\eta_F}{h_{T,F}} \int_F \varphi_i^{T_1}\,\varphi_j^{T_1} \quad \text{(diagonal block, row } T_1\text{)}$$

$$(\mathbf{A}_{T_2,T_2}^{IF,stab})_{i,j} = +\frac{\eta_F}{h_{T,F}} \int_F \varphi_i^{T_2}\,\varphi_j^{T_2} \quad \text{(diagonal block, row } T_2\text{)}$$

$$(\mathbf{A}_{T_1,T_2}^{IF,stab})_{i,j} = -\frac{\eta_F}{h_{T,F}} \int_F \varphi_i^{T_1}\,\varphi_j^{T_2} \quad \text{(off-diagonal block, row } T_1\text{, col } T_2\text{)}$$

$$(\mathbf{A}_{T_2,T_1}^{IF,stab})_{i,j} = -\frac{\eta_F}{h_{T,F}} \int_F \varphi_i^{T_2}\,\varphi_j^{T_1} \quad \text{(off-diagonal block, row } T_2\text{, col } T_1\text{)}$$

BR2 [Bassi and Rebay 1997].

$$a_h^{\mathrm{BR2}}(u_h, v_h) := \int_\Omega \left(\boldsymbol{\nabla}_h u_h - \mathbf{R}_h^k(u_h)\right) \cdot \left(\boldsymbol{\nabla}_h v_h - \mathbf{R}_h^k(v_h)\right) - \int_\Omega \mathbf{R}_h^k(u_h) \cdot \mathbf{R}_h^k(v_h)$$
$$+ \sum_{F \in \mathcal{F}_h} \eta_F \int_\Omega \mathbf{r}_F^k([\![u_h]\!]) \cdot \mathbf{r}_F^k([\![v_h]\!])$$

For all $F \in \mathcal{F}_h$ the (local) lifting operator $\mathbf{r}_F^k : L^2(F) \to [\mathbb{P}^k(\mathcal{T}_h)]^d$, reads

$$\int_\Omega \mathbf{r}_F^k(\phi) \cdot \boldsymbol{\tau}_h = \int_F \phi \{\!\{\boldsymbol{\tau}_h\}\!\} \cdot \mathbf{n}_F, \qquad \forall \boldsymbol{\tau}_h \in [\mathbb{P}^k(\mathcal{T}_h)]^d, \phi \in L^2(F).$$

The global lifting is $\mathbf{R}_h^k(v_h) := \sum_{F \in \mathcal{F}_h} \mathbf{r}_F^k([\![v_h]\!]), \qquad \forall\, [\![v_h]\!] \in \mathbb{P}^k(\mathcal{T}_h).$

η_F must be greater than the number of faces of the elements sharing F.

BR2 [Bassi and Rebay 1997].

Using the global lifting operator definition the bilinear form can be recasted as follows

$$a_h^{\mathrm{BR2}}(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h + \sum_{F \in \mathcal{F}_h} \eta_F \int_\Omega \mathbf{r}_F^k([\![u_h]\!]) \cdot \mathbf{r}_F^k([\![v_h]\!])$$
$$- \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h u_h\}\!\} \cdot \mathbf{n}_F \, [\![v_h]\!] + [\![u_h]\!] \, \{\!\{\nabla_h v_h\}\!\} \cdot \mathbf{n}_F$$

$$a_h^{\mathrm{IP}}(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h + \sum_{F \in \mathcal{F}_h} \frac{\eta_F}{h_{T,F}} \int_F [\![u_h]\!] \, [\![v_h]\!]$$
$$- \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h u_h\}\!\} \cdot \mathbf{n}_F \, [\![v_h]\!] + [\![u_h]\!] \, \{\!\{\nabla_h v_h\}\!\} \cdot \mathbf{n}_F$$

Only the stabilization terms differ.

BR2 Stabilization term local contribution.

Consider a face $F \in \mathcal{F}_h^i$

$$\int_\Omega \mathbf{r}_F([\![u_h]\!]) \cdot \mathbf{r}_F([\![v_h]\!]) \to \sum_{i=1}^{N_{\text{dof}}} \sum_{j=1}^{N_{\text{dof}}} \int_{T_1} \mathbf{r}_F^1 \left(\varphi_j^{T_1} - \varphi_j^{T_2} \right) \cdot \mathbf{r}_F^1 \left(\varphi_i^{T_1} - \varphi_i^{T_2} \right) + \sum_{i=1}^{N_{\text{dof}}} \sum_{j=1}^{N_{\text{dof}}} \int_{T_2} \mathbf{r}_F^2 \left(\varphi_j^{T_1} - \varphi_j^{T_2} \right) \cdot \mathbf{r}_F^2 \left(\varphi_i^{T_1} - \varphi_i^{T_2} \right)$$

or, using the local lifting definition

$$\int_\Omega \mathbf{r}_F([\![u_h]\!]) \cdot \mathbf{r}_F([\![v_h]\!]) = \int_F \{\!\{ \mathbf{r}_F([\![u_h]\!]) \}\!\} \cdot \mathbf{n}_F [\![v_h]\!] \to$$

$$\sum_{i=1}^{N_{\text{dof}}} \sum_{j=1}^{N_{\text{dof}}} \int_{T_1} 0.5 \left[\mathbf{r}_F^1 \left(\varphi_j^{T_1} - \varphi_j^{T_2} \right) + \mathbf{r}_F^2 \left(\varphi_j^{T_1} - \varphi_j^{T_2} \right) \right] \cdot \mathbf{n}_F \quad \left(\varphi_i^{T_1} - \varphi_i^{T_2} \right)$$

From the local lifting defintion we have

$$\int_\Omega \mathbf{r}_F[\![u_h]\!] \cdot \boldsymbol{\tau}_h = \int_F [\![u_h]\!]\{\!\{\boldsymbol{\tau}_h\}\!\} \cdot \mathbf{n}_F = 0.5 \int_F \left[u_h^{T_1} - u_h^{T_2}\right] \left[\boldsymbol{\tau}_h^{T_1} + \boldsymbol{\tau}_h^{T_2}\right] \cdot \mathbf{n}_F$$

$$\begin{aligned}
(\mathbf{R}^{11,d})_{ij} &= \int_{T_1} r_F^{11,d}\left(\varphi_j^{T_1}\right)\varphi_i^{T_1} &= 0.5\int_F \varphi_i^{T_1}\varphi_j^{T_1} n_{F,d}\\
(\mathbf{R}^{22,d})_{ij} &= \int_{T_2} r_F^{22,d}\left(-\varphi_j^{T_2}\right)\varphi_i^{T_2} &= -0.5\int_F \varphi_i^{T_2}\varphi_j^{T_2} n_{F,d}\\
(\mathbf{R}^{12,d})_{ij} &= \int_{T_1} r_F^{12,d}\left(-\varphi_j^{T_2}\right)\varphi_i^{T_1} &= -0.5\int_F \varphi_i^{T_1}\varphi_j^{T_2} n_{F,d}\\
(\mathbf{R}^{21,d})_{ij} &= \int_{T_2} r_F^{21,d}\left(\varphi_j^{T_1}\right)\varphi_i^{T_2} &= 0.5\int_F \varphi_i^{T_2}\varphi_j^{T_1} n_{F,d}
\end{aligned}$$

We need to solve four local problems in the form $\mathbf{M}\widehat{\mathbf{R}} = \mathbf{R}$.

Not required for orthonormal basis functions.

BR2 stabilization: Face integral form

$$\int_F \{\{\mathbf{r}_F([\![u_h]\!])\}\} \cdot \mathbf{n}_F [\![v_h]\!]$$

The local contribution are

$$(\mathbf{A}^{IF,stab}_{T_1,T_1})_{i,j} = 0.5 \int_F \left[\mathbf{r}^{11}_F \left(\varphi_j^{T_1}\right) + \mathbf{r}^{21}_F \left(\varphi_j^{T_1}\right)\right] \cdot \mathbf{n}_F \;\; \varphi_i^{T_1}$$

$$(\mathbf{A}^{IF,stab}_{T_2,T_2})_{i,j} = 0.5 \int_F \left[\mathbf{r}^{12}_F \left(-\varphi_j^{T_2}\right) + \mathbf{r}^{22}_F \left(-\varphi_j^{T_2}\right)\right] \cdot \mathbf{n}_F \;\; \left(-\varphi_i^{T_2}\right)$$

$$(\mathbf{A}^{IF,stab}_{T_1,T_2})_{i,j} = 0.5 \int_F \left[\mathbf{r}^{12}_F \left(-\varphi_j^{T_2}\right) + \mathbf{r}^{22}_F \left(-\varphi_j^{T_2}\right)\right] \cdot \mathbf{n}_F \;\; \varphi_i^{T_1}$$

$$(\mathbf{A}^{IF,stab}_{T_2,T_1})_{i,j} = 0.5 \int_F \left[\mathbf{r}^{11}_F \left(\varphi_j^{T_1}\right) + \mathbf{r}^{21}_F \left(\varphi_j^{T_1}\right)\right] \cdot \mathbf{n}_F \;\; \left(-\varphi_i^{T_2}\right)$$

Since

$$r_F^{11,d}\left(\varphi_j^{T_1}\right) = \mathbf{R}_{i,j}^{11,d}\varphi_i^{T_1}, \qquad r_F^{22,d}\left(-\varphi_j^{T_2}\right) = \mathbf{R}_{i,j}^{22,d}\varphi_i^{T_2}$$

$$r_F^{12,d}\left(-\varphi_j^{T_2}\right) = \mathbf{R}_{i,j}^{12,d}\varphi_i^{T_1}, \qquad r_F^{21,d}\left(\varphi_j^{T_1}\right) = \mathbf{R}_{i,j}^{21,d}\varphi_i^{T_2}$$

we get

$$(\mathbf{A}_{T_1,T_1}^{IF,stab})_{i,j} = 0.5\sum_d \int_F \left[\mathbf{R}_{q,j}^{11,d}\ \varphi_q^{T_1} + \mathbf{R}_{q,j}^{21,d}\ \varphi_q^{T_2}\right]\ n_{F,d}\ \ \varphi_i^{T_1}$$

$$(\mathbf{A}_{T_2,T_2}^{IF,stab})_{i,j} = -0.5\sum_d \int_F \left[\mathbf{R}_{q,j}^{12,d}\ \varphi_q^{T_1} + \mathbf{R}_{q,j}^{22,d}\ \varphi_q^{T_2}\right]\ n_{F,d}\ \ \varphi_i^{T_2}$$

$$(\mathbf{A}_{T_1,T_2}^{IF,stab})_{i,j} = 0.5\sum_d \int_F \left[\mathbf{R}_{q,j}^{12,d}\ \varphi_q^{T_1} + \mathbf{R}_{q,j}^{22,d}\ \varphi_q^{T_2}\right]\ n_{F,d}\ \ \varphi_i^{T_1}$$

$$(\mathbf{A}_{T_2,T_1}^{IF,stab})_{i,j} = -0.5\sum_d \int_F \left[\mathbf{R}_{q,j}^{11,d}\ \varphi_q^{T_1} + \mathbf{R}_{q,j}^{21,d}\ \varphi_q^{T_2}\right]\ n_{F,d}\ \ \varphi_i^{T_2}$$

In tensor notation

$$\mathbf{A}^{IF,stab}_{T_1,T_1} = 0.5 \sum_d \int_F \left[\left(\mathbf{R}^{11,d}\right)^t \varphi^{T_1} n_{F,d} + \left(\mathbf{R}^{21,d}\right)^t \varphi^{T_2} n_{F,d} \right] \otimes \varphi^{T_1}$$

$$\mathbf{A}^{IF,stab}_{T_2,T_2} = -0.5 \sum_d \int_F \left[\left(\mathbf{R}^{12,d}\right)^t \varphi^{T_1} n_{F,d} + \left(\mathbf{R}^{22,d}\right)^t \varphi^{T_2} n_{F,d} \right] \otimes \varphi^{T_2}$$

$$\mathbf{A}^{IF,stab}_{T_1,T_2} = 0.5 \sum_d \int_F \left[\left(\mathbf{R}^{12,d}\right)^t \varphi^{T_1} n_{F,d} + \left(\mathbf{R}^{22,d}\right)^t \varphi^{T_2} n_{F,d} \right] \otimes \varphi^{T_1}$$

$$\mathbf{A}^{IF,stab}_{T_2,T_1} = -0.5 \sum_d \int_F \left[\left(\mathbf{R}^{11,d}\right)^t \varphi^{T_1} n_{F,d} + \left(\mathbf{R}^{21,d}\right)^t \varphi^{T_2} n_{F,d} \right] \otimes \varphi^{T_2}$$

BR2 stabilization: Volume integral form

$$\int_{\Omega} \mathbf{r}_F([\![u_h]\!]) \cdot \mathbf{r}_F([\![v_h]\!])$$

The local contribution are

$$(\mathbf{A}^{IF,stab}_{T_1,T_1})_{i,j} = \int_{T_1} \mathbf{r}^{11}_F\left(\varphi_i^{T_1}\right) \cdot \mathbf{r}^{11}_F\left(\varphi_j^{T_1}\right) + \int_{T_2} \mathbf{r}^{21}_F\left(\varphi_i^{T_1}\right) \cdot \mathbf{r}^{21}_F\left(\varphi_j^{T_1}\right)$$

$$(\mathbf{A}^{IF,stab}_{T_2,T_2})_{i,j} = \int_{T_1} \mathbf{r}^{12}_F\left(-\varphi_i^{T_2}\right) \cdot \mathbf{r}^{12}_F\left(-\varphi_j^{T_2}\right) + \int_{T_2} \mathbf{r}^{22}_F\left(-\varphi_i^{T_2}\right) \cdot \mathbf{r}^{22}_F\left(-\varphi_j^{T_2}\right)$$

$$(\mathbf{A}^{IF,stab}_{T_1,T_2})_{i,j} = \int_{T_1} \mathbf{r}^{11}_F\left(\varphi_i^{T_1}\right) \cdot \mathbf{r}^{12}_F\left(-\varphi_j^{T_2}\right) + \int_{T_2} \mathbf{r}^{21}_F\left(\varphi_i^{T_1}\right) \cdot \mathbf{r}^{22}_F\left(-\varphi_j^{T_2}\right)$$

$$(\mathbf{A}^{IF,stab}_{T_2,T_1})_{i,j} = \int_{T_1} \mathbf{r}^{12}_F\left(-\varphi_i^{T_2}\right) \cdot \mathbf{r}^{11}_F\left(\varphi_j^{T_1}\right) + \int_{T_2} \mathbf{r}^{22}_F\left(-\varphi_i^{T_2}\right) \cdot \mathbf{r}^{21}_F\left(\varphi_j^{T_1}\right)$$

Since

$$r_F^{11,d}\left(\varphi_j^{T_1}\right) = \mathbf{R}_{i,j}^{11,d}\varphi_i^{T_1}, \qquad r_F^{22,d}\left(-\varphi_j^{T_2}\right) = \mathbf{R}_{i,j}^{22,d}\varphi_i^{T_2}$$

$$r_F^{12,d}\left(-\varphi_j^{T_2}\right) = \mathbf{R}_{i,j}^{12,d}\varphi_i^{T_1}, \qquad r_F^{21,d}\left(\varphi_j^{T_1}\right) = \mathbf{R}_{i,j}^{21,d}\varphi_i^{T_2}$$

we get

$$(\mathbf{A}_{T_1,T_1}^{IF,stab})_{i,j} = \sum_d \int_{T_1} \mathbf{R}_{p,i}^{11,d}\varphi_p^{T_1} \; \mathbf{R}_{q,j}^{11,d}\varphi_q^{T_1} + \sum_d \int_{T_2} \mathbf{R}_{p,i}^{21,d}\varphi_p^{T_2} \; \mathbf{R}_{q,j}^{21,d}\varphi_q^{T_2}$$

$$(\mathbf{A}_{T_2,T_2}^{IF,stab})_{i,j} = \sum_d \int_{T_1} \mathbf{R}_{p,i}^{12,d}\varphi_p^{T_1} \; \mathbf{R}_{q,j}^{12,d}\varphi_q^{T_1} + \sum_d \int_{T_2} \mathbf{R}_{p,i}^{22,d}\varphi_p^{T_2} \; \mathbf{R}_{q,j}^{22,d}\varphi_q^{T_2}$$

$$(\mathbf{A}_{T_1,T_2}^{IF,stab})_{i,j} = \sum_d \int_{T_1} \mathbf{R}_{p,i}^{11,d}\varphi_p^{T_1} \; \mathbf{R}_{q,j}^{12,d}\varphi_q^{T_1} + \sum_d \int_{T_2} \mathbf{R}_{p,i}^{21,d}\varphi_p^{T_2} \; \mathbf{R}_{q,j}^{22,d}\varphi_q^{T_2}$$

$$(\mathbf{A}_{T_2,T_1}^{IF,stab})_{i,j} = \sum_d \int_{T_1} \mathbf{R}_{p,i}^{12,d}\varphi_p^{T_1} \; \mathbf{R}_{q,j}^{11,d}\varphi_q^{T_1} + \sum_d \int_{T_2} \mathbf{R}_{p,i}^{22,d}\varphi_p^{T_2} \; \mathbf{R}_{q,j}^{21,d}\varphi_q^{T_2}$$

Since

$$r_F^{11,d}\left(\varphi_j^{T_1}\right) = \mathbf{R}_{i,j}^{11,d}\varphi_i^{T_1}, \qquad r_F^{22,d}\left(-\varphi_j^{T_2}\right) = \mathbf{R}_{i,j}^{22,d}\varphi_i^{T_2}$$

$$r_F^{12,d}\left(-\varphi_j^{T_2}\right) = \mathbf{R}_{i,j}^{12,d}\varphi_i^{T_1}, \qquad r_F^{21,d}\left(\varphi_j^{T_1}\right) = \mathbf{R}_{i,j}^{21,d}\varphi_i^{T_2}$$

we get

$$(\mathbf{A}_{T_1,T_1}^{IF,stab})_{i,j} = \sum_d \mathbf{R}_{p,i}^{11,d}\mathbf{R}_{q,j}^{11,d}\int_{T_1}\varphi_p^{T_1}\ \ \varphi_q^{T_1} + \sum_d \mathbf{R}_{p,i}^{21,d}\mathbf{R}_{q,j}^{21,d}\int_{T_2}\varphi_p^{T_2}\ \ \varphi_q^{T_2}$$

$$(\mathbf{A}_{T_2,T_2}^{IF,stab})_{i,j} = \sum_d \mathbf{R}_{p,i}^{12,d}\mathbf{R}_{q,j}^{12,d}\int_{T_1}\varphi_p^{T_1}\ \ \varphi_q^{T_1} + \sum_d \mathbf{R}_{p,i}^{22,d}\mathbf{R}_{q,j}^{22,d}\int_{T_2}\varphi_p^{T_2}\ \ \varphi_q^{T_2}$$

$$(\mathbf{A}_{T_1,T_2}^{IF,stab})_{i,j} = \sum_d \mathbf{R}_{p,i}^{11,d}\mathbf{R}_{q,j}^{12,d}\int_{T_1}\varphi_p^{T_1}\ \ \varphi_q^{T_1} + \sum_d \mathbf{R}_{p,i}^{21,d}\mathbf{R}_{q,j}^{22,d}\int_{T_2}\varphi_p^{T_2}\ \ \varphi_q^{T_2}$$

$$(\mathbf{A}_{T_2,T_1}^{IF,stab})_{i,j} = \sum_d \mathbf{R}_{p,i}^{12,d}\mathbf{R}_{q,j}^{11,d}\int_{T_1}\varphi_p^{T_1}\ \ \varphi_q^{T_1} + \sum_d \mathbf{R}_{p,i}^{22,d}\mathbf{R}_{q,j}^{21,d}\int_{T_2}\varphi_p^{T_2}\ \ \varphi_q^{T_2}$$

Since for orthogonal basis functions it holds

$$\int_T \varphi_p^T \varphi_q^T = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$$

$$(\mathbf{A}_{T_1,T_1}^{IF,stab})_{i,j} = \sum_d \mathbf{R}_{q,i}^{11,d} \;\; \mathbf{R}_{q,j}^{11,d} + \sum_d \mathbf{R}_{q,i}^{21,d} \;\; \mathbf{R}_{q,j}^{21,d}$$

$$(\mathbf{A}_{T_2,T_2}^{IF,stab})_{i,j} = \sum_d \mathbf{R}_{q,i}^{12,d} \;\; \mathbf{R}_{q,j}^{12,d} + \sum_d \mathbf{R}_{q,i}^{22,d} \;\; \mathbf{R}_{q,j}^{22,d}$$

$$(\mathbf{A}_{T_1,T_2}^{IF,stab})_{i,j} = \sum_d \mathbf{R}_{q,i}^{11,d} \;\; \mathbf{R}_{q,j}^{12,d} + \sum_d \mathbf{R}_{q,i}^{21,d} \;\; \mathbf{R}_{q,j}^{22,d}$$

$$(\mathbf{A}_{T_2,T_1}^{IF,stab})_{i,j} = \sum_d \mathbf{R}_{q,i}^{12,d} \;\; \mathbf{R}_{q,j}^{11,d} + \sum_d \mathbf{R}_{q,i}^{22,d} \;\; \mathbf{R}_{q,j}^{21,d}$$

In tensor notation

$$\mathbf{A}^{IF,stab}_{T_1,T_1} = \sum_d \left[(\mathbf{R}^{11,d})^t \, \mathbf{R}^{11,d} + (\mathbf{R}^{21,d})^t \, \mathbf{R}^{21,d} \right]$$

$$\mathbf{A}^{IF,stab}_{T_2,T_2} = \sum_d \left[(\mathbf{R}^{12,d})^t \, \mathbf{R}^{12,d} + (\mathbf{R}^{22,d})^t \, \mathbf{R}^{22,d} \right]$$

$$\mathbf{A}^{IF,stab}_{T_1,T_2} = \sum_d \left[(\mathbf{R}^{11,d})^t \, \mathbf{R}^{12,d} + (\mathbf{R}^{21,d})^t \, \mathbf{R}^{22,d} \right]$$

$$\mathbf{A}^{IF,stab}_{T_2,T_1} = \sum_d \left[(\mathbf{R}^{12,d})^t \, \mathbf{R}^{11,d} + (\mathbf{R}^{22,d})^t \, \mathbf{R}^{21,d} \right]$$

Advection

$$a_h^{\text{upw}} = \int_\Omega (\beta \cdot \nabla_h u_h) v_h + \int_{\partial\Omega} (\beta \cdot \mathbf{n})^\ominus u_h v_h$$
$$- \sum_{F \in \mathcal{F}_h^i} \int_F (\beta \cdot \mathbf{n}_F) [\![u_h]\!] \{\!\{v_h\}\!\} + \sum_{F \in \mathcal{F}_h^i} \int_F \frac{\eta}{2} |\beta \cdot \mathbf{n}_F| [\![u_h]\!][\![v_h]\!]$$

with $\eta > 0$ and $x^\ominus = \frac{1}{2}(|x| - x)$.

$$- \int_F (\beta \cdot \mathbf{n}_F) [\![u_h]\!] \{\!\{v_h\}\!\} \to - \int_F (\beta \cdot \mathbf{n}_F) 0.5 (\varphi^{T_1} + \varphi^{T_2}) \otimes (\varphi^{T_1} - \varphi^{T_2})$$

$$\int_F \frac{\eta}{2} |\beta \cdot \mathbf{n}_F| [\![u_h]\!][\![v_h]\!] \to \int_F \frac{\eta}{2} |\beta \cdot \mathbf{n}_F| (\varphi^{T_1} - \varphi^{T_2}) \otimes (\varphi^{T_1} - \varphi^{T_2})$$